ON THE FINITE SOLUTION OF NONLINEAR INEQUALITIES

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Abstract

We present an algorithm based on Newton's method and a systematic enlargement of a feasible region for solving finitely, systems of nonlinear inequalities. The method depends crucially on the superlinear rate of convergence of Newton's method.

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1. Introduction

An examination of the engineering literature (see for example [1]) shows that not infrequently the designer is not so much interested in optimizing performance, as in meeting specifications. Generally, such specifications can be expressed as a system of differentiable inequalities

$$g^j(x) \leq 0, \quad j = 1, 2, \ldots, m$$

(1.1)

which describe a set with a nonempty interior. An important special case in which a designer needs to solve a system of inequalities arises in problems of design centering, tolerancing and tuning (see [2]) [3]). In such a problem, a designer is required to minimize some performance index, subject to constraints on the form

$$\max_{\omega \in \Omega} \min_{\tau \in T} \max_{j \in J} \zeta_j^j(x, \omega, \tau) \leq 0,$$

where $x$ is the design vector (including tolerance and tuning range as components), $\omega$ is a tolerance parameter and $\tau$ is a tuning parameter. The optimization yields a nominal design $\hat{x}$ and the manufacturing process produces in a certain tolerance realization, $\hat{\omega} \in \Omega$. Should measurements show that

$$\max_{j \in J} \zeta_j^j(\hat{x}, \hat{\omega}, 0) > 0,$$

it now becomes necessary to compute a value $\hat{\tau} \in T$, for the tuning parameter, such that

$$\max_{j \in J} \zeta_j^j(\hat{x}, \hat{\omega}, \hat{\tau}) \leq 0.$$ 

Normally, $T$ has a simple description of the form

$$T = \{\tau | g^j(\tau) \leq 0, \quad j = 1, 2, \ldots, m_1\}$$

and hence the required $\hat{\tau}$ can be computed by solving a system of inequalities of the form (1), with $g^j(\tau) \triangleq \zeta_j^j(\hat{x}, \hat{\omega}, \tau)$ for all $j \in J$.

Now, as it is well known, under certain conditions, it is possible to find a solution to such a system of inequalities in a finite number of iterations by means of any one of the existing feasible directions algorithms (see [7])
Unfortunately, feasible directions algorithms are rather slow and the question arises whether it is not possible to adapt a faster method, such as the Newton method described in [4,5] to find a solution to (1.1) in a finite number of iterations. In this paper we obtain an affirmative answer to this question. Our scheme is based on applying Newton's method for a controlled number of iterations $\ell_1$ to a progression of inequalities:

$$g_j(x) + \varepsilon_i \leq 0, \quad j = 1, 2, \ldots, m \quad (1.2)$$

with $\varepsilon_i \downarrow 0$ and $\ell_{i+1} > \ell_i$, and on the fact that under certain assumptions Newton's method converges quadratically.

2. The Algorithm

Consider the problem of finding a point $\hat{x}$ satisfying

$$g(x) \leq 0 \quad (2.1)$$

where $g: \mathbb{R}^n \to \mathbb{R}^m$ is three times continuously differentiable. The first of the following assumptions is imposed by our desire to use Newton's method (see [ ]), while the second one is required to make finite solution of (2.1) possible. We shall use the notation $\mathbb{m} = \{1, 2, \ldots, m\}$.

**Assumption 2.1.** For any $x \in \mathbb{R}^n$, $0 \notin \operatorname{co} \nabla g_j^\gamma(x)$ where

$$I(x) \triangleq \{j \in \mathbb{m} | g_j(x) > 0\} \quad (2.2)$$

(i.e. the gradients $\nabla g_j^\gamma(x)$, $j \in I(x)$ satisfy the Robinson LI condition [6]).

**Assumption 2.2.** There exists an $\hat{x}$ such that $g(\hat{x}) < 0$. 

\[\square\]
Let
\[ v \triangleq (1,1,...,1)^T \in \mathbb{R}^n \]  
and let \( \varepsilon > 0 \) be arbitrary. Let
\[ g_\varepsilon(x) \triangleq g(x) + \varepsilon v \]  
and let \( g_\varepsilon^+(x) \) be defined by
\[ [g_\varepsilon^+(x)]^j \triangleq \max\{g_\varepsilon^j(x),0\}, \ j \in m \]  

Now, since by Assumption 2.2 there exists an \( \hat{x} \) such that \( g(\hat{x}) < 0 \), it is clear that there exists an \( \hat{\varepsilon} > 0 \) such that for all \( \varepsilon \in [0,\hat{\varepsilon}] \), there exists an \( \hat{x}_\varepsilon \) such that \( g_\varepsilon(\hat{x}_\varepsilon) \leq 0 \). If we knew such an \( \varepsilon \in (0,\hat{\varepsilon}] \), we could apply the following version of Newton's method described in [5] to find \( \hat{x}_\varepsilon \) (under the heading restoration iteration function a).

**Algorithm 2.1 (Newton Method - MP Version [5]).**

**Parameters:** \( \alpha \in (0,1/2) \), \( \beta \in (0,1) \), \( L \gg 1 \).

**Data:** \( x_0 \)

**Step 0:** Set \( i = 0 \).

**Step 1:** Solve the QP for \( v_i \)
\[ \min\{\|v\|^2 | g_\varepsilon(x_i) + \frac{\partial g_\varepsilon(x_i)}{\partial x} v < 0\} \]  

**Step 2:** If \( v_i \) exists and \( \|v_i\| \leq L \), set \( h_i = v_i \). Else set \( h_i = -\frac{\partial g_\varepsilon(x_i)}{\partial x} g_\varepsilon(x_i) + \) (i.e. set \( h_i = -\frac{\partial}{\partial x} \frac{1}{2}\|g_\varepsilon(x_i)\|_2^2 \)).

**Step 3:** Compute the smallest integer \( k \geq 0 \) such that
\[ \|g_\varepsilon(x_i + k h_i + \|h_i\|_2^2) \leq (1-2\alpha \beta^k) \|g_\varepsilon(x_i)\|_2^2 \]  

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Step 4: Set $x_{i+1} = x_i + \beta h_i$, set $i = i+1$ and go to step 1. □

We now collect from [4,5], the relevant results of this method.

Theorem 2.1: Suppose that Assumption (2.1) is satisfied and that $\varepsilon \in (0,\hat{\varepsilon} ]$ (i.e. there exists an $\hat{\varepsilon}$ such that $g_{\varepsilon}(\hat{\varepsilon}) < 0$).

a) If Algorithm 2.1 constructs a bounded sequence $\{x_i\}$, then $x_i \to \hat{x}_\varepsilon$ as $i \to \infty$, satisfying $g_{\varepsilon}(\hat{x}_\varepsilon) < 0$.

b) For any compact set $U$, there exists an $M \in (0,\infty)$, depending only on the values of the matrix $\frac{\partial g_{\varepsilon}(x)}{\partial x} = \frac{\partial g(x)}{\partial x}$ (and hence independent of $\varepsilon$) for $x \in U$, such that if for some $i_0$, $x_{i_0} \in U$ and $M \|g_{\varepsilon}(x_{i_0})\| < 1$, then for all $i \geq i_0$, $x_i \in U$ and hence $x_i \to \hat{x}_\varepsilon \in U$ as $i \to \infty$. Furthermore,

$$\|x_i - \hat{x}_\varepsilon\| \leq \frac{1}{M \varepsilon}$$

holds, with

$$\delta \in (0, M \|g_{\varepsilon}(x_{i_0})\|)$$

(2.9)

c) If $\varepsilon \in (0,\hat{\varepsilon} ]$ (i.e., with $\psi(\cdot)$ defined as in (2.10) below, min $\psi(\varepsilon(x)) > 0$), then $x_i \to \hat{x}_\varepsilon$ as $i \to \infty$, a minimizer of $\frac{1}{2} \|g_{\varepsilon}(x)\|^2$. □

Now, let

$$\psi_{\varepsilon}(x) \triangleq \max_{j \in m} g_{\varepsilon}^j(x)$$

(2.10)

and

$$\psi_{\varepsilon}(x)^+ \triangleq \max\{0, \psi_{\varepsilon}(x)\}$$

(2.11)

Then we have
\[
\psi_\varepsilon(x) = \psi_0(x) + \varepsilon
\]  

(2.12)

and, by the relation between \(L_\infty\) and \(L_2\) norms

\[
\frac{1}{\sqrt{m}} \|g_\varepsilon(x)_+\| \leq \psi_\varepsilon(x)_+ \leq \|g_\varepsilon(x)_+\|
\]  

(2.13)

Now, suppose that the conditions of Theorem 2.1 apply, that \(\varepsilon \in (0, \hat{\varepsilon})\), and that \(\{x_i\}\) is a sequence constructed by Algorithm (2.1), converging to \(\hat{x}_\varepsilon\).

Then, since \(\psi_\varepsilon(x)_+\) is locally Lipschitz continuous, with constant \(L\), say, in any compact neighborhood of \(\hat{x}_\varepsilon\), and \(\psi_\varepsilon(\hat{x}_\varepsilon)_+ = 0\), we obtain

\[
\psi_0(x_i)_+ + \varepsilon = \psi_\varepsilon(x_i)_+ - \psi_\varepsilon(\hat{x}_\varepsilon)_+ \leq L\|x_i - \hat{x}_\varepsilon\| \text{ for all } i
\]  

(2.14)

and hence, since \(g_\varepsilon(x_i)_+ \to 0\), there exists an \(i_0\) such that by (2.8) and (2.14)

\[
\psi_0(x_i)_+ \leq -\varepsilon + \frac{L}{M} \delta^2 \varepsilon \text{ for all } i \geq i_0
\]  

(2.15)

that is,

\[
\max_{j \in m} g_j(x_i)_+ \leq 0
\]  

(2.16)

for all \(i \geq i_0\) such that

\[
-\varepsilon + \frac{L}{M} \delta^2 \varepsilon \geq 0
\]  

(2.17)

This shows that if we knew a correct value for \(\varepsilon\), we would find a feasible point \(\bar{x}\) satisfying \(g(\bar{x}) \leq 0\) very rapidly. Thus, our attention must be directed towards constructing a procedure for finding a satisfactory \(\varepsilon\). We note in (2.17) that if we decrease \(\varepsilon\) suitably slowly, then because of the rapid decline of the term \(\frac{L}{M} \delta^2 \varepsilon\), we should be able to find a satisfactory \(\varepsilon\).
and still achieve (2.16) in a finite number of iterations. Next, we note that Algorithm 2.1 minimizes \( \frac{1}{2} \| g(x) \|_2^2 \). Since there is an \( \hat{x} \) such that \( g(\hat{x}) \leq 0 \), it follows that Algorithm 2.1 computes an \( \hat{x}_e \) such that, because of (2.12),
\[
\frac{1}{2} [\psi_e(\hat{x}_e) + ]^2 \leq \frac{1}{2} \| g_e(\hat{x}_e) \|_2^2 \leq \frac{m}{2} e^2
\]
\[ (2.18) \]
i.e.
\[
\psi_e(\hat{x}_e) + \leq \sqrt{m} e
\]
\[ (2.19) \]
Hence,
\[
\psi_e(\hat{x}_e) + \leq \sqrt{m} e
\]
\[ (2.20) \]
As a result, if Algorithm 2.1 is initialized at \( x_0 \), then for any \( \gamma \in (0,1) \), there exists a finite \( i \) such that
\[
\psi_0(x_i) - (\sqrt{m} - 1) e \leq \gamma \left[ \psi_0(x_0) - (\sqrt{m} - 1) e \right]
\]
\[ (2.21) \]
i.e.
\[
\psi_0(x_i) \leq \gamma \psi_0(x_0) + (1-\gamma)(\sqrt{m} - 1) e
\]
\[ (2.22) \]
The above observations form the basis for the algorithm below.

**Algorithm 2.2.**

**Parameters:** \( \alpha \in (0, \frac{1}{2}) \), \( \beta \in (0,1) \), \( L \gg 1 \), \( \gamma_1, \gamma_2 \in (0,1) \), \( \delta \in (0,1) \), a sequence of integers \( \{ \ell_k \}_{k=0}^{\infty} \) such that \( \ell_{k+1} > \ell_k \) for all \( k \).

**Data:** \( z_0 \in \mathbb{R}^n \), \( \varepsilon_0 > 0 \).

**Step 0:** Set \( k = 0 \).
Step 1: If \( \psi_0(z_k) \leq 0 \), stop. Else set \( i = 0, x_0 = z_k, \varepsilon = \varepsilon_k \).

Step 2: Solve QP (2.6) for \( v_i \).

Step 3: If \( v_i \) exists and \( \|v_i\| \leq L \), set \( h_i = v_i \). Else set \( h_i = \frac{\partial g(x_i)}{\partial x} e(x_i) \).

Step 4: Compute the smallest integer \( j \geq 0 \) such that

\[
\left\| g_{\varepsilon}(x_i + \beta^j h_i) \right\| \leq (1 - 2\alpha \beta^j) \left\| g_{\varepsilon}(x_i) \right\|^2 \tag{2.23}
\]

Step 5: Set \( x_{i+1} = x_i + \beta^j h_i \).

Step 6: If \( i \geq \ell_k \) and

\[
\psi_0(x_{i+1}) \leq \gamma_1 \psi_0(x_0) + (1 - \gamma_1) (\sqrt{m} - 1) \varepsilon_k \tag{2.24}
\]

Set \( z_{k+1} = x_{i+1}, \varepsilon_{k+1} = \gamma_2 \varepsilon_k \), set \( k = k+1 \) and go to step 1. Else, set \( i = i+1 \) and go to step 2.

Lemma 2.1: Suppose that Algorithm 2.2 constructs an infinite sequence \( \{z_k\} \).

Then any accumulation point \( \hat{z} \) of \( \{z_k\} \) satisfies \( \psi_0(\hat{z}) \leq 0 \) (i.e. \( g(\hat{z}) \leq 0 \)).

Proof: By construction, the sequence \( \{z_k\} \) satisfies (see (2.24))

\[
\psi_0(z_{k+1}) \leq \gamma_1 \psi_0(z_k) + (1 - \gamma_1) (\sqrt{m} - 1) \varepsilon_k, \quad k = 0, 1, 2, \ldots \tag{2.25}
\]

Since \( \gamma_1, \gamma_2 \in (0, 1) \), it follows from (2.25) that

\[
\lim \psi_0(z_k) \leq 0 \tag{2.26}
\]

and hence, if \( \hat{z} \) is an accumulation point of \( \{z_k\} \), then \( \psi_0(\hat{z}) \leq 0 \).

Theorem 2.2: Suppose that Assumptions 2.1 and 2.2 are satisfied. If Algorithm 2.2 constructs a bounded sequence \( \{z_k\} \) then there is a finite index \( s \geq 0 \) such that \( g(z_s) \leq 0 \).

Proof: Suppose, for the sake of contradiction that Algorithm 2.2 constructs
an infinite, bounded sequence \( \{z_k\} \). Then, by Lemma 2.1, there exists a subsequence, indexed by \( K \subset \{0,1,2,\ldots\} \) such that \( z_k \to \hat{z} \), with \( \psi_0(\hat{z}) \leq 0 \). Hence, since \( \epsilon_k \to 0 \) as \( k \to \infty \), there exists a \( k_0 \in K \) such that, for all \( k \geq k_0 \), the set \( \{x | \psi_\epsilon(x) \leq 0\} \neq \emptyset \) and, with \( M \) as in Theorem 2.1(b),

\[
\|g^{\epsilon_k}(z_k)_+\| \leq \sqrt{\psi}^{\epsilon_k}(z_k)_+ \leq \sqrt{m}(\psi_0(z_k)_++\epsilon_k) < \frac{1}{M}
\]  

(2.27)

Thus if \( \hat{x}_{\epsilon_k} \) is the limit of the infinite sequence \( \{x_1\} \) generated by algorithm 2.1 when it has been initialised at \( x_0 = z_k \), with \( k \geq k_0 \), then

\[
\|z_{k+1} - \hat{x}_{\epsilon_k}\| \leq \frac{1}{M} \delta^{\epsilon_k} \]  

(2.28)

for all \( k \in K, k \geq k_0 \), where \( \delta^{\epsilon_k} \in (0, M\|g^{\epsilon_k}(z_k)_+\|) \).

Since \( \{x | g^{\epsilon_k}_-(x) \leq 0\} \neq \emptyset \), it follows that \( \psi^{\epsilon_k}_-(\hat{x}_{\epsilon_k}) = 0 \) so that

\[
\psi_0(z_{k+1}) + \epsilon_k \leq \psi^{\epsilon_k}_-(z_{k+1}) - \psi^{\epsilon_k}_-(\hat{x}_{\epsilon_k}) + L_k \|z_{k+1} - \hat{x}_{\epsilon_k}\|
\]  

(2.29)

for \( k \geq k_0 \), where \( L_k \) is the Lipschitz constant associated with a compact set containing the bounded sequence \( \{x_1\} \) initiated at \( x_0 = z_k \). Since for each \( k \in K, k \geq k_0 \), the sequence \( \{x_1\} \) is contained in a sphere of radius at most \( 1/M \) centered on \( \hat{x}_{\epsilon_k} \), and since the sequence \( \{z_k\} \) is bounded, it follows that the collection \( \{L_k\} \) of Lipschitz constants can be bounded from above by an overall constant \( L \). Thus from (2.28) and (2.29), it follows that

\[
\psi_0(z_{k+1}) \leq \epsilon_k \gamma_1 + L_k \delta^2 \]  

(2.30)
for all \( k \in K \), \( k \geq k_0 \), where \( \delta_{\varepsilon_k} \in (0, M \| g_{\varepsilon_k}(z_k) \|) \). Now, by Lemma (2.1),

\[ \psi_0(z_k)_+ \to 0 \text{ as } k \to \infty, \quad k \in K \]

and by (2.13)

\[ \| g_{\varepsilon_k}(z_k) \| \leq \sqrt{m} (\psi_0(z_k)_+ + \varepsilon_k) \]

Consequently, \( \| g_{\varepsilon_k}(z_k) \| \to 0 \) as \( k \to \infty, \quad k \in K \), which shows that \( \delta_{\varepsilon_k} \to 0 \) as \( k \to \infty \), with \( k \in K \). Hence, since \( \varepsilon_k \to 0 \), there exists a \( k_1 \geq k_0, \quad k_1 \in K \) such that

\[ \psi_0(z_{k+1}) \leq -\varepsilon_0 \gamma_1 + \sum_{\varepsilon_k}^{k_1} \leq 0 \]

But then the algorithm must have stopped in Step 1 for \( k = k_1 + 1 \) and hence \( \{z_k\} \) cannot be infinite. This completes our proof. \( \square \)
References


