GLOBAL INVERSE FUNCTION THEOREM

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ABSTRACT

A simple proof of global inverse function theorem in $\mathbb{R}^n$ is given. A global homeomorphic version of the theorem is proved first. A global diffeomorphic version follows by an application of the classical local inverse function theorem.

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The problem of determining that a given function from \( \mathbb{R}^n \) into \( \mathbb{R}^n \) has an inverse is very useful in applications. In 1959, Palais established the necessary and sufficient condition for a function to be a diffeomorphism of \( \mathbb{R}^n \) onto itself, which appeared as an episode in a paper dealing with the determination of spaces of intertwining operators on differential forms. This global version of the classical inverse function theorem has been applied widely in nonlinear network theory and is generally referred to as Palais Theorem by circuit theorists. Palais originally stated it without proof as a corollary in [1]. We believe that a simple proof of this useful theorem will be helpful to the readers of this Journal.

This note presents a proof that is intuitively appealing and easily understood with a modest background in mathematical analysis. We first prove the necessary and sufficient condition for a global homeomorphism; the case of a global diffeomorphism follows easily by an application of the classical inverse function theorem.
THEOREM

Let $f$ be a map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then $f$ is a homeomorphism$^1$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ if and only if $f$ is

1. a local homeomorphism$^2$ and
2. a proper map$^3$.

Proof: $\Rightarrow$ By assumption $f$ is a (global) homeomorphism, hence it is a local homeomorphism. Because $f^{-1}$ is continuous, it maps any compact set into a compact set. [13, p. 78; 14, Theorem 4.1, p. 207].

$\Leftarrow$ We prove this in three steps: (1) $f$ is surjective (onto), (2) $f$ is injective (one-to-one), (3) $f^{-1}$ is continuous. To facilitate the presentation, we denote the domain of $f$ by $X$ and the range by $Y$; of course, $X = Y = \mathbb{R}^n$.

1. Surjective: Let $Y_1$ be the image of $f$, i.e., $Y_1 = f(X)$, or more specifically, $Y_1 = \{ y \in Y | f^{-1}(y) \text{ is a nonempty subset of } X \}$. We know that $Y$ is connected. If $Y_1$ is both open and closed, knowing also the fact that $Y_1$ is not empty, we can conclude that $Y_1 = Y$ [13, p. 59], i.e., $f$ is surjective.

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1. A homeomorphism of $X$ onto $Y$ is, by definition, a continuous bijective map $f$: $X \rightarrow Y$ such that $f^{-1}$ is also continuous.
2. A map $f$: $X \rightarrow Y$ is said to be a local homeomorphism if whenever $x \in X$ and $y \in Y$ are such that $f(x) = y$ then there exist open neighborhoods $U$ of $x$ and $V$ of $y$ such that $f$ restricted to $U$ is a homeomorphism of $U$ onto $V$.
3. A continuous map is said to be proper if the inverse image of any compact set is compact.
(i) \( Y_1 \) open: Let \( y_1 \in Y_1 \), then there exists an \( x_1 \in X \) such that \( f(x_1) = y_1 \).

Now \( f \) is a local homeomorphism means that there exist open neighborhoods \( U \) of \( x_1 \) and \( V \) of \( y_1 \) such that \( f \) is a homeomorphism from \( U \) onto \( V \). So \( V \subseteq Y_1 \) and \( Y_1 \) is thus open.

(ii) \( Y_1 \) closed: Let \( y \) be an accumulation point of \( Y_1 \), then there exists a sequence \( \{y_j\}_{j=1}^{\infty} \) with \( y_j \in Y_1 \) \( \forall j \), and \( y_j \to y \). Consider \( K = \{y_j\}_{j=1}^{\infty} \cup \{y\} \), which is clearly closed and bounded in \( Y \), hence compact. [13, p. 58; 14, Th. 4.5, p. 208]. By assumption, \( f^{-1}(K) \) is compact in \( X \). Now pick \( x_j = f^{-1}(y_j) \). \( \{x_j\}_{j=1}^{\infty} \) is a sequence in a compact set \( f^{-1}(K) \) in a metric space \( \mathbb{R}^n \), therefore \( \{x_j\}_{j=1}^{\infty} \) has a convergent subsequence, say \( \{x_{j_k}\}_{k=1}^{\infty} \to x \).

[13, p. 56; 14, Th. 4.4, p. 208]. But \( \{f(x_{j_k})\}_{k=1}^{\infty} \) is a subsequence of \( \{y_j\}_{j=1}^{\infty} \), therefore converges to the same limit \( y \). \( f \) is a continuous map because it is a local homeomorphism, hence \( f(x) = f(\lim x_{j_k}) = \lim f(x_{j_k}) = y \), therefore \( y \in Y_1 \). Hence \( Y_1 \) is closed. [13, p. 47; 14, p. 203].

(2) **Injective:** Suppose that \( f \) is not injective, hence there exist two distinct points \( x_1, x_2 \) such that \( f(x_1) = f(x_2) \). Without loss of generality, we can assume \( f(x_1) = f(x_2) = 0 \). Let \( \alpha: [0,1] \to X \) be defined by \( \alpha(t) = (1-t)x_1 + tx_2 \) and \( \beta = f \circ \alpha \). Geometrically, \( \alpha \) is the line segment joining \( x_1 \) to \( x_2 \) and \( \beta \), its image in \( Y \) under \( f \), is a closed curve through \( 0 \). (Fig. 1). Let \( B: [0,1] \times [0,1] \to Y \) be defined by \( B(t,\tau) = (1-\tau)\beta(t) \). Thus for each \( \tau \), \( B(\cdot,\tau) \) is obtained by shrinking the closed curve \( \beta \) toward the origin. The rough idea of the proof is to shrink the curve \( \beta \) toward the origin, the corresponding curve \( \alpha \) will be continuously deformed into some curve joining \( x_1 \) to \( x_2 \); the contradiction will be reached in the limit when \( \beta \) degenerates into a single point.
(i) Construction of the inverse image of B (Fig. 2): Let us define for each \( t \) a map \( A(t, \cdot) : [0,1] \rightarrow X \) by the following process of piecing together the local inverses of B. First let \( A(t, 0) = \alpha(t) \). Since \( f \) is a local homeomorphism, there exist homeomorphic neighborhoods of \( \alpha(t) \) and \( \beta(t) \), \( U_1 \) and \( V_1 \) respectively. Define \( A(t, \cdot) : [0, \tau_1] \rightarrow U_1 \) to be the local inverse image of \( B(t, \tau) \) for \( \tau \in [0, \tau_1] \) where \( \tau_1 \) is so chosen that \( B(t, \tau) \in V_1 \), for all \( \tau \in [0, \tau_1] \).

Thus we have \( f(A(t, \tau_1)) = B(t, \tau_1) \) and we can define \( A(t, \cdot) \) on \([\tau_1, \tau_2]\) with \( \tau_2 > \tau_1 \) as the local inverse of \( B(t, \cdot) \) around \( B(t, \tau_1) \). Repeat the same procedure; at each step, we extend \( \tau \) from \( \tau_k \) to \( \tau_{k+1} \) with \( \tau_{k+1} > \tau_k \). We are going to show by contradiction that the domain of \( A(t, \cdot) \) can always be extended to include \( 1 \). Suppose that the above process fails to do so. Then the increasing sequence \( \{\tau_k\} \) is bounded by \( 1 \) and has a least upper bound \( T \), so \( \{\tau_k\} \rightarrow T \leq 1 \).

But \( B(t, \cdot) \) is continuous, \( \lim_{k \rightarrow \infty} B(t, \tau_k) = B(t, T) \); and since \( \{A(t, \tau_k)\}_{k=1}^{\infty} \) is a sequence in a compact set \( f^{-1}(B(t, \tau) : \tau \in [0,1]) \), it has a subsequence converging to a limit \( A(t, T) \). Now because \( f \) is continuous, \( f(A(t, T)) = B(t, T) \), hence the domain of \( A(t, \cdot) \) is extended to include \( T \); moreover, in the case when \( T < 1 \), it can even be extended beyond \( T \) by local homeomorphism. Thus, we can define a map \( A : [0,1] \times [0,1] \rightarrow X \) with the property that \( f \circ A = B \), and also \( A(0, \tau) = x_1 \), \( A(1, \tau) = x_2 \), \( \forall \tau \).

(ii) Continuity of \( A(\cdot, \tau) \): We will show that for each \( \tau \), \( A(\cdot, \tau) : [0,1] \rightarrow X \) is continuous by open-set arguments [13, p. 70; 14, pp. 201-202]. Let \( \mathcal{O} \) be any open set in \( X \) and let its inverse image under \( A(\cdot, \tau) \) be denoted by \( \mathcal{J} \); equivalently, \( \mathcal{J} \) is the inverse image under \( A(\cdot, \tau) \) of the intersection of \( \mathcal{O} \) with the image of \( A(\cdot, \tau) \). Now for each \( t \), let the homeomorphic neighborhoods of \( A(t, \tau) \) and \( B(t, \tau) \) be \( U_t \) and \( V_t \), respectively. Note that
\[ (U_t \cap \emptyset) \text{ has } \mathcal{I} \text{ as its inverse image under } A(\cdot, \tau) \text{ and } \bigcup_{t \in [0,1]} f(U_t \cap \emptyset) \text{ also has } \mathcal{I} \text{ as its inverse image under } B(\cdot, \tau) \text{ since } B = f \circ A. \] 
But \( f(U_t \cap \emptyset) \) are open in \( Y \), so does \( \bigcup_{t \in [0,1]} f(U_t \cap \emptyset) \). Because \( B(\cdot, \tau) \) is continuous, \( \mathcal{I} \) is open in \( [0,1] \). This completes our proof that \( A(\cdot, \tau) \) is continuous.

Now for \( \tau = 1 \), \( B(t, 1) = 0, \forall t \). Geometrically, this is done by shrinking \( \beta \) to the origin. The corresponding \( A(t, 1) \) is still a continuous curve joining two distinct points \( x_1 \) and \( x_2 \). But the inverse image of a single point under a local homeomorphism \( f \) can not be a continuous curve.

To demonstrate this, suppose it were true, every neighborhood of \( x_1 \) would contain points of \( f^{-1}(0) \) other than \( x_1 \) itself, then it would be impossible for homeomorphic neighborhoods of \( x_1 \) and 0 to exist. Thus we have proved that \( f \) is injective.

Remark: Here we have in fact tacitly constructed a covering homotopy \( A \) of \( B \) (15, Th. 3, p. 59).

(3) Continuity of \( f^{-1} \): Recall that continuity is a local property. [13, p. 68; 14, pp. 201-202]. The fact that \( f^{-1} \) exists globally (by (1) and (2)) together with the local homeomorphism assumption asserts that \( f^{-1} \) is continuous. Q.E.D.

Lemma 1

Let \( f \) be a continuous map from \( \mathbb{R}^n \) into \( \mathbb{R}^n \), then \( f \) is a proper map if and only if \( \lim_{\|x\| \to \infty} \|f(x)\| = \infty \).

Proof: By contradiction. Suppose that there is a sequence \( \{x_k\} \) with \( \|x_k\| \to \infty \), yet \( \|f(x_k)\| < M < \infty \). Consider the closed and bounded ball
\[ B_M = \{ y \mid \|y\| \leq M \}, \text{ because } f \text{ is proper, } f^{-1}(B_M) \text{ is compact. However, } \{ x_k \} \text{ is contained in } f^{-1}(B_M), \text{ but } \|x_k\| \to \infty \text{ contradicts the compactness of } f^{-1}(B_M). \]

\[ \text{Therefore, } f \text{ is continuous implies that for each closed set } K, f^{-1}(K) \text{ is closed. Suppose } K \text{ is bounded yet } f^{-1}(K) \text{ is not, then there exists a sequence } \{ x_k \} \text{ in } f^{-1}(K) \text{ with } \|x_k\| \to \infty. \text{ Clearly } \{ f(x_k) \} \subset K. \text{ But by assumption } \|f(x_k)\| \to \infty, \text{ which contradicts boundedness of } K. \] Q.E.D.

**Lemma 2**

Let \( f \) be a \( C^k \) map \((k \geq 1)\) from \( \mathbb{R}^n \) into \( \mathbb{R}^n \), then \( f \) is a local \( C^k \)-diffeomorphism \(^4\) if and only if \( \det \left( \frac{\partial f}{\partial x} \right) \neq 0 \).

**Proof:** This is the well-known classical local inverse function theorem. [13, p. 211 and Ex. 17, p. 217; 14, p. 167].

**Corollary:**

Let \( f \) be a \( C^k \) map from \( \mathbb{R}^n \) into \( \mathbb{R}^n \), then \( f \) is a \( C^k \)-diffeomorphism if and only if

1. \[ \det \left( \frac{\partial f}{\partial x} \right) \neq 0 \ \forall x \]

2. \[ \lim_{\|x\| \to \infty} \|f(x)\| = \infty \]

**Proof:** It follows from the Theorem, Lemma 1 and 2, as well as the fact that differentiability is a local property. [13, p. 198; 14, p. 142]. Q.E.D.

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4. A \( C^k \)-map is, by definition, a map with continuous derivatives up to order \( k \). A \( C^k \)-diffeomorphism is, by definition, a bijective \( C^k \) map such that the inverse is also \( C^k \).
The authors would like to thank Prof. A. Weinstein for his helpful discussion.
REFERENCES


FOOTNOTES

1. A homeomorphism of $X$ onto $Y$ is, by definition, a continuous bijective map $f: X \rightarrow Y$ such that $f^{-1}$ is also continuous.

2. A map $f: X \rightarrow Y$ is said to be a local homeomorphism if whenever $x \in X$ and $y \in Y$ are such that $f(x) = y$ then there exist open neighborhoods $U$ of $x$ and $V$ of $y$ such that $f$ restricted to $U$ is a homeomorphism of $U$ onto $V$.

3. A continuous map is said to be proper if the inverse image of any compact set is compact.

4. A $C^k$-map is, by definition, a map with continuous derivatives up to order $k$. A $C^k$-diffeomorphism is, by definition, a bijective $C^k$ map such that the inverse is also $C^k$. 
FIGURE CAPTIONS

Fig. 1. For the proof by contradiction, it is assumed that \( x_1 \neq x_2 \) and that \( f(x_1) = f(x_2) = 0 \). The line segment \( \alpha \) which joins \( x_1 \) to \( x_2 \) is mapped by \( f \) onto the closed curve \( \beta \).

Fig. 2. As \( \tau \) goes from 0 to 1, \( B(\tau, T) = (1-\tau) \beta(t) \) travels in a straight line from \( \beta(t) \) to 0. For the same fixed \( t \), the corresponding curve \( A(\tau, T) \) is constructed from \( B(\tau, T) \) by successive local homeomorphisms.