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ON THE DEVELOPMENT OF CORRECT PROGRAMS WITH THE DOCUMENTATION

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Andrzej Blikle

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College of Engineering
University of California, Berkeley
94704
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Andrzej Blikle†

Institute of Computer Science
Polish Academy of Sciences
PKiN, P.O. Box 22
Phone: 20-38-88 Telex 813556

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†Currently visiting the Department of Electrical Engineering and Computer Sciences, Computer Science Division, University of California, Berkeley, California, 94720

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ABSTRACT. The paper presents a method of the systematic development of correct programs. A program is called correct if it is partially correct wrt given pre- and post-conditions and if it neither loops indefinitely nor aborts. The requirement of non-abortion makes our correctness stronger than the so called total correctness which is usually understood as partial correctness plus non-looping. In the described method programs are developed and transformed together with their documentation. The documentation consists of a precondition, a postcondition and a set of assertions. The assertions are chosen in such a way that they may be used in the correctness proof of the program. This provides an adequate description of the algorithm and may also be useful in program testing. The rules of program derivation are sound, i.e. if applied to correct programs they yield correct programs. The application of the method is explained on the example of the derivation of a bubblesort program.
1. INTRODUCTION

The motivation for the structured programming (Dijkstra 1968) was to help the programmer in developing, understanding, documenting and possibly also proving correct the program. The latter goal, however important and worth of effort - see Dijkstra (1976) for an interesting discussion of this problem - is still quite cumbersome at least if we first develop the program and only then try to prove it correct. The idea of developing and proving programs simultaneously stimulated many authors to formalize the process of programming by describing its steps as more or less formal transformations (Dijkstra (1975), Darlington (1975, 76), Spitzen, Levitt and Lawrence (1976), Wegbreit (1976), Bär (1977), Burstal and Darlington (1977) Björner (1978)). In such an approach every step of a proof of correctness has the same scheme: we prove the correctness of the current version of the program knowing that all former versions were correct. This observation rises immediately a new idea. Instead of checking each time that our programming step does not violates the correctness, we can prove once and for all that in a certain class of programs this step always preserves the correctness (Dershowitz and Manna (1975), van Emden (1975), Irlik (1976), Blikle (1977A, B, 78), Back (1978)). In this way the correctness proofs of programs are replaced by the soundness proofs of transformation rules.
In developing programs by sound transformations we successfully avoid the necessity of proving programs correct but at the same time we lose of course, the unique opportunity of learning from the proof of correctness about many relevant properties of the program (cf. Dijkstra (1976)). Even if these properties may be implicitly seen by the programmer through the way he has developed the program, they certainly will not be seen by the user since they are not reflected - or even not reflectable - in the specification of programs by pre- and postconditions.

In order to maintain all the advantages of programming by sound transformations without losing the advantages of having the proof of correctness in an explicite form we propose in this paper to enrich the input-output specification of programs by the specification of the proof of correctness. Technically, the proof of correctness is specified by a set of assertions nested in the appropriate places between the instructions of the program. Program correctness is understood here as partial correctness plus non-looping and non-abortion. Such correctness is stronger than so called total correctness which usually means (cf. Manna and Pnueli (1974) and Manna (1974)) partial correctness plus non-looping. The fact that we deal with the abortion problem makes the classical logic inadequate for the treatment of conditions and assertions in our method. We are using therefore McCarthy's (1967) partial logic which perfectly fits to that goal.

Another method of developing programs with assertions has been described by Lee, de Roever and Gerhard (1979). In that method, however, the underlined concept of program correctness is the
partial correctness and the assertions coincide with Floyd's invariants.

The fact that we extend the I/O specification of programs (the pre- and postconditions) by adding assertions seems to have the following advantages: First of all, assertions describe local properties of programs which may be helpful not only in program understanding but also in its maintenance and testing. Secondly, knowing assertions we can recheck program correctness in a nearly mechanical way. This option may be of interest in all these circumstances, where we need an extra high reliability of programs; i.e. in microprogramming. Finally, since our assertions satisfy the requirements of proofs of termination, they adequately describe the time complexity of all loops.

The paper is organized as follows. Sec.2 contains the description of an abstract programming language which provides the experimentation field for the method. Sec.3 is devoted to the particular logical framework which is needed in order to handle the problem of abortions. Sec.4 introduces the concept of the assertion-correctness of programs. The program development rules and the problem of their soundness is discussed in Sec.5. The last Sec.6 contains an example of the application of the method in the development of a bubblesort program. Another such example may be found in Blikle (1978) where an earlier version of the present method was applied to the development of an efficient program computing the integer square root.

2. THE LANGUAGE OF PROGRAMS' DEVELOPMENT AND SPECIFICATION

It is not the aim of this paper to concentrate on the technical details of a programming language suitable for our method of prog-
ram derivation. All we want to convey to the reader is the general idea of such a language along with some technical suggestions about the transformation rules and the programming techniques. The language which is described below should be considered as only an experimental version. Since it represents a certain method of programming and since it is the first approximation of what we may expect to have in the future, we shall call it PROMETH-1. (programming method, 1).

PROMETH-1 is a language of programs' development and documentation, rather than a simple language for coding algorithms. Consequently we allow there certain abstract constructions and data types which will be used only in the development and the documentation of programs but which are not intended for implementation. Secondly, our language represents, in fact, a family of languages with a common general syntax and semantics but with different abstract data types. In other words, we have the option of user-definable data types. Each time the user intends to derive a program, he starts from the design of an appropriate data type, thus establishing the primitives of the syntax and the semantics of his-problem-oriented version of PROMETH-1.

In this paper by an abstract data type (cf. Guttag (1977), Liskov and Zilles (1975)) we mean a relational system of the form

\[ DT = (D, f_1, \ldots, f_n, q_1, \ldots, q_m) \]

where \( D \) is a nonempty many-sorted carrier and \( f_i : [D] \to D \) and \( q_j : [D] \to \{ \text{true}, \text{false} \} \) are partial functions and partial predicates respectively. The partiality of functions and predicates is an essential point in our approach and is strongly connected with the fact that we are dealing with the problem of abortion (see Sec.3). The problem of
data-type specification in PROMETH-1 is skiped. For the sake of this paper we simply assume that our data type is always somehow defined - e.g. in the set theory. A very elegant formalism for data-type specification is provided by the initial-algebra approach (see ADJ (1975), Goguen (1978), Erig, Kreowski, Padawitz (1978) and papers referenced there).

Given DT we may establish the primitive syntactical components of PROMETH-1. First, with each \( f_i \) and \( q_j \) we associate the symbols \( F_i \) and \( Q_j \) respectively. For simplicity we assume that "=" will denote both, the identity relation in \( D \) and the corresponding predicate symbol. We also assume that for each sort in \( D \) there is in the set of \( q_j \)'s the corresponding sort predicate. This is a unary total predicate which gives the value true for arguments of the given sort and gives false for all other arguments. Typical sort predicates are integer \( n \), array \( a \), etc. We also want to have constant predicates true and false defined in an obvious way. Of course, these predicates are total as well.

Having introduced the data-type oriented syntax we establish the infinite set of identifiers (individual variables) IDE and we are ready to define the class EXP of expressions and the class CON of conditions. These classes are mutually recursive, i.e. each of them is defined recursively with respect to the other. Formally we should use here a set of BNF equations but for the sake of clarity we restrict ourselves to a more intuitive definition.

\[ EXP \text{ is the least syntactical class with the following properties:} \]

1) \( IDE \subseteq EXP \)

2) \( F_i(E_1, \ldots, E_a) \in EXP \) for any \( i \leq n \) and any \( E_1, \ldots, E_a \in EXP \)
3) \[ \text{if } c \text{ then } E_1 \text{ else } E_2 \text{ fi } \in \text{EXP} \] for any \( c \in \text{CON} \)
and any \( E_1, E_2 \in \text{EXP} \).

\( \text{CON} \) is the least syntactical class with the following properties:

1) \( \mu_j(E_1, \ldots, E_{b_j}) \in \text{CON} \) for any \( j \leq m \) and any
\( E_1, \ldots, E_{b_j} \in \text{EXP} \)

2) \( c_1 \rightarrow c_2, c_3 \in \text{CON} \) for any \( c_1, c_2, c_3 \in \text{CON} \)

3) \( (\forall x)c \in \text{CON} \) and \( (\exists x)c \in \text{CON} \) for any \( x \in \text{IDE} \) and \( c \in \text{CON} \)

Remark. In the applications we identify \( F_i \) with \( f_i \) and \( Q_j \)
with \( q_j \) and allow the infix notation. Typical elementary ex-
pressions are therefore \( x + \sqrt{y}, (x + y) - z, \max(k | k < 2^n) \), etc. and ty-

tical elementary conditions are of the form \( z < y \), a is sorted,
\( i < \text{length } a \), etc. □

Having defined \( \text{IDE} \), \( \text{EXP} \) and \( \text{CON} \) we can define subsequent syntac-
tical classes: \( \text{ASR} \) - of assertions, \( \text{TES} \) - of tests, \( \text{ASG} \) - of
assignments, \( \text{EIN} \) - of elementary instructions, \( \text{SIN} \) - of simple
instructions and \( \text{INS} \) - of instructions. We use the BNF formalism
for this purpose:

\[
\begin{align*}
\text{ASR} & ::= \text{as CON sa} \\
\text{TES} & ::= \text{if CON fi} \\
\text{ASG} & ::= \text{IDE := EXP} \\
\text{EIN} & ::= \text{skip} | \text{abort} | \text{TES} | \text{ASG} | \text{EIN;} \text{EIN} \\
\text{SIN} & ::= \text{EIN} \\
& \quad | \text{if CON then INS else INS fi} \\
& \quad | \text{while CON do INS as CON sa EXP od}
\end{align*}
\]
The class \texttt{SIN} has been introduced for technical reasons in order to have an unambiguous grammar. This is necessary for the definition of semantics. (Sec. 4)

So far we have defined rather usual programming concepts, although with somewhat extravagant syntax. The latter is the consequence of the assumption that our instructions (programs) are enriched by assertions. In the semantics of instructions these assertions play the role of comments and are simply skipped in the execution. Their role becomes essential in the class of \textit{assertion specified programs} (abbreviated \texttt{a.s. programs}). This class is denoted by \texttt{ASP} and is defined by the equation:

\[
\texttt{ASP ::= \text{pre CON; INS post CON}}
\]

In every a.s. program the conditions following \texttt{pre} and \texttt{post} are called the \textit{precondition} and the \textit{postcondition} respectively. In contrast to instructions, which describe algorithms and therefore their semantical meanings are I/O functions, the a.s. programs are claims about algorithms and therefore their semantical meanings are truth values. This is formalised in Sec. 4.

In order to define the semantics of our language we need to recall a few elementary facts from the calculus of binary relations. Let $D_1, D_2$ and $D_3$ be arbitrary nonempty sets. Given two binary relations $R_1 \subseteq D_1 \times D_2$ and $R_2 \subseteq D_2 \times D_3$ we define their
composition by the equation \( R_1 R_2 = \{(a, b) | (\exists c) (a R_1 c \& c R_2 b)\} \). This operation is associative, distributive over arbitrary unions and monotone (w.r.t. inclusion \( \subseteq \)) in both arguments. Instead of \( a R_1 c \& c R_2 b \) we frequently write, for short, \( a R_1 c R_2 b \). The operation of composition may be generalized to the cases where one of the arguments is a set. Let \( B \subseteq D_1 \), \( C \subseteq D_2 \) and \( R \subseteq D_1 \times D_2 \). Then

\[
BR = \{a | (\exists b) (b \in B \& bRa)\}
\]

\[
RC = \{a | (\exists c) (a Rc \& c \in C)\}
\]

Of course, \( BR \) is the image of \( B \) in \( R \) and \( RC \) is the coimage of \( C \) in \( R \). These new operations are also monotone and distributive in both arguments and are weakly associative in the following sense:

\[
R_1 (R_2 C) = (R_1 R_2) C \quad \text{for} \quad R_1 \subseteq D_1 \times D_2, \quad R_2 \subseteq D_2 \times D_3, \quad C \subseteq D_3
\]

\[
(BR_1) R_2 = B (R_1 R_2) \quad \text{for} \quad B \subseteq D_1, \quad R_1 \subseteq D_1 \times D_2, \quad R_2 \subseteq D_2 \times D_3
\]

By \( \emptyset \) we denote the empty set and the empty relation. Both of them (if at all they are different!) are zeros of the composition.

The case of particular interest is that where \( D_1 = D_2 = D \), i.e. where we are considering relations \( R \subseteq D \times D \). In this case the operation of composition has a neutral element which is the identity relation

\[
I = \{(a, a) | a \in D\}
\]
Given an arbitrary \( R \subseteq D \times D \) we define \( R^0 = I \), \( R^{i+1} = RR^i \) for \( i \geq 0 \) and \( R^* = \bigcup_{i=0}^{\infty} R^i \). The latter is called the iteration or the reflexive and transitive closure of \( R \).

The first semantical object which we define over \( DT \) is the set of states \( S = \{IDE \rightarrow D\} \). According to this equation states are total valuations of the set of identifiers, which means that in our model all the identifiers are global. This assumption can easily be relaxed and was adopted here for technical simplicity. It may be partly justified by the fact that we do not need the concept of a local variable in our example of Sec.6.

In this paper semantics is understood as a function (strictly speaking a many-sorted homomorphism) which assigns meanings to all the investigated syntactical entities. This function is denoted by \( [\ ] \) hence \([X]\) denotes the meaning of \( X \), where \( X \) may be an expression, a condition, an instruction etc. Of course, depending on the class where \( X \) belongs, \([X]\) will be of appropriate type:

1) \([\ ]\) : \( EXP \rightarrow [S \rightarrow D] \)
2) \([\ ]\) : \( CON \rightarrow [S \rightarrow \{true, false\}] \)
3) \([\ ]\) : \( INS \rightarrow [S \rightarrow S] \)
4) \([\ ]\) : \( ASP \rightarrow \{true, false\} \)

Here and in the sequel \([X \rightarrow Y]\) denotes the set of all partial functions from \( X \) to \( Y \).

The semantics of the class \( EXP \) is defined by the following recursive (schemes of) equations:

1) \([x](s) = s(x)\)
2) \([F_i(E_1, \ldots, E_{a_i})](s) = f_i([E_1](s), \ldots, [E_{a_i}](s))\)
This coincides with the usual understanding of expressions both in programming languages and in mathematical logic. In the semantics of CON we have to comply with a rather unusual assumption that our conditions represent partial functions too. Since this requires an additional discussion we postpone the description of the semantics of CON to Sec. 3.

Once we have established the semantics of EXP and CON the semantics of INS is defined by the denotational equations listed below. For the convenience of wording we assume that $x, E, c$ and $IN$ possibly with indices will always denote identifiers, expressions, conditions and instructions respectively.

(1) $[\text{as } c \text{ sa}] = I$  \hspace{1cm} (2.2)

In other words, assertions are semantically equivalent to \textit{skip} (e.g. as comments in ALGOL 60).

(2) $[\text{if } c \text{ fi}] = \{(s, s) | [c](s) = \text{true}\}$

This means that $\text{if } c \text{ fi}$ is a side-effect-free test which results a skip if $c$ is satisfied and which aborts the execution whenever either $\neg c$ is satisfied or the value of $c$ is undefined.

(3) $[x := E] = \{(s_1, s_2) | s_2(x) = [E](s_1) \text{ and } s_2(y) = s_1(y) \text{ for all } y \in \text{IDE-}\{x\}\}$
The semantics of the class ASP of assertion specifies programs strongly relates to the semantics of conditions and therefore is postponed to Sec. 4.

3. ON THE PARTIALITY OF CONDITIONS AND THE UNDERLINED LOGIC

In the majority of approaches to the problem of program correctness one may find the assumption that the expressions and conditions represent total functions. This assumption considerably simplifies the mathematical model but from the practical point of view is hardly acceptable. Every programmer knows that both expressions and conditions may lead to abortion if evaluated in an improper environment. For instance we frequently cannot evaluate division on integers and we certainly cannot evaluate the condition \( a(i) < a(j) \) whenever either \( i \) or \( j \) is outside of the scope of \( a \). Whereas the first case can easily be detected on the syntactical level (in compile time), the second requires the semantical analysis.

The partiality of expressions is something to which we already got used in mathematics, e.g. in the theory of recursive functions, and therefore it does not require any particular explanation. The
partiality of conditions, however, has not been so widely accept-
ed although the need of it in the theory of programs was recogniz-
ed as early as in 1961 by J. McCarthy (see McCarthy 1967). We take
the McCarthy's model as the base for our definition of the seman-
tics of CON.

Similarly as for the case of EXP (Sec. 2) the semantics of CON
is defined by a set of (schemes of) recursive equations

1) \[ Q_j(E_1, \ldots, E_{b_j}) (s) = q_j([E_1](s), \ldots, [E_{b_j}](s)). \]

2) \[ [c_1 \land c_2, c_3] (s) = \begin{cases} 
[c_2](s) & \text{if } [c_1](s) = \text{true} \\
[c_3](s) & \text{if } [c_1](s) = \text{false} \\
\text{undefined} & \text{if } [c_1](s) \text{ undefined} 
\end{cases} \]

3) \[ ((\forall x)c)(s) = \begin{cases} 
\text{true} & \text{if for any state } s_1 \text{ which differs} \\
& \text{from } s \text{ at most in } x, [c](s) = \text{true} \\
\text{false} & \text{if there exists a state } s_1 \text{ which} \\
& \text{differs from } s \text{ at most in } x, \text{ such} \\
& \text{that } [c](s_1) = \text{false} \\
\text{undefined} & \text{in all other cases, i.e. if there} \\
& \text{is no state } s_1 \text{ which differs from } s \\
& \text{at most in } x \text{ such that } [c](s) = \text{false} \\
& \text{but for some states } s_1 \text{ which differ} \\
& \text{from } s \text{ at most in } x, [c](s) \text{ is} \\
& \text{undefined.} 
\end{cases} \]
true if there exists a state $s_1$ which differs from $s$ at most in $x$, such that $[c](s) = true$

4) $[(\exists x)c](s) =$

false if for any state $s_1$ which differs from $s$ at most in $x$, $[c](s_1) = false$

undefined in all other cases

These equations require a few comments. First of all observe that the evaluation of the condition $c_1 \lor c_2 \land c_3$ is similar to the evaluation of the if-then-else expressions. If $c_1$ is undefined, then the whole condition is undefined. If $c_1$ is true, then we evaluate $c_2$ regardless whether $c_3$ is defined or not. The same concerns the symmetrical case. For instance $x>0 \land x+1>0, x^{-1}<0$ is true for any state $s$ such that $s(x) = 0$ despite the fact that $[x^{-1}<0](s)$ is undefined. Some important consequences of this property of $\land$ will be discussed later in this section. Now let us concentrate on a few examples with quantifiers. Suppose that the carrier $D$ of our data type contains two sorts - integers and integer arrays, and consider a state $s$ such that for a certain identifier $y$, $[\text{integer } y](s) = true$.

Then

1) $[(\forall x)(x+y)^2>0](s) = false$

2) $[(\forall x)(x+y)^2>0](s)$ is undefined

3) $[(\forall x)(\text{integer } x \rightarrow (x+y)^2>0, true)](s) = true$

4) $[(\exists x)(x+y)^2\leq0](s) = true$

5) $[(\exists x)(x+y)^2<0](s)$ is undefined

6) $[(\exists x)(\text{integer } x \rightarrow (x+y)^2<0, false)](s) = false$
In the examples 3) and 6) the quantifiers are restricted to a certain sort. Since this is a very common case, it is worth to extend the syntax of CON by allowing the conditions of the form $(\forall \text{sort } x)c$ and $(\exists \text{sort } x)c$, with the following semantics:

$$[(\forall \text{sort } x)c] = [(\forall x)(\text{sort } x \rightarrow c, \text{true})]$$
$$[(\exists \text{sort } x)c] = [(\exists x)(\text{sort } x \rightarrow c, \text{false})]$$

Now, 3) and 6) can be written in a more readable way:

7) $[(\forall \text{integer } x)(x+y)^2 \geq 0)(s) = \text{true}$
8) $[(\exists \text{integer } x)(x+y)^2 < 0)(s) = \text{false}$

For further convenience we may extend CON again by allowing the usual connectives such as $\lor$, $\land$, $\neg$ and $\Rightarrow$. We define their semantics after McCarthy (1967):

1) $[c_1 \lor c_2] = [c_1 \rightarrow \text{true}, c_2]$
2) $[c_1 \land c_2] = [c_1 \rightarrow c_2, \text{false}]$
3) $[\neg c_1] = [c_1 \rightarrow \text{false}, \text{true}]$  \hspace{1cm} (3.1)
4) $[c_1 \Rightarrow c_2] = [c_1 \rightarrow c_2, \text{true}]$

These connectives constitute a natural generalization of the classical case. Indeed, if the values of both $c_1$ and $c_2$ are defined, then the values of 1) - 4) are the same as in the classical logic. If $c_1$ is undefined, then each of 1)-4) is undefined but if $c_1$ is defined, then 1),2) and 4) may be defined even if $c_2$ is undefined. This asymmetry may be interpreted as the consequence of the fact that in our semantics we execute the conditions from left to right. E.g. if we execute $c_1 \lor c_2$ and the value of
c_1 \text{ turns out to be true, then we do not care about } c_2. \text{ Due to this principle neither } \lor \text{ nor } \land \text{ is commutative in McCarthy's logic.}

In our approach to programming we frequently have to describe certain relations which may hold between conditions. For this sake we first introduce an auxiliary notation. Let for any c

\[ \{c\} = \{s | [c](s) = \text{true} \} \]

As is easy to prove, for any \( c_1 \) and \( c_2 \)

\[ \{c_1 \land c_2\} = \{c_1\} \cap \{c_2\} \]
\[ \{c_1 \lor c_2\} \subseteq \{c_1\} \cup \{c_2\} \]

Now, we define four relations in the set CON:

- \( c_1 \equiv c_2 \) if \( [c_1] = [c_2] \) read: \( c_1 \) is strongly equivalent to \( c_2 \)
- \( c_1 \subseteq c_2 \) if \( [c_1] \subseteq [c_2] \) read: \( c_1 \) is less defined than \( c_2 \)
- \( c_1 \iff c_2 \) if \( \{c_1\} = \{c_2\} \) read: \( c_1 \) is equivalent to \( c_2 \)
- \( c_1 \implies c_2 \) if \( \{c_1\} \subseteq \{c_2\} \) read: \( c_1 \) implies \( c_2 \)

Our strong equivalence coincides with the McCarthy's strong equivalence but our equivalence is not his weak equivalence.

The set CON may be regarded as a relational system with the operations \( \equiv, \subseteq, \iff, \implies \). Below we sketch some properties of this system which we shall need in the applications. Proofs are left to the reader. Here and in the sequel we adopt the convention of using the words equivalent and implies homonymously: in the sense atta-
ched to $\iff$ and $\implies$ and in a colloquial sense, e.g. in saying that $c_1 \subseteq c_2$ implies $c_1 \implies c_2$. The appropriate meaning will be always defined by the context.

THEOREM 3.1 The relations $\sim$ and $\iff$ are equivalence relations in $\text{CON}$. Moreover $\sim$ is a congruence, but $\iff$ is not.

The relation $\iff$ is not a congruence since $c_1 \iff c_2$ does not imply $\sim c_1 \iff \sim c_2$.

THEOREM 3.2 The relations $\leq$ and $\implies$ are partial orderings in $\text{CON/}_{\sim}$ and $\text{CON/}_{\iff}$ respectively. The operations $\lor$ and $\land$ are monotone wrt both these orderings and the remaining operations are monotone only wrt $\leq$.

THEOREM 3.3 The equivalence $\sim$ is strictly stronger than $\iff$, i.e. $c_1 \sim c_2$ implies $c_1 \iff c_2$ but not vice versa. Also the ordering $\leq$ is strictly stronger than $\implies$, i.e. $c_1 \leq c_2$ implies $c_1 \implies c_2$ but not vice versa.

Below we are listing some important equivalences and inequalities of the propositional calculus in $\text{CON}$:

\begin{align*}
(1a) & \quad (c_1 \lor c_2) \lor c_3 \equiv c_1 \lor (c_2 \lor c_3) \\
(1b) & \quad (c_1 \land c_2) \land c_3 \equiv c_1 \land (c_2 \land c_3) \\
(2a) & \quad c_1 \lor c_1 \equiv c_1 \\
(2b) & \quad c_1 \land c_1 \equiv c_1 \\
(3a) & \quad c_1 \lor (c_1 \land c_2) \equiv c_1 \\
(3b) & \quad c_1 \land (c_1 \lor c_2) \equiv c_1 \\
(4a) & \quad c_1 \land (c_2 \lor c_3) \equiv (c_1 \land c_2) \lor (c_1 \land c_3)
\end{align*}
(4b)  \( c_1 \lor (c_2 \land c_3) \equiv (c_1 \lor c_2) \land (c_1 \lor c_3) \)

(5a)  \( c_1 \lor \text{false} \equiv c_1 \)

(5b)  \( c_1 \land \text{true} \equiv c_1 \)

(6)  \( \sim(-c_1) \equiv c_1 \)

(7a)  \( \sim(c_1 \lor c_2) \equiv \sim c_1 \land \sim c_2 \)

(7b)  \( \sim(c_1 \land c_2) \equiv \sim c_1 \lor \sim c_2 \)

(8a)  \( c_1 \lor \sim c_1 \equiv \text{true} \)

(8b)  \( c_1 \land \sim c_1 \equiv \text{false} \)

(9a)  \( \sim(\exists x) c \equiv (\forall x)(\sim c) \)

(9b)  \( \sim(\forall x) c \equiv (\exists x)(\sim c) \)

This proves that McCarthy's calculus with the strong equivalence is quite similar to the classical propositional calculus. So far we have discovered just two exceptions: (1) the lack of the commutativity of \( \lor \) and \( \land \), and (2) the inequalities in the place of equivalences in (8a) and (8b). On the strength of Theorem 3.3 we can replace \( \sim \) by \( \iff \) in (1a)-(7b) and \( \subseteq \) by \( \implies \) in (8a), (8b). There are also some laws which hold for \( \iff \) and \( \implies \) but does not hold for \( \sim \) and \( \subseteq \):

(10a)  \( c_1 \iff c_1 \lor c_2 \)

(10b)  \( c_1 \land c_2 \implies c_1 \)

(11)  \( c_1 \land c_2 \iff c_2 \land c_1 \)

Here the symmetry between \( \lor \) and \( \land \) is no more the case. In \( \text{CON/}\iff \), \( \land \) is commutative but \( \lor \) is not. In particular \( c_1 \iff c_2 \lor c_1 \) does not hold!

The discussion of McCarthy's logical calculus given in this section
is far from being complete. We only gave a general outline of
the approach restricted to our needs connected with the develop-
ment of the example of Sec.6. This subject definitely deserves an
independent investigation.

4. THE CORRECTNESS AND THE ASSERTION CORRECTNESS OF PROGRAMS
As was already mentioned in Sec.2 the semantical meanings of a.s.
programs are truth values. Accordingly to the traditional wording
of the field we shall say, however, that an a.s. program is correct
rather than true. Below we define two concepts of correctness. The
first is the strengthenning of Manna-Pnueli's total correctness
and may be understood as describing an auxiliary semntics. The
other, called assertion correctness, is the principal concept of
correctness in our method.

An assertion specified program \(\text{pre } c_{pr}; \text{IN post } c_{po}\) is called
correct if

\[
\{c_{pr}\} \subseteq [\text{IN}]{c_{po}}
\]  

(4.1)

This correctness means that for any state \(s\) which satisfies \(c_{pr}\)
the execution of \(\text{IN}\) terminates successfully - i.e. neither aborts
nor runs indefinitely - and the output state satisfies \(c_{po}\).
Observe that in the usual understanding of total correctness
(Manna and Pnueli (1974), Manna (1974)) the problem of abortion
is neglected: successful termination simply means no indefinite
execution. Consequently, the correctness defined by (4.1) is
stronger than the total correctness. For better explanation con-
sider the program
pre integer array A[0:n] \exists a = A \exists i = n

while a(i) < a(i-1) do a := swap (a, i, i-1);
    i := i-1 od

post a is a permutation of A

where swap (a, i, i-1) denotes the result of swapping the i-th element with the i-1 element in a. This program is totally correct (i.e. may be proved correct in the Manna-Pnueli's system) but it is not correct in our sense since i may reach the value of 0 in which case the execution aborts. For further discussion of (4.1) and the corresponding proof techniques see Blikle (1977C,79).

The above defined concept of correctness is restricted to global properties of programs. Below we define the assertion correctness which refers not only to the pre- and postcondition but also to the assertions of the program. Intuitively pre c_pr; IN post c_po is assertion correct if it is correct and if the assertions which occur in IN may be used in the proof of (4.1). The formal definition is inductive w.r.t. the syntax of INS:

(A) For any elementary instruction IN the a.s. program pre c_pr; IN post c_po is assertion correct if it is correct. Notice that elementary instructions contain no assertions.

(B) The a.s. program

\[
\text{pre c_pr; if c then IN}_1 \text{ else IN}_2 \text{ fi post c_po}
\]

is assertion correct if
(B1) \( c_{pr} \implies c \lor \neg c \)

(B2) \[ \text{pre } c_{pr} \land c; \text{ IN}_1 \text{ post } c_{po} \]
is assertion correct

(B3) \[ \text{pre } c_{pr} \land \neg c; \text{ IN}_2 \text{ post } c_{po} \]
is assertion correct

(C) The a.s. program

\[ \text{pre } c_{pr}; \text{ while } c \text{ do IN as } c_a \text{ sa } E \text{ od post } c_{po} \]
is assertion correct if

(C1) \( c_{pr} \implies c_a \land E \geq 0 \)

(C2) \[ \text{pre } c_a \land E \geq 1; \text{ if } c \text{ fi; IN post } c_a \land E \geq 0 \]
is a.c.

(C3) \[ \text{pre } c_a \land E < 1; \text{ if } \neg c \text{ fi post } c_{po} \]
is a.c.

(C4) \[ [\text{if } c_a \land E \geq 1 \text{ fi}][\text{IN}][E] \leq [E-1] \]

This definition requires a few comments. First of all, \( E \) is here the loop counter i.e. a real expression whose integer value gives the number of cycles through IN which must be performed in order to exit from the loop. This concept may be easily generalized using well founded sets (Floyd (1967)). We do not need, however, this generalization in our example of Sec.6 and moreover the arithmetical loop counter has the advantage of giving the explicit estimation of the time complexity of the loop (see the example in Sec.6). The condition \( c_a \) is called the loop assertion and loosely speaking describes the global effect of IN. Under this interpretation (C1) says that for any state which satisfies the precondition \( c_{pr} \), the loop assertion is satisfied and the number of cycles to be performed is defined. (C2) says that whenever the loop assertion is satisfied and the number of remaining cycles is not less than 1, then the body of the loop is executable, the
successive state satisfies $c_a$ again and the number of remaining cycles through the loop is defined. It also says that the above property may be proved using the assertions of $IN$. The conjunction of (C1) with (C2) guarantees that under the precondition $c_{pr}$ the loop will be executed without abortion and $c_a$ will be preserved in each cycle. Two remaining conditions imply that this execution will not continue indefinitely. Indeed, (C3) claims that if $c_a$ is satisfied and the remaining number of cycles is 0 then the control exits the loop and the postcondition is satisfied. The last condition (C4) guarantees that the value of $E$ will fall under 1 in a finite time since any execution of $IN$ in the environment where $c_a \& E > 1$ is satisfied decrements the value of $E$ by 1.

(D) If $IN_1 \in SIN$ then the a.s. program

$$\text{pre } c_{pr}; \text{IN}_1 \text{ as } c_a \text{ sa } \text{IN}_2 \text{ post } c_{po}$$

is assertion correct if

(D1) $\text{pre } c_{pr}; \text{IN}_1 \text{ post } c_a$ is assertion correct
(D2) $\text{pre } c_a; \text{IN}_2 \text{ post } c_{po}$ is assertion correct

(E) The a.s. program

$$\text{pre } c_{pr}; \text{inv } c_i; \text{IN vni post } c_{po} \quad (4.2)$$

is assertion correct if the a.s. program $\text{pre } c_{pr}; \text{IN}_1 \text{ post } c_{po}$
where \( IN_1 \) results in from \( IN \) by the substitution for each assertion as \( c_a \$ sa \) in \( IN \) the assertion as \( c_a \$ c_1 \$ sa \), is assertion correct.

The condition \( c_1 \) in (E) is called the permanent invariant in (4.2) and \( inv \ c_1 \) is called its declaration. The mirror key-word \( vni \) defines the scope of this declaration. Permanent invariants are used to "factorize" conditions which are permanently satisfied in a segment of a program. Typical factorizable conditions are these which describe the unchangable properties of the environment, e.g. the type of identifiers. More examples are provided in Sec.6.

To complete the definition of assertion correctness observe that every assertion in an assertion correct program is a Floyd's invariant but not vice versa. The critical point is that the Floyd invariants usually do not guarantee the executability and the termination of \( IN \). Indeed, consider the a.s. program

\[
\text{pre real } a \& a \geq 0 \& x=a \\
x:=x+1 \\
\text{as } x=a+1 \& x>0 \$ sa \\
x:=x^{-1} \\
\text{post } x=(a+1)^{-1}
\]

which is, of course, assertion correct. In the proof of partial correctness of this program we could use the invariant \( x=a+1 \). This invariant is, however, too weak to prove nonabortion and therefore it is not an assertion in our sense.

One of our motivations in defining the concept of assertion correct program was to formalize the property that a given set of
assertions can be used in proving a given program correct. To make sure that our goal has not been missed we must prove, first of all, that every assertion-correct program is correct. The proof of this theorem will also indicate in which way our assertions may be used in the proofs of program correctness.

THEOREM 4.1 Every a.s. program which is assertion correct is correct.

PROOF. This must be proved by induction on the syntactical complexity of a.s. programs. The first step (case (A) of the definition) is obvious. In the induction step we must consider the cases (B)-(E). Since the only nontrivial case is (C) consider the a.s. program

\[ \text{pre } c_{pr}; \text{while } c \text{ do IN as } c_a \text{ sa } E \text{ od post } c_{po} \]  

(4.3)

and assume that it is assertion correct. Now, let for any integer \( i \geq 0 \), \( A_i = \{c_a \& i \leq E < i+1\} \). We shall show the following

1) \( \{c_{pr}\} \subseteq \bigcup_{i=0}^{\infty} A_i \)

2) \( (\forall i \geq 1) (A_i \subseteq [\text{if } c \text{ fi; IN}]A_{i-1}) \)

3) \( A_0 \subseteq [\text{if } \neg c \text{ fi}]\{c_{po}\} \).

The conditions 1) and 3) are immediate from (C1) and (C3) respectively. Prove 2). Let \( i \geq 1 \) and let \( s \in A_i \). Then \( s \in \{c_a\} \) and \( [E](s) = d \) for some \( d \) with \( i \leq d < i+1 \). By (C2) and the induction assumption we have
hence there exists $s_1$ with $s \subseteq s_1$, $s_1 \in \{ c_a \}$ and $[E](s_1) = d_1$ for some $d_1 > 0$. Since, of course

$$s \subseteq [if \ c \ fi] \in \{ c_a \} \subseteq s_1 \subseteq [E]d_1$$

we get by (C4), $[E^{-1}](s) = d_1$. Therefore, $[E](s_1) = d_1 = [E^{-1}](s) = [E](s) - 1 = d - 1$. This implies the inequalities $i - 1 \leq [E](s_i) < i$ which implies $s_i \in A_i$ and terminates the proof of 2). Now, from 2) and 3) we prove by induction on $i \geq 0$

$$A_i \subseteq ([if \ c \ fi] \in \{ c \}) \subseteq \ldots \subseteq ([if \ c \ fi] \in \{ c \})^i$$

Therefore, by 1), \{ c_{pr} \} \subseteq \bigcup_{i=0}^{\infty} ([if \ c \ fi] \in \{ c \})^i = ([if \ c \ fi] \in \{ c \})^* = ([if \ c \ fi] \in \{ c \})^{c_{pr}} \subseteq \{ c_{pr} \}$ which completes the proof by the semantical axiom (8) of Sec.2. \[\square\]

As was mentioned in Sec.1 our assertions may be useful not only in the documentation and the mathematical verification of programs, but also in program testing. The latter follows from the fact that each assertion describes the local properties of programs, hence an a.s. program may be tested not only against a pre- and post conditions but also against the local assertions. Technically this may be done by the execution of a modified program which results in from the original one by the replacement of every assertion as $c_a \ sa$ (case D) by the test $if \ c_a \ fi$ and every loop assertion with the expression as $c_a \ sa \ E$ (case C) by the test $if \ c_a \ \& \ \& \ E \geq \ 0 \ fi$. If we call such a program a testing copy of the
original program then the following obvious theorem may be proved.

**THEOREM 4.2.** If an a.s. program is assertion correct, then the corresponding testing copy is correct. □

The obvious proof is left to the reader. Of course, we tacitly assume that the syntax of the language has been extended in such a way that the testing copies of programs belong to the language too. This requires also an obvious extension of semantics.

5. **THE RULES OF THE COMPOSITION AND THE TRANSFORMATION OF PROGRAMS.**

The main motivation for our method was to provide sound rules of programming. In this section we show a few such rules which seem to have a fairly broad field of applications. By no means, however, should our set of rules be regarded as complete. To get started we give three technical lemmas. Proofs are left to the reader.

**LEMMA 5.1.** For any identifier *k* and instruction IN the properties

(i) \([\text{IN}][k] \subseteq [k]\) and

(ii) \((\forall s_1, s_2)(s_1[\text{IN}]s_2 \implies [k](s_1) = [k](s_2))\)

are equivalent. □

This lemma says that the property \([\text{IN}][k] \subseteq [k]\) may be read as IN does not change the value of *k*. E.g. \([x:=x+y][y] \subseteq [y]\).
Of course, we can easily generalize this lemma to the case where \( k \) stands for an arbitrary expression.

**Lemma 5.2** For any function \( F: [S \rightarrow S] \) and any \( B_1, B_2, C_1, C_2 \subseteq S \) if

\[
B_1 \subseteq FC_1 \quad \text{and} \quad B_2 \subseteq FC_2
\]

then \( B_1 \cap B_2 \subseteq F(C_1 \cap C_2) \).

**Lemma 5.3** For any function \( F: [S \rightarrow S] \) and any \( C_1, C_2 \subseteq S \),

\[
FC_1 \cap FC_2 \subseteq F(C_1 \cap C_2)
\]

Returning to the sound rules of programming we may first of all observe that the definition of assertion correctness (A)-(E) in Sec. 4 already provides five such rules. The rule which follows from (D) is a bit too restricted since it requires that the first component of the composition be a simple instruction. This restriction was introduced only for the sake of the unambiguity of the definition (D) and may be relaxed now:

**Theorem 5.1** For any two instructions \( IN_1 \) and \( IN_2 \), if

\[
\text{pre } c_1; \; IN_1 \; \text{post } c_2
\]
\[
\text{pre } c_2; \; IN_2 \; \text{post } c_3
\]

are assertion correct, then
\[
\text{pre } c_1; \text{ IN}_1 \text{ as } c_2 \text{ sa } \text{ IN}_2 \text{ post } c_3
\]

is assertion correct. \(\square\)

**PROOF.** If \(\text{IN}_1\) is simple then the proof is done by the definition. Let then \(\text{IN}_1\) be arbitrary. In this case \(\text{IN}_1\) must be of the form

\[
\text{IN}_1 \text{ as } c^1 \text{ sa } \text{IN}_2 \text{ as } c^2 \text{ sa } \ldots \text{IN}_k \text{ as } c^k \text{ sa } \text{IN}_{k+1}
\]

for some \(k \geq 1\), where all \(\text{IN}_i\) are simple. This implies that the following programs as assertion correct

\[
\begin{align*}
\text{pre } c_1; \text{ IN}_1 & \text{ post } c^1 \\
\text{pre } c^1; \text{ IN}_2 & \text{ post } c^2 \\
& \ldots \\
\text{pre } c^k; \text{ IN}_{k+1} & \text{ post } c_2
\end{align*}
\]

Combining these a.s. programs according to the rule (D) we successively get the following assertion correct programs:

\[
\begin{align*}
\text{pre } c^k; \text{ IN}_{k+1} & \text{ as } c_2 \text{ sa } \text{IN}_2 \text{ post } c_3 \\
\text{pre } c^{k-1}; \text{ IN}_k & \text{ as } c^k \text{ sa } \text{IN}_{k+1} \text{ as } c_2 \text{ sa } \text{IN}_2 \text{ post } c_3 \\
& \text{etc.}
\end{align*}
\]

Besides the techniques of program development resulting from the rules (A)-(E) of Sec.4 there is another important class of techniques which we shall refer to as the introduction of an invariant. Generally speaking given an assertion correct program

\[
\text{pre } c_{pr}; \text{ IN } \text{ post } c_{po}
\]

and a condition \(c\) we say that we are introducing the invariant \(c\) into our program if we transform
IN into an \( \mathbf{IN} \), such that \( \mathbf{pre} \mathbf{c}_{\mathbf{pr}} \& \mathbf{c} \); \( \mathbf{IN} \) \( \mathbf{post} \mathbf{c}_{\mathbf{po}} \& \mathbf{c} \) is assertion correct. Below we describe two particular rules of the introduction of an invariant into a \texttt{while do} loop.

**THEOREM 5.2** *(the postfix enrichment of while do)* If

\[
\mathbf{pre} \mathbf{c}_{\mathbf{pr}} ; \texttt{while } \mathbf{c} \texttt{ do } \mathbf{IN} \texttt{ as } \mathbf{c}_{\mathbf{a}} \texttt{ sa } \mathbf{E} \texttt{ od } \mathbf{post} \mathbf{c}_{\mathbf{po}} \tag{5.1}
\]

is assertion correct, then for any \( \mathbf{c}_{1}, \mathbf{c}'_{1} \in \mathbf{CON} \) and any \( \mathbf{IN}_{1} \in \mathbf{INS} \) if

1) \( \mathbf{pre} \mathbf{c}_{a} \& \mathbf{c}_{1} \& \mathbf{E} \geq 1 \); \( \texttt{if } \mathbf{c} \texttt{ fi} \); \( \mathbf{IN} \mathbf{post} \mathbf{c}_{a} \& \mathbf{c}'_{1} \) is assertion correct

2) \( \mathbf{pre} \mathbf{c}_{a} \& \mathbf{c}'_{1} ; \mathbf{IN}_{1} \mathbf{post} \mathbf{c}_{a} \& \mathbf{E} \geq 0 \& \mathbf{c}_{1} \) is assertion correct

3) \( [\mathbf{if } \mathbf{c}_{a} \& \mathbf{c}'_{1} \& \mathbf{E} \geq 0 \texttt{ fi}] [\mathbf{IN}_{1}] [\mathbf{E}] \subseteq [\mathbf{E}] \)

then

\[
\mathbf{pre} \mathbf{c}_{\mathbf{pr}} \& \mathbf{c}_{1} ; \texttt{while } \mathbf{c} \texttt{ do } \mathbf{IN} \texttt{ as } \mathbf{c}_{a} \& \mathbf{c}'_{1} \texttt{ sa } \mathbf{IN}_{1} \texttt{ as } \mathbf{c}_{a} \& \mathbf{c}_{1} \texttt{ sa } \mathbf{E} \texttt{ od } \mathbf{post} \mathbf{c}_{\mathbf{po}} \& \mathbf{c}_{1}
\]

is assertion correct. \( \square \)

**COMMENT.** Since \( \mathbf{IN} \) violates the required invariant \( \mathbf{c}_{1} \) (assumption 1)), we have to supply the loop body with a recovery instruction \( \mathbf{IN}_{1} \), leading back to \( \mathbf{c}_{1} \) (assumption 2)). To make it sure that the alteration of the loop does not violate the termination property, we assume that \( \mathbf{IN}_{1} \) preserves the value of the loop counter \( \mathbf{E} \) (assumption 3)). \( \square \)
PROOF. We have to check that the resulting program satisfies the definition (C) of Sec. 4. First observe that (C1) and (C3) follow immediately from the assertion correctness of (5.1). Next, (C2) follows from 1), 2) and Theorem 5.1. It remains (C4) to be proved. By the assertion-correctness of (5.1)

\[
\text{if } c_a \land E > 1 \text{ fi}[\text{IN}][E] \subseteq [E-1]
\]

Therefore by 3) and the monotonicity of composition we get

\[
\text{if } c_a \land c_1 \land E > 1 \text{ fi}[\text{IN}][\text{if } c_a \land c_1 \land E > 0 \text{ fi}][\text{IN}][E] \subseteq [E-1] \quad (5.2)
\]

Now, by 1) (C2) and lemma 5.2 we have

\[
\{c_a \land c_1 \land E > 1\} \subseteq [\text{if } c \text{ fi}][\text{IN}] \{c_a \land c_1 \land E > 0\} \subseteq [\text{IN}] \{c_a \land c_1 \land E > 0\}
\]

This implies

\[
\text{if } c_a \land c_1 \land E > 1 \text{ fi}[\text{IN}][\text{if } c_a \land c_1 \land E > 0 \text{ fi}] = [\text{if } c_a \land c_1 \land E > 1 \text{ fi}][\text{IN}]
\]

By (5.2) we get therefore

\[
\text{if } c_a \land c_1 \land E > 1 \text{ fi}[\text{IN}][\text{IN}][E] \subseteq [E-1]
\]

\[
\text{THEOREM 5.3 (the prefix enrichment of while-do). If}
\]

\[
\text{pre } c_{pr}; \text{ while } c \text{ do IN as } c_a \text{ sa } E \text{ od post } c_{po}
\]
is assertion correct and

1) \( \text{pre } c_a \land c_1 \land E \geq 1; \text{ if } c \text{ fi } \text{ IN}_1 \text{ post } c_a \land c_1 \) is assertion correct

2) \( \text{pre } c_a \land c_1; \text{ IN post } c_a \land c_1 \land E > 0 \) is assertion correct

3) \([\text{if } c_a \land c_1 \land E \geq 1 \text{ fi}] [\text{IN}_1][\text{IN}] [E] \subseteq [E-1]\)

then

\( \text{pre } c_{pr} \land c_1; \text{ while } c \text{ do } \text{ IN}_1 \text{ as } c_a \land c_1 \text{ sa } \text{ IN as } c_a \land c_1 \text{ sa } E \text{ od post } c_{po} \land c_1 \)

is assertion correct. \( \square \)

COMMENT. This transformation is dual to the former. The new instruction \( \text{IN}_1 \), which is executed before \( \text{IN} \), violates \( c_1 \) and the old instruction \( \text{IN} \) provides the recovery. Since the nonmodification of \( E \) by \( \text{IN}_1 \) does not imply termination in this case, we have to assume that \( \text{IN}_1;\text{IN} \) has the property required in the definition. \( \square \)

PROOF. The case (C1), (C2) and (C3) as in Theorem 5.2. The case (C4) is obvious by 3). \( \square \)

The sound rules of programming described so far are either the rules of composition (B)-(D) or are transformations which change the structure of the program. Another large group of rules consists of transformations which only modify conditions in the program, but which do not change the control structure. Below we give three examples of such rules which are commonly used in program derivation.
THEOREM 5.4 If the a.s. program

\[ \text{pre } c_{pr}; \text{ IN post } c_{po} \]

is assertion correct and \( c'_{pr} \implies c_{pr} \) and \( c_{po} \implies c'_{po} \), then

\[ \text{pre } c'_{pr}; \text{ IN post } c'_{po} \]

is assertion correct. \( \square \)

The proof is obvious.

THEOREM 5.5. If in an arbitrary assertion correct program we replace:

1) any while-do or if-then-else condition \( c \) by \( c_1 \) such that \( c \cong c_1 \),

2) any precondition, postcondition or assertion \( c \) by \( c_1 \) such that \( c \iff c_1 \),

then the resulting program is assertion correct. \( \square \)

The proof follows immediately from the fact that in the semantics of assertion specified programs each branching condition is represented by the truth function \([c]\), whereas each precondition, postcondition or assertion is represented by the set of states \( \{c\} \). The essential point in this theorem is, however, that we cannot replace \( \cong \) by \( \iff \) in 1). An appropriate example is given in Sec.6. We shall also see in that section that many conditions, appearing in a.s. programs are of the form \( c_1 \& \ldots \& c_n \), where \( c_i \) are elementary. Since \( \& \) commutes in \( \text{CON/\iff} \) but does not commute in \( \text{CON/\cong} \) (Sec.3) our theorem indicates that
the ordering of $c_i$'s in $c_1 \& \ldots \& c_n$ is irrelevant whenever
the latter appears as a precondition, a postcondition or an as-
sertion but becomes relevant if it appears in while-do or
if-then-else.

THEOREM 5.6. If the a.s. program

$$\text{pre } c_{pr}; \text{ while } c \text{ do } \text{IN } as \ c_a \text{ sa } E \text{ od } \text{post } c_{po}$$

is assertion correct, then the a.s. program

$$\text{pre } c_{pr}; \text{ while } c \text{ do } \text{IN } as \ c_a \text{ sa } E \text{ od } \text{post } c_{po} \& c_a$$

is assertion correct. $\square$

The proof is immediate from the definition (C) of Sec.4.

6. AN EXAMPLE OF PROGRAM DERIVATION; BUBBLESORT

To get started we recall the intuitive idea of bubblesort. Suppose
that we are given a vertical column of bubbles, each bubble having
a certain weight. Suppose that our bubbles constitute an environ-
ment which satisfies the following Archimedes' principle: each
bubble which is lighter than its upper neighbor tends to swap with
this neighbor in moving up. At some initial moment all the bubbles
are glued together which prevents them from swapping. In the first
step of bubblesort we free the first bubble from the top. Of
course, nothing will happen since this bubble has no upper neigh-
bor. Next we free the second bubble. This time a swap may occur
if the second bubble is lighter than the first one. In each suc-
cessive step of our procedure we free the successive bubble which
immediately starts to move up in searching for such a position
in the column which does not violate the Archimedes' principle.
It is intuitively quite clear that in the last step of the procedure our column of bubbles will be ordered according to the increased weights.

The systematic development of the bubblesort program requires, first of all, the establishment of an appropriate data-type. It will be developed in a stepwise manner along with the development of the program. Since in this paper we skip the problem of the formal specification of data type, we are using below a mixture of formal and intuitive mathematics. In many cases we simply refer to a known mathematical concept (e.g. that of a permutation) rather than give an axiomatic definition. Despite this informality of our approach it still seems advisable to keep the many-sorted algebra style (ADJ 1975) in the specification of sorts and arities of functions. We start by the first approximation of our data-type and program.

SORTS

Int - integers
Arr - arrays; each array is a total function

\[ a: \{0, \ldots, n\} \rightarrow \text{Int}, \text{ where } n \geq 0 \]

Bol - \{true, false\}

FUNCTIONS

+, -, 0, 1 - the arithmetical functions and constants
**length**: \( \text{Arr} \rightarrow \text{Int} \) - the length of an array

**component**: \( \text{Arr} \times \text{Int} \rightarrow \text{Int} \) - the \( i \)-th component of an array;

according to the common style we shall write \( a(i) \)
in the place of \( \text{component}(a,i) \)

**seg**: \( \text{Arr} \times \text{Int} \rightarrow \text{Arr} \) - the initial segment;

\[
\text{seg} (a,j) = (a(0),\ldots,a(j)) \text{ for } 0 < j < \text{length } a
\]

**PREDICATES**

**integer, array** - the sort predicates (Sec.2)

\( \leq, < \) - The usual arithmetical inequalities

**is sorted**: \( \text{Arr} \rightarrow \text{Bol} \)

\[
a \text{ is sorted} : \neg (\forall \text{ integer } i) (0 < i < \text{length } a \Rightarrow a(i) < a(i+1))
\]

**perm**: \( \text{Arr} \times \text{Arr} \rightarrow \text{Bol} \)

\[
a_1 \text{ perm } a_2 :\equiv a_1 \text{ is a permutation of } a_2
\]

Now, we may establish the first approximation of our program
which we shall informally call the propulsion loop. Here and in
the sequel the operational part of the program will be framed in
order to distinguish it visually from the specification part.

**pre array** \( \forall a = A \land j = 0 \land k = \text{length } A \)

**inv** \( k = \text{length } a \land \forall a \text{ perm } A \land 0 < j < k \)

\[
\text{while } j < k \text{ do } j := j + 1
\]

as true sa k-j od \( (P_1) \)

**vni**

**post** \( j = k \)
This program only defines the framework of further approximations and is, obviously, assertion correct. Into this program we shall introduce the invariant \( \text{seq}(a,j) \) is sorted using the postfix enrichment of the loop (Theorem 5.2). Let

\[
\begin{align*}
c_1 & :\equiv \text{seq} (a, j) \text{ is sorted} \\
c'_1 & :\equiv \text{seq} (a, j-1) \text{ is sorted} \& j \geq 1
\end{align*}
\]

and let

\[
\begin{align*}
c & \equiv k = \text{length} a \& \text{a perm } A \& 0 \leq j \leq k
\end{align*}
\]

Of course, \( c \) is the permanent invariant declared in \( P_1 \). Now, accordingly to Theorem 5.2 we have to check that the program

\[
\begin{align*}
\text{pre } c \& c_1 \& k-j \geq 1; \\
& \text{if } j<k \text{ fi}; j:= j+1 \\
\text{post } c \& c'_1
\end{align*}
\]

is assertion correct and we have to construct an instruction \( \text{IN}_1 \) such that the following two conditions are satisfied:

\[
\begin{align*}
\text{pre } c \& c'_1; \text{ IN}_1 & \text{ post } c \& c_1 \& k-j \geq 0 \text{ is a.c.} \quad (6.1) \\
[\text{if } c \& c'_1 \& k-j \geq 0 \text{ fi}][\text{IN}_1][k-j] & \subseteq [k-j] \quad (6.2)
\end{align*}
\]

The first requirement is, of course, satisfied. Therefore, on the strength of Theorem 5.2, for any \( \text{IN}_1 \) which satisfies (6.1) and (6.2) the subsequent program is assertion correct. We write it already in a simplified form removing \( c_1 \) from the precondition -
since for \( j=0 \) it is always true – and replacing \( j=k \& c_1 \) in the postcondition by \( j=k \& a \text{ is sorted}, \) since \( j=k \& k = \text{length } a \) 
\& \( \text{seg}(a,j) \text{ is sorted} \) implies \( j=k \& a \text{ is sorted}. \) Formally we apply here the Theorems 5.4 and 5.6.

\[
\text{pre array } A \& a=A \& j=0 \& k=\text{length } A
\]

\[
\text{inv c}
\]

\[
\begin{align*}
\text{while } j<k \text{ do} \\
\quad j:=j+1
\end{align*}
\]

\[
\text{as } \text{seg } (a,j-1) \text{ is sorted } \& j \geq 1 \text{ sa}
\]

\[
\text{IN}_1
\]

\[
\text{as } \text{seg } (a,j) \text{ is sorted } \text{sa } k-j \text{ od}
\]

\[
\text{vni}
\]

\[
\text{post } j=k \& a \text{ is sorted}
\]

Since there are many \( \text{IN}_1 \) which satisfy the conditions (6.1) and (6.2), our \( P_2 \) represents a class of sorting procedures organized accordingly to the following iterative scheme: given an array \( a \) where \( \text{seg}(a,j) \) has already been sorted, increase \( j \) by 1 and permute \( a \) in such a way that the new \( \text{seg}(a,j) \) is sorted again. Our prospective bubblesort belongs to this class. In order to describe it we extend our data type by two new sorts, four new functions and one new predicate.
SORTS

Vec - vectors; each vector is a total function \( v: \mathbb{N} \rightarrow \text{Int} \)
where \( \mathbb{N} \) is an arbitrary finite set of integers

Set - finite subsets of Int

FUNCTIONS

\textbf{swap:} \quad \text{Arr} \times \text{Int} \times \text{Int} \rightarrow \text{Arr};

\text{swap} (a, i, j) is, for \( 0 < i, j < \text{length} \ a \), the result of swapping the \( i \)-th with the \( j \)-th element in \( a \)

\textbf{but:} \quad \text{Arr} \times \text{Int} \rightarrow \text{Vec}

\text{a but} \ i \ \text{is, for} \ 0 < i < \text{length} \ a, \ \text{the restriction of array} \ a \ \text{to the domain}
\{0, \ldots, \text{length} \ a\} - \{i\}

\textbf{max:} \quad \text{Set} \rightarrow \text{Int}

\text{max} \ B \ is \ the \ maximal \ element \ of \ the \ set \ B

\textbf{bd:} \quad \text{Arr} \times \text{Int} \rightarrow \text{Int}; \ \text{read: bubbledepth}

\text{bd}(a, i) = \begin{cases} 0 & \text{if} \ i < 0 \lor a(i) > a(i-1) \ \text{then} \ 0 \\ \text{else} \ \text{max} \ \{d | a(i) < a(i-d)\} & \end{cases}

PREDICATES

First we extend the formerly defined predicate \textbf{is sorted} to the sort of vectors. We also assume that the empty vector satisfies this predicate. Now, we define the new predicate.
bubbles in \text{seg}(\cdot) : \text{Int} \times \text{Int} \times \text{Arr} \rightarrow \text{Bol}

\begin{align*}
\text{i bubbles in } \text{seg}(a, j) & := 0 < i < j < \text{length } a \\
\text{seg}(a, j) & \text{ but } i \text{ is sorted} \\
i < j \Rightarrow a(i+1) > a(i)
\end{align*}

The following may be proved easily:

\begin{align*}
\text{bd}(a, i) \geq 1 & \equiv i > 0 \land a(i) < a(i-1) \quad (6.3) \\
i = j \land j \geq 1 \land i \text{ bubbles in } \text{seg}(a, j) & \iff \\
& \iff i = j \land j \geq 1 \land \text{seg}(a, j-1) \text{ is sorted} \quad (6.4) \\
\text{bd}(a, i) = 0 \land i \text{ bubbles in } \text{seg}(a, j) & \implies \\
& \implies \text{seg}(a, j) \text{ is sorted} \quad (6.5)
\end{align*}

Using the predicate \(i \text{ bubbles in } \text{seg}(a, j)\) we may construct the assertion-specified program which describes the bubbling process:

\begin{enumerate}
\item \text{pre } c \land i = j \land j \geq 1 \land i \text{ bubbles in } \text{seg}(a, j)
\item \text{inv } c
\item \text{while } \text{bd}(a, i) \geq 1 \text{ do}
\begin{enumerate}
\item \text{while } \text{bd}(a, i) \geq 1 \text{ do}
\item a := \text{swap}(a, i-1, i)
\item as i-1 bubbles in \text{seg}(a, j) sa
\item i := i-1
\item as i bubbles in \text{seg}(a, j) sa \text{bd}(a, i) od
\end{enumerate}
\item \text{vni}
\item \text{post } c \land \text{bd}(a, i) = 0 \land i \text{ bubbles in } \text{seg}(a, j)
\end{enumerate}

The assertion correctness of this program may be proved directly from the definitions (C) and (D) of Sec.4. This proof is left to the reader. Now, we modify (P3) into the form required by the
conditions (6.1) and (6.2). This is done in the following steps.

(1) The pre- and postcondition are modified on the strength of (6.4) and (6.5); cf. Theorem 5.4.

(2) The while condition \( \text{bd}(a,i) > 1 \), which is unacceptable from the practical viewpoint, is replaced by \( i > 0 \land a(i) < a(i-1) \); cf. (6.3) and Theorem 5.5.

(3) The program which results in from (1) and (2) is combined sequentially (rule (D) of Sec.4) with the program

\[
\text{pre } c \land j > 1 \land \text{seg}(a,j-1) \text{ is sorted} \\
\begin{array}{l}
\quad i := j \\
\end{array}
\]

\[
\text{post } c \land i = j \land j > 1 \land \text{seg}(a,j-1) \text{ is sorted}
\]

We get

\[
\text{pre } c \land j > 1 \land \text{seg}(a,j-1) \text{ is sorted} \\
\text{inv } c \\
\begin{array}{l}
\quad i := j \\
\quad \text{as } i = j \land j > 1 \land \text{seg}(a,j-1) \text{ is sorted} \quad \text{sa} \\
\quad \text{while } i > 0 \land a(i) < a(i-1) \text{ do} \\
\quad \quad a := \text{swap}(a,i-1,i) \\
\quad \quad \text{as } i-1 \text{ bubbles in seg}(a,j) \text{ sa} \\
\quad \quad i := i-1 \\
\quad \quad \text{as } i \text{ bubbles in seg}(a,j) \text{ sa } \text{bd}(a,i) \text{ od} \\
\end{array}
\]

\[
\text{vni}
\]

\[
\text{post } c \land \text{seg}(a,j) \text{ is sorted}
\]
This program is assertion correct since it has been derived from another assertion correct programs using sound transformations. Since \( c \implies k-j \geq 0 \), the latter condition may be added to the post-condition of \((P_4)\). Therefore, the instruction of \((P_4)\) satisfies (6.1). It also satisfies (6.2) since neither \( j \) nor \( k \) is modified in \((P_4)\). Consequently, the instruction of \((P_4)\) has all the properties required from \( IN_1 \) of \((P_2)\) and may be substituted there. In this way we get the final version of our program:

```plaintext
pre array A \& a=A \& j=0 \& k = length A
inv k = length A \& a perm A \& 0<j<k
while j<k do
  j := j+1
  as j>1 \& seg(a,j-1) is sorted sa
  i := j
  as i=j \& j>1 \& seg(a,j-1) is sorted sa
  while i>0 \& a(i)<a(i-1) do
    a := swap(a,i-1,i)
    as i-1 bubbles in seg(a,j) sa
    i := i-1
    as i bubbles in seg(a,j) sa bd(a,i) od
  as seg(a,j) is sorted sa k-j od
vn
post j=k \& a is sorted
```

This program is, of course, assertion correct. Observe that if in the inner loop we replace the while condition \( i>0 \& a(i)<a(i-1) \) by the condition \( a(i)<a(i-1) \& i>0 \) which is equivalent - but not strongly equivalent - to the former, then we get a program
which is no longer correct. That new program aborts whenever the value of \( i \) reaches 0 since in that case \( a(i) < a(i-1) \) cannot be evaluated.

7. FINAL REMARKS

Practically all the issues discussed in this paper has only been sketched and require further development. First of all we should learn more about McCarthy's logic. Secondly, we should better develop the set of program derivation rules. Thirdly, the language PROMETH-1 should be extended by new programming constructions and by the appropriate subset of data-type specification language. Finally, some attention should be given to the methodology of programming in PROMETH.

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