QUALITATIVE THEORY OF DYNAMIC NONLINEAR NETWORKS

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# QUALITATIVE THEORY OF DYNAMIC NONLINEAR NETWORKS

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## ABSTRACT

Several new results concerning the qualitative properties of a large class of dynamic nonlinear networks are presented. These results are all stated in terms of the network topology and the elements' constitutive relations. Among other things, simple conditions are derived which guarantee that an autonomous nonlinear network has a globally, asymptotically stable operating point. In the case of nonautonomous nonlinear networks, fairly general sufficient conditions are given which guarantee that the network will have either a bounded, or an eventually uniformly bounded output for a bounded input. Other conditions are shown to guanantee that the network will have a unique steady state response. Moreover, the frequency spectrum of almost periodic responses are expressed in terms of the spectrum of the input signals. Estimates of the transient decay are derived for a large class of dynamic nonlinear networks and the concept of "time constant" for linear networks is generalized to high order nonlinear networks.

To demonstrate that the hypotheses of our results are rather weak, counter-examples are presented showing violation of each "network-topology hypothesis" and "element-constitutive-relation hypothesis" will invariably lead to significantly different qualitative behaviors -- such as the existence of subharmonic oscillations.

The results to be presented should contribute toward the formulation of a unified theory of nonlinear networks.

## I. Introduction

Much of the analysis of dynamic electrical networks has traditionally been directed towards the study of <u>linear networks</u>. Specifically, results have been developed concerning the analysis and synthesis of linear networks and systems [1]-[3], the linear amplification of "small signal" voltage and current waveforms [4], and the use of linearized models of nonlinear networks in computer-aided analysis and design [5], [6]. In the last decade, however, the technological advances in device engineering has made it possible to synthesize highly nonlinear electrical elements [7] and, at the same time there has been a phenomenal increase in the use of large-signal digital and analog networks. It has thus become necessary to examine nonlinear networks and to find ways to predict their <u>qualitative behavior</u>.

Most of the literature dealing with the study of nonlinear networks involves the <u>formulation</u> of network equations, and the analysis of their inherent structure [8]-[12]. There are relatively few results dealing with behavior of dynamic nonlinear networks [8], [13]-[17]. These are usually concerned with networks containing specific nonlinear elements such as transistors or iron-core inductors, or they deal with more general non-linearities, but their hypotheses are in terms of <u>mathematical</u> conditions on the overall network equations.

In this paper we present a series of results concerning the qualitative analysis of dynamic nonlinear networks. This paper is a summary of theorems developed and presented in three papers [18]-[20]. We give conditions for determining a large variety of types of network behavior; e.g., conditions are given which guarantee that networks have no finite-escape time solutions, that the solutions are bounded or eventually uniformly bounded, that the solutions converge to a globally asymptotically stable equilibrium point or to a unique steady-state solution (these concepts are defined in the following sections of this paper). We also examine the periodic nature of network solutions. An important aspect of these theorems is that, unlike previous results of a similar type, in their final form the hypotheses of the theorems involve examining the nonlinearity of each network element, and examining the interconnection of the elements in the network. That is, the overall network equations need not be solved, or even formed. Furthermore,

the nonlinearities of the network elements are described in <u>circuit-theoretic</u> terms rather than by a purely mathematical description. Hence, these results are both practical and easily verifiable.

In Sec. II we characterize the class of nonlinear networks which will be examined, and we define the various types of nonlinearities used to characterize the elements. In Sec. III we motivate and discuss the graph-theoretic results of [18]. In Sec. IV we present the theorems concerning the behavior of autonomous and nonautonomous networks given in [19] and [20] respectively. In the concluding Sec. V we present two tables which summarize nearly all of the results of [18], [19] and [20].

It is assumed throughout this paper that the voltages and currents of any network satisfy the <u>Kirchoff Voltage Law</u> (KVL) and the <u>Kirchoff Current Law</u> (KCL), and that all network elements are <u>lumped</u> and <u>time-invariant</u> (except time-varying sources) [1].

## II. <u>Characterization of the Dynamic Nonlinear Network</u>

We shall examine the behavior of a dynamic nonlinear network $\mathcal{N}$ containing nonlinear capacitors, inductors and resistors. Linear two-terminal and nonlinear two-terminal capacitors, inductors and resistors are shown respectively in Fig. 1(a) and 1(b). The voltages ($v_C$, $v_L$, and $v_R$, resp.) and currents ($i_C$, $i_L$ and $i_R$, resp.) of each element are measured as illustrated in the figure.

We let $q_C$ and $\phi_L$ denote respectively the capacitor charge and inductor flux linkage, where $\frac{d}{dt}q_C(t) \triangleq i_L(t)$, and $\frac{d}{dt}\phi_L(t) \triangleq v_L(t)$. Capacitors and inductors are traditionally viewed as two-terminal elements which may be "coupled." For example, if $\mathcal{N}$ contains $n_C$ linear capacitors, they are described by the equation

$$q_C = \underset{\sim}{C}v_C \tag{1}$$

where the off-diagonal elements of the $n_C \times n_C$ matrix $\underset{\sim}{C}$ denote the coupling. This is discussed in further detail in the next paragraphs of this section.

The term <u>resistor</u> is used to denote any electrical element whose behavior is specified completely by its voltage and currents, i.e., there is no third variable such as charge or flux-linkage needed to prescribe the element behavior. In this sense, the set of resistors contains almost every important electrical element other than capacitors or inductors. A

common example of a nonlinear resistor is the _transistor_ shown in Fig. 1(c). It is a three-terminal resistor which is viewed as a _grounded two-port_ with voltages $v_E$ and $v_C$, and currents $i_E$ and $i_C$. In general, any (n+1)-terminal resistor may be modeled as a grounded n-port resistor.

There is an important class of resistors which we shall view as separate from other resistors. These are the _voltage sources_ and _current sources_ shown respectively in Fig. 1(d). Voltage sources are resistors whose voltages are prescribed _a priori_ but whose currents are arbitrary. A dual definition applies to current sources. As shown in Fig. 1(d) the voltage of the voltage sources and the current of the current sources may vary with time.

Consider the dynamic nonlinear network $\mathcal{N}$ shown in Fig. 2. It is formed using the elements of Fig. 1. Specifically, it contains $n_C$ (possibly coupled) one-port capacitors, and $n_L$ (possibly coupled) one-port inductors. Let $v_C, i_C, q_C \in \mathbb{R}^{n_C}$ and $v_L, i_L, \phi_L \in \mathbb{R}^{n_L}$ denote respectively the capacitor voltages, currents, charges, and the inductor voltages, currents and fluxes. The constitutive relations of a charge-controlled capacitor and a flux-controlled inductor are defined respectively by:

$$v_C = h_C(q_C)$$
$$i_L = h_L(\phi_L) \tag{2}$$

where $h_C: \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and $h_L: \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$. The constitutive relations of a voltage-controlled capacitor and current-controlled inductor are defined respectively by

$$q_C = f_C(v_C)$$
$$\phi_L = f_L(i_L) \tag{3}$$

Equation (1) is thus a special case of (3). Define the $n_p$-vectors ($n_p = n_C + n_L$);

$$v_p = \begin{bmatrix} v_C \\ v_L \end{bmatrix} ; \quad i_p = \begin{bmatrix} i_C \\ i_L \end{bmatrix} ; \quad x_p = \begin{bmatrix} v_C \\ i_L \end{bmatrix}; \quad y_p = \begin{bmatrix} i_C \\ v_L \end{bmatrix}; \quad z_p = \begin{bmatrix} q_C \\ \phi_L \end{bmatrix} \tag{4}$$

then (2) and (3) become respectively

$$x_p = h_p(z_p) ; \quad z_p = f_p(x_p) \tag{5}$$

$h_p \triangleq (h_C^T, h_L^T)^T$, $f_p \triangleq (f_C^T, f_L^T)^T$ (where "T" denotes transpose). Often the capacitors and inductors may be described by either (2) or (3), in which case $h_p = f_p^{-1}$.

We view the $n_C$ capacitors and the $n_L$ inductors as attached to an $(n_p + n_S)$-port N. Time-varying,[1] independent voltage and current sources are attached to the remaining $n_S$ ports of N. Let $u_S \in \mathbb{R}^{n_S}$ denote the voltages of the independent voltage sources, and currents of the independent current sources. Let $w_S \in \mathbb{R}^{n_S}$ denote the currents of the independent voltage sources, and voltages of the independent current sources. The vectors $x_p, y_p, u_S$ and $w_S$ are port variables of N as well as capacitor, inductor and source variables. The multiport N may contain (nonlinear) one-port resistors, (nonlinear) multi-port resistors,[2] and constant independent voltage and current sources.

Assume resistor $R_\alpha$ of N is an $n_\alpha$-port resistor. Its voltage and current are, respectively, $v_{R_\alpha}, i_{R_\alpha} \in \mathbb{R}^{n_\alpha}$. In defining its constitutive relations (when it exists) we assume that for each port of the $n_\alpha$-port resistor either the port voltage or the port current is an independent resistor variable, and the remaining port variable is a dependent resistor variable. Let $x_{R_\alpha}, y_{R_\alpha} \in \mathbb{R}^{n_\alpha}$ denote respectively the independent and dependent resistor vectors. The constitutive relation is therefore

$$y_{R_\alpha} = g_{R_\alpha}(x_{R_\alpha}) \tag{6}$$

Let $m_R$ be the number of resistors of N, and let $n_R$ be the number of all internal resistor ports of N ($m_R = n_R$ if, and only if, all resistors are two-terminal elements). The composite resistor vectors are $v_R, i_R \in \mathbb{R}^{n_R}$ representing respectively all internal voltages and currents. Let the $m_R$ resistors be described by their constitutive relation $g_{R_1}(\cdot), g_{R_2}(\cdot), \ldots, g_{R_{m_R}}(\cdot)$,

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[1] Here, a source is considered time-varying if it indeed varies with time, or if it is a constant source which is to be represented by the source vector $u_S$.

[2] N may also contain <u>controlled sources</u> (i.e., voltage and current sources whose voltages and currents respectively are dependent upon the voltage and currents of other elements of N) in the sense that most practical controlled sources can be described as "coupling" with multi-port resistors. For example, although transistors, FET, and operational amplifiers are multi-terminal elements which are often modeled using controlled sources, they can also be represented as multi-port resistors.

and let $x_R, y_R \in \mathbb{R}^{n_R}$ denote, respectively, the independent and dependent resistor vectors, then

$$y_R = g_R(x_R) \tag{7}$$

is the composite resistor constitutive relation representing all internal resistors, where $g_R(\cdot) = \left[ g_{R_1}^T(\cdot), g_{R_2}^T(\cdot), \ldots, g_{R_\alpha}^T(\cdot), \ldots g_{R_{m_R}}^T(\cdot) \right]^T$. The constitutive relation of the "overall resistor" $(n_p + n_S)$-port N (when it exists) is given by

$$y_p = -g_p(x_p, u_S) \tag{8}$$

$$w_S = -g_S(x_p, u_S) \tag{9}$$

where $g_p(\cdot, \cdot): \mathbb{R}^{n_p + n_S} \to \mathbb{R}^{n_p}$ and $g_S(\cdot, \cdot): \mathbb{R}^{n_p + n_S} \to \mathbb{R}^{n_S}$.

<u>Remarks</u>: 1. If N is an autonomous $n_p$-port, i.e. there are a no time-varying sources, then (8) becomes $y_p = -g_p(x_p)$.

    2. Equations (8) and (9) have a negative sign because the port currents (in Fig. 2) are directed away from the ports on "voltage-driven" (i.e., capacitor and voltage source) ports, and the port voltages are reversed on the "current-driven" (i.e., inductor and current source) ports. These reference directions and polarities are chosen to be consistent with those assigned to the capacitors, inductors, and sources.

    Using (5) with (8) and (9), we can write the <u>dynamical system representation</u> [3] of $\mathcal{N}$. Note first that $\frac{d}{dt} z_p(t) = \dot{z}_p(t) = y_p(t)$; hence

$$\dot{z}_p = -g_p\!\left(h_p(z_p), u_S\right) \tag{10}$$

$$w_S = -g_S\!\left(h_p(z_p), u_S\right) \tag{11}$$

These equations describe the input-output system where $u_S(\cdot)$ is the input, $w_S(\cdot)$ is the output, and $z_p(t)$ denotes the state at time t. An alternative way to view $\mathcal{N}$ is to assume that the source waveform $u_S(t)$ represents fixed time-varying sources, in which case we are interested only in the capacitor and inductor waveforms described by the state equation (10). In all cases, it is (10) which is of primary importance in determining the behavior of $\mathcal{N}$, and to this differential equation and its autonomous counterpart

$$\dot{\underset{\sim}{z}}_p = -\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right) \qquad (12)$$

we devote our attention in the sequel.

The following definitions characterize the various types of n-ports considered here, and the form of the behavior of $\mathcal{N}$ which we shall examine. Let $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary function. The function $\underset{\sim}{f}$ is _passive_ if, and only if, for all $\underset{\sim}{x} \in \mathbb{R}^n$,

$$\underset{\sim}{x}^T \underset{\sim}{f}(\underset{\sim}{x}) \geq 0 \qquad (13)$$

It is _strictly passive_ if, and only if, (13) is true where the left side is positive for all $\underset{\sim}{x} \neq \underset{\sim}{0}$. It is _eventually passive_ if, and only if, there exists a constant $k > 0$ such that (13) is true for all[3] $\|\underset{\sim}{x}\| > k$, and _eventually strictly passive_ if there exists $k > 0$ such that the left side of (13) is strictly positive for all $\|\underset{\sim}{x}\| > k$.

Let $\underset{\sim}{f}: D \subset \mathbb{R}^n \to \mathbb{R}^n$, where D is convex in $\mathbb{R}^n$.

The function $\underset{\sim}{f}$ is _increasing_ in D if, and only if, for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$

$$(\underset{\sim}{x}' - \underset{\sim}{x}'')^T\left(\underset{\sim}{f}(\underset{\sim}{x}') - \underset{\sim}{f}(\underset{\sim}{x}'')\right) \geq 0 \qquad (14a)$$

It is _strictly increasing_ in D if, and only if, the left side of (14) is positive for all $\underset{\sim}{x}' \neq \underset{\sim}{x}''$. The function $\underset{\sim}{f}$ is _uniformly increasing_ in D if, and only if, there exists $\gamma > 0$ such that for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$

$$(\underset{\sim}{x}' - \underset{\sim}{x}'')^T[\underset{\sim}{f}(\underset{\sim}{x}') - \underset{\sim}{f}(\underset{\sim}{x}'')] \geq \gamma\|\underset{\sim}{x}' - \underset{\sim}{x}''\|^2 \qquad (14b)$$

_Remarks_: The passive property and the increasing property defined above with their extensions reflect the intrinsic nature of nonlinear functions. Thus, in using these concepts in Sec. IV to determine the behavior of dynamic nonlinear networks, the results are completely general and not dependent upon the particular _mathematical form_ of the nonlinearities. Note that these definitions satisfy our intuitive physical concepts. For example, a resistor described by (6) is _passive_ (i.e., the resistor absorbs and never delivers energy) if, and only if, $g_{R_\alpha}$ is passive by the above definition. A case in point is the Ebers-Moll equation [21] describing

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[3]The norm $\|\cdot\|$ we have used in this paper is the Euclidean norm $\|\underset{\sim}{x}\| = [(x_1)^2 + \ldots + (x_n)^2]^{1/2}$. Of course, the following results remain valid for any choice of norm in $\mathbb{R}^n$.

the transistor in Fig. 1(c);

$$\begin{bmatrix} i_E \\ i_C \end{bmatrix} = \underset{\sim}{g}_{tr}\begin{pmatrix} v_E \\ v_C \end{pmatrix} \triangleq \begin{bmatrix} 1 & -\alpha_R \\ -\alpha_F & 1 \end{bmatrix}\begin{bmatrix} I_{ES}(e^{v_E/V_T}-1) \\ I_{CS}(e^{v_C/V_T}-1) \end{bmatrix} \qquad (15)$$

where the subscript "tr" denotes transistor. In (15), $I_{ES}$, $I_{CS}$, $\alpha_R$, $V_T$, and $\alpha_F$ are positive constants, and furthermore $\alpha_R < 1$, $\alpha_F < 1$, and $\alpha_R I_{CS} = \alpha_F I_{ES}$. Now, it can easily be shown that $\underset{\sim}{g}_{tr}$ is a strictly passive $C^\infty$-surjective diffeomorphism [19], [21]. We shall discuss this transistor equation

## III. Closure Properties of Resistive n-Ports

Let us examine the state equation of the autonomous network (12); namely $\dot{\underset{\sim}{z}}_p = -\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p)\right)$. As we shall see in Theorem 6 of the next section, if $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing diffeomorphic state function[4] mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, and if $\underset{\sim}{g}_p$ is strictly passive, then $\underset{\sim}{0} \in \mathbb{R}^{n_p}$ is the globally asymptotically stable equilibrium point of the differential equation. That is, for every solution $\underset{\sim}{z}_p(\cdot)$, $\lim_{t\to\infty} \underset{\sim}{z}_p(t) = \underset{\sim}{0}$. This leads us to the following:

Conjecture: Let $\mathcal{N}$ be an autonomous network. Assume its capacitor-inductor function $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ and each "internal" resistor function $\underset{\sim}{g}_{R_\alpha}$, $\alpha = 1,\ldots,$ $m_R$ is strictly passive. Then $\underset{\sim}{g}_p$ is strictly passive and $\underset{\sim}{0} \in \mathbb{R}^{n_p}$ is the globally asymptotically stable equilibrium point of the autonomous state equation (12). ∎

The key point of this Conjecture is the seemingly reasonable assumption that if each $\underset{\sim}{g}_{R_\alpha}$ is strictly passive, then the "overall" $n_p$-port resistor function $\underset{\sim}{g}_p$ is also strictly passive. Let us examine the validity of the Conjecture. Let us first examine the network of Fig. 3(a),[5] where

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[4] The $C^1$-function $\underset{\sim}{h}_p$ is a state function if, and only if, its Jacobian is symmetric everywhere in $\mathbb{R}^{n_p}$; i.e., it is an exact 1-form. Capacitors and inductors described by state functions are said to be reciprocal. The condition of reciprocity is weak and is satisfied by most capacitors and inductors of practical interest. We assume throughout this paper that $\underset{\sim}{h}_p$ is a state function.

[5] In this figure and in all other figures, any element voltage or current not explicitly shown is measured according to the convention illustrated by Fig. 1. Thus, for example in Fig. 3(a) the current $i_R$ of resistor $R_1$ is measured in the same way as the inductor current $i_L$.

we assume resistor $R_1$ is <u>current-controlled</u> ($v_{R_1} = g_{R_1}(i_{R_1})$) and resistor
$R_2$ is <u>voltage controlled</u> ($i_{R_2} = g_{R_2}(v_{R_2})$). Then

$$\begin{bmatrix} i_L \\ v_L \end{bmatrix} = -\underset{\sim}{g}_p \begin{pmatrix} v_C \\ i_L \end{pmatrix} = - \begin{bmatrix} g_{R_2}(v_C) + i_L \\ g_{R_1}(i_L) - v_C \end{bmatrix} \tag{16}$$

and it is easy to see that if $g_{R_1}$ and $g_{R_2}$ are strictly passive, then $\underset{\sim}{g}_p$
is strictly passive. Hence, for this example, the <u>Conjecture</u> is true.
We next examine Fig. 3(b) assuming the resistor is voltage-controlled.

$$\begin{bmatrix} i_{C_1} \\ i_{C_2} \\ i_{C_3} \\ v_L \end{bmatrix} = -\underset{\sim}{g}_p \begin{pmatrix} v_{C_1} \\ v_{C_2} \\ v_{C_3} \\ i_L \end{pmatrix} = - \begin{bmatrix} g_R(v_{C_2} - v_{C_1}) \\ g_R(v_{C_2} - v_{C_1}) + i_L \\ -i_L \\ -v_{C_2} + v_{C_3} \end{bmatrix} \tag{17}$$

Here, if $g_R$ is strictly passive, then $g_p$ is passive but <u>not strictly</u>
passive. Furthermore, the network <u>cannot</u> have a globally asumptotically
stable equilibrium point, because if

$$\begin{bmatrix} v_{C_1} \\ v_{C_2} \\ v_{C_3} \\ i_L \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ I \end{bmatrix} \tag{18a}$$

denotes one equilibrium point, then

$$\begin{bmatrix} v_{C_1} \\ v_{C_2} \\ v_{C_3} \\ i_L \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ I \end{bmatrix} + \begin{bmatrix} E \\ E \\ E \\ 0 \end{bmatrix} \tag{18b}$$

denotes another equilibrium point, for $E \in \mathbb{R}^1$. Hence, the <u>Conjecture</u>
is false.

Similarly, for the <u>linear</u> network of Fig. 3(c) where the (linear)
resistor is strictly passive, it can easily be shown that $\underset{\sim}{g}_p$ is <u>not</u> strictly

passive. Furthermore, the network does not have a globally asymptotically stable equilibrium point because

$$\begin{pmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{pmatrix} = \begin{pmatrix} \beta \sin 1/\sqrt{LC} \; t \\ -\beta \sin 1/\sqrt{LC} \; t \end{pmatrix} \qquad (19)$$

is a solution of the network for _any_ $\beta \in \mathbb{R}^1$.

We conclude that in general the _Conjecture_ is false; for some networks such as that of Fig. 3(a) its conclusions are valid, while for other networks such as those of Figs. 3(b) and 3(c), the conclusion is false. We further conclude that the problem does not lie in the choice of the resistor functions $g_R$ or the capacitor or inductor function $h_p$; e.g., if the network of Fig. 3(b) has _one_ equilibrium point such as (18a), it will have an infinite number of equilibrium points given by (18b), _regardless of the element constitutive relations_. Rather the problem lies with the way the network elements are interconnected. This observation has motivated the graph-theoretic research discussed and presented in [18]. In particular a series of theorems are derived in [18] which give conditions such that properties possessed by the resistor functions $g_{R_\alpha}$ are "inherited" by the composite $n_p$-port function $g_p$. These results are summarized in Table 1 and discussed below. In general, they can be called _closure property theorems_.

Let N be an autonomous $n_p$-port equation $y_p = -g_p(x_p)$.

_Theorem 1_:

(i) If each function $g_{R_\alpha}$ of each internal resistor is passive, then $g_p$ is passive.

(ii) If each function $g_{R_\alpha}$ of each internal resistor is increasing, then $g_p$ is increasing. ∎

We will not prove this theorem, or any of the others presented here. It is instructive, however, to illustrate the main points in the proof of (i) of _Theorem 1_. The proof is based on the application of _Tellegen's Theorem_:[6] Noting that the voltages and currents of the $n_p$-port N are

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[6]_Tellegen's Theorem_ states that for any network containing b branches, the linear subspace $\mathcal{V} \subset \mathbb{R}^b$ containing the voltage vectors such that KVL is satisfied is _orthogonal_ to the subspace $\mathcal{J} \subset \mathbb{R}^b$ containing the current vectors such that KCL is satisfied. See, for example, [1], [8].

measured as shown in Fig. 2, we have

$$x_{\sim p}^T g_{\sim p}(x_{\sim p}) = -v_{\sim p}^T i_{\sim p} = v_{\sim R}^T i_{\sim R} = \sum_{\alpha=1}^{m_R} (x_{\sim R_\alpha}^T) g_{R_\alpha}(x_{\sim R_\alpha}) \geq 0 \qquad (20)$$

which proves that $g_{\sim p}$ is passive. An equation similar to (20) can be written to prove (ii) of <u>Theorem 1</u>.

We have already shown with the counterexamples of Fig. 3 that if each $g_{\sim R}$ is strictly passive, then $g_{\sim p}$ is not necessarily strictly passive. The condition used to guarantee that $g_{\sim p}$ is strictly passive is the following:

<u>Fundamental Topological Hypothesis</u>: <u>There is no loop and no cutset formed exclusively by the ports of N.</u>

A more general form of the Fundamental Topological Hypothesis is given in Table 2, and Footnote 7, where it is assumed as in Fig. 2 that capacitors, inductors and sources are attached to the ports of N.

<u>Theorem 2</u>: Assume the autonomous network N satisfies the Fundamental Topological Hypothesis. Then there is a matrix $P_{\sim} \in \mathbb{R}^{2n_p \times 2n_R}$ which has elements +1, -1, and 0 such that for any set of network voltages and currents $v_{\sim p}$, $v_{\sim R}$, $i_{\sim p}$, and $i_{\sim R}$ satisfying KCL and KVL, we have

$$\begin{bmatrix} v_{\sim p} \\ i_{\sim p} \end{bmatrix} = P_{\sim} \begin{bmatrix} v_{\sim R} \\ i_{\sim R} \end{bmatrix} \qquad \blacksquare \qquad (21)$$

<u>Remarks</u>: 1. This theorem and its extensions form the basis of a large number of closure property theorems of [18]. These are listed in Table 1 and discussed to some extent in this section.

2. It is possible to elaborate further on this theorem. For example, using the constitutive relations of the resistors, it is possible under certain topological conditions given in [18] to find an <u>inverse</u> of the equation (21); i.e., there exists a function $h_{\sim N}: \mathbb{R}^{2n_p} \to \mathbb{R}^{2n_R}$ such that for any set of voltages and currents $v_{\sim p}$, $v_{\sim R}$, $i_{\sim p}$, and $i_{\sim R}$ satisfying KCL and KVL, we have

$$\begin{bmatrix} v_{\sim R} \\ i_{\sim R} \end{bmatrix} = h_{\sim N} \begin{pmatrix} v_{\sim p} \\ i_{\sim p} \end{pmatrix} \qquad (22)$$

When independent sources, capacitors and inductors are attached to N to form $\mathcal{N}$ as in Fig. 2, we can use the following condition to write a functional-relationship similar to (21):

<u>Inductor-Capacitor Loop-Cutset Hypothesis (L.C. Hypothesis)</u>

Let the dynamic nonlinear network $\mathcal{N}$ contain capacitors, inductors, resistors and constant sources. The capacitors and inductors are described by $\underset{\sim}{h}_p$ in (5), where $\underset{\sim}{h}_p$ is a $C^3$-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Let each resistor function $\underset{\sim}{g}_{R_\alpha}$ be a $C^3$-function, and let $\underset{\sim}{u}_S(\cdot)$ satisfy a global Lipschitz condition. Assume that the state equation (8) has at least one bounded solution and assume there is no loop (resp. cutset) formed exclusively by capacitors and voltage sources (resp., inductors and current sources). Furthermore, let $\mathcal{S}$ be any set of capacitors and inductors such that any capacitor or inductor in $\mathcal{S}$ forms a loop and/or cutset exclusively with any combination of independent voltage and current sources, and other capacitors and inductors of $\mathcal{S}$. For each such set $\mathcal{S}$, assume that at least one of the following conditions is satisfied:

(a) There is a capacitor $C_j$ in $\mathcal{S}$ which is in a loop formed exclusively with any combination of independent sources and other elements of $\mathcal{S}$, but not in a cutset formed exclusively with any combination of current sources and elements of $\mathcal{S}$. This capacitor is not coupled to any other capacitor of $\mathcal{S}$.

(b) There is an inductor $L_j$ in $\mathcal{S}$ which is in a cutset formed exclusively with any combination of independent sources and other elements of $\mathcal{S}$ but not in a loop formed exclusively with any combination of voltage sources and elements of $\mathcal{S}$. This inductor is not coupled to any other inductor of $\mathcal{S}$. ∎

Using the L.C. Hypothesis we show in Theorem 12 of [18] that there is a continuous function $\underset{\sim}{h}_{\mathcal{N}}$ such that

$$\begin{pmatrix} \underset{\sim}{v}_p(\cdot) \\ \underset{\sim}{i}_p(\cdot) \end{pmatrix} = \underset{\sim}{h}_{\mathcal{N}} \begin{pmatrix} \underset{\sim}{v}_R(\cdot) \\ \underset{\sim}{i}_R(\cdot) \\ \underset{\sim}{u}_S(\cdot) \end{pmatrix} \tag{23}$$

where $\underset{\sim}{v}_p(\cdot)$, $\underset{\sim}{i}_p(\cdot)$, $\underset{\sim}{v}_R(\cdot)$, $\underset{\sim}{i}_R(\cdot)$ and $\underset{\sim}{u}_S(\cdot)$ are $C^1$-functions of time, and KCL and KVL are satisfied. In Sec. IV we will show that there is a non-trivial difference between the application of the <u>Fundamental Topological Hypothesis</u> and the <u>L.C. Hypothesis</u> in determining the behavior of dynamic nonlinear networks.

The closure properties derived in [18] using <u>Theorem 2</u> and Eq. (21)

are summarized in Table 1. Also included in Table 1 is a pair of results involving equivalent networks. Of particular interest is the equivalent transformation of networks containing loops of capacitors, and/or cutsets of inductors.

In Theorem 3 below we present some of the closure property results which are of particular interest in the next section.

Theorem 3: Let N be an $n_p$-port satisfying the Fundamental Topological Hypothesis.[7]

(i) If each internal resistor function $g_{R_\alpha}$ is strictly-passive (resp., strictly increasing, $C^\mu$—strictly increasing, surjective and diffeomorphic) then $g_p(\cdot, u_S)$ in (8) is strictly passive (resp., strictly increasing, $C^\mu$-strictly increasing, surjective and diffeomorphic) for all $u_S \in \mathbb{R}^{n_S}$.

(ii) If each $g_{R_\alpha}$ is eventually strictly passive and satisfies the "growth condition"

$$\lim_{\|x_{R_\alpha}\| \to \infty} \frac{1}{\|x_{R_\alpha}\|} (x_{R_\alpha})^T g_{R_\alpha}(x_{R_\alpha}) = +\infty \tag{24}$$

then $g_p(\cdot, u_S)$ is eventually strictly passive, for all $u_S \in \mathbb{R}^{n_S}$.  ∎

Remark: In [18] we note that (24) is true if $g_{R_\alpha}$ is uniformly increasing or is a $C^1$-strictly increasing and surjective diffeomorphism. Furthermore, (24) may be relaxed to

$$\lim_{\|x_{R_\alpha}\| \to \infty} (x_{R_\alpha})^T g_{R_\alpha}(x_{R_\alpha}) = +\infty \tag{25}$$

if N contains no sources, or if all sources are voltage sources (resp., current sources) and all resistors are voltage-controlled (resp., current-controlled).

## IV. The Behavior of Dynamic Nonlinear Networks

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[7] When N contains independent sources, the Fundamental Topological Hypothesis should be modified as follows: There is no loop (resp., cutset) formed exclusively by capacitors, inductors, and/or independent voltage sources (resp., current sources). Observe that under this hypothesis, the set defined in the preceding L.C. Hypothesis becomes an empty set and hence the topological conditions of the L.C. Hypothesis are automatically satisfied.

In this section, the theorems of [19] and [20] prescribing the behavior of autonomous and nonautonomous dynamic nonlinear networks are discussed. These results are summarized in Table 2 of Sec. V. The hypotheses take two forms; first, purely mathematical conditions of the network state equations (10) and (12), and second, conditions on the network element constitutive relations and on their interconnection. Theorems of the latter type use the graph theoretic results of the previous section. These results are discussed and illustrated in the theorems below. Let us begin by examining networks which have no finite-forward escape time solutions; i.e., networks whose solutions exist for all "forward" times.

A. <u>Networks with No Finite Escape-Time Solutions</u>

A well-known theorem of A. Wintner [22] states that if there exists a continuous function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\int^{\infty} \frac{du}{\psi(u)} = +\infty$$

such that for $g_{\sim p}$ and $h_{\sim p}$ given in (8) and (10), we have

$$\left\| g_p\left( h_{\sim p}(z_{\sim p}), u_{\sim S} \right) \right\| \leq \psi(\|z_{\sim p}\|), \tag{26}$$

$\forall \, \|z_{\sim p}\| > k$, $\forall \, u_{\sim S} \in \mathbb{R}^{n_S}$, where k is any positive constant, then the solutions $z_{\sim p}(\cdot)$ of state Eq. (10) are well-defined <u>for all</u> $t \in \mathbb{R}^1$.

This conclusion offers more than is really needed in the study of dynamic nonlinear networks and, as a consequence, the conditions (26) is prohibitively strong. This observation is illustrated with the network of Fig. 4 which is formed with two of the most common elements used in electrical networks; namely, the linear capacitor and the silicon diode. The diode constitutive relation is given in Fig. 4, where $I_S$ and $V_T$ are positive constants. The state equation of this network is given by

$$\dot{v}_C = -\frac{I_s}{C} \left[ e^{v_C/V_T} - 1 \right] \tag{27}$$

As we shall see (<u>Theorem 6</u>), $v_C = 0$ is the globally asymptotically stable equilibrium point of (27); i.e., every solution $v_C(\cdot)$ satisfies $\lim_{t \to \infty} v_C(t) = 0$. Every other solution has the form

$$v_C(t) = v_C(0) \, \ell n \left( \frac{e^{f(t)}}{e^{f(t)} - \text{sgn } v_C(0)} \right)^{V_T/v_C(0)} \tag{28a}$$

where

$$f(t) \triangleq \left(I_S/CV_T\right)t + \ln\left(\frac{\text{sgn } v_C(0)}{1 - e^{-v_C(0)/V_T}}\right) \qquad (28b)$$

and

$$\text{sgn } v_C(0) \triangleq v_C(0)/|v_C(0)| \qquad (28c)$$

where, of course, $v_C(0) \neq 0$. Observe that when $v_C(0) > 0$, (27) has a finite "backward" escape-time solution. For example, if we choose the time

$$t_1 \triangleq -\left(\frac{CV_T}{I_S}\right)\ln\left(\frac{1}{1 - e^{-v_C(0)/V_T}}\right) < 0 \qquad (29)$$

then it follows from (28a) that $v_C(t_1) = +\infty$. In [19] criteria are presented which guarantee that for any initial time $t_0 \in \mathbb{R}^1$ and any initial condition $z_p(t_0)$ of (10) the solutions $z_p(\cdot)$ are well-defined for all "forward time" $t \geq t_0$. Furthermore, the criteria are more practical for our purposes than that of Wintner because most networks of practical interest, such as that of Fig. 4, satisfy the conditions. An example of these results is the following:

Theorem 4: Assume there exists a continuous function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (25) such that

$$z_p^T g_p\left(h_p(z_p), u_S\right) \geq -\psi\left(\|z_p\|^2\right) \qquad (30)$$

$\forall \|z_p\| > k$, $\forall u_S$, where $k$ is a positive constant. Then for any initial time $t_0$ and initial condition $z_p(t_0)$, the solutions of (10) are well-defined for all $t \geq t_0$. ∎

Two theorems based on variations of these criteria and particular choices of the function $\psi(\cdot)$ are given in Table 2.

B. Networks with Bounded or Eventually Uniformly Bounded Solutions

The solutions of state equations (10) and (12) are said to be bounded if, and only if, for every solution $z_p(\cdot)$ there exists a constant $k_0$ and time $t_0$ such that $\|z_p(t)\| < k_0$ for all $t \geq t_0$. The solutions are said to be eventually uniformly bounded if, and only if, there exists $k_0$ such that for every solution $z_p(\cdot)$ there is a time $t_0$ such that $\|z_p(t)\| < k_0$ for all $t \geq t_0$.

A series of theorems are presented in [19] and [20] and summarized in Table 2 in Sec. V. They give conditions such that the solutions of state equations (10) and (12) are eventually uniformly bounded. The theorems are based on a well-known application of Lyapunov's Direct Method [23], [24]. Furthermore, using Brouwer's Fixed Point Theorem, we show in [19] and [20] that if the solutions of (12) are eventually uniformly bounded then (12) has an equilibrium point, and if the solutions of (10) are eventually uniformly bounded, then there is a periodic solution $z_p(\cdot)$ with period $T > 0$ whenever $u_S(\cdot)$ is periodic with period T.

The theorems concerning the eventual uniform boundedness of solutions of (10) require that $g_p(\cdot, u_S)$ be eventually strictly passive for each $u_S \in \mathbb{R}^{n_S}$. We can therefore use the methods of the previous section to apply conditions to the resistor functiong $g_{R_\alpha}$ and to invoke the Fundamental Topological Hypothesis to assure that this criterion is satisfied. Several such results are presented in Table 2. We can form a similar theorem for transistor networks.[8] It is based on the observation that the transistor constitutive relation $g_{tr}$ given in (15) is strictly passive, and satisfies (25).

Theorem 5: Let $\mathcal{N}$ be a network whose state equation is given by (10). This network may contain capacitors, inductors, transistors (described by (15)), voltage-controlled resistors which are eventually strictly passive and which satisfy (25), and independent voltage sources. Assume the capacitor-inductor function $h_p$ is a state function and its associated scalar function $H_p: \mathbb{R}^{n_p} \to \mathbb{R}^{1_p}$ is such that $\nabla H_p = h_p$, and

$$\lim_{\|z_p\| \to \infty} \| h_p(z_p) \| = +\infty$$

$$\lim_{\|z_p\| \to \infty} H_p(z_p) = +\infty \qquad (31)$$

Assume further that $\mathcal{N}$ satisfies the Fundamental Topological Hypothesis.

Under these conditions, for any bounded and continuous set of time-varying voltage sources (i.e., $u_S(\cdot)$ is continuous and bounded), the

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[8] Of course, any theorem of Table 2 may be applied to transistor networks if $g_{tr}$ in (15) satisfies the appropriate conditions. Theorem 5 is specifically designed to apply to transistor networks.

solutions of (10) are eventually uniformly bounded. Furthermore, if $\underset{\sim}{u}_S(\cdot)$ is periodic with period $T > 0$, then (10) has a solution $\underset{\sim p}{z}(\cdot)$ which is periodic with period T. ∎

C. **Networks with Unique Steady-State Solutions**

The state equations (10) and (12) are said to have a unique steady-state solution if, and only if, regardless of the initial conditions, for any two solutions $\underset{\sim p}{z}'(\cdot)$ and $\underset{\sim p}{z}''(\cdot)$, both solutions are bounded and

$$\lim_{t \to \infty} \| \underset{\sim p}{z}'(t) - \underset{\sim p}{z}''(t) \| = 0 \tag{32}$$

Note that if the autonomous state equation (12) has a unique steady-state solution, then the solutions of (12) are eventually uniformly bounded and hence (12) has a constant solution $\underset{\sim p}{z}(t) \equiv \underset{\sim p}{z}^*$. Then, from (32) we conclude that $\underset{\sim p}{z}^*$ is the globally asymptotically stable equilibrium point of (12).

It is often both useful and necessary to determine if an autonomous network has a globally asymptotically stable equilibrium point, or to determine if a nonautonomous network has a unique steady-state solution. Consequently we have developed a number of results dealing with these forms of network behavior in [19] and [20], respectively. The results are listed in Table 2. In the remainder of this section we discuss two of the theorems with their extensions. The following theorm has been discussed in conjunction with the <u>Conjecture</u> of the previous section.

<u>Theorem 6</u>: Let $\mathcal{N}$ be an autonomous network described by state equation (12). Assume $\underset{p}{h}$ is a $C^1$-strictly increasing and surjective diffeomorphic state function, and assume $\underset{\sim p}{g}$ is strictly passive. Then $\underset{\sim p}{z}^* \overset{\Delta}{=} \underset{p}{h}^{-1}(\underset{\sim}{0})$ is the globally asymptotically stable equilibrium point of (12). ∎

From <u>Theorem 3</u> we conclude that the condition that $\underset{\sim p}{g}$ is strictly passive is satisfied if each resistor constitutive relation $\underset{\sim}{g}_{R_\alpha}$ is strictly passive, <u>and</u> if $\mathcal{N}$ satisfies the <u>Fundamental Topological Hypothesis</u>.

We have already noted that if the nonautonomous state equation (10) has a unique steady-state solution, then the solutions of (10) are eventually uniformly bounded. If $\underset{\sim}{u}_S(\cdot)$ is <u>periodic</u> with period T, then (10) has a periodic solution $\underset{\sim p}{z}^*(\cdot)$ with period T. Then, from Eq. (32) we

conclude that every other solution $z_p(\cdot)$ converges to $z_p^*(\cdot)$ as $t \to +\infty$; i.e., every solution $z_p(\cdot)$ is "asymptotically periodic." We can extend this observation to the case where $u_S(\cdot)$ is asymptotically almost periodic[8] by using the following result of Yoshizawa [24]: If the state equation (10) has a unique steady-state solution and $u_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then each solution $z_p(\cdot)$ of (10) is asymptotically almost periodic. Furthermore, let $S_{u_S}$ and $S_{z_p}$ -- called the Spectrum of $u_S(\cdot)$ and $z_p(\cdot)$ -- denote respectively the countable modules formed by integer combinations of the Fourier exponents of the almost periodic functions to which $u_S(\cdot)$ and $z_p(\cdot)$ converge. Then,

$$S_{z_p} \subset S_{u_p}.$$

We shall use this result in the following theorem. First, let us define the ultimate range of a time-varying function $\xi(\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ as

$$\mathcal{R}_\infty\bigl(\xi(\cdot)\bigr) \triangleq \overline{\bigcap_{T \in \mathbb{R}^1} \{\xi \in \mathbb{R}^n : \exists\, t_0 \geq T \text{ such that } \xi(t_0) = \xi\}} \qquad (33)$$

**Theorem 7:** Assume in the dynamic nonlinar network $\mathcal{N}$ that the capacitor-inductor function $h_p$ is a $C^1$-strictly increasing and diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, and its Jacobian is Lipschitz continuous. Assume $\mathcal{N}$ satisfies the **Fundamental Topological Hypothesis** and let each resistor function $g_{R_\alpha}$ be a $C^1$-strictly increasing diffeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$, satisfying

$$\lim_{\|x_{R_\alpha}\| \to \infty} \frac{1}{\|x_{R_\alpha}\|} \left[ (x_{R_\alpha})^T g_{R_\alpha}(x_{R_\alpha}) \right] = +\infty \qquad (34)$$

Under these conditions, for every $u_S^* \in \mathbb{R}^{n_S}$ there exists $\delta > 0$ such that for any Lipschitz continuous and bounded $u_S(\cdot)$ satisfying[9]

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[8] A continuous function $\xi(\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ is **asymptotically almost periodic** if, and only if, there is a continuous almost periodic function $\xi_0(\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ such that $\lim_{t \to \infty} \xi(t) - \xi_0(t) = 0$. Note that the almost periodic function $\xi_0(\cdot)$ may be uniformly approximated by a Fourier series in the same sense that a periodic function can be represented by a Fourier series [24].

[9] That is, if $\hat{u}_S \in \mathcal{R}_\infty\bigl(u_S(\cdot)\bigr)$, then $\|u_S - u_S^*\| < \delta$.

$$\left\| \mathcal{R}_\infty\left(u_S(\cdot)\right) - u_S^* \right\| < \delta \tag{35}$$

$\mathcal{N}$ has a unique steady-state solution. Furthermore, if $u_S(\cdot)$ is asymptotically almost periodic, then every solution of $\mathcal{N}$ is asymptotically almost periodic, and in the steady-state $\mathcal{S}_{z_p} \subset \mathcal{S}_{u_S}$. ∎

In many of the theorems of this type presented in [19] and [20] it is possible to extend the conclusions in the following way: If (10) has a unique steady-state solution in the sense of Eq. (32), the solution $z_p'(\cdot)$ and $z_p''(\cdot)$ may converge _exponentially_ to each other. That is, there may exist constants $\bar{\gamma} \geq \underline{\gamma} > 0$ and times $\tau_{max} \geq \tau_{min} > 0$ such that

$$\underline{\gamma}\|z_p'(0) - z_p''(0)\|e^{-t/\tau_{min}} \leq \|z_p'(t) - z_p''(t)\|$$

$$\leq \bar{\gamma}\|z_p'(0) - z_p''(0)\|e^{-t/\tau_{max}}, \quad \forall \, t \geq 0 \tag{36}$$

It is useful to establish inequalities such as (36) in order to estimate "transient decay times" of dynamic networks. An _algorithm_ for computing the two constants $\bar{\gamma} > 0$ and $\tau_{max} > 0$ of (36) is presented in [19].

It is also possible to use _weaker hypotheses_ in these theorems. Specifically, the _Fundamental Topological Hypothesis_ may be replaced by the weaker _L.C. Hypothesis_ in many cases. For example, the linear network of Fig. 5(a) has a globally asymptotically stable equilibrium point $\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Similarly, it can be shown (assuming $I(\cdot)$ is continuous and bounded) that if each resistor constitutive relation in Fig. 5(b) is a $C^\infty$-diffeomorphism, then the network has a unique steady-state solution. In both cases, the _Fundamental Topological Hypothesis_ is not satisfied (there is a loop formed by the capacitor and inductor in Fig. 5(a), and a cutset formed by the capacitor, inductor, and current source in Fig. 5(b)). The _L.C. Hypothesis_ can be applied, however, to show that these networks converge to a unique steady-state solution.

## V. Conclusions

A number of results concerning the qualitative behavior of dynamic nonlinear networks have been presented and discussed. In their final form, the hypotheses of these results involve conditions on the individual

elements' interconnections. These conditions are simple, easy to verify, and thus quite practical. The graph-theoretic theorems of [18], and the network behavior theorems of [19], [20] are summarized respectively in Table 1 and Table 2 which follow.

We conclude by noting that these theorems are often the best possible. A case in point is the application of Theorem 7 to the network of Fig. 6. This theorem is directly applicable, and it can be easily seen that there exists some "sufficiently small" $\delta > 0$, such that for any $\delta_0 > 0$, $0 \leq \delta_0 < \delta$ the network state equation has a unique steady-state solution. On the other hand, we have shown in [20] that for $\delta_0 = 1$ this circuit does not have a unique steady-state solution. Instead, we found a periodic solution with frequency $\omega = 1$, and a subharmonic solution with frequency $\omega = 1/10$. Similar counterexamples for the other theorems of [19] and [20] can be found to show that the conditions given are often necessary as well as sufficient.

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<div align="center">

## Table 1: GRAPH-THEORETIC THEOREMS
## OF NETWORK $\mathcal{N}$ AND N-PORT N

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### A. Definitions and Notation

| Network Elements | Notation and Dimension | Constitutive Relation |
|---|---|---|
| $n_C$ capacitors<br>$\quad q_C$ = charge<br>$\quad \tilde{v}_C$ = voltage<br>$n_L$ inductors<br>$\quad \phi_L$ = flux<br>$\quad i_L$ = current<br>$n_C + n_L = n_p$ | $q_C, v_C \in \mathbb{R}^{n_C}$<br><br>$\phi_L, i_L \in \mathbb{R}^{n_L}$<br><br>$z_p = \begin{bmatrix} q_C \\ \phi_L \end{bmatrix};\ x_p \begin{bmatrix} v_C \\ i_L \end{bmatrix}$ | $q_C = f_C(v_C)$ ; $v_C = h_C(q_C)$<br><br>$\phi_L = f_L(i_L)$ ; $i_L = h_L(\phi_L)$<br><br>$z_p = f_p(x_p)$ ; $x_p = h_p(z_p)$ |
| $m_R$ resistors<br>$\quad v_R$ = voltage<br>$\quad i_R$ = current<br>each resistor<br>$R_\alpha(\alpha=1,..,m_R)$ is<br>an $n_\alpha$-port<br><br>$\sum\limits_{\alpha=1}^{m_R} n_\alpha = n_R$ | $v_{R_\alpha}, i_{R_\alpha} \in \mathbb{R}^{n_\alpha}$<br><br>$x_{R_\alpha}, y_{R_\alpha} \in \mathbb{R}^{n_\alpha}$<br><br>$v_R, i_R \in \mathbb{R}^{n_R}$<br><br>$x_R, y_R \in \mathbb{R}^{n_R}$ | $y_{R_\alpha} = g_{R_\alpha}(x_{R_\alpha})$<br><br><br>$y_R = g_R(x_R)$ |
| Independent voltage and current sources:<br><br>$\quad n_S$ | $u_S \in \mathbb{R}^{n_S}$ | $u_S = u_S(\cdot)$ |
| Autonomous n-port | $v_p, i_p \in \mathbb{R}^{n_p}$ | $y_p = -g_p(x_p)$ |
| Nonautonomous n-port | $x_p, y_p \in \mathbb{R}^{n_p}$ | $y_p = -g_p(x_p, u_S)$ |

### B. Equivalent Networks

From network $\mathcal{N}$ we may form an equivalent network $\tilde{\mathcal{N}}$ such that:

1. There is no loop and no cutset formed exclusively by voltage and current sources.

2. There is no loop (resp., cutset) formed exclusively by capacitors and constant voltage sources (resp., inductors and constant current sources).

Moreover, each of the following properties of the capacitors and inductors of $\mathcal{N}$ are also properties of the capacitors and inductors of $\tilde{\mathcal{N}}$:

(i) $f_p$ is a state function

(ii) $f_p$ is an increasing (strictly increasing, uniformly increasing) function.

(iii) $f_p$ is a strictly increasing, surjective, $C^\mu$-diffeomorphic state function, $\mu \geq 0$.

Table 1 Continued

## C. Closure Properties of n-Ports

| Resistor Property of each $g_{R_\alpha}, \alpha = 1,2,\ldots m_R$ | Additional Conditions (If Any) Under which the n-Port Constitutive Relation $g_p$ Has This Property | |
|---|---|---|
| | Graph-Theoretic Conditions | Resistor Conditions |
| **Autonomous n-Ports** | | |
| Passive | none | none |
| Increasing | none | none |
| Strictly Passive | There is no loop and no cut-set formed exclusively by the ports. | none |
| Strictly Increasing | There is no loop and no cut-set formed exclusively by the ports. | none |
| Strictly Increasing, Surjective $C^\mu$-Diffeo-morphic ($\mu \geq 0$) | There is no loop and no cut-set formed exclusively by the ports. | none |
| Eventually Strictly Passive | There is no loop and no cut-set formed exclusively by the ports. | For each $\alpha = 1,\ldots,m_R$ $$\lim_{\|x_{R_\alpha}\| \to \infty} (x_{R_\alpha})^T g_{R_\alpha}(x_{R_\alpha}) = +\infty$$ |
| Uniformly Increasing | There is no loop and no cut-set formed exclusively by the ports. | Each $g_{R_\alpha}$ and $g_{R_\alpha}^{-1}$ is uniformly increasing |
| **Nonautonomous n-Ports** | | |
| (Here, $g_p(\cdot, u_S)$ has the appropriate property for each $u_S \in \mathbb{R}^{n_S}$) | | |
| Increasing | none | none |
| Strictly Increasing | There is no loop (resp., cut-set) formed exclusively by the ports and voltage (resp., current) sources. | none |
| Strictly Increasing Surjective, $C^\mu$-diffeomorphic ($\mu \geq 0$) | There is no loop (resp., cut-set) formed exclusively by the ports and voltage (resp., current) sources. | none |
| Eventually Strictly Passive | There is no loop (resp., cut-set) formed exclusively by the ports and voltage (resp., current) sources. | $$\lim_{\|x_{R_\alpha}\| \to \infty} \frac{1}{\|x_{R_\alpha}\|}(x_{R_\alpha})^T g_{R_\alpha}(x_{R_\alpha}) = +\infty$$ |
| Uniformly Increasing | There is no loop (resp., cut-set) formed exclusively by the ports and voltage (resp., current) sources. | Each $g_{R_\alpha}$ and $g_{R_\alpha}^{-1}$ is uniformly increasing |

Table 1 Continued

D. Relationship between Internal Resistor and External Port Variables

| Network or N-port Relationship | Properties of the Relationship | Criteria for the Relationship to exist |
|---|---|---|
| Autonomous n-port equation $$\begin{bmatrix} \underset{\sim}{v}_p \\ \underset{\sim}{i}_p \end{bmatrix} = \underset{\sim}{P} \begin{bmatrix} \underset{\sim}{v}_R \\ \underset{\sim}{i}_R \end{bmatrix}$$ $$\underset{\sim}{x}_p = \underset{\sim}{P}_1 \begin{bmatrix} \underset{\sim}{x}_R \\ \underset{\sim}{y}_R \end{bmatrix}$$ | The elements of matrices $\underset{\sim}{P}$ and $\underset{\sim}{P}_1$ are +1, -1, and 0. There is no all zero row. $$\underset{\sim}{P} \in \mathbb{R}^{2n_p \times 2n_R}$$ $$\underset{\sim}{P}_1 \in \mathbb{R}^{n_p \times 2n_R}$$ | Necessary and Sufficient Condition: There is no loop and no cutset formed exclusively by the ports. |
| Autonomous n-port equation $$\begin{bmatrix} \underset{\sim}{v}_R \\ \underset{\sim}{i}_R \end{bmatrix} = \underset{\sim}{h}_N \begin{pmatrix} \underset{\sim}{v}_p \\ \underset{\sim}{i}_p \end{pmatrix}$$ | When each resistor function $g_{R_\alpha}$ is $C^\mu$, $\mu \geq 0$, then $\underset{\sim}{h}_N$ is $C^\mu$. | Theorem 4, Reference [18]. |
| Network equation $$\begin{bmatrix} \underset{\sim}{v}_p(\cdot) \\ \underset{\sim}{i}_p(\cdot) \end{bmatrix} = \underset{\sim}{h}_{\mathcal{N}} \begin{pmatrix} \underset{\sim}{v}_R(\cdot) \\ \underset{\sim}{i}_R(\cdot) \\ \underset{\sim}{u}_S(\cdot) \end{pmatrix}$$ | For any compact interval $D_t = [\tau_1, \tau_2] \subseteq \mathbb{R}^1$, $\tau_1 < \tau_2$, $\underset{\sim}{h}_{\mathcal{N}}$ is a continuous map of a subset of $C^1$-functions mapping $D_t$ into $\mathbb{R}^{2n_R + n_S}$ into the space of $C^1$-functions mapping $D_t$ into $\mathbb{R}^{2n_p}$ | The L.C. Hypothesis Also, $\underset{\sim}{v}_p(\cdot)$, $\underset{\sim}{v}_R(\cdot)$, $\underset{\sim}{i}_p(\cdot)$, $\underset{\sim}{i}_R(\cdot)$ and $\underset{\sim}{u}_S(\cdot)$ satisfy KVL and KCL and are $C^1$-functions of time. |

# TABLE 2. CRITERIA FOR DETERMINING THE BEHAVIOR OF DYNAMIC NONLINEAR NETWORKS

The notation is the same as in Table 1, with the following additions:

1] Let $\xi(\cdot): \mathbb{R}^1 \to \mathbb{R}^m$ be continuous. Define the <u>ultimate range</u> of $\xi(\cdot)$; $\mathcal{R}_\infty\left(\xi(\cdot)\right) \triangleq \overline{\bigcap_{T\in\mathbb{R}^1} \{\xi \in \mathbb{R}^m: \exists\, t_0 \geq T, \text{ with } \xi(t_0) =}$

2] Let $\xi(\cdot): \mathbb{R}^1 \to \mathbb{R}^m$ be <u>asymptotically almost periodic</u> and let $\{\omega_k\}$ be the set of Fourier exponents of the almost periodi function to which $\xi(\cdot)$ converges. Define the <u>spectrum of $\xi(\cdot)$</u> $\mathcal{S}_\xi$ to be the countable module formed by integer combinations of the $\omega_k$.

3] Network state equations: Autonomous network $\dot{z}_p = -g_p\left(h_p(z_p)\right)$. Nonautonomous network $\dot{z}_p = -g_p\left(h_p(z_p), u_S\right)$. The functions $h$ and $g_p$ are $C^1$. The autonomous network contains no voltage or current sources other than constant sources which are absorbed into the capacitor, inductor, and resistor constitutive relations.

The different forms of network behavior are listed beginning with the weakest conclusion that there are no finite escape time solutions to the strongest conclusion that there is a unique steady-state solution. In general, the criteria of the stronger conclusions are also sufficient to guarantee the weaker conclusions.

The hypotheses are of two types: first, mathematical conditions on $h_p$ and $g_p$, and second, conditions on $h_p$, the resistor functions $g_{R_\alpha}$, and on the interconnection of elements.

| Network Behavior | Conditions on $h_p$; $h_p$ is a State Function and $H_p$ is a Functional Satisfying $\nabla H_p \equiv h_p$ | Conditions on $g_p$ and on $u_S$ | Additional Conclusions and Comments |
|---|---|---|---|
| <u>Autonomous or nonautonomous network</u>: there exists no finite forward escape time solution. | $\exists$ constants $k_1 \geq 0$, $\overline{\gamma} \geq \underline{\gamma} > 0$ such that $\underline{\gamma}\|z_p'-z_p''\|^2 \leq (z_p'-z_p'')^T[h_p(z_p')-h_p(z_p'')] \leq \overline{\gamma}\|z_p'-z_p''\|^2$, $\forall\ \|z_p'\| > k_1$ $\quad \|z_p''\| > k_1$ | $\exists$ arbitrary matrix $G_p$, vector $\hat{y}_p$, constants $k_1 \geq 0$, $k_2 \geq 0$ such that $x_p^T[g_p(x_p,u_S) + G_p x_p + \hat{y}_p] \geq -k_1$, $\forall\ u_S,\ \forall\ \|x_p\| > k_2$ | 1. Solution may become unbounded as $t \to \infty$. 2. Solution may become unbounded at some <u>finit</u> but <u>backward</u> time. |
| <u>Autonomous or nonautonomous network</u>: there exists no finite forward escape time solution. | $\displaystyle\lim_{\|z_p\|\to\infty} \|h_p(z_p)\| = +\infty$ $\displaystyle\lim_{\|z_p\|\to\infty} H_p(z_p) = +\infty$ | $\exists$ constants $k_1 \geq 0$ and $k_2 \geq 0$ such that $x_p^T g_p(x_p,u_S) \geq -k_1$, $\forall\ u_S,\ \forall\ \|x_p\| > k_2$ | The conditions on $h_p$ and $H_p$ are satisfied if $h_p$ is a $C^1$-strictly increasing diffeomorphic surjective state function. |
| <u>Autonomous or nonautonomous network</u>: solutions are bounded | $\displaystyle\lim_{\|z_p\|\to\infty} \|h_p(z_p)\| = +\infty$ $\displaystyle\lim_{\|z_p\|\to\infty} H_p(z_p) = +\infty$ | $\exists\ k_1 > 0$ such that $\|u_S(t)\| < k_1$, $\forall\ t \in \mathbb{R}^1$, and $g_p(\cdot,u_S)$ is eventually passive, $\forall\ \|u_S\| < k_1$ | Solution may have arbitrarily large amplitude, depending on the initial condition. Lossless networks have this property. |
| <u>Autonomous network</u>: solutions are eventually uniformly bounded | $\displaystyle\lim_{\|z_p\|\to\infty} \|h_p(z_p)\| = +\infty$ $\displaystyle\lim_{\|z_p\|\to\infty} H_p(z_p) = +\infty$ | $g_p$ is eventually strictly passive | The network has an equilibrium point. Lossless networks do not have this property. |

TABLE 2 CONTINUED

| Network Behavior | Conditions on $\underset{\sim}{h}_p$; $\underset{\sim}{h}_p$ is a State Function and $H_p$ is a Functional Satisfying $\nabla H_p \equiv \underset{\sim}{h}_p$ | Conditions on $\underset{\sim}{g}_p$ and on $\underset{\sim}{u}_S$ | Additional Conclusions and Comments |
|---|---|---|---|
| Nonautonomous network: solutions are eventually uniformly bounded | $\lim_{\\|\underset{\sim}{z}_p\\| \to \infty} \\| \underset{\sim}{h}_p(\underset{\sim}{z}_p) \\| = +\infty$ <br><br> $\lim_{\\|\underset{\sim}{z}_p\\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty$ | $\exists\, k_1 > 0$ such that $\\|\underset{\sim}{u}_S(t)\\| < k_1$, $\forall\, t \in \mathbb{R}^1$, and $g_p(\cdot, \underset{\sim}{u}_S)$ is eventually strictly passive, $\forall\, \\|\underset{\sim}{u}_S\\| < k_1$. | If $\underset{\sim}{u}_S(\cdot)$ is periodic with period $T > 0$, then there is a solution $\underset{\sim}{z}_p^*(\cdot)$ which is periodic with period $T$. |
| Autonomous network: there is a globally asymptotically stable equilibrium point $\underset{\sim}{z}_p^*$ | $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing surjective and diffeomorphic state function. | $\underset{\sim}{g}_p$ is strictly passive with respect to $\underset{\sim}{x}_p^* = \underset{\sim}{h}_p(\underset{\sim}{z}_p^*)$, i.e., $(\underset{\sim}{x}_p - \underset{\sim}{x}_p^*)^T \underset{\sim}{g}_p(\underset{\sim}{x}_p) > 0$, $\forall\, \underset{\sim}{x}_p \neq \underset{\sim}{x}_p^*$. | If $\dfrac{\partial g_p(\underset{\sim}{x}_p^*)}{\partial \underset{\sim}{x}_p}$ is positive definite, solutions converge <u>exponentially</u> to the equilibrium point. |
| Autonomous network: there is a globally asymptotically stable equilibrium point $\underset{\sim}{z}_p^*$ | $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing surjective and diffeomorphic state function | $\underset{\sim}{g}_p$ is a strictly increasing and surjective homeomorphism. | If $\underset{\sim}{g}_p$ is in addition a $C^1$-diffeomorphism, solutions converge exponentially to the equilibrium point. |
| Nonautonomous network: for every $\underset{\sim}{u}_S^*$, and $\varepsilon > 0$ $\exists\, \delta > 0$ and a <u>unique</u> $\underset{\sim}{z}_p^*$ such that $g_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*), \underset{\sim}{u}_S^*\right) = 0$, $\left[\\|\mathcal{R}_\infty\left(\underset{\sim}{u}_p(\cdot)\right) - \underset{\sim}{u}_S^*\\| < \delta\right] \Rightarrow$ $\left[\\|\mathcal{R}_\infty\left(\underset{\sim}{z}_p(\cdot)\right) - \underset{\sim}{z}_p^*\\| < \varepsilon\right]$ for every solution $\underset{\sim}{z}_p(\cdot)$. | $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing surjective and diffeomorphic state function. | $g_p(\cdot, \underset{\sim}{u}_S)$ is a strictly increasing eventually strictly passive, and surjective $C^1$-diffeomorphism, $\forall\, \underset{\sim}{u}_S$. | If $\lim_{t \to \infty} \underset{\sim}{u}_S(t) = \underset{\sim}{u}_S^*$, then $\lim_{t \to \infty} \underset{\sim}{z}_p(t) = \underset{\sim}{z}_p^*$ for every solution $\underset{\sim}{z}_p(\cdot)$. |
| Nonautonomous network: there exists a unique steady-state solution | $\underset{\sim}{h}_p$ is linear; i.e., $\underset{\sim}{h}_p(\underset{\sim}{z}_p) = \underset{\sim}{\Gamma}_p \underset{\sim}{z}_p$ where $\underset{\sim}{\Gamma}_p$ is positive-definite and symmetric. | $g_p(\cdot, \underset{\sim}{u}_S)$ is a strictly increasing and eventually strictly passive homeomorphism, $\forall\, \underset{\sim}{u}_S$. | If $\underset{\sim}{u}_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then every solution $\underset{\sim}{z}_p(\cdot)$ is asymptotically almost periodic and $\mathcal{S}_{z_p} \subset \mathcal{S}_{u_S}$. If $g_p(\cdot, \underset{\sim}{u}_S)$ is in addition a $C^1$-diffeomorphism, solutions converge exponentially to the steady-state |

TABLE 2 CONTINUED

| Network Behavior | Conditions on $\underset{\sim}{h}_p$; $\underset{\sim}{h}_p$ is a State Function and $H_p$ is a Functional Satisfying $\nabla H_p \equiv \underset{\sim}{h}_p$ | Conditions on $\underset{\sim}{g}_p$ and on $\underset{\sim}{u}_S$ | Additional Conclusions and Comments |
|---|---|---|---|
| <u>Nonautonomous network</u>: there is a unique steady-state solution, and all other solutions converge exponentially to the steady-state. | $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing and surjective diffeomorphic state function. | $\underset{\sim}{g}_p$ is linear; $$\underset{\sim}{g}_p(\underset{\sim}{x}_p,\underset{\sim}{u}_S) = \underset{\sim}{G}_p\underset{\sim}{x}_p + \underset{\sim}{G}_S\underset{\sim}{u}_S$$ where $\underset{\sim}{G}_p$ is positive-definite and symmetric. | If $\underset{\sim}{u}_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then every solution $\underset{\sim}{z}_p(\cdot)$ is asymptotically almost periodic, and $\mathcal{S}_{z_p} \subset \mathcal{S}_{u_S}$. |
| <u>Nonautonomous network</u>: for every $\underset{\sim}{u}_S^*$ $\exists\, \delta > 0$ such that if $$\left\| \mathcal{R}_\infty\!\left(\underset{\sim}{u}_S(\cdot)\right) - \underset{\sim}{u}_S^* \right\| < \delta$$ then there is a unique steady state solution | $\underset{\sim}{h}_p$ is a $C^1$-strictly increasing and surjective diffeomorphic state function, and its Jacobian is Lipschitz continuous. | $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ is a strictly-increasing eventually strictly passive and surjective $C^1$-diffeomorphism, $\forall\, \underset{\sim}{u}_S$. | If $\underset{\sim}{u}_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then every solution $\underset{\sim}{z}_p(\cdot)$ is asymptotically almost periodic, and $\mathcal{S}_{z_p} \subset \mathcal{S}_{u_S}$. |

In the remainder of this table, the criteria of network behavior is given in terms of conditions on the resistor constitutive relations $\underset{\sim}{g}_{R_\alpha}$ and element interconnection rather than on the function $\underset{\sim}{g}_p(\cdot)$. Note that the conditions on $\underset{\sim}{u}_S(\cdot)$ and the Additional Conclusions and Comments apply to the different types of network behavior below.

<u>Fundamental Topological Hypothesis</u>. There is no loop (resp., cutset) formed exlusively by capacitors, inductors and/or voltage (resp., current) sources.

<u>L.C. Hypothesis</u>: See page 11.

| Network Behavior | Conditions on $\underset{\sim}{h}_p$; $\underset{\sim}{h}_p$ is a State Function and $H_p$ is a Functional Satisfying $\nabla H_p \equiv \underset{\sim}{h}_p$ | Resistor Conditions | Graph-Theoretic Conditions |
|---|---|---|---|
| <u>Autonomous network</u>: solutions are eventually uniformly bounded | $\lim_{\|\underset{\sim}{z}_p\| \to \infty} \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| = +\infty$ $\lim_{\|\underset{\sim}{z}_p\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty$ | Each resistor function $\underset{\sim}{g}_{R_\alpha}$ is eventually strictly passive, and $$\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} (\underset{\sim}{x}_{R_\alpha})^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) = +\infty$$ | Fundamental Topological Hypothesis |
| <u>Nonautonomous network</u>: solutions are eventually uniformly bounded | $\lim_{\|\underset{\sim}{z}_p\| \to \infty} \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| = +\infty$ $\lim_{\|\underset{\sim}{z}_p\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty$ | Each $\underset{\sim}{g}_{R_\alpha}$ is eventually strictly passive, and $$\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} \frac{1}{\|\underset{\sim}{x}_{R_\alpha}\|} (\underset{\sim}{x}_{R_\alpha})^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) = +\infty$$ | Fundamental Topological Hypothesis |

TABLE 2 CONTINUED

| Network Behavior | Conditions on $h_{\sim p}$; $h_{\sim p}$ is a State Function and $H_p$ is a Functional Satisfying $\nabla H_p \equiv h_{\sim p}$ | Resistor Conditions | Graph-Theoretic Conditions |
|---|---|---|---|
| Autonomous network: there is a globally asymptotically stable equilibrium point $z_{\sim p}^*$ | $h_{\sim p}$ is a $C^1$-strictly increasing and surjective diffeomorphic state function. | Each $g_{R_\alpha}$ is strictly passive (Except for contrived situations, this condition excludes independent sources) | Fundamental Topological Hypothesis, or the L.C. Hypothesis. |
| Autonomous network: there is a globally asymptotically stable equilibrium point $z_{\sim p}^*$ | $h_{\sim p}$ is a $C^1$-strictly increasing and surjective diffeomorphic state function. | Each $g_{R_\alpha}$ is a strictly increasing, surjective homeomorphism. | Fundamental Topological Hypothesis, or the L.C. Hypothesis. |
| Nonautonomous network: for every $u_{\sim S}^*$, and $\epsilon > 0$ $\exists \delta > 0$ and a <u>unique</u> $z_{\sim p}^*$ such that $g_p\left(h_{\sim p}(z_{\sim p}^*), u_{\sim S}^*\right) = 0$, $\left[\left\|\mathcal{R}_\infty\left(u_{\sim S}(\cdot)\right) - u_{\sim S}^*\right\| < \delta\right] \Rightarrow$ $\left[\left\|\mathcal{R}_\infty\left(z_{\sim p}(\cdot)\right) - z_{\sim p}^*\right\| < \epsilon\right]$ for every solution $z_{\sim p}(\cdot)$. | $h_{\sim p}$ is a $C^1$-strictly increasing and surjective diffeomorphic state function. | Each $g_{R_\alpha}$ is a strictly increasing $C^1$-diffeomorphism satisfying $\lim\limits_{\|x_{\sim R_\alpha}\| \to \infty} \frac{1}{\|x_{\sim R_\alpha}\|}(x_{\sim R_\alpha})^T g_{R_\alpha}(x_{\sim R_\alpha}) = +\infty$ | Fundamental Topological Hypothesis. |
| Nonautonomous network: there exists a unique steady-state solution | $h_{\sim p}$ is linear; i.e., $h_{\sim p}(z_{\sim p}) = \Gamma_{\sim p} z_{\sim p}$ where $\Gamma_{\sim p}$ is positive-definite symmetric | Each $g_{R_\alpha}$ is a strictly increasing homeomorphism satisfying $\lim\limits_{\|x_{\sim R_\alpha}\| \to \infty} \frac{1}{\|x_{\sim R_\alpha}\|}(x_{\sim R_\alpha})^T g_{R_\alpha}(x_{\sim R_\alpha}) = +\infty$ | Fundamental Topological Hypothesis, or the L.C. Hypothesis. |
| Nonautonomous network: there is a unique steady-state solution, and all other solutions converge exponentially to the steady-state | 1. The network contains only one type of energy storage elements; i.e., either all capacitors, or all inductors. 2. $h_p$ is a $C^1$-strictly increasing and surjective diffeomorphic state function. | Each $g_{R_\alpha}$ is linear; i.e., $g_{R_\alpha}(x_{R_\alpha}) = G_\alpha x_{R_\alpha}$ where $G_\alpha$ is positive definite and symmetric. | Fundamental Topological Hypothesis, or the L.C. Hypothesis (in this case, transients do not necessarily decay exponentially). |
| Nonautonomous network: for every $u_{\sim S}^*, \exists \delta > 0$ such that if $\|\mathcal{R}_\infty\left(u_{\sim S}(\cdot)\right) - u_{\sim S}^*\| < \delta$ then there is a unique steady state solution | $h_{\sim p}$ is a $C^1$-strictly increasing surjective diffeomorphic state function, and its Jacobian is Lipschitz continuous. | Each $g_{R_\alpha}$ is a strictly increasing $C^1$-diffeomorphism satisfying $\lim\limits_{\|x_{\sim R_\alpha}\| \to \infty} \frac{1}{\|x_{\sim R_\alpha}\|}(x_{\sim R_\alpha})^T g_R(x_{\sim R_\alpha}) = +\infty$ | Fundamental Topological Hypothesis. |

Fig. 1



Fig. 2

Fig. 3



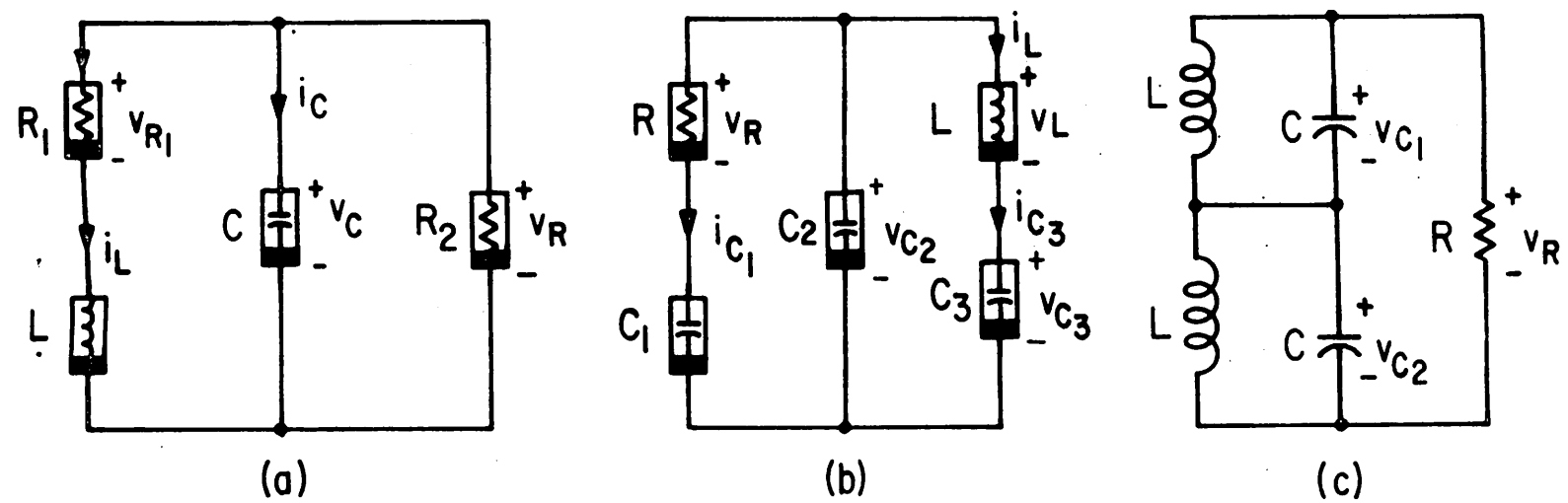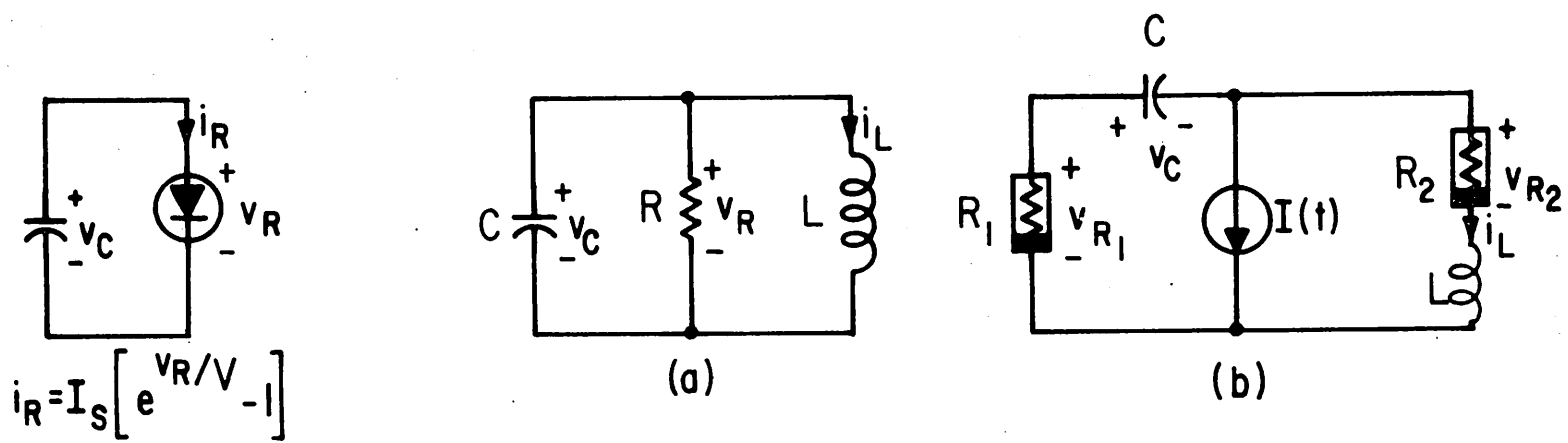$$i_R = I_S \left[ e^{v_R/V} - 1 \right]$$

Fig. 4



Fig. 5



Fig. 6