A QUADRATICALLY CONVERGENT PRIMAL–DUAL ALGORITHM WITH GLOBAL CONVERGENCE PROPERTIES FOR SOLVING OPTIMIZATION PROBLEMS WITH EQUALITY CONSTRAINTS (Revised)

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Memorandum No. ERL-M455
24 June 1974

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ABSTRACT

This paper presents a globally convergent multiplier method which utilizes an explicit formula for the multiplier. The algorithm solves finite dimensional optimization problems with equality constraints. A unique feature of the algorithm is that it automatically calculates a value for the penalty coefficient, which, under certain assumptions, leads to global convergence.

Research sponsored by the Joint Services Electronics Program, Contract F44620-71-C-0087 and the National Science Foundation, Grant GK-37672.
1. **Introduction**

In the 1930's and 1940's (see for example [9]) it was customary to convexify the Lagrangian of equality constrained variational problems by adding a penalty term. Later in 1958, Arrow and Solow [1], in considering gradient methods for solving the problem \( P, \min\{f(x)|g(x) = 0\} \) with \( f: \mathbb{R}^n \to \mathbb{R}^1, \ g: \mathbb{R}^n \to \mathbb{R}^m \ (m < n) \), proposed to solve \( P \) by finding a saddle point of the convexified Lagrangian, \( \bar{\psi}_c(x,y) = f(x) + \langle y, g(x) \rangle + \frac{1}{2c} \|g(x)\|^2 \), for \( c \) sufficiently large. In 1969, Hestenes [10] and Powell [16] and in 1970, Haarhoff and Buys [8] proposed related methods for solving \( P \), based on the convexified Lagrangian. Within a short period of time, these methods became known as methods of multipliers and generated a great deal of interest [2], [3], [4], [5], [6], [7], [13], [14] in the form of specific refinements and extensions. There are two basic types of multiplier methods: Those that compute estimates to the Lagrange multipliers \( \bar{y} \) after each inner iteration (e.g. as described in [10], [16], etc.), and those that estimate the multipliers \( y(x) \) continuously, as first described in [5]. The basic deficiency common to all these schemes is that they do not incorporate a method for selecting a finite value of \( c \) adequate to ensure convergence of the sequences constructed.

In this paper we present a multiplier method which uses an explicit formula for \( y(x) \). Our algorithm appears to be the first in the multiplier methods family, to incorporate an automatic scheme for determining a finite value of \( c \) compatible with convergence, and it converges quadratically under mild assumptions. The algorithm is partly based on ideas due to Fletcher [7], which we gratefully acknowledge.
2. Preliminary Results

We shall consider the following minimization problem:

\[
\min \{ f(x) | g(x) = 0 \}
\]  

(1)

where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \), with \( m \leq n \). We shall denote the components of a vector by superscripts, the elements of a sequence by subscripts and we shall make use of the following hypotheses.

**Assumption 1:** The functions \( f \) and \( g \) are three times continuously differentiable.

**Assumption 2:** For every \( x \in \mathbb{R}^n \) the vectors \( \nabla g_i(x) \), \( i = 1, 2, \ldots, m \), are linearly independent.

We recall that the Lagrangian \( \lambda : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \), for problem (1) is defined by

\[
\lambda(x, y) = f(x) + \langle y, g(x) \rangle
\]  

(2)

where \( \langle , \rangle \) denotes the Euclidean scalar product.

Given any \( x \in \mathbb{R}^n \), it is possible to define \( y(x) \in \mathbb{R}^m \) as the unique minimizer of \( \| \nabla_x \lambda(x, y) \|^2 \) over \( \mathbb{R}^m \), i.e.,

\[
y(x) = \arg \min \{ \| \nabla_x \lambda(x, y) \|^2 | y \in \mathbb{R}^m \}
\]  

(3)

where \( \| \| \) denotes the Euclidean norm and we use the notation

\( \nabla_x \lambda(x, y) \triangleq \left[ \frac{\partial}{\partial x} \lambda(x, y) \right]^T \). The symbol \( \nabla \) will always denote a column vector of first partial derivatives. We shall use a subscript on \( \nabla \) only when confusion is possible. We shall make use of the following two properties of the multiplier function \( y(\cdot) \)

**Proposition 1:** For every \( x \in \mathbb{R}^n \),
\[
\frac{\partial g(x)}{\partial x} \mathcal{L}(x, y(x)) = 0
\]  (4)

**Proof:** It follows immediately from (3) that

\[
y(x) = -\left( \frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T \right)^{-1} \frac{\partial g(x)}{\partial x} \nabla f(x)
\]  (5)

and hence

\[
\frac{\partial g(x)}{\partial x} \mathcal{L}(x, y(x)) = \frac{\partial g(x)}{\partial x} \left( \nabla f(x) + \frac{\partial g(x)}{\partial x} y(x) \right) = 0
\]  (6)

**Proposition 2:** The function \( y \) is twice continuously differentiable and its Jacobian matrix is given by

\[
\frac{\partial y(x)}{\partial x} = \left( \frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T \right)^{-1} \left[ \frac{\partial g(x)}{\partial x} \frac{\partial^2 \mathcal{L}(x, y(x))}{\partial x^2} + \sum_{j=1}^{m} e_j \mathcal{L}(x, y(x))^T \frac{\partial^2 g_j(x)}{\partial x^2} \right]
\]  (7)

where \( e_j \) denotes the \( j \)th column of the \( m \times m \) identity matrix.

**Proof:** That \( y \) is twice continuously differentiable follows directly from Assumption 1 and (5). To obtain formula (7), we make use of a dyadic expansion of (4) as follows.

\[
\frac{\partial g(x)}{\partial x} \mathcal{L}(x, y(x)) = \sum_{j=1}^{m} e_j \frac{\partial g_j(x)}{\partial x} \mathcal{L}(x, y(x)) = 0
\]  (8)

Hence, differentiating (8), we obtain,

\[
\sum_{j=1}^{m} e_j \left( \mathcal{L}(x, y(x))^T \frac{\partial^2 g_j(x)}{\partial x^2} + \frac{\partial g_j(x)}{\partial x} \left[ \frac{\partial^2 \mathcal{L}(x, y(x))}{\partial x^2} \right]
\]

\[
+ \frac{\partial^2 \mathcal{L}(x, y(x))}{\partial y \partial x} \frac{\partial y(x)}{\partial x} \right) = 0
\]  (9)

-3-
Since \( \frac{\partial^2 \ell(x,y(x))}{\partial y \partial x} = \frac{\partial g(x)^T}{\partial x} \), (7) follows immediately from (9).

To obtain a penalty function type method which does not require infinite penalization for the infraction of constraints in (1) to yield a solution of (1), Hestenes [9] and Powell [13] have independently proposed the use of the following, parametrized, augmented Lagrangian

\[
\psi_c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^1, \text{ with}
\]

\[
\psi_c(x,y) = \ell(x,y) + \frac{1}{2} c \|g(x)\|^2
\]

where \( c > 0 \) is a parameter. We shall make use of a simple modification of (10), also used by Fletcher [7], viz., \( \psi_c : \mathbb{R}^n \rightarrow \mathbb{R}^1, \) defined for \( c \geq 0 \), by

\[
\psi_c(x) = \ell(x,y(x)) + \frac{1}{2} c \|g(x)\|^2
\]

We shall depend on the fact that there is an important relationship between stationary points of \( \psi_c \) and of \( \ell \), as we now show.

**Proposition 3** If a pair \((\bar{x}, \bar{y})\) is a stationary point for \( \ell(\cdot, \cdot) \), then \( \bar{y} = y(\bar{x}) \) and, for any \( c \geq 0 \), \( \bar{x} \) is a stationary point for \( \psi_c(\cdot) \).

**Proof:** Since \((\bar{x}, \bar{y})\) satisfy \( \nabla_x \ell(\bar{x}, \bar{y}) = 0 \), \( \nabla_y \ell(\bar{x}, \bar{y}) = 0 \), it follows from (3) that \( \bar{y} = y(\bar{x}) \) and from (2) that \( g(\bar{x}) = 0 \). Hence, for any \( c \geq 0 \),

\[
\nabla_x \psi_c(\bar{x}) = \nabla_x \ell(\bar{x}, y(\bar{x})) = 0.
\]

The following result shows to what extent is the converse of proposition 3 true.

**Proposition 4** For every compact subset \( S \) of \( \mathbb{R}^n \), there exists a \( c_S \geq 0 \) such that for all \( c \geq c_S \), if \( \bar{x} \in S \) is a stationary point of \( \psi_c(\cdot) \), then \((\bar{x}, y(\bar{x}))\) is a stationary point of \( \ell(\cdot, \cdot) \).
is a stationary point of \( \ell(\cdot, \cdot) \).

**Proof:** Let \( S \) be a compact subset of \( \mathbb{R}^n \) and suppose that for some \( c > 0 \), \( \vec{x} \in S \) satisfies \( \nabla \psi_c(\vec{x}) = 0 \). Hence, making use of (4),

\[
0 = \frac{\partial g(\vec{x})}{\partial x} \nabla \psi_c(\vec{x}) = \frac{\partial g(\vec{x})}{\partial x} \left( \nabla \ell(\vec{x}, y(\vec{x})) + \frac{\partial y(\vec{x})}{\partial x} g(\vec{x}) + c \frac{\partial g(\vec{x})}{\partial x} g(\vec{x}) \right)
\]

\[
= \left[ c \frac{\partial g(\vec{x})}{\partial x} \frac{\partial g(\vec{x})}{\partial x}^T + \frac{\partial g(\vec{x})}{\partial x} \frac{\partial y(\vec{x})}{\partial x} \right] g(\vec{x})
\]

(12)

Consequently, we must have

\[
\left[ c I + \left( \frac{\partial g(\vec{x})}{\partial x} \frac{\partial g(\vec{x})}{\partial x}^T \right)^{-1} \frac{\partial g(\vec{x})}{\partial x} \frac{\partial y(\vec{x})}{\partial x} \right] g(\vec{x}) = 0
\]

(13)

Since \( S \) is compact and all the matrices in (13) and continuous, there exists a \( c_S \geq 0 \) such that for all \( c \geq c_S \) and any \( \vec{x} \in S \), the matrix in (13) multiplying \( g(\vec{x}) \) is nonsingular. Hence, if \( c \geq c_S \), \( g(\vec{x}) = 0 \) and therefore, \( g(\vec{x}) = \nabla \ell(\vec{x}, y(\vec{x})) = 0 \) and \( \nabla \ell(\vec{x}, y(\vec{x})) = \nabla \psi_c(\vec{x}) = 0 \), which completes our proof. \( \square \)

Since the algorithm in the next section is designed to find local minima of \( \psi_c(\cdot) \), for adequately large values of \( c \), it is interesting to establish the relationship between the local minima of \( \psi_c(\cdot) \) and \( f(\cdot) \).

**Corollary 1:** For every compact subset \( S \) of \( \mathbb{R}^n \) there exists a \( c_S \geq 0 \) such that for all \( c \geq c_S \), if \( \vec{x} \in S \) is a local minimum of \( \psi_c(\cdot) \), then \( \vec{x} \) is a local minimum of \( f(\cdot) \) on \( \Omega = \{ x | g(x) = 0 \} \).

**Proof:** By proposition 4, there exists a \( c_S \geq 0 \) such that if \( c \geq c_S \) and \( \vec{x} \) is a local minimum of \( \psi_c(\cdot) \), then \( g(\vec{x}) = 0 \). But \( \psi_c(x) = f(x) \) for all \( x \in \Omega \), and hence the corollary follows. \( \square \)
3. The Algorithm: Convergence

The algorithm which we are about to describe is related to the Gauss–Newton method and makes use of an approximate Hessian of \( \psi_c(\cdot) \), defined as follows. For any \( c \geq 0 \) and \( x \in \mathbb{R}^n \), let \( H_c(x) \) be an \( n \times n \) matrix defined by

\[
H_c(x) = \frac{\partial^2 \psi_c(x,y(x))}{\partial x^2} + \frac{\partial g(x)^T}{\partial x} \frac{\partial y(x)}{\partial x} + \frac{\partial y(x)^T}{\partial x} \frac{\partial g(x)}{\partial x}
\]

\[ + c \frac{\partial g(x)^T}{\partial x} \frac{\partial g(x)}{\partial x}
\]

(14)

Proposition 5: For all \( \bar{x} \) satisfying \( g(\bar{x}) = 0 \) and any \( c \geq 0 \),

\[
H_c(\bar{x}) = \frac{\partial^2 \psi_c(\bar{x})}{\partial x^2}
\]

(15)

Proof: Eq. (15) follows directly by calculation from the fact that for any \( c \geq 0 \) and \( x \in \mathbb{R}^n \),

\[
\frac{\partial^2 \psi(x)}{\partial x^2} = H_c(x) + \sum_{j=1}^{m} g^j(x) \left( \frac{\partial^2 y^j(x)}{\partial x^2} + c \frac{\partial^2 g^j(x)}{\partial x^2} \right)
\]

(16)

The algorithm below makes use of a preselected monotonically increasing sequence \( \{c_j\}_{j=0}^{\infty} \), with \( c_{j+1} > c_j \geq 0 \), \( j = 0,1,2,\ldots \), and \( c_j \to \infty \) as \( j \to \infty \). For example, one could use sequences defined by \( c_j = c_0 + j\rho \), \( j = 0,1,2,\ldots \), \( \rho > 0 \), or \( c_j = c_0 \nu^j \), \( j = 0,1,2,\ldots \), with \( \nu > 1 \) and \( c_0 > 0 \).

Algorithm: Parameters: \( \alpha \in (0,\frac{1}{2}) \), \( \beta \in (0,1) \), \( 0 < \varepsilon_0 \ll 1 \), \( 0 < \varepsilon_1 \ll 1 \), \( \gamma \geq 1 \), \( \{c_j\}_{j=0}^{\infty} \). Initial guess: \( z_0 \).

Step 0: Set \( j = 0 \).
Step 1: Set $i = 0$, and set $x_0 = z_j$.

Step 2: If $\nabla \psi_j (x_i) \neq 0$, go to step 4; else go to step 3.

Step 3: If $g(x_i) = 0$, stop; else go to step 10.

Step 4: If $\langle \frac{\partial g(x_i)}{\partial x^T}, \frac{\partial g(x_i)}{\partial x} \rangle \frac{\partial g(x_i)}{\partial x} \frac{\partial g(x_i)}{\partial x^T} \psi_j (x_i) \rangle > \|g(x_i)\|^2$, go to step 5; else go to step 10.

Comment: The test in step 4, roughly, is on the angle between $\nabla \psi_j (x_i)$ and the Newton direction $v(x_i) \triangleq \frac{\partial g(x_i)}{\partial x} \frac{\partial g(x_i)}{\partial x^T} \psi_j (x_i)$, for solving $g(x) = 0$, defined by $v(x_i) = \arg\min\{\|v\|^2 |g(x_i) + \frac{\partial g(x_i)}{\partial x} v = 0\}$.

Step 5: If $|\det H_j (x_i)| \geq \epsilon_0$, go to step 6; else go to step 7.

Step 6: If $\langle \nabla \psi_j (x_i), H_j (x_i) \psi_j (x_i) \rangle - \|g(x_i)\|^2 \psi_j (x_i) \|H_j (x_i) \psi_j (x_i) \|$,

set $h(x_i) = -H_j (x_i) \psi_j (x_i)$ and go to step 8; else go to step 7.

Step 7: Set $h(x_i) = -\psi_j (x_i)$.

Step 8: Compute the smallest nonnegative integer $\lambda_i \geq 0$ such that $\psi_j (x_i + \beta \lambda_i h(x_i)) - \psi_j (x_i) \leq \beta \alpha \langle \nabla \psi_j (x_i), h(x_i) \rangle$ (18)

Step 9: Set $x_{i+1} = x_i + \beta \lambda_i h(x_i)$, set $i = i + 1$, and go to step 2.
Step 10: Set $z_{j+1} = x_i$, set $j = j + 1$ and go to step 1.

Lemma 1: Suppose that $x_i$ is such that $\nabla \psi_{c_j} (x_i) \neq 0$, and $h(x_i)$ is as constructed by the algorithm, in step 6 or in step 7, then there exists an $\epsilon_i > 0$ such that (18) holds.

Proof: Note that

$$\langle \nabla \psi_{c_j} (x_i), h(x_i) \rangle \leq \delta_j (x_i) \| \nabla \psi_{c_j} (x_i) \| \| h(x_i) \| < 0$$

(19)

where

$$\delta_j (x_i) \triangleq \max \{-1, -\min \{\epsilon_i, \max \{c_j, \max \| h(x_i) \| \} \} \}$$

(20)

The lemma now follows from the Mean Value Theorem.

Lemma 2: Suppose that the algorithm has constructed an infinite sequence $\{x_i\}_{i=0}^{\infty}$ (i.e., step 10 is reached a finite number of times only), then any accumulation point $x^*$ of $\{x_i\}_{i=0}^{\infty}$ satisfies $\nabla \psi_{c_j^*} (x^*) = 0$, and $g(x^*) = 0$, where $j^*$ is the last value of $j$.

Proof: Suppose that $x^*$ is an accumulation point of $\{x_i\}_{i=0}^{\infty}$. Then, because of the test in step 5, given any $\epsilon^* > 0$ there exists an $M^* \in (0, \infty)$ such that $\| h(x_i) \| \leq M^* \| \nabla \psi_{c_j^*} (x_i) \|$ for all $\| x_i - x^* \| < \epsilon^*$. Making use of this fact, of (19) and some of the results in section 2.1 of [12], we conclude that $\nabla \psi_{c_j^*} (x^*) = 0$ must hold.

Next, denoting by $k \subset \{0, 1, 2, \ldots\}$ the indices of the convergent subsequence, since $j$ never grows beyond $j^*$, we must have, according to step 4, for $i=0, 1, 2, \ldots$

$$\| g(x_i) \| \leq \left( \frac{\partial g(x_i)}{\partial x} \left( \frac{\partial g(x_i)}{\partial x} \left( \frac{\partial g(x_i)}{\partial x} \right)^T \right)^{-1} g(x_i), \nabla \psi_{c_j^*} (x_i) \right)$$

(21)
But \( \nabla_{x} c_{j} \rightarrow 0 \) as \( i \rightarrow \infty \), \( i \in K \), and \( g(\cdot) \), \( \frac{\partial g}{\partial x} (\cdot) \) are continuous, and

\[
\frac{\partial g(x)}{\partial x} \quad \frac{\partial g(x)}{\partial x}^T
\]

is nonsingular for all \( x \). Hence we must have \( g(x_i) \rightarrow 0 \) as \( i \rightarrow \infty \), \( i \in K \) and hence, \( g(x^*) = 0 \).

**Lemma 3:** Suppose that the algorithm has constructed an infinite sequence of points \( \{z_j\}_{j=0}^{\infty} \). Then \( \{z_j\}_{j=0}^{\infty} \) has no accumulation points.

**Proof:** Suppose that the algorithm has constructed an infinite sequence \( \{z_j\}_{j=0}^{\infty} \) which has an accumulation point \( z^* \). We shall show that this leads to a contradiction. Let \( K \subset \{0,1,2,\ldots\} \) be such that \( z_j \rightarrow z^* \) as \( j \rightarrow \infty \), \( j \in K \). Obviously, the set \( S = \{z_j\}_{j \in K} \) is compact. Now consider the test in step 4, whose failure results in the construction of a new \( z_j \). Let \( \theta : \mathbb{R}^1 \times \mathbb{R}^n + \mathbb{R}^1 \) be defined by

\[
\theta(c,x) = \left( \frac{\partial g(x)}{\partial x} \right)^{-1} g(x), \quad \frac{\partial \psi_c(x)}{\partial x} - \|g(x)\|^2 (22)
\]

Then, making use of the right hand side of (12), we find that

\[
\theta(c,x) = \left( \frac{\partial g(x)}{\partial x} \right)^{-1} g(x), \quad \frac{\partial g(x)}{\partial x} \psi_c(x) - \|g(x)\|^2 \]

\[
= \left( \frac{\partial g(x)}{\partial x}, \frac{\partial g(x)}{\partial x}^T \right)^{-1} g(x), \left( c \frac{\partial g(x)}{\partial x} + \frac{\partial g(x)}{\partial x} \frac{\partial y(x)}{\partial x} \right) g(x) \]

\[
- \|g(x)\|^2
\]

\[
= \left( g(x), \left( c-1 \right) I + \left( \frac{\partial g(x)}{\partial x}, \frac{\partial g(x)}{\partial x}^T \right)^{-1} \frac{\partial g(x)}{\partial x} + \frac{\partial g(x)}{\partial x} \frac{\partial y(x)}{\partial x} \right) g(x) \quad (23)
\]

Since \( S \) is compact and \( \theta \) is continuous, it now follows that there exists a \( c^* > 0 \), such that \( \theta(c,x) > 0 \) for all \( c \geq c^* \) and for all \( x \in S \). Let \( j^* \) be such that \( c_j \geq c^* \) for all \( j \geq j^* - 1 \), then the algorithm could not
have constructed any point \( z_j \), \( j \geq j^* \), \( j \in K \), on account of a failure to satisfy the test in step 4. Hence, it must have constructed the points \( z_j \), \( j \in K \), \( j \geq j^* \) because of the transfer in step 3, i.e., we must have \( \nabla \psi_{c_{j-1}}(z_j) = 0 \), \( g(z_j) \neq 0 \) for all \( j \in K \), \( j \geq j^* \). But according to proposition 4, there exists a \( c_S \) such that if \( \nabla \psi_{c}(z) = 0 \) then \( g(z) = 0 \) for all \( z \in S \), for all \( c \geq c_S \). Let \( j^{**} \geq j^* \) be such that \( c_j \geq c_S \) for all \( j \geq j^{**} - 1 \), then we see that \( z_j \), \( j \in K \), \( j \geq j^{**} \) could not have been constructed. Thus we have a contradiction.

We can now collect our results and state them as a theorem.

Theorem 1: (a) If the algorithm constructs a finite sequence \( \{x_i\}_{i=0}^\nu \) and stops, then \( \nabla l(x_\nu, y(x_\nu)) = 0 \) and \( g(x_\nu) = 0 \). (b) If the algorithm constructs an infinite sequence \( \{x_i\}_{i=0}^\infty \), then any accumulation point \( x^* \) of \( \{x_i\}_{i=0}^\infty \) satisfies \( \nabla l(x^*, y(x^*)) = 0 \), \( g(x^*) = 0 \). (c) If the algorithm constructs an infinite sequence \( \{z_i\}_{j=0}^\infty \) then this sequence has no accumulation points.

4. Rate of Convergence of the Algorithm

We shall now investigate the rate of convergence of the algorithm described in Section 3, when it converges to a local nonsingular minimum point of the problem (1). We recall that a point \( \bar{x} \) is a local nonsingular minimum point of (1) if (i) \( g(\bar{x}) = 0 \), (ii) there exists a multiplier \( \bar{y} \in \mathbb{R}^m \) such that

\[
\nabla l(\bar{x}, \bar{y}) = 0
\]

(24)

and

\[
\langle x, \frac{\partial^2 l(\bar{x}, \bar{y})}{\partial x^2} x \rangle > 0 \text{ for all } x \in \{x | \frac{\partial g(\bar{x})}{\partial x} x = 0, x \neq 0\}
\]

(25)
Proposition 6: If \( \bar{x} \) is a nonsingular local minimum point of (1), then there exists a \( \bar{c} > 0 \) such that for all \( c > \bar{c} \), \( \bar{x} \) is a strong local minimum point of \( \psi_c(\cdot) \), i.e. \( \nabla\psi_c(\bar{x}) = 0, \frac{\partial^2 \psi_c(\bar{x})}{\partial x^2} > 0 \).

Proof: By propositions 3 and 5, for any \( c > 0 \), \( \Delta \psi_c(\bar{x}) = 0 \) and
\[
\frac{\partial^2 \psi_c(\bar{x})}{\partial x^2} = H_c(\bar{x}).
\]
From (14), summing the first three matrices and calling the result \( A(\bar{x}) \), we have
\[
H_c(\bar{x}) = A(\bar{x}) + c \frac{\partial g(\bar{x})}{\partial x}^T \frac{\partial g(\bar{x})}{\partial x} = 0 \tag{26}
\]
Now, because of (25), \( \langle x, A(\bar{x}) x \rangle > 0 \) for all \( x \neq 0 \) such that \( \frac{\partial g(\bar{x})}{\partial x} x = 0 \) and hence there exists a \( \bar{c} > 0 \) such that \( H_c(\bar{x}) \), and consequently \( \frac{\partial^2 \psi_c(\bar{x})}{\partial x^2} \), is positive definite for all \( c > \bar{c} \).

The converse of this result follows trivially from (26) and proposition 4.

Proposition 7: For every compact subset \( S \) of \( \mathbb{R}^n \) there exists a \( \bar{c} > 0 \) such that for any \( c \geq \bar{c} \), if \( \bar{x} \in S \) is a strong local minimum of \( \psi_c(\cdot) \), then \( \bar{x} \) is a nonsingular local minimum of (1).

Corollary 2: Suppose that the algorithm has constructed a sequence \( \{x_i\}_{i=0}^{\infty} \) which converges to \( x^* \) a strong local minimum of \( \psi_c(\cdot) \), where \( j^* \) is the last value of \( j \). Then \( x^* \) is a nonsingular local minimum of (1).

Proof: By theorem 1, \( g(x^*) = 0 \), i.e., \( x^* \) is feasible and \( \nabla_x \mathcal{L}(x^*, y(x^*)) = 0 \).
Since \( H_{j^*}(x^*) \) is positive definite, we must have \( \frac{\partial^2 \mathcal{L}(x^*, y(x^*))}{\partial x^2} \) positive definite on the linear manifold \( \{x | \frac{\partial g(x^*)}{\partial x} x = 0\} \), hence we are done.

Lemma 4: Suppose that the algorithm has constructed a sequence \( \{x_i\}_{i=0}^{\infty} \)
which converges to \( x^* \), a strong local minimum of \( \psi_{j^*}(\cdot) \) and suppose that
\( \varepsilon_0 < \frac{1}{2} |\det H_{c_j^*}(x^*)| \), then there exists an \( j^* \) such that for all \( i > j^* \),
\[
 h(x_i) = -H_{c_j^*}(x_i)^{-1} \nabla \psi_{c_j^*}(x_i) \text{ and } l_i = 0.
\]

**Proof:** Since \( H_{c_j^*}(x^*) > 0 \) by assumption, and \( H_{c_j^*}(\cdot) \) is continuous, there exists a closed ball \( B(x^*) \) with center at \( x^* \) such that \( H_{c_j^*}(x) > 0 \) and \( H_{c_j^*}(\cdot) \) is continuous, for all \( x \in B(x^*) \). Since \( x_i \to x^* \), there exists an \( i_1 \) such that for all \( i > i_1 \), \( x_i \in B(x^*) \) and \( |\det H_{c_j^*}(x_i)| \geq \varepsilon_0 \). Next, because \( H_{c_j^*}(x)^{-1} > 0 \) for all \( x \in B(x^*) \) and \( B(x^*) \) is compact and because \( \nabla \psi_{c_j^*}(x_i) \to 0 \), there exists an \( m \in (0,\infty) \) and an \( i_2 > i_1 \), such that for all \( i \geq i_2 \), (since \( i \geq 1 \))

\[
\langle \nabla \psi_{c_j^*}(x_i), H_{c_j^*}(x_i)^{-1} \nabla \psi_{c_j^*}(x_i) \rangle =
\min \left\{ \varepsilon_1, \| \nabla \psi_{c_j^*}(x_i) \|, \| \nabla \psi_{c_j^*}(x_i) \|, H_{c_j^*}(x_i)^{-1} \nabla \psi_{c_j^*}(x_i) \| \right\}
\geq \| \nabla \psi_{c_j^*}(x_i) \|^2 (m - \| \nabla \psi_{c_j^*}(x_i) \|^{-1} \| H_{c_j^*}(x_i)^{-1} \nabla \psi_{c_j^*}(x_i) \|)
\geq 0
\]

Hence, for all \( i \geq i_2 \), the test (17) in step 6 is satisfied and the algorithm sets \( h(x_i) = -H_{c_j^*}(x_i)^{-1} \nabla \psi_{c_j^*}(x_i) \).

Next, we consider the test (18) in step 8 for \( i \geq i_2 \). For \( x \in B(x^*) \) let

\[
\phi(x) = \psi_{c_j^*}(x+h(x)) - \psi_{c_j^*}(x) - \alpha \langle \nabla \psi_{c_j^*}(x), h(x) \rangle \tag{28}
\]

where \( \alpha \in (0,\frac{1}{2}) \) is as in the algorithm. Then making use of the second order Taylor formula, for all \( i \geq i_2 \), we obtain
\[
\phi(x_1) = (1-\alpha)(\mathbf{\nabla} \psi_{c_{j^*}}(x_1), h(x_1)) + \int_0^1 (1-t) h(x_1), \frac{\partial^2 \psi_{c_{j^*}}}{\partial x^2} (x_1 + \mathbf{th}(x_1)) h(x_1) \, dt
\]

\[
= (\frac{1}{2-\alpha}) \langle \mathbf{\nabla} \psi_{c_{j^*}}(x_1), h(x_1) \rangle
\]

\[
+ \int_0^1 (1-t) \langle h(x_1), \left[ \frac{\partial^2 \psi_{c_{j^*}}}{\partial x^2} (x_1 + \mathbf{th}(x_1)) - H_{c_{j^*}}(x_1) \right] h(x_1) \rangle \, dt
\]  

(29)

But \( H_{c_{j^*}}(\cdot)^{-1} \) is continuous on \( B(x^*) \) and hence there exists an \( M = \sup_{x \in B(x^*)} \) such that

\[
\phi(x_1) \leq \left[ -\frac{1}{2-\alpha} m + \frac{1}{2} M^2 \sup_{t \in [0,1]} \left| \frac{\partial^2 \psi_{c_{j^*}}}{\partial x^2} (x_1 + \mathbf{th}(x_1)) - H_{c_{j^*}}(x_1) \right| \right] \parallel \mathbf{\nabla} \psi_{c_{j^*}}(x_1) \parallel^2
\]

(30)

Since \( \sup_{t \in [0,1]} \left| \frac{\partial^2 \psi_{c_{j^*}}}{\partial x^2} (x_1 + \mathbf{th}(x_1)) - H_{c_{j^*}}(x_1) \right| \to 0 \) as \( i \to \infty \), there exists an \( i^* \) such that \( \phi(x_1) \leq 0 \) for all \( i \geq i^* \), i.e., the test (18) is passed with \( \lambda_i = 0 \) for all \( i \geq i^* \).

**Theorem 2:** Suppose that the algorithm has constructed a sequence \( \{x_i\}_{i=0}^{\infty} \) which converges to \( x^* \), a strong local minimum of \( \psi_{c_{j^*}}(\cdot) \), and that \( \varepsilon_0 < \frac{1}{2} |\text{det} H_{c_{j^*}}(x^*)| \). Then

\[
\frac{\parallel x_{i+1} - x^* \parallel}{\parallel x_i - x^* \parallel} \to 0 \quad \text{as} \quad i \to \infty
\]

(31)

Furthermore, if the functions \( f \) and \( g \) are three times Lipschitz continuously differentiable on a convex neighborhood of \( x^* \) then there exist an \( i^* \) and a \( \nu \in (0, \infty) \) such that

\[
\parallel x_{i+1} - x^* \parallel \leq \nu \parallel x_i - x^* \parallel^2 \text{ for all } i \geq i^*
\]

(32)
Proof: Let \( i^* \) be as in Lemma 4. Then for all \( i \geq i^* \), \( x_{i+1} = x_i \)

\[- H_{c_{j^*}^*} (x_i)^{-1} \psi_{c_{j^*}} (x_i), \]

and hence,

\[
\|x_{i+1} - x^*\| = \| (x_i - x^*) - H_{c_{j^*}^*} (x_i)^{-1} (\psi_{c_{j^*}} (x_i) - \nabla \psi_{c_{j^*}} (x^*)) \|
\]

\[
\leq \| H_{c_{j^*}^*} (x_i)^{-1} \int_0^1 \| H_{c_{j^*}^*} (x_i) - \frac{\partial^2 \psi_{c_{j^*}^*}}{\partial x^2} (x^* + t(x_i - x^*)) \| \, dt
\]

\[
\|x_i - x^*\| \quad (33)
\]

But for all \( i \geq i^* \), \( \| H_{c_{j^*}^*} (x_i)^{-1} \| \leq M < \infty \), where \( M \) is as in lemma 5, and hence

\[
\|x_{i+1} - x^*\| \leq M \sup_{t \in [0,1]} \| H_{c_{j^*}^*} (x_i) - \frac{\partial^2 \psi_{c_{j^*}^*}}{\partial x^2} (x^* + t(x_i - x^*)) \| (34)
\]

\[
\|x_i - x^*\| \leq \sup_{t \in [0,1]} \| H_{c_{j^*}^*} (x_i) - \frac{\partial^2 \psi_{c_{j^*}^*}}{\partial x^2} (x^* - t(x_i - x^*)) \|
\]

Since \( \sup_{t \in [0,1]} \| H_{c_{j^*}^*} (x_i) - \frac{\partial^2 \psi_{c_{j^*}^*}}{\partial x^2} (x^* - t(x_i - x^*)) \| \to 0 \) as \( i \to \infty \), (31) follows from (34).

Next, suppose that \( f \) and \( g \) are three times Lipschitz continuously differentiable on a convex neighborhood \( B \) of \( x^* \). Without loss of generality, we may assume that \( x_i \in B \) for all \( i \geq i^* \). Hence, (34) yields

\[
\|x_{i+1} - x^*\| \leq M \sup_{t \in [0,1]} \| H_{c_{j^*}^*} (x_i) - H_{c_{j^*}^*} (x^*) \|
\]

\[
\|x_i - x^*\| \leq 2ML \|x_i - x^*\| \quad \text{for all} \quad i \geq i^*, \quad (35)
\]
where \( L \) is a Lipschitz constant which is valid both for \( H_c (\cdot) \) and for \( \frac{\partial^2 \psi_c (\cdot)}{\partial x^2} \), and where we have made use of the fact that \( \frac{1}{2} \frac{\partial^2 \psi_c (x^*)}{\partial x^2} = H_c (x^*) \).

Setting \( v = 2ML \) in (35), we obtain (32).

Conclusion:

We have shown in this paper that there is at least one scheme for automatically finding an adequate penalty for use in a multiplier method. This scheme ensures convergence, but it does not guarantee quadratic convergence, since it does not ensure that the matrices \( H_{e_j^*} (x_i) \) do in fact become positive definite. One could, of one so desired, add a test to check if the matrices \( H_{c_j^*} (x_i) \) are positive definite and increase \( c_j \) to make them positive definite. However, it is not clear that the extra work is justified, since in practice it has been found that the \( c_j \) resulting from a straightforward use of the algorithm is, usually, large enough to ensure quadratic convergence.
References


