Two Way Deterministic Finite Automata are Exponentially More Succinct than Sweeping Automata

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Keywords.

One way deterministic finite automaton (ldfa).

Two way deterministic finite automaton (2dfa).

One way non deterministic finite automaton (lnfa).

Definition

Sweeping automaton (sa): a 2dfa which can halt or change the direction of its head motion only at the ends of the input tape.

Problem Setup and Main Result.

This note is a refinement of a work of M. Sipser [1]. His main result is:

(*) For all \( n \) there is a language \( B_n \) which is accepted by an \( n \)-state lnfa but not by any sa with less than \( 2^n - 1 \) states.

i.e. lnfa are exponentially more succinct than sa. We add the following contribution:

(**) 2dfa are are exponentially more succinct than sa.

The following lemma is fundamental in our proof.

Lemma: Let \( L \) be a language on a finite alphabet \( \Sigma \) such that

1) If \( w \) belongs to \( L \) then all substrings of \( w \) belong to \( L \).

2) In \( L \) there is at least one word \( x \) such that for all words \( u \) and \( v \) in \( L \) \( uxv \) belongs to \( L \).

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3) There exists a string $d$ over such that

i) $d$ has length $2^n$

ii) $d$ does not belong to $L$

iii) After the removal of any of its non-empty substrings $d$ belongs to $L$.

Then $L$ cannot be accepted by an $sa$ with less than $2^n - 1$ states.

PROOF: The proof can be carried out by following, step by step, the demonstration given in [1] that $B_n$ cannot be accepted by an $sa$ with less than $2^n - 1$ states. In fact properties (1), (2) and (3) hold for $B_n$ and they are the only features of $B_n$ involved in that proof.

In order to prove (***) we define the language $A_n$. Say that a bipartite graph is of type 1 if no two edges meet at a right node and of type 2 if it is a complete bipartite graph. The alphabet $\Sigma_n$ of $A_n$ consists of all bipartite graphs satisfying these two properties:

i) The graphs have $n$ left nodes and $n$ right nodes

ii) The graphs are either of type 1 or of type 2. A sequence of such symbols constitutes a string by identifying right and left nodes of adjacent bipartite graphs.

\[
\begin{array}{c}
\begin{array}{c}
\text{some members of } \Sigma_3 \\
\end{array} \\
\begin{array}{c}
\text{a string over } \Sigma_3 \\
\end{array}
\end{array}
\]

A string $s$ on $\Sigma_n$ is a word of $A_n$ iff

a) There is a path leading from a leftmost node of $s$ to a rightmost one.

Definition: A chain of a string $s$ is a maximal substring of $s$ consisting of symbols of type 1.

Say a chain is good iff there is a path from one of its leftmost nodes to one of its rightmost nodes. Then $s$
belongs to \( A_n \) iff

b) all chains of \( s \) are good.

Note that a chain is circuit free and thus (b) can be checked using depth first search by a 2dfa with \( O(n^2) \) states. We summarize these observations in the following theorem 1.

**THEOREM 1:** \( A_n \) is accepted by an \( O(n^2) \)-state 2dfa.

**THEOREM 2:** \( A_n \) cannot be accepted by an sa with less than \( 2^n - 1 \) states.

**PROOF:** We show that \( A_n \) satisfies the conditions of the lemma. Let \( w \) belong to \( A_n \) and \( s \) be a substring of \( w \). As there is a path from a left most node of \( w \) to a rightmost one, such a path must also connect a leftmost node of \( s \) with a rightmost node of \( s \). Thus property (1) holds for \( A_n \). The word consisting of a single complete bipartite graph is a valid \( x \) for property (2). Let's now construct \( d_n \) (a valid \( d \) for \( A_n \)): write down \( 2^n \) columns of \( n \) nodes numbered 1 through \( n \) (top-down). Order the subsets of \( In=\{1,\ldots,n\} \) first by cardinality and then lexicographically. In the \( i \)th column from the left, mark the nodes that correspond to the \( i \)th subset of \( In \). For \( i=1 \) to \( 2^n - 1 \), connect the 1st marked node of column \( i \) with the 1st marked node of column \( i+1 \), 2nd with 2nd and so on; let last marked node connect to all remaining marked nodes (at most one) of column \( i+1 \). For unmarked nodes, connect the last of them in column \( i \) with all unmarked nodes of column \( i+1 \). \( d \) for \( n=3 \) looks as follows

![Diagram of \( d \) for \( n=3 \)](image)

\( d_n \) has length \( 2^n \). In \( d_n \) no path runs from a leftmost node to a rightmost one but the removal of any non empty substring will create one. Thus \( d_n \) has all the properties required in (3); this completes the proof.

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**References**

[1] M. Sipser, Lower bounds on the size of sweeping automata,
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