SOJOURN TIMES AND THE OVERTAKING CONDITION
IN JACKSONIAN NETWORKS

by

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ABSTRACT

Consider an open multiclass Jacksonian network in equilibrium and a path such that a customer travelling along it cannot be overtaken by subsequent arrivals. Then the sojourn times of this customer in the nodes constituting the path are all mutually independent and so the total sojourn time is easily calculated. Two examples are given to suggest that the non-overtaking condition may be necessary to insure independence when there is a single customer class.

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1. Introduction

In 1957, Reich [1] proved that, in equilibrium, the sojourn times of a customer in each of two M/M/1 queues in tandem are independent, and in 1963 [2], he extended this result to an arbitrary number of such queues in tandem. This result was very recently extended by Lemoine [3] to the case of Jacksonian networks which are trees. Since trees have no parallel paths, and since the service discipline is FCFS, every path in a tree network has the non-overtaking property: a customer travelling along the path cannot be overtaken by a subsequent arrival.

The main result of this paper is to show that in any open multiclass Jacksonian network, the sojourn times of a customer at the various nodes of a non-overtaking path are all mutually independent. Since the distribution of the sojourn times at each node is known, it is easy to calculate the sojourn times for non-overtaking paths.

The paper shows that in a single-class three node network which has two parallel paths (Figure 3) the sojourn times at the various nodes are not all independent. Thus the non-overtaking condition cannot be generally relaxed. It is also shown that for any network with a single customer class the sojourn times along any path which permits overtaking cannot be independent at least under light traffic.

The paper is organized as follows. In section 2 the notion of a marked customer is made precise and some technical results are recalled. The definition of overtaking is given in section 3 which also contains two basic lemmas. The main result occupies section 4 and the "negative" examples section 5. Some concluding remarks are collected in section 6.

2. The marked customer

It is convenient to first recall some results about Poisson processes.
Let \((S, \Sigma, P_N)\) be a probability space and \((\mathcal{F}_t), t \in \mathbb{R}_+\), an increasing family of sub-\(\sigma\)-fields. For \(n = 1, \ldots, d\) let \(N^n = (N^n_t), t \geq 0,\) be independent \((\mathcal{F}_t)\)-Poisson processes with rates \(\lambda^n\). Let \(N = (N^1_t, \ldots, N^d_t), t \geq 0.\) We say that \(N\) is a (vector) \((\mathcal{F}_t)\)-Poisson process with rate \(\lambda = (\lambda^1, \ldots, \lambda^d).\) The following statement is the strong Markov property for Poisson processes (see e.g. [4].)

**Lemma 2.1.** Let \(T\) be an a.s. finite \((\mathcal{F}_t)\)-stopping time. Then \(\hat{N}\) is a \((\mathcal{F}_t)\)-Poisson process with rate \(\lambda\), where

\[
\hat{N}_t = N_{T+t} - N_T, \quad \mathcal{F}_t = \mathcal{F}_{T+t}, t \geq 0.
\]

Consequently \(\mathcal{F}_T\) and \(\mathcal{G}_T = \sigma(\hat{N}_t, t \geq 0)\) are independent.

The result will be used below in a slightly different form.

Consider the space \((S, \mathcal{F}_N, P_N)\). Recall that we can always assume that \(S\) consists of the space of sample paths of the Poisson process and \(N_t\) is just the coordinate map (see e.g. [5, Ch. XIII.]). In the remainder such a canonical representation will always be used. This allows us to define the translation operators \(\{\theta_t\}, \{\hat{\theta}_t\}, t \geq 0,\) by

\[
N_s \cdot \theta_t(\sigma) = N_{s+t}(\sigma), \quad N_s \cdot \hat{\theta}_t(\sigma) = N_{s+t}(\sigma) - N_t(\sigma), \sigma \in S, s \geq 0.
\]

Simple calculations show that \(\{\theta_t\}, \{\hat{\theta}_t\}\) are semigroups. Now let \((X, \mathcal{X}, P_0)\) be another probability space, where \(X\) is a countable set.

Define

\[
(\Omega, \mathcal{X}, P) = (X \times S, \mathcal{X} \times \mathcal{F}_N, P_0 \times P_N), \quad \mathcal{F}_t = \mathcal{X} \times \mathcal{F}_N^t.
\]

Let \((X_t), t \in \mathbb{R}_+\), be a \((\mathcal{F}_t)\)-adapted, right-continuous process with values in \(X\).

**Lemma 2.2.** Let \(T\) be an a.s. finite \((\mathcal{F}_t)\)-stopping time. Then for all bounded \(\mathcal{F}_t\)-measurable functions \(\phi\)
\[ E\{\psi(x_T, N \cdot \delta_T) | \mathcal{F}_T\} = \psi(x_T), \]

where

\[ \psi(x) = \int_S \psi(x, \omega) P_N(\text{d}\omega), \quad x \in X. \]

**Proof.** Under \( P, X \) and \( \mathcal{F}^N \) are independent so that on \( (\Omega, \mathcal{F}, P) \cdot N \) is a \( (\mathcal{F}_T, P) \)-Poisson process and Lemma 2.1 applies. Now, if \( \psi \) is of the form \( \psi(x, \omega) = 1_A(x)1_B(N_t(\omega)) \) where \( A \subset X, B \subset N, t \geq 0 \) are fixed and \( 1_A, 1_B \) are indicator functions, then

\[ E\{\psi(x_T, N \cdot \delta_T) | \mathcal{F}_T\} = 1_A(x_T) P\{N_t \in B | \mathcal{F}_T\}, \]

since \( X_T \) is \( \mathcal{F}_T \)-measurable,

\[ = 1_A(x_T) P\{N_t \in B \} \]

and this is indeed equal to \( \psi(x_T) \) since, in this case,

\[ \psi(x) = \int_S 1_A(x)1_B(N_t) P_N(\text{d}\omega) = 1_A(x) P\{N_t \in B \} \]

The general case now follows by a monotone class argument. \( \Box \)

Consider now a Jacksonian network \( \mathcal{N} \) with \( N \) nodes and \( L \) customer types. The arrivals of external customers of class \( \ell \) at node \( i \) form an independent Poisson process with rate \( \lambda^{\ell}_i \). Node \( i \) is an \( \text{M/M/1} \) queuing system with FCFS discipline and service rate \( \mu_i \) independent of customer class. A class \( \ell \) customer who completes service at \( i \) changes into a class \( m \) customer and either immediately joins the queue at \( j \) with probability \( r_{im} \) or leaves the network with probability \( r_{i0} \).

Naturally \( \sum_{j=0}^{N} \sum_{m=1}^{L} r_{jm} = 1 \). Let \( \lambda^{\ell}_i \) be the solution, assumed unique, to the equations

\[ \lambda^{\ell}_i = \gamma_i + \sum_{j=0}^{N} \sum_{m=1}^{L} \lambda^{m}_{j} r_{jm}, \quad i = 1, \ldots, N, \quad \ell = 1, \ldots, L. \]

Set \( \lambda_i = \sum_{\ell=1}^{L} \lambda^{\ell}_i \), and assume \( \rho_i = \lambda_i \mu_i^{-1} < 1 \).
Following [6] or [7] we give a precise description of the Markov process describing the evolution of the state of the network. Let \( \bar{X} = \{1, \ldots, L\}^* \) be the set of all finite sequences of elements in \( \{1, \ldots, L\} \) including the null sequence \( \phi \). The state space is \( X = \bar{X}^\mathbb{N} \), so that a state is an \( n \)-tuple \( x = (x_1, \ldots, x_n) \) where \( x_i \) represents the customers in queue at node \( i \) with the right-most element in \( x_i \), being the class of the customer in service and the left-most element, the class of the customer who arrived most recently.

We adopt the following notation. \( a \cdot b \) is the concatenation of two elements in \( \bar{X} \). Also, for \( x_i \in \bar{X} \),

- \( a(x_i) \) is the left-most element in \( x_i \), \( a(\phi) = 0 \),
- \( d(x_i) \) is the right-most element in \( x_i \), \( d(\phi) = 0 \),
- \( |x_i|_\ell \) is the number of customers in \( x_i \) of class \( \ell \),
- \( |x_i| = \sum |x_i|_\ell \) is the number of customers in \( x_i \),

and if \( |x_i| > 0 \) let \( \hat{x}_i \) be obtained from \( x_i \) by deleting its right-most element.

There are three types of possible state transitions and associated Poisson processes:

(i) Internal Transitions. For \( 1 \leq i, j \leq N, 1 \leq \ell, m \leq L \), let

\[
E_{ij}^{\ell m} = \{x \in X | d(x_i) = \ell \}, \quad T_{ij}^{\ell m} : E_{ij}^{\ell m} \rightarrow X \text{ with } T_{ij}^{\ell m}(x_1, \ldots, x_N) = (x_1, \ldots, \hat{x}_i, \ldots, m, x_j, \ldots, x_N),
\]

and let \( N_{ij}^{\ell m} \) be an independent Poisson process with rate \( \mu_{ij}T_{ij}^{\ell m} \).

(ii) External arrivals. For \( 1 \leq i \leq N, 1 \leq \ell \leq L \), let \( U_i^\ell = X \),

\[
A_i^\ell : U_i^\ell \rightarrow X \text{ with } A_i^\ell(x_1, \ldots, x_N) = (x_1, \ldots, \hat{x}_i, \ldots, x_N),
\]

and let \( N_i^\ell \) be an independent Poisson process with rate \( \gamma_i^\ell \).
(iii) External departures. For $1 \leq i \leq N, 1 \leq \ell, m \leq L$, let

$$v^* = \{x \in X | d(x_i) = \ell\}, D^\ell_m : v^* \rightarrow X \text{ with } D^\ell_m(x_1, \ldots, x_N) = (x_1, \ldots, x_i, \ldots, x_N)^{1,\ell},$$

and let $N^\ell_m$ be an independent Poisson process with rate $\mu^\ell_m$.

Let $X_0$ be an independent random variable with values in $X$; $X_0$ is the initial state. Then the state process $(X_t), t \geq 0$ is the unique right-continuous piecewise constant solution of the differential equation (2.1), (2.2) below. For $E \subset X$ let $\xi_t(E) = 1_{E}(X_t), \xi_t(x) = 1_{\{x\}}(X_t)$.

$$d\xi_t(x) = \sum_{i,j,m} [\xi_t(-(T^\ell_m)^{-1}x_i), \xi_t(\eta_i)] \xi_t(\delta_i) dN^\ell_m(t)$$

$$+ \sum_{i,j} [\xi_t(-(A_i)^{-1}x_i), \xi_t(\eta_i)] \xi_t(\delta_i) dN^\ell_i(t)$$

$$+ \sum_{i,j,m} [\xi_t(-(D^\ell_mA_i)^{-1}x_i), \xi_t(\eta_i)] \xi_t(\delta_i) dN^\ell_m(t), \quad (2.1)$$

$$\xi_0(x) = 1_{\{x\}}(X_0). \quad (2.2)$$

Let $N = (N^1_1, \ldots, N^d_1), t \geq 0$, with rate $\lambda = (\lambda^1, \ldots, \lambda^d)$, be the collection of Poisson processes introduced above. $(N_t)$ is given its canonical representation. The state process $(X_t)$ is then $\sigma(X_t) \vee T^N_t$ adapted, strong Markov process and $\sigma(X_t)$ is independent of $T^N_t$. Assume that $X_0$ is given the equilibrium distribution $P_0$ given by

$$P_0(x_1, \ldots, x_N) = P^1(x_1, \ldots, x_N), \quad (2.3)$$

$$P^i(x_i) = P^i_1(1-P^i_1)^{L-i} \frac{P^i_1}{1-P^i_1}, \quad (2.4)$$

where $p^i_1 = \lambda_i^{\ell_1}$. (See [6] or [8]). The state process $(X_t)$ is now stationary.

Suppose now that at time 0 a customer of type $\ell_1$, call him C, is introduced at the end of the queue at node 1 and that after he leaves node 1 he proceeds in sequence through nodes 2, 3, ..., $n$, maintaining class identity $\ell_1$ while he is at node 1. We wish to analyze the
sojourn times of C at these nodes. To do this we need to augment
the state description of the network from \((X_t)\) given above to \((\tilde{X}_t)\) say,
so that at \(t\) the position of C in the network is given by \(\tilde{X}_t\). We do
this simply by increasing the number of customer classes to \(2L\) with zero
external arrival rates for the new classes \(L+1, \ldots, 2L\), and by agreeing
to "mark" the customer C as being of class \(L+\ell\) whenever "unmarked" he
would be of class \(\ell\). The process \((\tilde{X}_t)\) now satisfies a differential
equation analogous to (2.2) and with an initial distribution \(\tilde{P}_0\) of \(\tilde{X}_0\)
which is obtained from the steady-state distribution \(P_0\) of \(X_0 = (X_{10}, \ldots, X_{N0})\),
by simply adding of type a customer of type \(L+\ell\) (namely C) to the left
of \(X_{10}\). The process \((\tilde{X}_t)\) is then also a right-continuous, strong Markov
process. It is, however, not in equilibrium and so we may not directly
apply known equilibrium results of Jacksonian networks. (This point is
somewhat overlooked in [1]-[3].)

**Definition 2.1.** Let \(T_0 = 0\) and \(T_i\) the departure time of C from node
\(i, \; i = 1, \ldots, n\). \(S_i = T_i - T_{i-1}\) is the sojourn time of C at \(i\).

Evidently \(T_i\) is a stopping time of the process \((\tilde{X}_t)\). We show in
section 4 that \(S_1, \ldots, S_n\) are all independent if the path \((1, \ldots, n)\)
does not permit overtaking.

3. The non-overtaking condition

Recall the routing probabilities \(\{r_{ij}^{2m}\}\) introduced previously.

**Definition 3.4.** For \(1 \leq i, j \leq N\), write \(i \rightarrow j\) if for some \(\ell, m\)
\(r_{ij}^{2m} > 0\). Let

\[
P_{ij} = \{\pi = (i, i_1, \ldots, i_m, j) | i \rightarrow i_1, i_1 \rightarrow i_2, \ldots, i_m \rightarrow j, \text{ for some } i_1, \ldots, i_m\}
\]

\[
P_{ij}^k = \{\pi \in P_{ik}, \pi' \in P_{kj}\}.
\]
Thus $P_{ij}$ is the set of all paths in the network going from $i$ to $j$ and $P_{ij}^k$ consists of those paths which in addition go via $k$. Note that the path $(1,\ldots,n)$ taken by $C$ is in $P_{1n}$.

**Definition 3.2.** The path $\pi = (i_1, i_2, \ldots, i_m) \in P_{i_1i_m}$ permits no overtaking if

$$P_{i_1i_m} \subseteq P_{i_1i_{n+1}} \text{ for } 1 \leq n < v \leq m.$$  

The condition means that all paths from $i_n$ to $i_v$ must go through $i_{n+1}$; hence a customer who traverses $i_1,\ldots,i_m$ cannot be "overtaken" by any customer who enters $i_1$ after him. In Figure 3, the path $(1,3)$ permits no overtaking, but the path $(1,2,3)$ does because $(1,3)$ does not go through 2.

**Assumption.** In the remainder of the section and the next section it is assumed that the path $(1,2,\ldots,n)$ permits no overtaking and the nodes $i = 1,\ldots,n-1$ are without self-loop, i.e., $r^i_{ii} = 0$, all $i,m$.

**Definition 3.3.** For $i = 1,\ldots,n-1$, let

$$P_i = \{ j \mid P_{j}^i \subseteq P_{i}^j \text{ for some } k \in \{i+1,\ldots,n\} \}.$$  

Thus $P_i$ consists of all nodes $j$ from which there is a path reaching $\{i+1,\ldots,n\}$ without going through $i$. See Figure 1.

**Lemma 3.1.** (i) $j \not\in P_i$ for $1 \leq j < i \leq n-1$

(ii) $\{i\} \cup P_i \subseteq P_{i-1}$, $2 \leq i \leq n-1$

**Proof** (i) By the assumption every path from $j$ to $k$ for $j \leq i < k \leq n$ must go through $i$; that is, $P_{jk} = P_{jk}^i$.

(ii) $i \in P_{i-1}$ since $(i,i+1)$ is a path. Next let $j \in P_{i+1}$. We show that $j \in P_i$. Let $k \in \{i+2,\ldots,n\}$ and $\pi \in P_{jk}$ such that $\pi \not\in P_{i+1}^j$. It is enough to show that $\pi \in P_{jk}^i$. If $\pi \in P_{jk}^i$ then $\pi' = (\pi',\pi'')$ with
π' ∈ P_j it and π'' ∈ P_{jk}. But then π'' ∈ P_{jk}^{i+1} since (1,2,...,n) permits no overtaking, and so π ∈ P_{jk}^{i+1} which is a contradiction.

In the remainder of this section and the next we shall denote the augmented state process X by X_t. There should be no confusion since we will not need to refer to the unaugmented process. For any subset of nodes J ⊂ {1,...,N} let \( X_t^J = \{X_t \mid t \in J\} \) denote those components of \( X_t \) which correspond to the nodes in J. It is convenient to introduce a new node 0 to which go all the customers who leave the original network. (This is in keeping with the notation \( T_0, N_0 \).)

For \( i = 1,...,n-1 \), let

\[
\begin{align*}
Q_i &= P_i \cup \{i\}, \quad P_i^C = \{j \mid 1 \leq j < N, j \notin P_i\}, \\
D_i &= P_i^C \cup \{0\}, \\
\end{align*}
\]

For \( i = 1,...,n-1 \) define the vector processes \( N_i, \tilde{N}_i \) as follows:

\[
\begin{align*}
N_i^1 &= \{N_{k,j}^{l,m}, N_r^{l,m} \mid j \in Q_i, r \in P_i, 0 < j < N, 1 \leq l, m < N\} \\
\tilde{N}_i &= \sum_{j' \in D_i} N_{k,j'}^{l,m}, N_{r,0}^{l,m} \quad (34)
\end{align*}
\]

Thus \( N_i^1 \) consists of the Poisson processes associated with internal transitions and departures from nodes in \( Q_i \) together with external arrivals into \( P_i \); \( \tilde{N}_i \) consists of these same processes except that customers leaving \( P_i \) are not distinguished by their destination. Later \( \tilde{N}_i \) will be used to construct a simpler network equivalent to \( N \).

Recall Definition 2.1. The next lemma summarizes the crucial observation that after \( T_{i-1} \) the progress of \( C \) depends only upon the state of the queues in nodes \( Q_i \) at \( T_{i-1} \) and the processes \( \tilde{N}_i \) after \( T_{i-1} \).

**Lemma 3.2.** For \( i = 1,...,n-1 \), there is a measurable mapping \( \phi_i \) depending only on \( P_i \) such that
\[ (S_1, X^1_{T_1}) = Q_1(X^i_{T_1}, \hat{N}^1_i, \hat{T}_{i-1}) \]

"depending only on \( P_i \)" means that \( \phi_i \) is the same for all networks with \( N \) nodes \( L \) classes for which \((1,...,n)\) permits no overtaking and which have the same set \( P_i \).

**Proof.** Evidently, there is a function \( f_i \), depending only on \( P_i \), such that

\[
X^i_{T_1} = f_i(S_1, X^i_{T_1}, \hat{N}^i_i, \hat{T}_{i-1}, 1[0,S_i](\cdot) F^i_{T_{i-1}}),
\]

(3.2)

where \( F_i \) is the set of flows of customers going from \( P_i \) to \( P_i \) and 

\[ 1[0,S_i](\cdot) \]

is the indicator of the random time interval \([0,S_i]\). By Definition 3.3, \( F_i \) is simply the flows from \( i \) to \( P_i \) (See Figure 1). Hence there exist functions \( g_i \) and \( h_i \), depending only on \( P_i \), such that

\[
S_i = g_i(X^i_{T_{i-1}}, \hat{N}^i_i, \hat{T}_{i-1})
\]

(3.3)

\[
1[0,S_i] F^i_{T_{i-1}} = h_i(S_1, X^i_{T_{i-1}}, \hat{N}^i_i, \hat{T}_{i-1})
\]

(3.4)

The assertion follows from (3.2), (3.3), (3.4). \( \Box \)

**Corollary 3.1.** There is a mapping \( \phi \), depending only on \((P_1,...,P_{n-1})\), such that

\[
(S_1,...,S_n) = \phi(X_0^1, \hat{N}^1)
\]

**Proof.** Assume, as induction hypothesis, that for some \( f_{i-1} \) depending only on \((P_1,...,P_{i-1})\),

\[
(S_1,\ldots,S_{i-1},X^i_{T_{i-1}}) = f_{i-1}(X_0^1, \hat{N}^1)
\]

(3.5)

By Lemma 3.2, for \( i \leq n-1, \)

\[
(S_i,X^i_{T_i}) = \phi_i(X^i_{T_{i-1}}, \hat{N}^i_i, \hat{T}_{i-1}).
\]
By Lemma 3.1, $Q_i \subseteq P_{i-1}$ and $\tilde{N}_i$ is a subvector of $\tilde{N}_{i-1}$. Hence (3.5) is proved for $i = 2, \ldots, n$. For $S_n$, observe that

$$S_n = \phi_n (\chi_{T_{n-1}}^n, \tilde{N}_n \cdot \delta_{T_{n-1}})$$

where $\tilde{N}_n$ is obtained as in (3.1) by setting $P_n = \phi$, $Q_n = \{n\}$, $D_n = \{0,1,\ldots,N\}$. The assertion now follows since $n \in P_{n-1}$ and $\tilde{N}_n$ is a subvector of $\tilde{N}_{i-1}$.

4. The main result.

We show here that the sojourn times $S_1, \ldots, S_n$ of $C$ are all independent. Moreover $S_i$ has the same distribution as the sojourn time of a customer in an M/M/1 queue (in equilibrium) with arrival rate $\lambda_i$ and service rate $\mu_i$. Recall that $\lambda_i$ is the total average arrival rate into node $i$.

We first give an outline of the proof (see Figure 2). The idea is to reduce the problem to the path $(2, \ldots, n)$. To do this it will be shown that in $\mathcal{M}$.

a) $S_1$ and $(S_2, \ldots, S_n)$ are independent;

b) $(S_2, \ldots, S_n)$ has the same distribution as that of the sojourn times of $C$ if $C$ were introduced into the network at node 2 with $\mathcal{N}$ being in equilibrium.

To prove a) we introduce a simpler network $\tilde{\mathcal{M}}$ (Definition 4.1) such that $(S_1, \ldots, S_n)$ have the same distribution in $\mathcal{M}$ and $\tilde{\mathcal{M}}$ (Lemma 4.2) and for which $S_1$ and $(S_2, \ldots, S_n)$ are independent (Lemma 4.4). To prove b) we show that in $\mathcal{M}$, $X_{T_1}^1$ and $X_{T_1}^c$ both have the equilibrium distribution and then b) follows from the fact that $(X_t)$ is a strong Markov process and $T_1 = S_1$ is a stopping time of $(X_t)$. $(X_{T_1}^1$ and $X_{T_1}^c$ do not contain $C$ so that the equilibrium distribution is well defined and given by (2.3), (2.4).)
Definition 4.1. (See Fig. 2) Let $\tilde{N}$ be the network obtained from $N$ by changing the Poisson processes associated with the latter in the following way:

For $1 \leq i \leq N$, $0 \leq j \leq N$, $1 \leq \ell, m \leq L$,

$$\tilde{N}_{ij} = \begin{cases} 0 & \text{if } (i,j) \in Q_1 \times P^c_1 \\ \sum_{j \in D_1} \tilde{N}_{ij} & \text{if } (i,j) \in Q_1 \times \{0\} \\ \tilde{N}_{ij} & \text{otherwise}, \end{cases}$$

and for $1 \leq i \leq N$, $1 \leq \ell \leq L$,

$$\tilde{N}_i^\ell = \begin{cases} N_i^\ell & \text{if } i \in P_1 \\ \text{an independent Poisson process with rate } \lambda_i^\ell, & \text{if } i \in P^c_1. \end{cases}$$

Essentially, $\tilde{N}$ is obtained by forcing customers who, in $N$, moved from $Q_1$ and $P^c_1$ to leave the network and then "compensating" the nodes in $P^c_1$ by external Poisson arrivals with the same average rate. Thus in $\tilde{N}$ the average rate of arrivals of any class at any node is the same as the corresponding rate in $N$.

Lemma 4.1. The following elements in $N$ and $\tilde{N}$ are the same:

(i) the vector process $N^1$ (and hence $N_i^1, i=2, \ldots, n-1$).

(ii) the subsets $P_1, \ldots, P_{n-1}$

(iii) the equilibrium distribution of the unmarked state.

Proof. (ii) is immediate and (i) follows by checking that (3.1) yields the same process $N^1$ whether it is obtained from $N^1$ or $\tilde{N}^1$. (iii) follows from (2.3), (2.4) since the $\lambda_i^\ell$ are the same for $N$, $\tilde{N}$.

Suppose now that at time $0^-$ the unmarked state of $\tilde{N}$, which we continue to denote by $X_{0^-}$, has the equilibrium distribution given by (2.3), (2.4). At time $0$ we introduce customer $C$ at the end of the queue in node 1, i.e., to the left of $X_{0^-}$. Let $T_i^1, S_i^1$ continue to denote the various departure and sojourn time of $C$ in $\tilde{N}$.
Lemma 4.2. \( \{S_i, X^i_T | i=1, \ldots, n-1\} \) and \( \{S_1, \ldots, S_n\} \) have the same distribution in \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \).

**Proof.** Follows from Corollary 3.1, and Lemma 4.1.

Lemma 4.3. Let \( Z^1, Z^2 \) be independent random variables. Then
\[
E\{h(f(Z^1), Z^2) | f(Z^1), g(Z^1)\} = E\{h(f(Z^1), Z^2) | f(Z^1)\}
\]
for any bounded measurable function \( h \) and measurable functions \( f, g \).

**Proof.** This is obvious if \( h(f(Z^1), Z^2) = h_1(f(Z^1))h_2(Z^2) \). The result follows from this using a monotone class argument.

Lemma 4.4. In \( \tilde{\mathcal{N}} \), \( S_1 \) and \( (S_2, \ldots, S_n) \) are independent.

**Proof.** It is convenient to denote
\[
P = P_1, \quad Q = Q_1 = P_1 \cup \{1\}, \quad R = \{i | 1 < i < N, i \notin Q\}.
\]
The proof is divided into several steps.

**Step 1.** \( P(X^1_S | X^P_S, S_1) = P(X^1_S | S_1) \). (4.1)

To see this define the following vector processes:

\[
\begin{align*}
\alpha &= \text{flows of customers from } R \text{ to } 1, \\
\beta &= \text{external arrivals into } 1, \\
\gamma &= \text{external arrivals and service processes in } R.
\end{align*}
\]

By Lemma 3.2, since \( T_1 = S_1 \),
\[
(S_1, X^P_S) = \phi_1(X^1_0, X^P_0, \tilde{N}^1).
\]

By definition of \( S_1 \), we have (see Fig. 2), \( X^1_S \) is measurable with respect to \( \sigma(S_1, \alpha, \beta) \). Also \( \sigma(\alpha) \subseteq \sigma(X^R_0, \gamma) \). Hence
\[
X^1_S \text{ is measurable with respect to } \sigma(S_1, X^R_0, \beta, \gamma).
\]

But
\[
\sigma(X^R_0, \beta, \gamma) \text{ and } \sigma(X^1_0, X^P_0, \tilde{N}^1) \text{ are independent}
\]
and so (4.1) follows by applying Lemma 4.3 to (4.2), (4.3), (4.4).

**Step 2.** $X^{1}_{S_1}$ and $X^{P}_{S_1}$ are independent (4.5)

By construction of $\tilde{\mathcal{N}}$ the link joining nodes 1 and 2 is not part of any loop. Hence, by the output theorem for Jacksonian networks (see e.g. [6] or [7]), for the unmarked state $X^{1}_{S_1}$, $X^{P}_{S_1}$ are independent. Now at time $S_1 = T_1$, the position of $C$ is known as a function of $X^{P}_{S_1}$ since $C$ is at the end of queue at node 2. Hence $X^{1}_{S_1}$, $X^{P}_{S_1}$ are independent for the marked state as well.

**Step 3.** $X^{1}_{S_1}$ and $X^{P}_{S_1}$ are independent (4.6)

This is proved by using a technique of Reich [1]. Let $W = X^{P}_{S_1}$ and let $V$ be the number of customers in node 1 at $S_1$. For any complex number $z$, we find

$$E(z^V|W = w) = \int_0^\infty E(z^V|W = w, S_1 = t) dP(S_1 < t|W = w)$$

$$= \int_0^\infty E(z^V|S_1 = t) dP(S_1 < t|W = w), \text{ by (4.1)}.$$

Now, again by the output theorem, the arrivals into node 1 form a Poisson process with rate $\lambda_1$ and so

$$E(z^V|S_1 = t) = \int_0^\infty z^V \exp(-\lambda_1 t) \frac{(\lambda_1 t)^V}{V!} = \exp(z-1)\lambda_1 t.$$

Also, by (4.5), $z^V$ and $W$ are independent. Hence

$$E(z^V) = \int_0^\infty \exp(z-1)\lambda_1 t \ dP(S_1 < t|W = w)$$

Since the left-hand side does not depend on $w$ it follows that $S_1$ and $W$ are independent, and (4.6) is proved.

**Step 4.** $S_1$ and $(S_2, \ldots, S_n)$ are independent.
First observe that by the output theorem for Jacksonian networks, the arrivals into node 1 form independent Poisson processes. Hence (see Figure 2) \((X_t^Q)\) is a strong Markov process and \(S_1\) is its stopping time. Therefore, by the strong Markov property,

\[
P\{S_2, \ldots, S_n | S_1, X_{S_1}^Q\} = P\{S_2, \ldots, S_n | X_{S_1}^Q\}.
\] (4.7)

Next we claim that

\[
P\{S_2, \ldots, S_n | X_{S_1}^P, X_{S_1}^P\} = P\{S_2, \ldots, S_n | X_{S_1}^P\}.
\] (4.8)

To prove this it is evidently enough to show that

\(X_{S_1}^P\) and \(\{S_2, \ldots, S_n, X_{S_1}^P\}\) are independent.

But by repeated use of Lemma 3.2 and the fact that \(P_2 \cup \{2\} \subset P_1 = P\) we see that

\((S_2, \ldots, S_n)\) measurable with respect to \(\sigma(X_{S_1}^P, X_{S_1}^P)\); (4.10)

Moreover

\(X_{S_1}^P\) and \(X_{S_1}^P\) independent.

(4.11)

From (4.5), (4.10), (4.11) it is easy to conclude (4.9). From (4.7), (4.8)

\[
P\{S_2, \ldots, S_n | S_1, X_{S_1}^P\} = P\{S_2, \ldots, S_n | X_{S_1}^P\},
\]

and conditioning both sides with respect to \((S_1, X_{S_1}^P)\) gives

\[
P\{S_2, \ldots, S_n | S_1, X_{S_1}^P\} = P\{S_2, \ldots, S_n | X_{S_1}^P\}.
\] (4.12)

Finally, denoting \(W = X_{S_1}^P\),

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\[
P(S_1, \ldots, S_n) = \sum_w P(S_1, \ldots, S_n, W=w) = \sum_w P(S_2, \ldots, S_n | S_1, W=w) P(S_1, W=w)
\]
\[
= \sum_w P(S_2, \ldots, S_n | W=w) P(S_1, W=w), \text{ by (4.12)}
\]
\[
= \sum_w P(S_2, \ldots, S_n | W=w) P(S_1) P(W=w), \text{ by (4.6)}
\]
\[
= P(S_2, \ldots, S_n) P(S_1).
\]

The lemma is proved.

**Theorem 4.1.** In \( \mathcal{N} \) the sojourn times \( S_1, \ldots, S_n \) are independent. Moreover \( S_i \) has the same distribution as the sojourn time in an \( M/M/1 \) queue with arrival rate \( \lambda_i \) and service rate \( \mu_i \).

**Proof.** By Lemmas 4.2, 4.4 \( S_1 \) and \( \{S_2, \ldots, S_n\} \) are independent. Moreover \( X_0^1 \) has the equilibrium distribution given by (2.4) and so \( S_1 \) is distributed as asserted. Now in the network \( X^1_{S_1} = X^1_{T_1} \) also has the equilibrium distribution given by (2.3), (2.4) [9]. We can apply the argument above to \( \mathcal{N} \) and construct an equivalent network \( \mathcal{N} \) to conclude that \( S_2 \) and \( \{S_3, \ldots, S_n\} \) are independent with \( S_2 \) distributed as asserted. The result follows by successive repetitions.

**5. Paths which permit overtaking**

Consider the network of Figure 3 with 3 nodes and only one class of customers. In terms of the notation of Section 2, and dropping the superscript since there is only class, we have the following parameters:

- arrival rates: \( \gamma_1 = \lambda, \gamma_2 = 0, \gamma_3 = 0, \)
- routing probabilities: \( r_{12} = p = 1-r_{13} = 1-q; r_{23} = 1; r_{30} = 1, \)
- service rate at node \( i \) is \( \mu_i, i = 1,2,3. \)

We assume that \( \lambda < \mu_1, \lambda < \mu_3, \lambda \cdot p < \mu_2 \). Observe that the path \( (1,2,3) \) permits overtaking since the path \( (1,3) \) does not go through node 2.
As before let \( X_{0-} = (X_{0-}^1, X_{0-}^2, X_{0-}^3) \) be given the equilibrium distribution defined by (2.3), (2.4) and obtain \( X_0 \) by adding \( C \) to the left of \( X_{0-}^1 \). Suppose \( C \) takes the path \((1,2,3)\) and let \( T_1, S_1 \) be the corresponding departure and sojourn time.

Theorem 5.1. (i) \( S_1 \) and \( S_2 \) are independent; \( S_2 \) and \( S_3 \) are independent
(ii) \( S_1 \) and \( S_3 \) are not independent.

Proof. (i) The paths \((1,2)\) and \((2,3)\) do not permit overtaking and so the result follows from Theorem 4.1.

(ii) The idea behind the proof is this. If \( S_1 \) is large, then \( C \) is likely to leave behind him many customers in node 1. Therefore it is likely that some of these will overtake \( C \), by using the path \((1,3)\), and arrive at node 3 before \( C \). Thus at node 3 \( C \) will find a larger queue, thereby increasing \( S_3 \). This reasoning suggests that \( S_1 \) and \( S_3 \) are positively correlated. We now give a formal proof.

By the strong Markov property (applied to the augmented state process and its stopping time \( S_1 \)) it follows that

\[
a(n; x_2, x_3) = E[S_3 | X_{s_1}^2 = x_2, X_{s_1}^3 = x_3, X_{s_1}^1 = n]
\]
is well-defined. We claim that for all \((x_2, x_3, n)\)

\[
a(n+1; x_2, x_3) > a(n; x_2, x_3) \tag{5.1}
\]
To see this consider Figure 4 and denote by \( \omega \) a realization of all the independent Poisson processes involved in the description of the system and of \((X_{0-}^2, X_{0-}^3)\). Let \( \nu \) denote a realization of \( X_{0-}^1 \). Now introduce \( C \) to the left of \( X_{0-}^1 \). Let \( P \) denote the probability obtained by giving \( X_{0-} \) the equilibrium distribution. Then it should be clear that

\[
S_3(\omega, \nu + 1) \geq S_3(\omega, \nu) \text{ for all } \omega, \nu. \quad \text{(Indeed, since } S_2(\omega, \nu + 1) \equiv S_2(\omega, \nu) \text{ it suffices to show that } X^3_t(\omega, \nu + 1) \geq X^3_t(\omega, \nu), \text{ } t \geq 0 \text{ and this obvious.)}
\]
Moreover, it is easy to exhibit a set of realizations with positive probability for which \( S_3(\omega, v+1) > S_3(\omega, v) \); and then (5.1) follows using the strong Markov property.

Now it is known [9] that the unmarked state at time \( S_1 \) also has the equilibrium distribution. Hence \( X_{S_1}^1 \) and \( (X_{S_1}^2, X_{S_1}^3) \) are independent. Also

\[
a(n) = E[S_3 | X_{S_1}^1 = n] = \sum_{x_2, x_3} E[S_3 | X_{S_1}^2 = x_2, X_{S_1}^3 = x_3, X_{S_1}^1 = n] P(X_{S_1}^2 = x_2, X_{S_1}^3 = x_3)
\]

Since \( p(x_2, x_3) > 0 \) for all \( x_2 \geq 0, x_3 \geq 0 \), it follows from (5.1) that

\[
a(n+1) > a(n), n \geq 0.
\]

Next observe that

\[
b(n | t) = P(X_{S_1}^1 \geq n | S_1 = t) = \sum_{m=n}^{\infty} \frac{(\lambda t)^m}{m!} \exp(-\lambda t),
\]

is such that

\[
b(n | t') > b(n | t), n \geq 0, t' > t > 0.
\]

Finally, let

\[
c(t) = E[S_3 | S_1 = t] = \sum_{n=0}^{\infty} E[S_3 | X_{S_1}^1 = n, S_1 = t] P(X_{S_1}^1 = n | S_1 = t)
\]

\[
= \sum_{n=0}^{\infty} E[S_3 | X_{S_1}^1 = n] P(X_{S_1}^1 = n | S_1 = t), \text{ by the strong Markov property,}
\]

\[
= \sum_{n=0}^{\infty} a(n) [b(n | t) - b(n+1 | t)]
\]

We claim that

\[
c(t') > c(t), t' > t > 0
\]
To prove this observe that $\sum_{n=0}^{\infty} a(n) = ES_3 = \infty$ and $\sum_{n=0}^{\infty} b(n|t)$

$= E(x_{S_1}^1|S_1 = t) = \lambda t < \infty$ so that the following calculations are justified.

Let $t' > t > 0$. Then

$$c(t') - c(t) = \sum_{n=0}^{\infty} a(n)[b(n|t') - b(n+1|t')] - [b(n|t) - b(n+1|t)]$$

$$= \sum_{n=0}^{\infty} a(n)[b(n|t') - b(n|t)] - a(n+1)[b(n|t') - b(n+1|t)]$$

$$< \sum_{n=0}^{\infty} a(n)[b(n|t') - b(n|t)] - a(n+1)[b(n+1|t') - b(n+1|t)]$$

$$= a(0)[b(0|t') - b(0|t)] > 0.$$

This shows that

$$E(S_3|S_1 = t') > E(S_3|S_1 = t)$$

and so $S_1, S_3$ are not independent.

For our second example we return to the N node network discussed earlier. Suppose there is a single customer class. Let $\xi = \xi(\lambda_1, \ldots, \lambda_N)$, $0 < \xi < 1$, be the external arrival rates, $\{r_{ij}\}$ the routing probabilities and $\xi \lambda = \xi(\lambda_1, \ldots, \lambda_N)$ the average total arrival rates.

Suppose the path $(1, \ldots, n)$ permits overtaking. Let $i \geq 2$ be the largest integer such that $(1, \ldots, i)$ permits no overtaking but $(1, \ldots, i+1)$ does. From now on let $i+1 = n$. And let $\pi_1 = (1', 2', \ldots, m', n)$ be a path "parallel" to $(1, \ldots, n)$ as in Figure 6.

As before let $C$ enter node 1 at $T_0 = 0$ when the network is in equilibrium. Suppose $C$ leaves $i$ at $T_i$ and let $S_i = T_i - T_{i-1}$. $S_i$ has the same distribution as the sojourn time for a M/M/1 queue with input rate $\xi \lambda_i$ and service rate $\mu_i$, and so

$$P(S_i \leq s) = 1 - \exp(-\mu_i(1-\rho_i)s), \quad s \geq 0 \quad (5.2)$$
where $p = e^{\lambda_1^{-1}}$. Also
\begin{equation}
P\{T_{n-1} - T_1 < s\} = P \sum_{n-1}^{s} S_{n-1} < s, \tag{5.3}
\end{equation}
where the $S_i$ are distributed as in (5.2) and moreover they are independent
by Theorem 4.1 since $(1, \ldots, n-1)$ permits no overtaking.

We shall show that $S_1$ and $S_n$ are dependent, at least for $\varepsilon > 0$
sufficiently small. We adapt an argument of Burke [10].

Let $E$ be the event that there is a customer, say $C'$, in service at
node $1 = 1'$ at time $T_1$ which is when $C$ leaves $1$. The probability that
$C'$ takes path $\pi'$ is
\[ r' = r_1' r_2' r_3' \cdots r_{m-1}' r_m' n \]
r' > 0 since $\pi'$ is a path (Definition 3.1).

For each node $n$ let $M_n$ be an independent random variable with the
service time distribution,
\[ P\{M_n < y\} = 1 - \exp(-\mu_n y) \tag{5.4} \]
and let
\[ Q = M_1 + \ldots + M_m. \]

The probability that $C$ will encounter a busy server at node $n$
conditioned on the event $E$ is $P\{X_n^{n-1} > 0 | E\}$.

Lemma 5.1. $P\{X_n^{n-1} > 0 | E\} > r'P\{Q < T_{n-1} - T_1 < Q + M_n\}$.

Proof. $C'$ is in service at $1'$ at $T_1$, and so will complete service at
$T_1 + M_1$, and enter $2'$ with probability $r_1'r_2'$. If $C'$ finds $2'$ empty,
he will immediately enter service. If $C'$ finds another customer in
service call the latter $C'$. Then $C'$ will complete service at $2'$ at
time $T_1 + M_1 + M_2$, and move to $3'$ with probability $r_2'r_3'$. In any case,
conditioned on $E$, $C'$ will enter 3' at $T + M_1 + M_2$, with probability $r_1^2 r_2^3$. Continuing in this way, and renaming customers if necessary, $C'$ will enter node $n$ at time $T_1 + M_1 + \ldots + M_n = T_1 + Q$ with probability $r'$. If $C'$ encounters a customer in service at $n$ call the latter $C'$. Then $C'$ will leave $n$ at $T_1 + Q + M_n$ so that $C'$ is in service at $n$ during the random interval $(T_1 + Q, T_1 + Q + M_n)$ and so he will block $C$ if the latter's arrival time at $n$, $T_{n-1}$, falls in this interval. The assertion is proved.

**Lemma 5.2.** There is $\delta > 0$ such that for all $0 < \epsilon < 1$,

$$b(\epsilon) = r' \mathbb{P}\{Q < T_{n-1} - T_1 < Q + M_n\} > \delta.$$  

**Proof.** The "blocking" probability is given by

$$b(\epsilon) = r' \mathbb{P}\{\sum_{k=1}^{m} M_k < \sum_{i=1}^{m-1} S_i < \sum_{k=1}^{m} M_k + M_n\}$$

where $M_k$ is distributed as in (5.4), $S_i$ as in (5.2), and all of them are independent. For any $\epsilon$ $b(\epsilon) > 0$, since, for instance, for fixed numbers $0 < a < b$,

$$\mathbb{P}\{\sum_{k=1}^{m} M_k < \sum_{i=1}^{m-1} S_i < \sum_{k=1}^{m} M_k + M_n\} > \mathbb{P}\{\sum_{k=1}^{m} M_k < a\} \mathbb{P}\{\sum_{i=1}^{m-1} S_i < b\} \mathbb{P}\{M_n > b\} > 0.$$  

Moreover $b(\epsilon)$ varies continuously with $\epsilon$ and so the assertion follows.

Customers arrive into node 1 at an average rate $\lambda > 0$. They need not arrive in a Poisson stream, but since the unmarked process is stationary, there is $\tau(\epsilon) < \infty$ such that with probability at least one-half a customer arrives at node 1 if $S_1 = T_1 > \tau(\epsilon)$, i.e.,

$$\mathbb{P}(E|S_1 > \tau(\epsilon)) = \mathbb{P}(X_1^{T_1 > 0}|S_1 > \tau(\epsilon)) \geq 1/2.$$  

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Combining this with Lemmas 5.1, 5.2 gives the estimate

$$P(X_{T_n-1} > 0 | S_i > n. n. n. n. \tau(\varepsilon)) > 1/2 \delta.$$  \hspace{1cm} (5.5)

Now $S_n$ is the service time for $1 + X_{T_n-1}^n$ customers. Hence from (5.5)

$$E(S_n | S_i > \tau(\varepsilon)) > (1/2 \delta + 1) \mu_n^{-1}.$$  

On the other hand the unconditional distribution of $X_{T_n}^n$ is just its equilibrium distribution [9] and so,

$$E_S = (1 - \rho_n)^{-1} \mu_n^{-1},$$

and so for $\varepsilon$, equivalently $\rho_n$, sufficiently small

$$E_S | S_i > \tau(\varepsilon)) > E_S$$

so that $S_i$ and $S_n$ are dependent.

**Theorem 5.2.** Let be a Jacksonian network with a single class. Along any path which permits overtaking the various sojourn times are not all independent for sufficiently low traffic intensities.

6. Concluding Remarks

In networks which are trees every path permits no overtaking and so the sojourn times at the various nodes are independent. Thus the results of [3] follow from Theorem 4.1. One interesting example to consider is the so-called "full duplex" system of Figure 6 in which there are two classes of customers, the first travelling right to left and the second travelling in the opposite direction. By Theorem 4.1 the sojourn times of each customer class at various nodes are independent. It may be worth recalling here that by the output thereom [6,7] customers of each class leave the system in a Poisson stream, although by the example in [7] the flow of customers between any two adjacent nodes is not Poisson.
Theorem 5.2 establishes a strong presumption in favor of the conjecture that the independence holds only along paths which permit no overtaking. Suppose that of a node in the network has an M/M/m queuing system with $m \geq 2$. The existence of parallel servers clearly permits overtaking and this suggests that independence will not hold along a path containing such a node. This has been shown by Burke [10] for light traffic in a tandem connection of 3 nodes in which the middle node is M/M/m, $m \geq 2$ and the extreme nodes are M/M/1. It should be possible to extend this result.
REFERENCES


FIGURE CAPTIONS

Figure 1. $(1,2,3,\mathcal{F})$ permits no overtaking.

Figure 2. Sojourn times in $\mathcal{N},\mathcal{F}$ are identically distributed.

Figure 3. $(1,2,3)$ permits overtaking

Figure 4. Equivalent network for example.

Figure 5. $(1,...,n)$ permits overtaking.

Figure 6. The full-duplex system.
Figure 1
\[ Q = P \cup \{ 1 \} , \quad R = \{ 1, \ldots, N \} / Q \]
Figure 3
Figure 4
Figure 5
Figure 6