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A MEMO ON MARDEN'S INVESTIGATIONS
OF THE ZERO-DISTRIBUTION WITHIN
THE UNIT CIRCLE

by

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The problem of stability of linear discrete system and zero-distribution within the unit circle has occupied my attention and effort in the last three years.[5-9] At the completion of this study it has been my contention and belief that considerable progress has been made and that simplified methods of testing for the zero-dix-tribution has been evolved. These simplified methods which are extensively discussed in the literature, are now known as the determinant method,[5] the table form[8] and the division method.[9]

My progress on this problem has been greatly aided by the early work of Schur-Cohn.[4,11] and later by the works of Marden.[1-3] The latter work is the first account in the English literature and in a form of a book of the problem of the zero-distribution within unit circle. This account is presented in Ch. X of Ref. 1, entitled "The Number of Zeros in a Given Circle." While the work of Marden is useful and significant it does suffer in a few places from errors, inconsistencies and mininterpretations of Cohn's work on which some of the material is based. It is the purpose of this short memoir to point out these errors and to indicate the correct statements and modifications required. These corrections were made possible as a result of my work on this interesting problem. This memoir is not intended for publication but rather to be distributed internally and privately for all those who are interested in the satisfactory solution of this problem.

Starting with Ch. X, p. 148, I will itemize the points to be discussed as follows:

1. Problem 3 on p. 155, should read:

"The determinant Δ_n is the resultant of the two polynomials $F(z)$ and $F^*(z)$ and hence vanishes when $F(z)$ has a zero on the unit circle $|z| = 1$."

To show the above, we present $F(z)$ as follows:

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^2 = a_n \prod_i^{1 \ldots n} (z - z_i) = 0, a_n > 1. \quad (1)$$

In an earlier work the following has been established[5, 6]

$$\Delta_n = (A_{n-1} - B_{n-1})^2 F(1) F(-1) = (A_{n-1} - B_{n-1})^2 a_n^2 \prod_i^{1 \ldots n} (1 - z_i)^2$$

$$(2)$$

and

$$A_{n-1} - B_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_n^{n-1} \prod_{i < k}^{1 \ldots n} (1 - z_i z_k). \quad (3)$$

Substituting (3) into (2) we obtain

$$\Delta_n = a_n^{2n} \prod_{i < k}^{1 \ldots n} (1 - z_i z_k)^2 \prod_i^{1 \ldots n} (1 - z_i)^2. \quad (4)$$

It is readily evident from (4) that Δ_n vanishes when $|z_i| = 1$ and also when $|z_i z_k| = 1$. The latter indicates the case of reciprocal zeros with respect to the unit circle.[7]

2. Theorem (44.1) p. 157 should read:[7]

"For a given polynomial $F(z) = a_0 + a_1 z \ldots + a_n z^n$, let the sequence of polynomials $F_{i+1}(z) = a_0^{(i)} F_i(z) - a_{n-i}^{(i)} F_i^*(z)$, $i = 0, 1, \ldots n - 1$, be constructed. Then if for some $k < n$, $P_k = \delta_1 \delta_2 - -\delta_k \neq 0$, $\ldots k = 1, 2, \ldots$ but $F_{k+1} \equiv 0$, then $F(z)$ has either $n - k$ zeros on the unit circle $|z| = 1$ or $n - k$

reciprocal zeros with respect to the unit circle or $n - k$ zeros of both reciprocal form and on the unit circle; it has p zeros in the circle, where p is the number of negative P_i for $i = 1, 2, \ldots, k$ and it has $q = k - p$ zeros outside the unit circle."

The proof of the above theorems can be constructed similarly as Sec. 44 p. 155 of Marden's book except that $\psi(z)$ of Eq. (44.1) should be modified to include also reciprocal zeros with respect to the unit circle. A simplified test for the theorem discussed in this item is represented in Ref. 7.

3. Theorem (45.1), pp. 158-159, should read:

If the coefficient of the polynomial $g(z) = b_0 + b_1 z + \ldots b_m z^m$ satisfy the relations:

$$b_m = u\overline{b}_0, \quad b_{m-1} = u\overline{b}_1, \quad \ldots, \quad b_{m-q+1} = u\overline{b}_{q-1}, \quad b_{m-q} \neq u\overline{b}_q$$

where

$q \leq m/2$ and $|u| = 1$, then $g(z)$ has $\underline{\text{in}}$ and $\underline{\text{on}}$ the circle $|z| = 1$ as many zeros as the polynomial indicated in Eq. (45.6) and so on $\ldots$ (see Marden's, p. 158).

The proof of the above extension of Marden's theorem lies in the following two facts:

1. Since the factor $(z^q + 2b/|b|)$ does not vanish in and on the unit circle $|z| = 1$, $g(z)$ has as many zeros in and on $|z| = 1$ as $G(z)$. This fact has been also shown by Cohn.[4]

2. The Lemma (42.1) on p. 149 can be modified for $\delta_{j+1} > 0$, and noting Theorem (44.1) or its modified version stated above to read.

If $F_j(z)$ has P_j zeros in the unit circle C and if $\delta_{j+1} > 0$, then F_{j+1} has

$$P_{j+1} = \frac{1}{2} \{n - j - [(n-j) - 2P_j]\} = P_j$$

zeros in C and the same number and location of zeros on the unit circle as $F_j(z)$ has.

Based on these two facts and following the proof of Marden of Theorem (45.1) on p. 159, we can readily establish the above stated extension. It should be noted that with this extension we can readily determine the zero-distribution when both singular cases occur.

4. Although problem (2) is stated correctly on p. 181, it does indicate an inconsistency with Marden Theorem (44.1) on p. 157.

5. Problem 3 p. 161 should read:

"A necessary and sufficient condition for all the zeros of of $g(z)$ to be on the unit circle is that $g(z)$ satisfy condition (45.4) and that all the zeros of $g'(z)$ should lie in, on or in and on the unit circle."

6. Problem 4 p. 161, should read:

"A necessary and sufficient condition that all the zeros of $f(z) = a_0 z + a_1 z + \ldots + a_n z^n$ which are different from each other (no multiple zeros) lie on the circle $|z| = 1$ is that in Eq. (42.8) all $a_k^{(1)} = 0$ and that also $f'(z)$ have all its zeros in this circle."

The discussion of the above modification can be readily ascertained from Ref. 7 or from a restudy of Cohn's[4] and Schur's works.[11]

7. In reading Marden's discussions in Sec. 45, p. 158, one gets the impression the $f(z)$ or $g(z)$ has no zero on the unit circle but a certain δ_k vanishes. This impression is not precise for in the extension of theorem (45.1) (discussed in item 3 of this memoir) it is indicated that $g(z)$ can also have zeros on the unit circle. Furthermore from the modification of item (2) and Theorem (45.1), p. 159 the possibility that $g(z)$ has zeros on the unit circle is definitely included.

Actually the discussion of Theorem 45.2 by Marden should be in Sec. 44. Also the discussion of Theorem 45.1 should belong to a special section by itself, because it deals with the situation when both singular cases occur in one example. This is illustrated in the following example.

8. Example:

Let $g(z)$ be given as follows:

$$g(z) = 1 - 2.55z - .5z^2 + .85z^3 + 4.2z^4 + z^5 = (z - \frac{1}{2})^2 (z + 4)[(z + .6)^2 + .8^2] = 0.$$

In the above polynomial, $b_m = b_0$, hence the singular case of Theorem 45.1 occurs. We obtain first $G_1(z)$ using Eq. (45.6) as follows:

$$G_1(z) = 3 - 14.4z - 16.35z^2 + 1.2z^3 + 22.05z^4 + 16.5z^5.$$

According to item (3) the above polynomial should have the same zeros on the unit circle as $g(z)$ and also the same number of zeros inside the unit circle as $g(z)$. To obtain the zero distribution of $G_1(z)$ we use the following table:[8]

$G_1(z)$		3	-14.4	-16.26	1.2	22.05	16.5
		16.5	22.05	1.2	-16.26	-14.4	3
$G_2(z)$	$\delta_1 =$	-263.25	-407	-68.58	271.9	303.75	
		303.75	271.9	-68.58	-407	-263.25	
$G_3(z)$	$\delta_2 =$	-2.31	2.46	3.88	5.20		
		5.20	3.88	2.46	-2.31		

$G_4(z)$ $\delta_3 =$ -21.70   -25.89   -21.70

-21.70   -25.89   -21.70

$G_5(z)$ 0   0   0

Note: $G_4(z) \cong 1 + 1.2z + z^2 = [(z + .6)^2 + .8^2]$.

From the above table we determine that $G_1(z)$ has two zeros inside the unit circle and two zeros on the unit circle. Hence $g(z)$ has two zeros inside the unit circle and two zeros on the unit circle, the remaining zero should then lie outside the unit circle.

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