A NOTE ON THE EXTENSION PRINCIPLE FOR FUZZY SETS

by

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1. Introduction

The extension principle described by L. A. Zadeh [5] provides a natural way for extending the domain of a mapping or a relation defined on a set U to fuzzy subsets of U. It is particularly useful in connection with the computation of linguistic variables [5], the calculus of linguistic probabilities ([3], [5]), arithmetic of fuzzy numbers ([1], [5]), and, more generally, in applications which call for an extension of the domain of a relation. Furthermore, as shown in [1], in the analysis of fuzzy numbers, the set-method (i.e. the use of $\alpha$-level sets of a fuzzy set) is simpler than the functional approach (i.e. the use of the membership function of a fuzzy set.)

In this note, we examine the resolution of identity [5], i.e. the set-representation of fuzzy sets, and we prove that the application of the extension principle to a fuzzy set may be viewed as the application of this principle to the $\alpha$-level sets of the set in question. However, in general, if

$$f : X \times Y \rightarrow Z$$

and A, B are fuzzy subsets of X and Y, respectively, we do not have:

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\[ [f(A,B)]_\alpha = f(A_\alpha, B_\alpha) \]  \hspace{1cm} (1.1)

where \( A_\alpha \) and \( B_\alpha \) are the \( \alpha \)-level sets of \( A \) and \( B \), respectively, and 
\( [f(A,B)]_\alpha \) is the \( \alpha \)-level set of \( f(A,B) \). We shall give a necessary and
sufficient condition for obtaining this equality, and shall define a
class of fuzzy numbers in which this equality holds for all continuous \( f \).

2. The Resolution of Identity

The collection of all fuzzy subsets of a set \( X \) is denoted by
\( \mathcal{P}(X) \). If \( A \in \mathcal{P}(X) \), its membership function is denoted by \( \mu_A : X \rightarrow [0,1] \).
We write \( 1_A \) if \( A \) is nonfuzzy.

For \( \alpha \in [0,1] \), recall that the \( \alpha \)-level set of \( A \) is defined by
\[ A_\alpha = \{ x \in X : \mu_A(x) > \alpha \} \]

If, \( A, B \in \mathcal{P}(X) \), then by definition, \( A = B \) iff \( \mu_A(x) = \mu_B(x) \),
\( \forall x \in X \). It is easy to verify that
\[ A = B \Rightarrow A_\alpha = B_\alpha, \forall \alpha \in (0,1), \]

It is also obvious that \( S_A = \bigcup A_\alpha \), where \( S_A \) is the support of the
fuzzy set \( A \), defined by
\[ S_A = \{ x : \mu_A(x) > 0 \} \]

On the other hand, we have
\[ \forall x \in X, \ \mu_A(x) = \sup_{\alpha \in [0,1]} \sup_{\alpha \in [0,1]} [\alpha 1_A(x)] \] \hspace{1cm} (2.1)

and thus \( A \) may be represented as in the following form, called the resolution of
identity [5]
A = \int_{0}^{1} \alpha A_\alpha \quad (2.2)

where \( \int_{0}^{1} \) represents the union over \( \alpha \in [0,1] \), and \( \alpha A_\alpha \) is the fuzzy set whose membership function is

\[
l_{\alpha A_\alpha}(x) = \begin{cases} 
\alpha & \text{if } x \in A_\alpha \\
0 & \text{if } x \notin A_\alpha
\end{cases}
\]

**Proposition 2.1.** If \( A', \alpha \in [0,1] \), is a family of subsets of \( x \) such that:

\[
A = \int_{0}^{1} \alpha A'\alpha
\]

then

(i) \( A' \subseteq A_\alpha \), \( \forall \alpha \in [0,1] \)

(ii) \( \bigcup_{\alpha} A_\alpha = \bigcup_{\alpha} A'\alpha \)

\( \alpha \in [0,1] \) \( \alpha \in [0,1] \)

**Proof**

(i) Let \( x \in A'_{\alpha_0} \), then \( \alpha_0 A_\alpha \), \( (x) = \alpha_0 \), and thus:

\[
\mu_A(x) = \sup_{\alpha \in [0,1]} [\alpha l_{A_\alpha}(x)] \geq \alpha_0 \Rightarrow x \in A_\alpha \]

(ii) The equality in (ii) follows from the fact that the right and left hand sides of (ii) are both equal to the support \( S_A \) of \( A \).

3. **The extension principle**

Recall that if \( f : X \to Y \), and \( A \in \mathcal{P}(X) \), then the fuzzy set \( f(A) \) is defined, via the extension principle, by

\[
f(A) \in \mathcal{P}(Y), \quad \mu_{f(A)}(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \quad (3.1)
\]

-3-
Remark

In order to apply this principle to fuzzy mapping, we rewrite (3.1) under the following equivalent form:

\[ \mu_{f(A)}(y) = \sup_{x \in X} [\mu_A(x) \wedge l_f(x)(y)] \]  
(3.2)

where \( l_f(x)(y) = 1 \) or 0 according as \( y = f(x) \) or \( y \neq f(x) \).

If \( f \) is a multi-valued mapping, i.e. \( f : X \rightarrow \mathcal{P}(Y) \), and \( A \in \mathcal{P}(X) \), then (3.1) leads to:

\[ \mu_{f(A)}(y) = \sup_{x \in f^*(y)} \mu_A(x) \]  
(3.3)

where

\[ f^* : Y \rightarrow \mathcal{P}(X) \]

\[ f^*(y) = \{x \in X : y \in f(x)\} \]

It is easy to see that (3.3) is the same as:

\[ \mu_{f(A)}(y) = \sup_{x \in X} [\mu_A(x) \wedge l_f(x)(y)] \]  
(3.4)

For \( B \neq \phi \) and \( B \subseteq Y \), we have:

\[ f^*(B) = \{x \in X : f(x) \cap B \neq \phi\} \]

Now let \( X^* \) be the domain of \( f \), i.e.

\[ X^* = \{x \in X : f(x) \neq \phi\} \]

and

\[ f_* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) : f_*(B) = \{x \in X^* : f(x) \subseteq B\} \]
then
\[ f_*(B) \subseteq f*(B), \forall B \neq \emptyset \]
and
\[ f_*(B) = [f*(B')]', \]
where the prime stands for set-complement, thus
\[ 1_{f_*(B)}(x) = 1 - 1_{f*(B')} (x) \]
but
\[ 1_{f*}(y)(x) = 1_{f}(x)(y) \]
and hence
\[ 1_{f_*(B)}(x) = 1 - \sup_{x \in B'} 1_{f}(x)(y) \]
Note also that
\[ 1_{f*}(B)(x) = \sup_{x \in B} 1_{f}(x)(y) \]
If \( f \) is a fuzzy mapping, i.e. \( f : X \rightarrow \mathcal{P}(Y) \), and \( A \in \mathcal{P}(X) \), then (3.2) leads to
\[ \mu_{f}(A) (y) = \sup_{x \in X} [\mu_{A}(x) \land \mu_{f}(x)(y)] \quad (3.5) \]
Define \( f^* : Y \rightarrow \mathcal{P}(X) \) by:
\[ \mu_{f^*}(y)(x) = \mu_{f}(x)(y) \quad (3.6) \]
For $B \subseteq Y$, we have, by (3.5):

$$
\mu_{f^*}(B)(x) = \sup_{y \in Y} [1_B(y) \land \mu_{f^*}(y)(x)]
$$

$$
= \sup_{y \in Y} [1_B(y) \land \mu_f(x)(y)]
$$

$$
= \sup_{y \in B} \mu_f(x)(y)
$$

**Proposition 3.1.**

Let $A \in \mathcal{P}(X)$, and $f : X \rightarrow Y$, then:

$$
f(A) = \int_0^1 \alpha f(A_\alpha) \quad (3.7)
$$

**Proof:**

$$
\mu_{f(A)}(y) = \sup_{x \in f^{-1}(y)} \mu_A(x)
$$

$$
= \sup_{x \in f^{-1}(y)} \left[ \sup_{\alpha \in [0,1]} \alpha 1_A(x) \right] \text{ by (2.1)}
$$

$$
= \sup_{\alpha \in [0,1]} \left[ \alpha 1_A(x) \right]
$$

$$
\left\{ x \in f^{-1}(y) \right\}
$$

$$
(3.8)
$$

On the other hand, let $B = \int_0^1 \alpha f(A_\alpha)$, then:

$$
\mu_B(y) = \sup_{\alpha \in [0,1]} \alpha 1_f(A_\alpha)(y)
$$
= \sup_{\alpha \in [0,1]} \sup_{x \in f^{-1}(y)} \alpha \cdot 1_{A_\alpha}(x)
= \sup_{x \in f^{-1}(y)} [\sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(x)]
= \sup_{x \in f^{-1}(y)}[\alpha \cdot 1_{A_\alpha}(x)] = \mu_{f(A)}(y).
\begin{cases}
\{x \in f^{-1}(y) \} \\
\{\alpha \in [0,1]\}
\end{cases}

\textbf{Remark} \quad \text{From the above it follows that}

f(A) = \int_0^1 \alpha [f(A)]_\alpha = \int_0^1 \alpha f(A_\alpha)

\text{with } f(A_\alpha) \subseteq [f(A)]_\alpha, \quad \forall \alpha \in [0,1]

\text{But in general,}

f(A_\alpha) \neq [f(A)]_\alpha

\textbf{Proposition 3.2.} \quad \text{Let } f : X \times Y \to Z, \text{ and } A \in \mathcal{P}(X), B \in \mathcal{P}(Y); \text{ then}

f(A,B) = \int_0^1 \alpha f(A_\alpha, B_\alpha)

\text{(3.9)}

\textbf{Proof:}

(\text{i}) \quad \mu_{f(A,B)}(z) = \sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \land \mu_B(y)]

(3.10)
(ii) Let $T = \int_0^1 \alpha f(A_\alpha, B_\alpha)$, then:

$$
\mu_1(z) = \sup_{\alpha \in [0,1]} \alpha \int_{f^{-1}(z)} (x, y) \in f^{-1}(z)
$$

$$
= \sup_{\alpha \in [0,1]} \left( \sup_{(x, y) \in f^{-1}(z)} \{ \alpha l_{A_\alpha}(x) \land \alpha l_{B_\alpha}(y) \} \right)
$$

$$
(\alpha l_{A_\alpha}(x) \land \alpha l_{B_\alpha}(y)) \in f^{-1}(z)
$$

$$
= \sup_{\alpha \in [0,1]} [\alpha l_{A_\alpha}(x) \land \alpha l_{B_\alpha}(y)]
$$

(3.11)

To prove that (3.10) and (3.11) are equivalent, it is sufficient to show that:

$$
[\sup_{\alpha \in [0,1]} \alpha l_{A_\alpha}(x)] \land [\sup_{\alpha \in [0,1]} \alpha l_{B_\alpha}(y)] = \sup_{\alpha \in [0,1]} [\alpha l_{A_\alpha}(x) \land \alpha l_{B_\alpha}(y)]
$$

(3.12)

To this end, let:

$$
\begin{cases}
\alpha_0 = \sup_{\alpha \in [0,1]} \alpha l_{A_\alpha}(x) \\
\beta_0 = \sup_{\alpha \in [0,1]} \alpha l_{B_\alpha}(y)
\end{cases}
$$

If $\alpha_0 \land \beta_0 = 0$, say $\alpha_0 = 0$, then $\alpha l_{A_\alpha}(x) = 0$ for all $\alpha \in [0,1]$, thus (3.12) is verified.

Suppose now that $\alpha_0 \land \beta_0 > 0$.

We have

$$
\begin{cases}
x \in A_\alpha \text{ for all } \alpha < \alpha_0 \\
x \notin A_\alpha \text{ for all } \alpha > \alpha_0
\end{cases}
$$

Since if there exists $\alpha'$ such that:

$$
\begin{cases}
\alpha' < \alpha_0 \\
x \in A_{\alpha'}
\end{cases}
$$
then $x \notin A_\alpha$ for all $\alpha < \alpha'$ (this follows from the fact that $\alpha \leq \beta \Rightarrow A_\alpha \supseteq A_\beta$), thus:

$$\sup_{\alpha \in [0,1]} A_\alpha (x) < \alpha' < \alpha_0,$$

which is a contradiction; and if there exists $\alpha''$ such that:

\[
\begin{cases}
\alpha'' > \alpha_0 \\
x \in A_{\alpha''}
\end{cases}
\]

then $\sup A_\alpha (x) \geq \alpha_0$, which is also a contradiction.

In the same way, we have:

\[
\begin{cases}
y \in B_\alpha, \text{ for all } \alpha < \beta_0 \\
y \notin B_\alpha, \text{ for all } \alpha > \beta_0.
\end{cases}
\]

Thus: $\alpha_1 A_\alpha (x) \land \alpha_1 B_\alpha (y) = \alpha$ for $\alpha < \alpha_0 \land \beta_0$ and $= 0$ for all $\alpha \geq \alpha_0 \land \beta_0$

and hence:

$$\sup_{\alpha \in [0,1]} [\alpha_1 A_\alpha (x) \land \alpha_1 B_\alpha (y)] = \alpha_0 \land \beta_0.$$

Remark. We have then $f(A_\alpha, B_\alpha) \subseteq [f(A, B)]_\alpha$, $\forall \alpha \in [0,1]$ but in general,

$f(A_\alpha, B_\alpha) \neq [f(A, B)]_\alpha$.

Proposition 3.3. With the notation of Proposition 3.2, a necessary
and sufficient condition for the equality:

$$[f(A, B)]_\alpha = f(A_\alpha, B_\alpha), \forall \alpha \in [0,1]$$

is: $\forall z \in Z, \sup (x, y) \in f^{-1}(z)$ is attained.
Proof.

(i) **Necessity**: Let \( z \in Z \) and

\[
\sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \land \mu_B(y)] = t.
\]

i.e.,

\[
\mu_{f(A,B)}(z) = t \Rightarrow z \in [f(A,B)]_t
\]

\[
\Rightarrow z \in f(A_t,B_t).
\]

i.e. \( \exists \hat{x} \in A_t \) and \( \hat{y} \in B_t \) such that \( f(\hat{x},\hat{y}) = z \).

For \( (\hat{x},\hat{y}) \in f^{-1}(z) \) and \( \mu_A(\hat{x}) \geq t \),

\[
\mu_B(\hat{y}) \geq t \Rightarrow \mu_A(\hat{x}) \land \mu_B(\hat{y}) \geq t
\]

But

\[
\sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \land \mu_B(y)] \geq \mu_A(\hat{x}) \land \mu_B(\hat{y})
\]

and thus

\[
\mu_A(\hat{x}) \land \mu_B(\hat{y}) = t.
\]

(ii) **Sufficiency** By Proposition 3.2 and Proposition 2.1, we have:

\[
f(A_\alpha,B_\alpha) \subseteq [f(A,B)]_\alpha, \ \forall \alpha \in [0,1].
\]

Now let \( z \in [f(A,B)]_\alpha \), i.e.

\[
\mu_{f(A,B)}(z) = \sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \land \mu_B(y)] \geq \alpha.
\]
If $\mu_f(A,B)(z) > \alpha$, then by definition of $\sup$ there exists $(\hat{x}, \hat{y}) \in f^{-1}(z)$ such that:

$$\alpha < \mu_A(\hat{x}) \land \mu_B(\hat{y}) \leq \mu_f(A,B)(z) \Rightarrow \hat{x} \in A_\alpha \text{ and } \hat{y} \in B_\alpha.$$ 

Thus $z = f(\hat{x}, \hat{y}) \in f(A_\alpha, B_\alpha)$.

If $\mu_f(A,B)(z) = \alpha$ then by hypothesis, there exists $(x', y') \in f^{-1}(z)$ such that:

$$\mu_A(x') \land \mu_B(y') = \sup_{(x,y) \in f^{-1}(z)} [\mu_A(x) \land \mu_B(y)] = \alpha$$

$$\Rightarrow x' \in A_\alpha \text{ and } y' \in B_\alpha.$$ 

Thus $z = f(x', y') \in f(A_\alpha, B_\alpha)$.

4. On convexity of fuzzy numbers

By a fuzzy number we mean a fuzzy subset of the real line $\mathbb{R}$. Interval analysis [2] deals with closed bounded intervals (compact convex sets of $\mathbb{R}$) as an extension of numbers. Fuzzy numbers can be regarded as an extension of closed bounded intervals, thus the definition of a fuzzy number seems too general [see Section 5 for a smaller class of fuzzy numbers]. However, the arithmetic for fuzzy numbers can be defined via the extension principle. Since the relation (1.1) is not satisfied for general fuzzy numbers, the function method is the main tool of analysis. To illustrate this point, we shall review in what follows the concept of convexity and prove some properties of fuzzy numbers.*

*Many interesting results in the arithmetic of fuzzy numbers are contained in a recent paper by M. Mizumoto and K. Tanaka [1], e.g. convexity, algebraic structures, ordering of fuzzy numbers.
Let $X$ be the space $\mathbb{R}^n$ (or more generally a real linear space).

To define the convexity for fuzzy subsets of $X$, we start with the following remark. A subset $A$ of $X$ is convex iff

$\forall \alpha \in \mathbb{R}, A_\alpha = \{x : l_A(x) \geq \alpha\}$ is convex. This leads to

**Definition 4.1** [5] a fuzzy subset $A$ of $X$ is convex if its membership function $\mu_A$ is quasi-concave.

**Remarks:**

(i) A useful characterization of fuzzy convexity is the following:

A convex $\iff$ \begin{align*}
\forall x, y \in X, \forall \lambda \in [0,1] \quad \mu_A[\lambda x + (1-\lambda)y] &\geq \mu_A(x) \land \mu_B(y)
\end{align*}

(ii) If $A$ is convex, so is its support $S_A$.

**Proposition 4.2.** The following are equivalent:

(i) $A \in \mathcal{P}(X)$ is convex.

(ii) $\forall x, y \in X,$ the function $\lambda \mapsto \mu_A[\lambda x + (1-\lambda)y]$ is quasi-concave on $[0,1]$.

**Proof.** Denote by $\phi$ the function $\lambda \mapsto \mu_A[\lambda x + (1-\lambda)y]$.

(i) $\Rightarrow$ (ii), Let $\lambda', \lambda'' \in [0,1]$ and $\lambda \in [\lambda', \lambda'']$. (we suppose $\lambda' < \lambda''$).

For $x, y \in X$, let:

\[ \begin{cases} 
\hat{x} = \lambda' x + (1-\lambda')y \\
\hat{y} = \lambda'' x + (1-\lambda'')y
\end{cases} \]

Then: $\lambda = \alpha \lambda' + (1-\alpha)\lambda''$ for some $\alpha \in [0,1]$ and

$\alpha \hat{x} + (1-\alpha)\hat{y} = [\alpha \lambda' + (1-\alpha)\lambda'']x + [\alpha(1-\lambda') + (1-\alpha)(1-\lambda'')]y$

$= [\alpha \lambda' + (1-\alpha)\lambda'']x + [1 - (\alpha \lambda' + (1-\alpha)\lambda'')]y$

$= \lambda x + (1-\lambda)y = \hat{z}$.
By quasi-concavity of $\mu_A$, we have then:

$$\mu_A(\hat{z}) \geq \mu_A(\hat{x}) \wedge \mu_A(\hat{y})$$

i.e.

$$\phi(\lambda) \geq \phi(\lambda') \wedge \phi(\lambda'').$$

(ii) $\Rightarrow$ (i) Let $x, y \in X$ and $z = \lambda x + (1-\lambda)y$, $\lambda \in [0,1]$

$$\phi(0) = \mu_A(y), \phi(1) = \mu_A(x)$$

since $0 \leq \lambda \leq 1 \Rightarrow \phi(\lambda) \geq \phi(0) \wedge \phi(1)$ by quasi-convexity of $\phi$ on $[0,1]$, i.e.,

$$\mu_A[\lambda x + (1-\lambda)y] \geq \mu_A(x) \wedge \mu_A(y)$$

Q.E.D.

**Definition 4.3** A fuzzy subset $A$ of $X$ is said to be strongly convex if $A$ is convex and its membership function $\mu_A$ is pseudo-concave.

**Remarks**

(i) A function $f : X \rightarrow \mathbb{R}$ is said to be pseudo-concave [4] if

$$\forall x, y \in X \text{ such that } f(x) \neq f(y)$$

$$\forall z \text{ } z = \lambda x + (1-\lambda)y, \text{ with } \lambda \in (0,1)$$

we have

$$f(z) > f(x) \wedge f(y).$$

(ii) This notion of convexity is useful for fuzzy mathematical programming. Note that a local maximum of a quasi-concave function is not necessarily a global one, but for pseudo-concave function, a local maximum is also a global one.
Proposition 4.4 A convex fuzzy subset $A$ of $X$ is strongly convex if its membership function $\mu_A$ is injective on $\{\mu_A < 1\}$.

**Proof.** By quasi-convexity of $\mu_A$, we have:

$$\forall x, y \in X, \forall \lambda \in [0,1], z = \lambda x + (1-\lambda)y$$

$$\mu_A(z) \geq \mu_A(x) \land \mu_A(y).$$

(4.1)

We have to verify that strict inequality holds in (4.1) for $(x,y)$ such that $\mu_A(x) \neq \mu_A(y)$, and for $\lambda \in ]0,1[$. Consider two cases:

(i) $\mu_A(x) = 1$ and $\mu_A(y) < 1$.

a) If $\mu_A(z) = 1 \Rightarrow \mu_A(z) > \mu_A(x) \land \mu_A(y)$

b) If $\mu_A(z) < 1$, then by injectivity of $\mu_A$ on $\{\mu_A < 1\}$, we have: $\mu_A(z) \neq \mu_A(y)$, but $\mu_A(z)$ verifies (4.1), i.e.

$$\mu_A(z) > \mu_A(x) \land \mu_A(y) = \mu_A(y).$$

Thus $\mu_A(z) > \mu_A(y)$

(ii) $x, y \in \{\mu_A < 1\}$.

a) If $z \in \{\mu_A = 1\} \Rightarrow \mu_A(z) > \mu_A(x) \land \mu_A(y)$.

b) If $\mu_A(z) < 1$, then from (4.1):

$$\mu_A(z) \geq \mu_A(x) \land \mu_A(y)$$

but $z \neq x \neq y \Rightarrow \begin{cases} \mu_A(z) \neq \mu_A(x) \\ \mu_A(z) \neq \mu_A(y) \end{cases}$

and hence $\mu_A(z) > \mu_A(x) \land \mu_A(y)$

Q.E.D.
Remark. It should be noted that quasi-convexity and pseudo-concavity are
two distinct notions. If \( f \) is pseudo-concave, then \( \forall x, y \) such that \( f(x) \neq f(y) \), and \( \lambda \in (0,1] \), we have

\[
f[\lambda x + (1-\lambda)y] > f(x) \land f(y)
\]

but for \( (x,y) \) such that \( f(x) = f(y) \), it can happen that

\[
f[\lambda x + (1-\lambda)y] < f(x) \land f(y).
\]

Fuzzy convex sets of \( \mathbb{R}^n \) have most of the algebraic properties of
ordinary convex sets. The following proposition is an extension to
fuzzy sets in the case of a sum.

**Proposition 4.5**

If \( A, B \in \mathcal{P}(\mathbb{R}^n) \) are convex, then so is \( A+B \).

**Proof:** Note that \( A+B \) is a fuzzy subset of \( \mathbb{R}^n \) defined via the
extension principle,

\[
\mu_{A+B}(z) = \sup \{ \mu_A(x) \land \mu_B(y) \ |
\begin{cases} 
(x,y) \\
x + y = z 
\end{cases} \}
\]

(i) Denote \( \phi(x,y) = \mu_A(x) \land \mu_B(y) \)

let \( (x',y'), (x'',y'') \in \mathbb{R}^n \times \mathbb{R}^n \), and \( \lambda \in [0,1] \)

\[
\begin{cases} 
x = \lambda x' + (1-\lambda)x'' \\
y = \lambda y' + (1-\lambda)y'' 
\end{cases}
\]

We have:

\[
\phi(x,y) \geq [\mu_A(x') \land \mu_A(x'')] \land [\mu_B(y') \land \mu_B(y'')].
\]

thus \( \phi \) is quasi-concave on \( \mathbb{R}^n \times \mathbb{R}^n \).
(ii) Let \( Z', Z'' \in \mathbb{R}^n, \lambda \in [0,1], Z = \lambda Z' + (1-\lambda)Z'' \).

Let \( \varepsilon > 0 \), and assume that there exists \( (x_1, y_1) \) (depending of \( \varepsilon \)) such that \( x_1 + y_1 = Z' \) and

\[
\phi(x_1, y_1) \geq \sup_{(x,y) \mid x + y = Z'} \phi(x,y) - \varepsilon \tag{4.2}
\]

There also exists \( (x_2, y_2) \) such that \( x_2 + y_2 = Z'' \) and \( \phi(x_2, y_2) \geq \sup_{(x,y) \mid x + y = Z''} \phi(x,y) - \varepsilon \tag{4.3} \)

since \( Z = \lambda Z' + (1-\lambda)Z'' \)

\[
\Rightarrow Z = \lambda (x_1 + y_1) + (1-\lambda) (x_2 + y_2) = \hat{x} + \hat{y}
\]

with

\[
\begin{align*}
\hat{x} &= \lambda x_2 + (1-\lambda) x_1 \\
\hat{y} &= \lambda y_2 + (1-\lambda) y_1.
\end{align*}
\]

We have \( \sup_{(x,y) \mid x + y = Z} \phi(x,y) \geq \phi(\hat{x}, \hat{y}) \geq \phi(x_1, y_1) \land \phi(x_2, y_2) \).

by quasi-concavity of \( \phi \).

Hence, by (4.2) and (4.3), we have:

\[
\sup_{(x,y) \mid x + y = Z} \phi(x,y) \geq \left[ \sup_{(x,y) \mid x + y = Z'} \phi(x,y) - \varepsilon \right] \land \left[ \sup_{(x,y) \mid x + y = Z''} \phi(x,y) - \varepsilon \right]
\]

\[
\geq \left[ \sup_{(x,y) \mid x + y = Z'} \phi(x,y) \land \sup_{(x,y) \mid x + y = Z''} \phi(x,y) \right] - \varepsilon
\]

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and this holds for all \( \epsilon > 0 \), thus:

\[
\mu_{A+B}(z) \geq \mu_{A+B}(z') \land \mu_{A+B}(z'')
\]

Q.E.D.

Remark. A fuzzy convex set \( A \) is said to be strongly convex on its support if the restriction of \( \mu_A \) to \( S_A \) is pseudo-concave. Thus, as a consequence of the Proposition 4.5, a fuzzy convex set is strongly convex on its support if \( \mu_A \) is injective on \( S_A - \{ \mu_A = 1 \} \). Bounded convex sets of \( \mathbb{R}^n \) are not strongly convex on their support.

5. A class of fuzzy numbers

Let \( A \in \mathcal{P}(\mathbb{R}) \), the support of \( A \) is denoted by \( S_A \). The topological support of its membership function \( \mu_A \) is \( \overline{S_A} = \{ x : \mu_A(x) \neq 0 \} \), i.e., the closure of \( S_A \).

We consider the following class of fuzzy numbers:

\[
A \in \mathcal{P}(\mathbb{R}, \mathcal{K}) \iff A \in \mathcal{P}(\mathbb{R}) \land \mu_A \text{ is upper semicontinuous (u.s.c.)} \land \overline{S_A} \text{ compact.}
\]

This class contains all singletons, as well as closed bounded intervals.

**Proposition 5.1** If \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous, then

\[
\forall A, B \in \mathcal{P}(\mathbb{R}, \mathcal{K}), \text{ and we have:}
\]

\[
[f(A,B)]_{\alpha} = f(A_{\alpha}, B_{\alpha}), \ \forall \alpha \in [0,1].
\]

**Proof:** In virtue of Proposition 3.3, it is sufficient to prove that:

\[
\forall z \in \mathbb{R}, \text{ sup}_{(x,y) \in f^{-1}(z)} [\mu_A(x) \land \mu_B(y)]
\]

is attained.
Let $\phi(x,y) = \mu_A(x) \land \mu_B(y)$, then $\phi(x,y) \geq 0$, and $\phi$ is u.s.c.

Thus, $\sup_{(x,y) \in f^{-1}(z)} \phi(x,y) = \sup_{(x,y) \in f^{-1}(z) \cap (\overline{S}_A \times \overline{S}_B)} \phi(x,y)$

Since $\phi = 0$ outside of $\overline{S}_A \times \overline{S}_B$.

But $\overline{S}_A \times \overline{S}_B$ is compact, and $f^{-1}(z)$ is closed by continuity of $f$; hence $f^{-1}(z) \cap (\overline{S}_A \times \overline{S}_B)$ is compact.

Thus $\phi$, being U.S.C., assumes its maximum on the compact set $f^{-1}(z) \cap (\overline{S}_A \times \overline{S}_B)$, $\forall z \in \mathbb{R}$.

Remark:

It should be observed that the following equalities (for $A, B \in \mathcal{P}(\mathbb{R})$)

$$(A + B)_\alpha = A_\alpha + B_\alpha$$

$$(A \land B)_\alpha = A_\alpha \land B_\alpha$$

which appeared in [1], hold only under the additional assumptions noted above.
REFERENCES;


