LIMITATIONS ON THE TRANSIENT RESPONSE
OF POSITIVE REAL FUNCTIONS

by

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Because the impulse response of a physical network,

\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega, \]

is uniquely determined by the real part Fourier integral\(^1\)

\[ h(t) = \frac{2}{\pi} \int_{0}^{\infty} \left[ \text{Re}\{ H(j\omega) \} \right] \cos \omega t \, d\omega, \quad (1) \]

it and its integrals can be bounded in many ways. The integrals of the impulse response (step, ramp, \ldots) can be denoted by

\[ h_k(t) = \int_{0}^{t} \cdots \int_{0}^{t} h(t) \, dt \cdots \, dt; \]

\[ \text{k times} \quad \text{k times} \]

hence, the impulse response is \( h_0(t) \), the ramp response is \( h_2(t) \), etc. The even order responses are given by

\[ h_{2n}(t) = \frac{2}{\pi} \int_{0}^{\infty} (-1)^n \frac{1}{\omega^{2n}} \left[ \text{Re}\{ H(j\omega) \} \right] \cos \omega t \, d\omega. \quad (2) \]

The cosine inequalities

\[ 1 - \cos \omega t \geq \frac{1}{4^n} \left[ 1 - \cos (2^n \omega t) \right] \quad (3) \]
used by Papoulis\textsuperscript{2} to bound the frequency attenuation for filters with monotonic step response may be applied here to bound the transient response of positive real network functions. Since for a p.r. function

\[
\Re \left\{ H(j\omega) \right\} \geq 0,
\]

we have for \( n \) even

\[
h_{2n}(0) - h_{2n}(t) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\omega^{2n}} \left[ \Re \left\{ H(j\omega) \right\} \right] (1 - \cos \omega t) d\omega
\]

\[
\geq \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\omega^{2n}} \left[ \Re \left\{ H(j\omega) \right\} \right] \frac{1}{4a} \left[ 1 - \cos (2^{a} \omega t) \right] d\omega
\]

\[
= \frac{1}{4a} \left[ h_{2n}(0) - h_{2n}(2^{a} t) \right]
\]

or

\[
4^{a} \left[ h_{2n}(0) - h_{2n}(t) \right] \geq h_{2n}(0) - h_{2n}(2^{a} t), \; n \text{ even.} \quad (4)
\]

Similarly,

\[
4^{a} \left[ h_{2n}(0) - h_{2n}(t) \right] \leq h_{2n}(0) - h_{2n}(2^{a} t), \; n \text{ odd.} \quad (5)
\]

Taking \( n = 0 \) and \( a = 1 \), we have the impulse response bound from (4),
\[ 4 \left[ h(0) - h(t) \right] \geq h(0) - h(2t) \]

or

\[ h(2t) \geq h(t) - 3h(0). \quad (6) \]

An even more interesting bound, however, is that obtained on the ramp response by taking \( n = 1 \) and \( a = 1 \) and employing (5):

\[ 4 \left[ h_2^2(0) - h_2^2(t) \right] \leq h_2^2(0) - h_2^2(2t). \]

Under the obvious condition \( h_2(0) = 0 \), we have

\[ h_2(2t) \leq 4h_2(t). \quad (7) \]

But the ramp response is the integral of step response, consequently

\[
\begin{align*}
\int_0^{2t} h_1(t) \, dt &\leq 4 \int_0^t h_1(t) \, dt \\
\int_0^{2t} h_1(t) \, dt + \int_0^t h_1(t) \, dt &\leq 4 \int_0^t h_1(t) \, dt \\
\int_0^{2t} h_1(t) \, dt &\leq 3 \int_{t_0}^t h_1(t) \, dt.
\end{align*}
\]

\[ (8) \]
From the graphical presentation of a typical step response involved in the realization of a delay function, we may interpret the ripple as the shaded area in Fig. 1, represented by the integral on the right-hand side of (8). Moreover, by one of the usual definitions of rise-time, the area represented by the integral on the left-hand side of (8) may be taken as the delay minus the rise-time. Hence, we see clearly the interchange which may be obtained between the rise-time, delay, and ripple of a network function with a positive real part,

$$t_d - t_r \leq 3 \text{ (ripple area).}$$

(9)

This is but one of many such bounds on the transient response which may be obtained from (4) and (5).
Fig. 1    Delay Function Step Response
REFERENCES

