Copyright © 1963, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

Lectures on

LINEAR, TIME-VARYING CIRCUITS and MULTITERMINAL RC NETWORKS

by

Dr. Sidney Darlington

Visiting Mackay Lecturer in
Electrical Engineering
University of California, Berkeley

from

Bell Telephone Laboratories
Murray Hill, New Jersey

A series of lectures given between
April 15 and May 24, 1963 at
Berkeley

Notes by: Ronald A. Rohrer (Lectures 1-6)
Samuel C. Lee (Lecture 7)
Typed by: Mrs. Marilyn Thirion

TABLE OF CONTENTS

UNIVERSITY OF CALIFORNIA
Electrical Engineering

April, 1963

Linear, Time-varying Circuits

Notes on lectures delivered by S. Darlington
of Bell Telephone Laboratories.

Lecture 1: Circuits of R's and C's which can amplify.

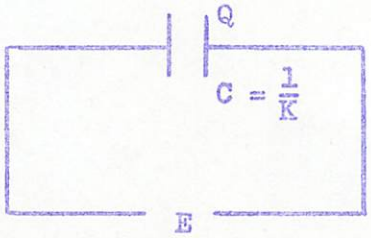
The analysis of time-varying networks may be approached from two standpoints:

1) By using the mathematical theory of linear differential equations;

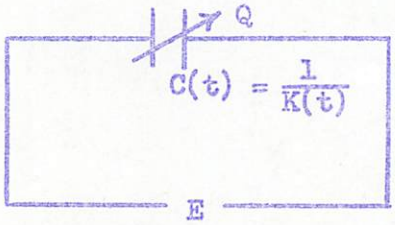
2) By extending the theory of fixed circuits.

The problem with using the first approach is that the equations which arise from circuits only constitute a small class of all linear differential equations. Hence, the second approach will be employed to ascertain which concepts of fixed circuits generalize, and what modifications must be made to obtain analogous quantities for time-varying circuits.

A time-varying network element may be obtained by mechanically varying a linear element or as a small signal approximation for a pumped nonlinear element.



Fixed



T-V

$$E = KQ \qquad\qquad E = K(t)Q$$

$$Q = CE \qquad\qquad Q = C(t)E$$

$$K = \frac{1}{C} \qquad\qquad K(t) = \frac{1}{C(t)}$$

For a nonlinear capacitor,

$$Q = F(E) \ .$$

Suppose now that the voltage E driving this capacitor is composed of two elements - the signal voltage E_s and the pump voltage E_p:

$$E = E_s + E_p .$$

Suppose further that

$$|E_s| << |E_p|$$

(this is a small signal approximation); then

$$F(E) \doteq F(E_p) + \left.\frac{\partial F}{\partial E}\right|_{E_p} E_s = Q_p + Q_s .$$

Since

$$F(E_p) = Q_p$$

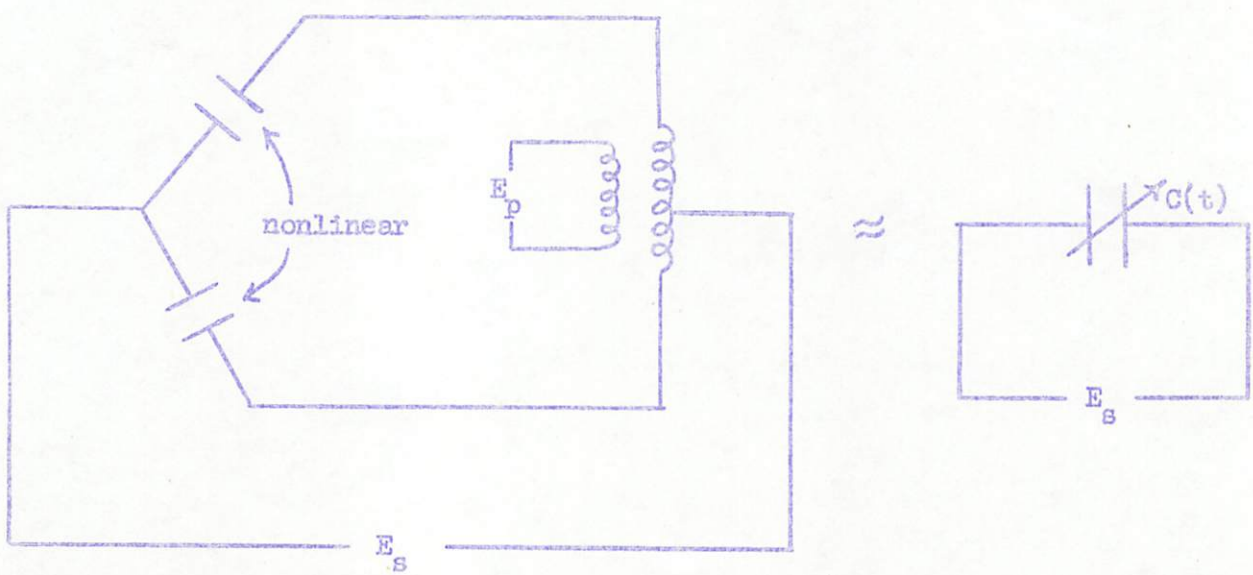
for no signal (i.e., $E_s = 0$), one can say

$$Q_s = C(t)\ E_s ,$$

where

$$C(t) = \left.\frac{\partial F}{\partial E}\right|_{E_p} .$$

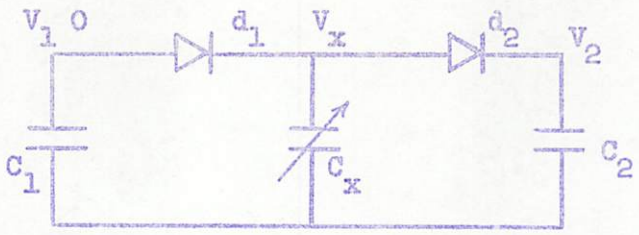
Such a time-varying capacitor might be obtained as shown below.



Since the energy stored in a capacitor is

$$\frac{1}{2} KQ^2 ,$$

if K increases, then the stored energy increases. A simple network which utilizes a time-varying capacitance to obtain voltage amplification is shown below.

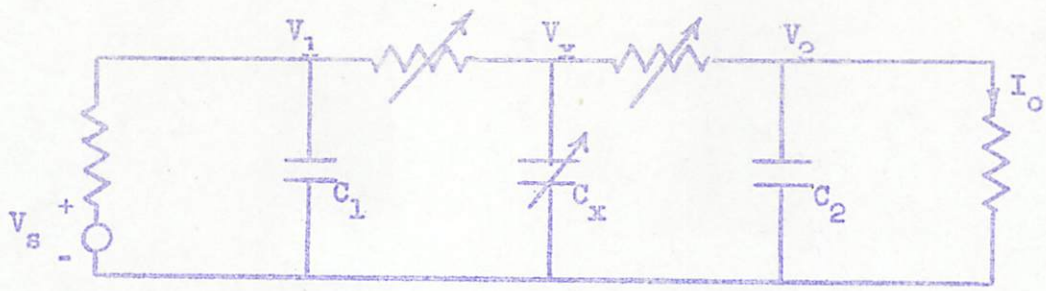


When C_x is largest (thus V_x is smallest) charge will be transfered from C_1 to C_x until V_1 and V_x are equal. Then, as C_x decreases, V_x increases and there is no current flow through d_1, but charge is transferred through d_2 to C_2 until V_2 and V_x are equal. Thus, the signal V_1 at C_1 is carried to C_2 at the higher level V_2. In the steady state, the amplification approaches the limit

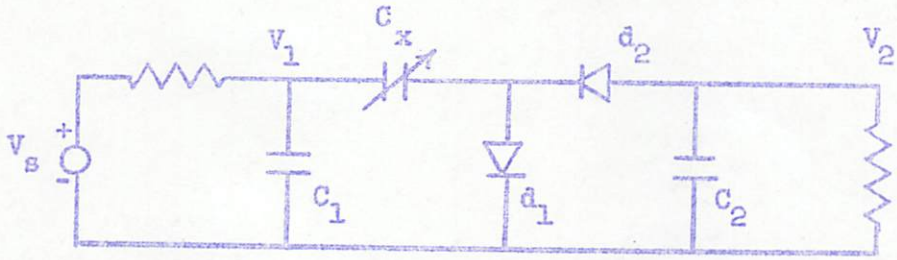
$$\frac{V_2}{V_1} = \frac{C_{x\ max}}{C_{x\ min}} .$$

If V_1 were initially negative, however, no such amplification would occur because of the diode orientation. This problem can be overcome by using a time-varying resistor as a switch (each properly synchronized) in place of the diode. Ultimately, one might obtain linear, time-varying networks, and then replace the time-varying resistors with diodes.

Pictured below is a low frequency amplifier which takes advantage of the effects described above.



A disadvantage is that only voltage gain is provided (as with grounded base transistor). To carry the transistor analogy further, one may consider the time-varying network analog of the grounded emitter transistor shown below.



In the steady state, the voltages approach the limit

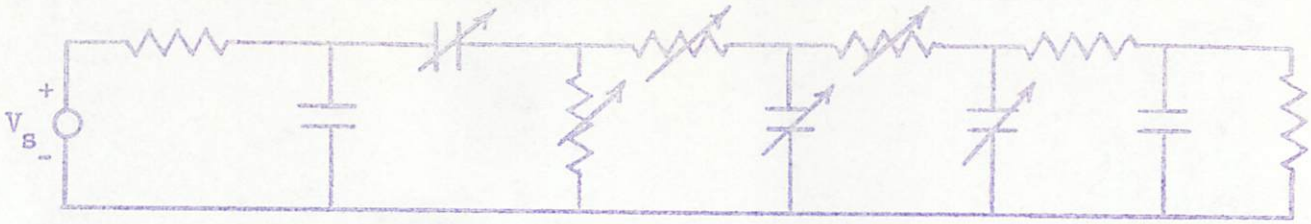
$$\frac{V_1 - V_2}{V_1} = \frac{C_{x\ max}}{C_{c\ min}} ;$$

hence,

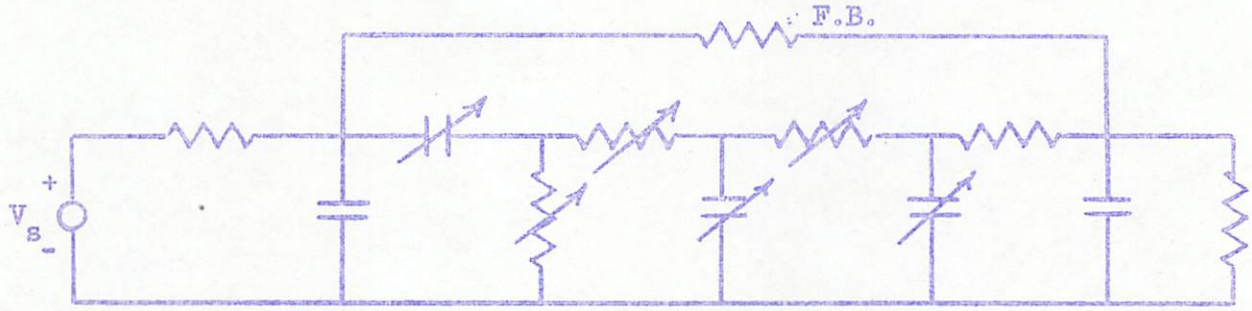
$$\frac{V_2}{V_1} = -\left(\frac{C_{x\ max}}{C_{x\ min}} - 1\right) .$$

On the average the current input is zero; besides being an analog of the grounded emitter transistor, this circuit is an analog of a magnetic amplifier.

One might obtain a multistage amplifier with both current and voltage gain by cascading a number of the above described amplifiers as shown below.

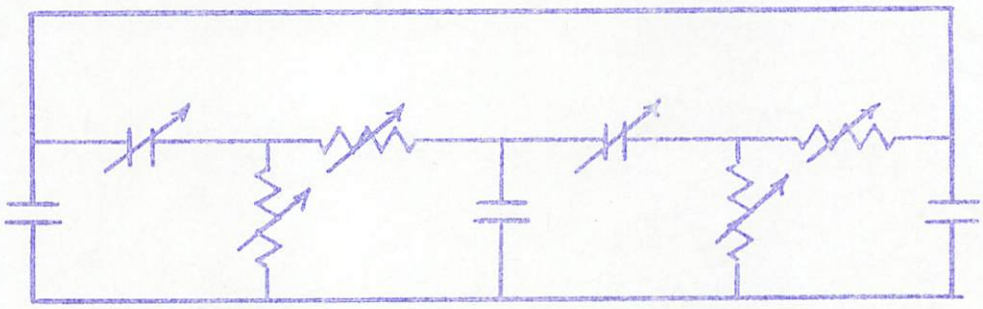


This might even be made into a feedback amplifier by the addition of a feedback element as shown below.

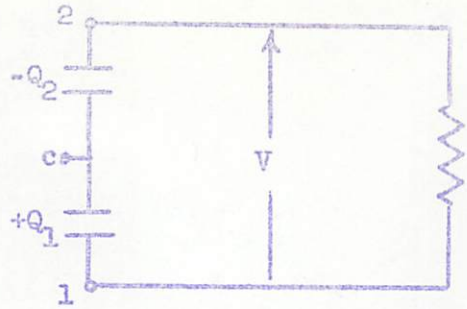


By considering the simple network shown below with positive feedback, one can easily see how it is possible to obtain real mode instability with time-varying R's and C's.

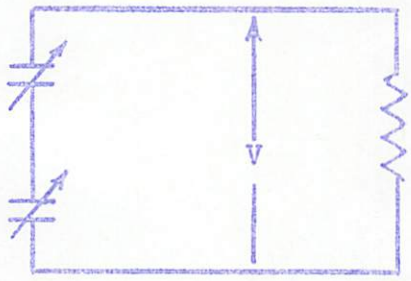
Positive Feedback



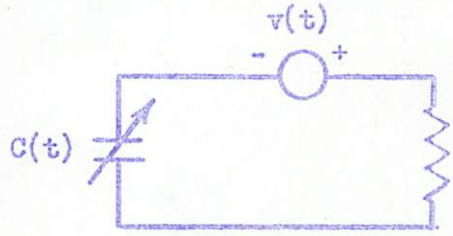
Time-varying capacitors may also be used as transformers. Consider the stationary network shown below; current flows through the resistor until the



voltage V between terminals one and two goes to zero. But the voltage at node c is not necessarily zero; one cannot tell externally (without using c) whether there is hidden charge on the capacitors. If the capacitors are time-varying (as below) the voltage V goes to zero on the average, but the voltage never becomes



identically zero. If one wished to create a Thevenin equivalent for this situation, it would take the form shown below. The only instance in which the



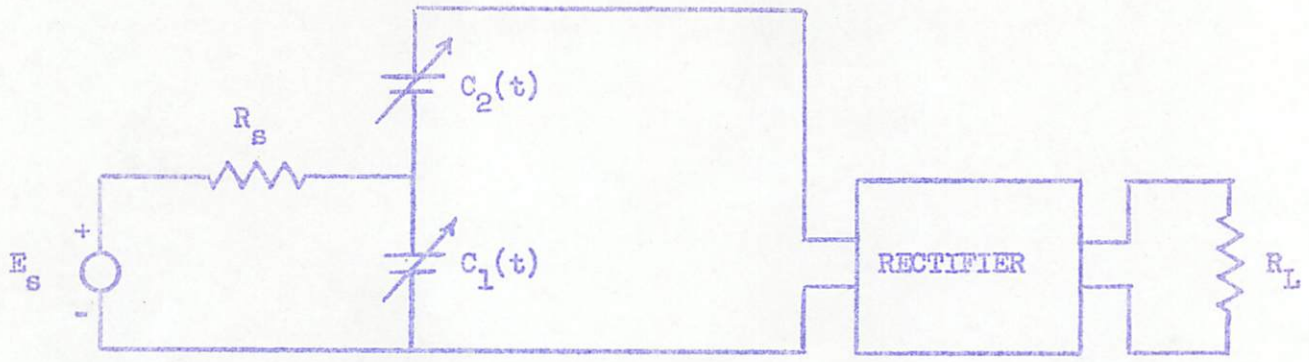
capacitor C(t) would be time-invariant would be when

$$\frac{1}{C_1(t)} + \frac{1}{C_2(t)} = \text{constant.}$$

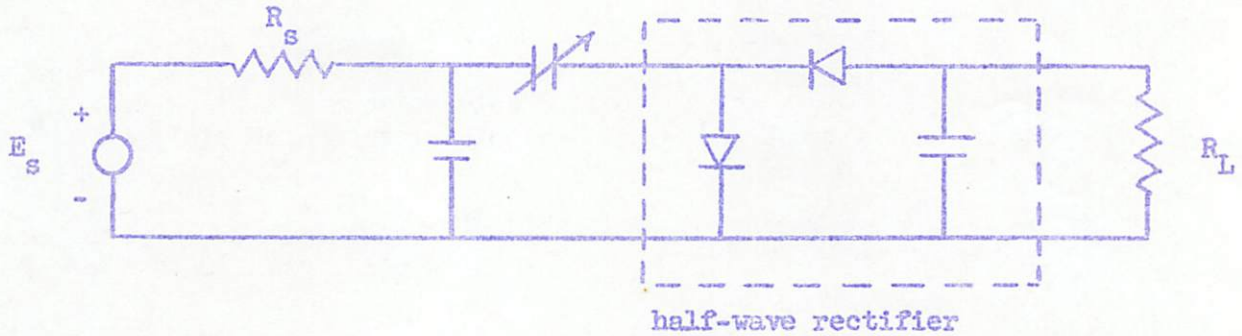
The generator v(t) compensates for the voltage difference created when

$$C_1(t) \neq C_2(t).$$

The principle described above may be used to obtain an amplifier as shown below. $C_1(t)$ and $C_2(t)$ are in phase opposition (i.e., one is large when the other is small) and vary much faster than the signal voltage E_s. It is not necessary



that both of the capacitors vary; replacing one of the time-varying capacitors by a stationary capacitor and utilizing a half-wave rectifier, one obtains the following network which has been previously discussed.



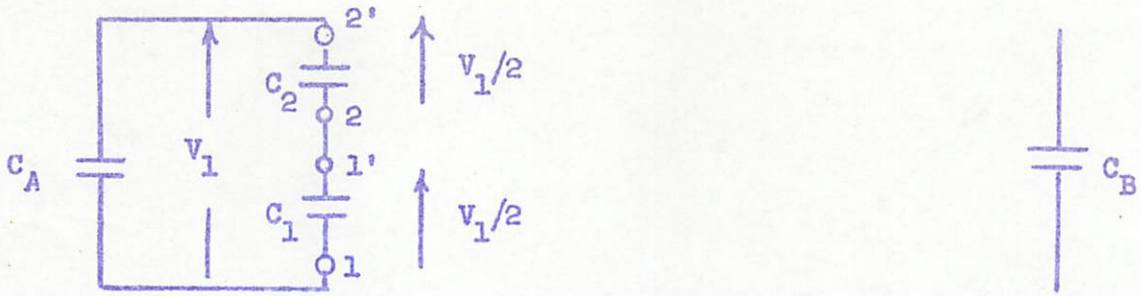
Similar circuits have been described by W. P. Mason and R. F. Wick, and also by J. R. Baird:

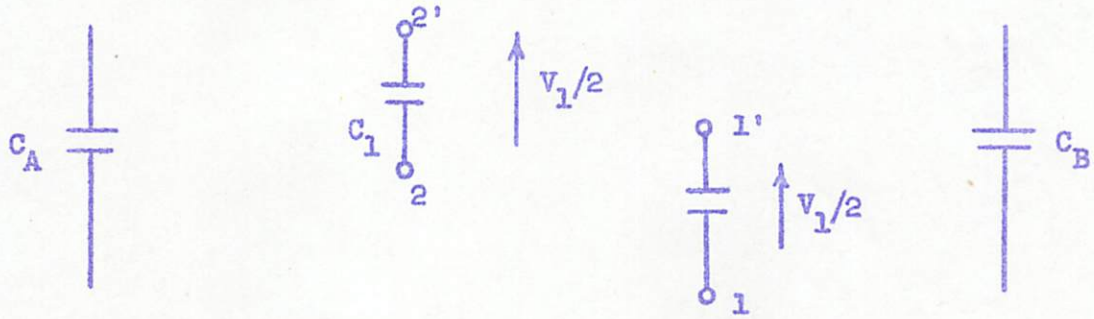
Mason, W. P. and Wick, R. F., "Ferroelectrics and the dielectric amplifier", Proc. IRE, Vol. 42, pp. 1606-1620, Nov., 1954.

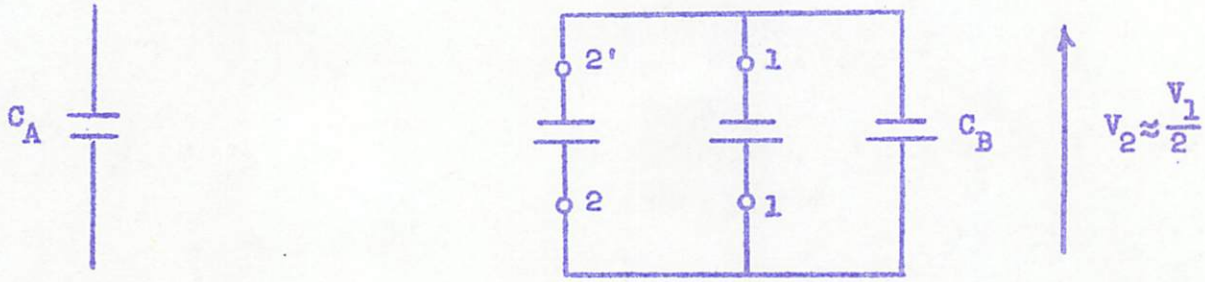
Mason, W. P., "Ferroelectrics", Proc. of the Symposium on the Role of Solid State Phenomena in Electric Circuits, Polytechnic Institute of Brooklyn, April, 1957.

Baird, J. R, "Low frequency reactance amplifier", Proc. IRE, Vol. 51, pp. 298-303, Feb., 1963.

Variable networks containing capacitors can also be used as transformers. Consider the sequence of events shown below.



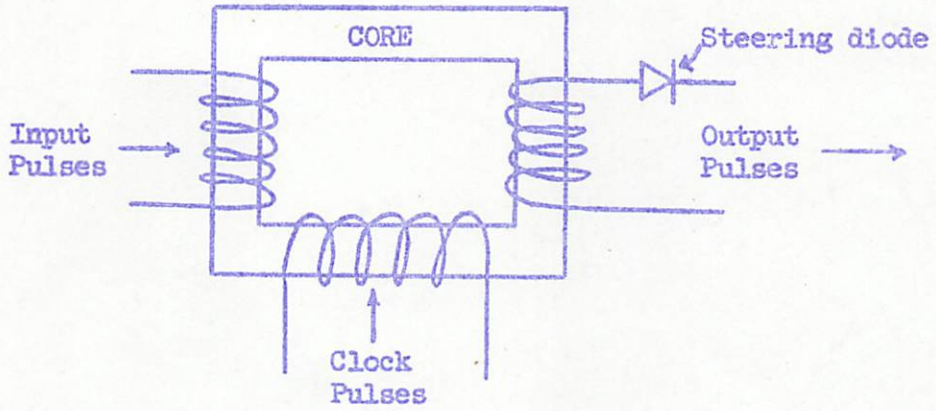




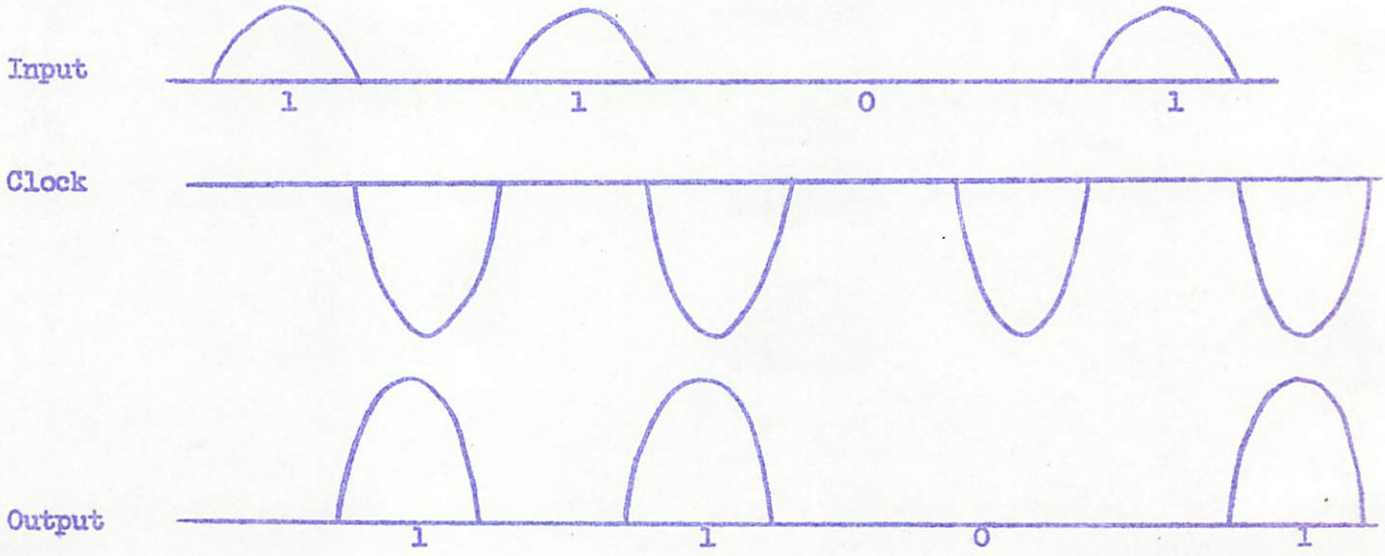
If this cycle of switching the series combination of C_1, C_2 and C_A to the parallel combination of C_1, C_2 and C_B and vice-versa is repeated the voltage on C_B approaches half of that on C_A, but C_B receives twice the charge as that taken from

C_A. Thus, the circuit acts like a transformer with a 2:1 voltage ratio which works even at d.c.!

In the above analyses the tacit assumption that the signal frequency is much smaller than the pump frequency has been used. It might be possible, however, to amplify signals of frequencies comparable to that of the pump. As a starting point, one might consider the magnetic pulse regenerator shown below.

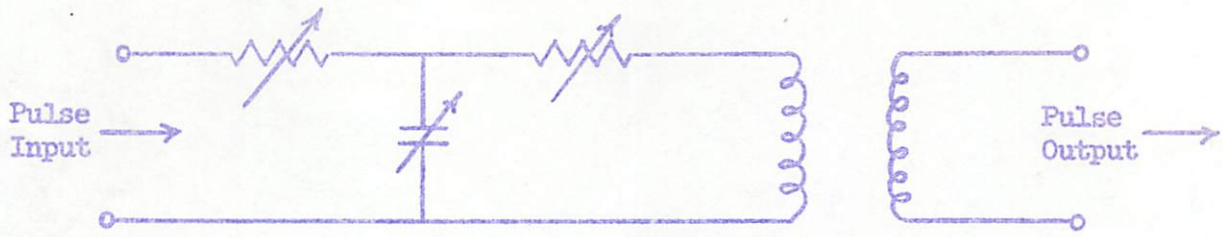


Typical time relations are shown below.



The same effect can be obtained by using a nonlinear capacitor as a time-varying capacitor. For the circuit shown below where C(t) is large during the signal and

small during the output (the pump is equivalent to the clock), there is possible retiming, reshaping and amplification.



MEVV:mt
4-26-63

Dr. S. Darlington Spring 1963

Lecture 2: Properties of linear, time-varying transducers of finite order.

I. References:

1. C. A. Desoer and A. Paige, "Linear Time-Varying G-C Networks: Stable and Unstable," IEEE Trans. on Circuit Theory; to appear June, 1963.

2. L. A. Zadeh, "Time-varing Networks--I," Proc. IRE, Vol. 49, pp. 1488-1501; Oct. 1961.

3. S. Darlington, "Time Variable Transducers," Polytechnic Institute of Brooklyn, Proc. of the Symposium on Active Networks and Feedback Systems, pp. 621-633; April, 1960.

II. Background.

Much of the mathematics to be presented here appears in texts or classical references. The purpose here will be to investigate the external properties of linear transducers of finite order (characterized by variable coefficient differential equations of finite order), and to attempt to point up the similarities and differences between them and lumped stationary networks.

III. Time-varying Linear Transducers.

Consideration of the operator notation

$$p \equiv \frac{d}{dt},$$

$$p^n \equiv \frac{d^n}{dt^n}$$

leads to many useful analogies between fixed and time-varying networks.

A. Network characterization.

1. Fixed circuits:

$$Y(s) = \frac{A(s)}{B(s)},$$

where A(s) and B(s) are polynomials of finite order.

2. Time-varying circuits:

$$Ay(t) = Bx(t),$$

where

$$A = \sum_{k=1}^{n} a_k(t)p^k$$

$$B = \sum_{k=1}^{m} b_k(t)p^k .$$

3. The essential difference is that the a_k and b_k are functions of time for time-varying networks.

B. Solutions of network equations.

1. Fixed circuits:

$$Ay(t) = 0 \Longrightarrow y(t) = \sum_{k=1}^{n} K_k e^{s_k t} ;$$

$\phi_k(t) = e^{s_k t}$ are natural modes.

2. Time-varying circuits:

$$Ay(t) = 0 \Longrightarrow y(t) = \sum_{k=1}^{n} K_k \phi_k(t) ;$$

$\phi_k(t)$ are basis functions.

3. Since for a fixed network

$$\phi_k(t) = e^{s_k t} ,$$

there is an analogy between poles (s_k) and basis functions (ϕ_k). The problem is that the s_k's are relatively easy to find compared to the ϕ_k's in the general case.

C. Zero response.

1. Fixed circuits:

$$0 = Bx(t) \Longrightarrow x(t) = \sum_{j=1}^{m} C_j e^{s_j t} .$$

2. Time-varying circuits:

$$0 = Bx(t) \Longrightarrow x(t) = \sum_{j=1}^{m} C_j \, e^{s_j t}$$

$z_j(t)$ are zero response functions.

3. These zero response functions are those input functions which yield the same effect as no input at all. In this respect they are analogous to zeros of stationary network functions; in fact, for a fixed network

$$z_j(t) = e^{s_j t} .$$

where the s_j's are the zeros.

IV. Solution for the Driven Time-varying Network.

A. Impulse response:

Suppose that the input is

$$x(t) = \delta(t - \tau),$$

an impulse applied at time , then the impulse response takes the form

$$y(t) = \sum_{k=1}^{n} \phi_k(t) \, W_k(\tau) .$$

B. General response:

The response to an arbitrary function, $x(t)$, is

$$y(t) = \int_{-\infty}^{t} \sum_{k=1}^{n} \phi_k(t) \, W_k(\tau) \, x(\tau) \, d\tau$$

(for a causal network).

C. Fixed circuits:

$$y(t) = \int_{-\infty}^{t} \sum_{k=1}^{n} \phi_k(t) \, W_k(\tau) \, x(\tau) \, d\tau ,$$

where

$$\phi_k(t) = e^{s_k t}$$

and

$$W_k(\tau) = J_k \, e^{-s_k}$$

(J_k is the residue).

D. Explicit forcing function:

Suppose that the forcing function is f(t), i.e., the network equation is

$$Ay(t) = f(t);$$

then one has the solution

$$y(t) = \int_{-\infty}^{t} \sum_{k=1}^{n} \phi_k(t) \, W_k^{o}(\tau) \, f(\tau) \, d \quad .$$

Here, the solution for the special case

$$Ay(t) = \delta(t - \tau)$$

would be

$$y(t) = \sum_{k=1}^{n} \phi_k(t) \, W_k^{o}(\tau).$$

1. Classical Green's function theory, coupled with the constraint of physical realizability, gives

$$\sum_{k=1}^{n} \phi_k(t) \, W_k^{o}(t) = 0$$

$$\sum_{k=1}^{n} \phi_k'(t) \, W_k^{o}(t) = 0$$

- - - - - - - - - - - -

$$\sum_{k=1}^{n} \phi_k^{(n-1)}(t) \, W_k^{o}(t) = 0$$

$$\sum_{k=1}^{n} \phi_k^{(n)}(t)\, W_k^{o}(t) = \frac{1}{a_n(t)} .$$

2. For the input $x(t)$ the solution takes the form

$$y(t) = \int_{-\infty}^{t} \sum_{k=1}^{n} \phi_k(t)\, W_k^{o}(\tau) \left[\sum_{j=1}^{m} b_j(\tau) \frac{d^j}{d\ \ j} x(\tau) \right] d\tau .$$

By repeated application of integration by parts,

$$\int u(\tau)\, v'(\tau)\, d\tau = u(\tau)\, v(\tau) - \int u'(\tau)\, v(\tau)\, d\tau ,$$

one ultimately arrives at the expression

$$W_k(\tau) = \sum_{j=1}^{m} (-1)^j \frac{d^j}{d\ \ j} \left[b_j(\tau)\, W_k^{o}(\tau) \right] .$$

The expression (in terms of t)

$$\sum_{j=1}^{m} (-1)^j\, p^j b_j(t) ,$$

which operates on W_k^{o}, is of course the adjoint of the original operator

$$B = \sum_{j=1}^{m} b_j(t)\, p^j .$$

3. References:

Friedman, Coddington and Levinson, Hildebrand (Meth. of Appl. Math.).

V. Inverse problem: given the basis functions, find the operator A.

$$Ay(t) = K(t) \det \begin{bmatrix} y(t) & \dot{y}(t) & \cdots & y^{(n)}(t) \\ \phi_1(t) & \dot{\phi}_1(t) & \cdots & \phi_1^{(n)}(t) \\ \cdot & \cdot & \cdots & \cdot \\ \phi_n(t) & \dot{\phi}_n(t) & \cdots & \phi_n^{(n)}(t) \end{bmatrix}$$

where $K(t)$ is an arbitrary nonzero multiplier. Note that

$$y(t) = \phi_k(t) \Longrightarrow Ay(t) = 0$$

as two rows of the matrix are the same -- this goes for any $y(t)$ which is a linear combination of $\phi_k(t)$'s.

A. Fixed circuits:

1. Distinct poles,

$$\phi_k(t) = e^{s_k t},$$

$$\dot{\phi}_k(t) = s_k e^{s_k t},$$

etc.

$$Ay(t) = K(t)\prod_{k=1}^{n} e^{s_k t} \begin{bmatrix} y & \dot{y} & \cdots & y^{(n)} \\ 1 & s_1 & \cdots & s_1^{\,n} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & s_n & \cdots & s_n^{\,n} \end{bmatrix}$$

If one takes

$$K(t) = \prod_{k=1}^{n} e^{-s_k t},$$

$Ay(t)$ becomes a differential equation with constant coefficients.

2. Multiple poles,

$$\phi_k(t) = e^{s_k t}$$

$$\phi_{k+1}(t) = te^{s_k t}$$

$$\vdots$$

$$\phi_{k+\alpha}(t) = t^{\alpha} e^{s_k t}.$$

The above method easily generalizes.

B. Time-varying circuits:

Assume

$$\phi_k(t) = H_k(t)\, e^{s_k t},$$

where $H_k(t)$ is periodic; this would be the case for periodically variable networks from Floquet theory. Actually in the most general case one might have

$$\phi_k(t) = (H_{k1}(t) + tH_{k2}(t))\, e^{s_k t} ,$$

etc. The form of $H_k(t)$ might be as complicated as

$$H_k(t) = \sum_{\sigma=0}^{\infty} \sum_{\sigma=0}^{\infty} H_{k\sigma\rho} \cos\,(\sigma\omega_1 t + \rho\omega_2 + \beta_{\sigma\rho}) \quad .$$

Under the first assumption, one can derive the differential equation from the basis functions. For the case of multiple poles, this cannot always be done.

D. Open problem:

What is a general class of such bounded coefficients for differential equations?

VI. Operator Manipulations

A. Operator identity:

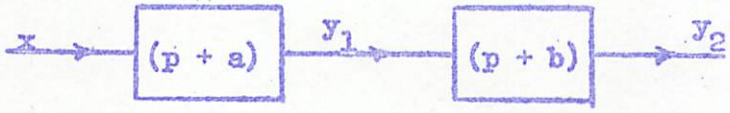
$$p \equiv \frac{d}{dt} \ , \ a \equiv a(t)$$

$$\begin{aligned} pax &= \frac{d}{dt}(ax) \\ &= \dot{a}x + a\dot{x} \\ &= (\dot{a} + ap)x \end{aligned}$$

$$\therefore \ pa = \dot{a} + ap$$

Note: $a = \text{const} \Longrightarrow a$ and p commutes.

B. Cascade of operators:



$$\left.\begin{aligned} y_1 &= (p + a)x \\ y_2 &= (p + b)y_1 \end{aligned}\right\} \text{ assume 1st order}$$

$$\therefore y_2 = (p + b)\left[(p + a)x\right]$$
$$= (p + b)(p + a)x$$
$$= (p^2 + bp + pa + ba)x$$
$$= \left[p^2 + (b + a)p + a + ba\right] x$$

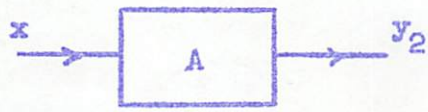
Reverse order

$$(p + a)(p + b) = p^2 + (b + a)p + b + ba$$
$$\neq (p + b)(p + a)$$

These two operators do not commute unless

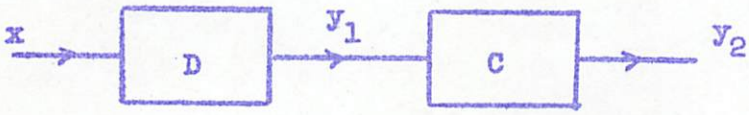
$$a = b + \text{const.}$$

C. Factoring:



$$y_2 = Ax$$

Attempt to put this in the form



$$A = CD$$

If $\phi_1, \ldots, \phi_n$ are basis functions of A, choose $\phi_1, \ldots, \phi_m$, $m < n$ to be basis functions of D (Say the first m -- the order of the original ϕ_k's can be arbitrarily rearranged); then,

$$A\,\phi_k = 0 \Longrightarrow CD\,\phi_k = 0\ ,\ k = 1, \ldots, n$$
$$C(D\,\phi_k) = 0 = C(0),\ k = 1, \ldots, m,$$

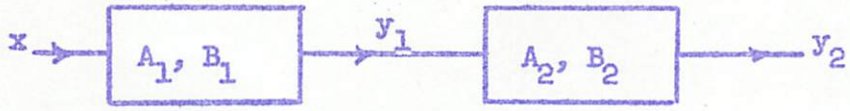
as $D\,\phi_k = 0$, $k = 1, \ldots, m$.

But

$$C(D\,\phi_k) = 0,\ k = m+1, \ldots, n;$$

$\therefore$ $D\,\phi_k$ must be the basis functions of C for $k = m+1, \ldots, n$. D can be found as in V from $\phi_1, \ldots, \phi_m$ as an mth order operator; C can be found as in V from $D\,\phi_{m+1}, \ldots, D\,\phi_n$ as an (n-m)th order operator.

D. More general cascade:



$$A_1 y_1 = B_1 x$$

$$A_2 y_2 = B_2 y_1 \,.$$

The problem is to find an expression

$$A y_2 = B x \,.$$

Introduce arbitrary operators M and N; then,

$$M A_1 y_1 = M B_1 x,$$

$$N A_2 y_2 = N B_2 y_1 \,.$$

Let

$$M A_1 = N B_2,$$

then

$$N A_2 y_2 = N B_2 y_1 = M A_1 y_1 = M B_1 x \,.$$

Hence

$$A = N A_2 \text{ and } B = M B_1 \,,$$

where

$$M A_1 = N B_2 \,.$$

MEVV:mt
5-13-63

UNIVERSITY OF CALIFORNIA
Electrical Engineering

Dr. S. Darlington Spring 1963

Lecture 3: Stability of Networks of Resistors and Capacitors.

I. Review of Some Matrix Algebra Formulae.

ϕ = scalar ; A, B, X, Y = matrices

A. Operations on matrices.

$X(A + B)Y = XAY + XBY$ (distributitive)

$\phi XY = X\phi Y = XY\phi$ (scalar commutes)

$(XY)^t = Y^t X^t$ (transposition)

$(XY)^{-1} = Y^{-1}X^{-1}$ (inversion)

B. Derivation of some matrix functions.

$pX^2 = \dot{X}X + X\dot{X} \neq 2X\dot{X}$

$pXYZ = \dot{X}YZ + X\dot{Y}Z + XY\dot{Z}$

$p(X^{-1}) = - X^{-1}\dot{X}X^{-1}$

$pX^t = (pX)^t$

$pA\phi = \dot{\phi}A + \phi pA$

$pAe^{\phi} = e^{\phi}(\dot{\phi}A + pA)$

C. Some operator identities.

$pA = \dot{A} + Ap$

$p^2A = \ddot{A} + 2\dot{A}p + Ap^2$

$pAp = \dot{A}p + Ap^2 = - p\dot{A} + p^2A$

D. Matrices of order one (scalars).

W,V = column matrices

$W^tV = \sum_{k=1}^{n} W_k V_k = V^tW$

If $Y^t = Y$, then $W^tYV + V^tYW$, $\dot{V}^tYV = V^tY\dot{V} = \frac{1}{2} P(V^tYV) - \frac{1}{2} V^t\dot{Y}V$.

If $Y^t = - Y$, then $V^tYV = - V^tYV = 0$.

II. Background.

A. Assume all time-varying R's and C's are positive and not discontinuous.

B. Network equations are

$$I = (G + pC)E,$$

where

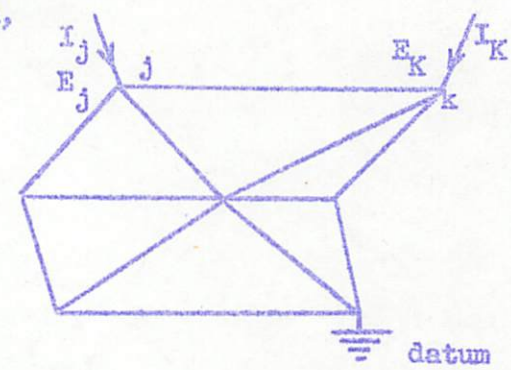
$$I = \begin{pmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ I_n \end{pmatrix},$$

the matrix of branch currents, and

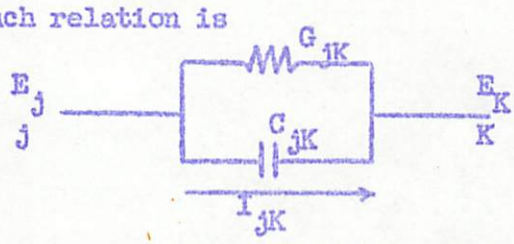
$$E = \begin{pmatrix} E_1 \\ \cdot \\ \cdot \\ \cdot \\ E_n \end{pmatrix},$$

the matrix of node voltages; G and C are time-dependent nxn square matrices.

C. For the given graph,



the simple branch relation is



$$I_{jK} = (G_{jK} + pC_{jK})(E_j - E_K),$$

since for the resistance

$$I_{jK} = G_{jK}(E_j - E_K),$$

and for the capacitance

$$Q_{jK} = C_{jK}(E_j - E_K)$$

or

$$I_{jK} = pC_{jK}(E_j - E_K).$$

D. Further assumptions.

1. $G^t = G$, $C^t = C$ for the network matrices.

2. G is a positive definite or semi-definite matrix. C is a positive definite or semi-definite matrix. These conditions can be assured by a suitable choice of network variables (see Desoer and Paige).

III. Formulation.

A. Pure mathematical approach.

Let $Q = CE$, $E = C^{-1}Q$; then $I = (A + p)Q$, where

$$A = GC^{-1}.$$

The mathematical approach degenerates at this point to a study of this specific first order differential equation which characterizes the network.

B. Circuit theory approach.

$$I = (G + pC)E = (G + \dot{C} + Cp)E$$

IV. Stability Conditions by Studying the Power for the Unexcited Network.

A. Power into network:

$$\begin{aligned} P = \sum_{k=1}^{n} E_k I_k &= E^t I \\ &= E^t(G + \dot{C} + Cp)E \\ &= E^t GE + E^t \dot{C}E + E^t C\dot{E} \\ &= E^t GE + \tfrac{1}{2}E^t \dot{C}E + \tfrac{1}{2}pE^t CE \quad (\text{since } C = C^t) \end{aligned}$$

$$\therefore P = E^t(G + \tfrac{1}{2}\dot{C})E + \tfrac{1}{2}pE^t CE .$$

B. A "symmetrical" expression for the network current can be obtained from the following two expressions for I:

$$I = (G + C + Cp)E \qquad (1)$$

$$I = (G + pC)E . \qquad (2)$$

$$I = \frac{1}{2}\left[(1) + (2) \right] = \left[G + \frac{1}{2}\dot{C} + \frac{1}{2}Cp + \frac{1}{2}pC \right] E$$

C. Transformation of variables:

Define the instantaneous transformation

$$E = N\hat{E} \quad , \quad \hat{I} = N^t I \ .$$

Then

$$\hat{I} = \left\{ N^t \left[G + \frac{1}{2}\dot{C} + \frac{1}{2}(Cp + pC) \right] N \right\} E \ .$$

Expanding this, one obtains

$$\hat{I} = N^t(G + \frac{1}{2}\dot{C})N\hat{E} + \frac{1}{2}N^t CpN\hat{E} + \frac{1}{2}N^t pCN\hat{E}.$$

But

$$\frac{1}{2}N^t CpN = \frac{1}{2}N^t CNp + \frac{1}{2}N^t C\dot{N}$$

and

$$\frac{1}{2}N^t pCN = \frac{1}{2}pN^t CN - \frac{1}{2}\dot{N}^t CN \ .$$

Define

$$\hat{G} = N^t(G + \frac{1}{2}\dot{C})N$$

$$\hat{C} = N^t CN$$

$$J = N^t C\dot{N} - \dot{N}^t CN \ .$$

Note: J is an antimetrical matrix; i.e.,

$$J^t = - J \ .$$

Moreover,

$$\hat{G}^t = \hat{G}$$

and

$$\hat{C}^t = \hat{C} \ .$$

A nonsingular transformation matrix N implies $\hat{G}$ and $\hat{C}$ are positive-definite. Pick the matrix N so as to simultaneously diagonalize the matrices $\hat{G}$ and $\hat{C}$ (this is well-known from math); i.e.,

$$\hat{C} = N^t CN = U \text{ (the identity matrix)}$$

$$\hat{G} = N^t(G + \tfrac{1}{2}\dot{C})N = -\Lambda = -\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

D. Power in the new variables:

$$\hat{I} = (-\Lambda + pU + J)\hat{E}$$

$$P = E^t I = \hat{E}^t N^t I = \hat{E}^t \hat{I}$$

$$\therefore P = \hat{E}^t(-\Lambda + pU + J)\hat{E} .$$

Since $J^t = -J$,

$$\hat{E}^t J \hat{E} = 0 ;$$

therefore,

$$P = \hat{E}^t(-\Lambda + pU)\hat{E}.$$

Or

$$\boxed{P = -\sum_{k=1}^{n} \lambda_k \hat{E}_k^2 + \frac{1}{2}p \sum_{k=1}^{n} \hat{E}_k^2}$$

V. Stability of R-C Networks with Periodically Variable Elements.

A. Unexcited network

$$0 = (G + pC)E_\sigma$$

$$E_\sigma = N\hat{E}_\sigma$$

$$\hat{I} = N^t I = 0$$

B. N will be periodic, as will be λ_k.

C. Suppose

$$E_\sigma = H_\sigma e^{s_\sigma t},$$

where H_σ is periodic; then

$$\hat{E}_\sigma = \hat{H}_\sigma e^{s_\sigma t},$$

where

$$H_\sigma = N\hat{H}_\sigma .$$

Define

$$F_\sigma = \hat{H}_\sigma e^{\Theta},$$

where θ is a scalar and is periodic in time;

then

$$\hat{E}_\sigma = F_\sigma e^{s_\sigma t - \theta} .$$

The equation governing the unexcited network is

$$0 = (-\Lambda + pU + J)F_\sigma e^{s_\sigma t - \theta} .$$

But

$$pF_\sigma e^{s_\sigma t - \theta} = \left[(s_\sigma - \dot{\theta})F_\sigma + \dot{F}_\sigma\right] e^{s_\sigma t - \theta} .$$

Therefore,

$$0 = \left[(s_\sigma - \theta)U - \Lambda + p + J\right] F_\sigma .$$

One can premultiply this equation by F_σ^t to obtain

$$0 = F_\sigma^t \left[(s_\sigma - \dot{\theta})U - \Lambda + pU + J\right] F_\sigma .$$

Since

$$F_\sigma^t JF_\sigma = 0,$$

this becomes

$$0 = \sum_{k=1}^{n} (s_\sigma - \theta - \lambda_k)F_k^2 + \frac{1}{2}p \sum_{k=1}^{n} F_k^2$$

Everything is periodic, so one can take the average; e.g.

$$\text{ave } \dot{X} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \dot{X}\, dt = \frac{X(t_2) - X(t_1)}{t_2 - t_1} .$$

Over one period then

$$\text{ave } \frac{1}{2}p \sum_{k=1}^{n} F_k^2 = 0 .$$

If F_k is bounded, but not necessarily periodic, this average can be taken over a long interval; then

$$\text{ave } \frac{1}{2}p \sum_{k=1}^{n} F_k^2 \longrightarrow 0 .$$

Finally one obtains

$$0 = \text{ave}\left[\sum_{k=1}^{n}(s_\sigma - \dot{\theta} - \lambda_k)F_k^{\,2}\right],$$

where the λ_k's are the roots of

$$\det\left[\lambda C + \frac{1}{2}\dot{C} + \dot{G}\right] = 0 .$$

Consider the instantaneous minimum and maximum eigenvalues

$$\lambda_{min} = \min\left\{\lambda_1, \ldots, \lambda_n\right\}$$

and

$$\lambda_{max} = \max\left\{\lambda_1, \ldots, \lambda_n\right\},$$

then from the above (by adjusting θ so that

$$\lambda_{min} + \theta_1 = \text{const.} = \text{ave}\ \lambda_{min}$$

and

$$\lambda_{max} + \theta_2 = \text{const.} = \text{ave}\ \lambda_{max}$$

in each instance) one obtains

$$\underline{\underline{\text{ave}\ \lambda_{min} \leq s_\sigma \leq \text{ave}\ \lambda_{max}}} .$$

If

$$s_\sigma > \dot{\theta} + \lambda_{max}\ , \text{ for all time,}$$

then

$$0 = \text{ave}\left[\sum_{k=1}^{n}(s_\sigma - \dot{\theta} - \lambda_k)F_k^{\,2}\right]$$

would be impossible (similarly for $s_\sigma < \theta + \lambda_{min}$).

MEVV:mt
5-14-63

UNIVERSITY OF CALIFORNIA
Electrical Engineering

Dr. S. Darlington Spring 1963

Lecture 4: Stability of Time-Varying, Two-Element-Kind Networks.

I. Complex basis functions.

For a fixed network the basis functions take the form

$$e^{s_\sigma t},\ te^{s_\sigma t},\ t^2 e^{s_\sigma t},\ \ldots$$

In the non-degenerate case, for time-varying networks, one might have

$$H_\sigma e^{s_\sigma t}$$

or

$$(H_{\sigma_1} + H_{\sigma_2} t)\, e^{s_\sigma t},$$

where H_σ, H_{σ_1} are periodic. Could use complex notation or let

$$s_\sigma = (s_\sigma + j\omega_\sigma t),$$

then

$$H_\sigma^* e^{(s_\sigma + j\omega_\sigma)t}$$

would be "basis functions". These would represent the actual basis functions:

$$H_{\sigma_{real}} e^{s_\sigma t}$$

$$H_{\sigma_{im}} e^{s_\sigma t},$$

where $H_{\sigma_{real}}$ and $H_{\sigma_{im}}$ are not necessarily periodic but are still bounded for large t. Hence, F_σ is still bounded.

II. Second-order example.

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$F_\sigma = \begin{pmatrix} F_{\sigma_1} \\ F_{\sigma_2} \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & J_{12} \\ -J_{12} & 0 \end{pmatrix}$$

$$\therefore\ 0 = (s_\sigma - \dot{\theta} - \lambda_1)F_{\sigma_1} + \dot{F}_{\sigma_1} + J_{12}F_{\sigma_2}$$

and

$$0 = (s_\sigma - \dot{\theta} - \lambda_2)F_{\sigma_2} + \dot{F}_{\sigma_2} - J_{12}F_{\sigma_1}$$

$$0 = \text{ave}\left[(s_\sigma - \dot{\theta} - \lambda_1)F_{\sigma_1}^2 + (s_\sigma - \dot{\theta} - \lambda_2)F_{\sigma_2}^2\right] .$$

Suppose

$$s_\sigma = \lambda_1 + \dot{\theta}$$

and

$$\lambda_1 > \lambda_2 ,$$

then

$$0 = \text{ave}\left[(\lambda_1 - \lambda_2)F_{\sigma_2}^2\right] .$$

Since

$$\lambda_1 - \lambda_2 > 0 ,$$

$$F_{\sigma_2} \equiv 0 ;$$

then

$$0 = - J_{12}F_{\sigma_1}$$

implies

$$F_{\sigma_1} = 0 .$$

But this is an incompatibility. Thus,

$$s_\sigma \neq \lambda_1 + \dot{\theta} .$$

III. Analysis of time-varying R-C network on mesh basis.

$$E = (K + Rp)Q$$

$$K \equiv C^{-1}$$

$$E = \left[K - \frac{1}{2}\dot{R} + \frac{1}{2}(Rp + pR) \right] Q$$

Through the equivalent of a frequency transformation, the mesh analysis of R-L networks is equivalent to the nodal analysis of G-C networks, etc.

IV. Time-varying L-C networks.

Let

$$E_k = p\,\Phi_k$$

(Φ_k is the flux); then the network equations become

$$I = (S + pCp)\,\Phi \ .$$

where

$$S \equiv L^{-1} \ .$$

The power is

$$P = E^t I = \dot{\Phi}^t I \ ;$$

hence,

$$P = -\frac{1}{2}\Phi^t \dot{S}\,\Phi + \frac{1}{2}\dot{\Phi}^t C\,\dot{\Phi} + \frac{1}{2}p(\Phi^t S\,\Phi) + \frac{1}{2}p(\dot{\Phi} C\,\dot{\Phi}).$$

For stability bounds, transform the power relation (compare to R-C case).

$$I = 0 \Longrightarrow P = 0 \ .$$

Basis functions:

$$\Phi = \Phi_\sigma = H_\sigma e^{s_\sigma t} \quad \text{(neglect degenerate case),}$$

or

$$\Phi_\sigma = F_\sigma e^{s_\sigma t + \theta}$$

$$\dot{\Phi}_\sigma = p\Phi_\sigma = \left[\dot{F}_\sigma + (s_\sigma + \dot{\theta}) \right] e^{s_\sigma t + \theta}$$
$$= F_\sigma' e^{s_\sigma t + \theta} \ .$$

Substituting these in power relation,

$$0 = F_\sigma^t \left[(s_\sigma + \dot{\theta})S - \frac{1}{2}\dot{S} \right] F_\sigma + F_\sigma'^t \left[(s_\sigma + \dot{\theta})C + \frac{1}{2}\dot{C} \right] F_\sigma'$$
$$+ \frac{1}{2}pF^t S F_\sigma + \frac{1}{2}pF_\sigma'^t C F_\sigma' \ .$$

As before, one obtains

$$0 = \text{ave}\left\{ F_\sigma^t \left[(s_\sigma + \dot{\theta})S - \frac{1}{2}\dot{S}\right] F_\sigma + F_\sigma'^t \left[(s_\sigma + \theta)C + \frac{1}{2}\dot{C}\right] F_\sigma' \right\}.$$

This expression is still not in diagonal form; hence, let

$$F_\sigma = N\hat{F}_\sigma ,$$

and pick N so that

$$NSN^t = U$$

and

$$-\frac{1}{2} N\dot{S}N^t = -\Lambda \quad .$$

Furthermore, introduce still another transformation

$$F_\sigma' = N'\hat{F}_\sigma'$$

so that

$$N'CN'^t = U$$

and

$$\frac{1}{2} N'CN'^t = -\Lambda \quad .$$

Hence,

$$0 = \text{ave}\left\{ \sum_{k=1}^{n} (s_\sigma + \dot{\theta} - \lambda_k)F_k^2 + \sum_{k=1}^{n} (s_\sigma + \theta - \lambda_k')F_k^2 \right\}.$$

Again $(s_\sigma + \dot{\theta} - \lambda_k)$ and $(s_\sigma + \dot{\theta} - \lambda_k')$ cannot be positive or negative for all k and t.

<u>Theorem:</u>

$$\text{ave}\ \lambda_{min} \leq s_\sigma \leq \text{ave}\ \lambda_{max} ,$$

where λ_{min} (λ_{max}) is the instantaneous minimum (maximum) of the eigenvalues in the combined set

$$\{\lambda_1, \ldots, \lambda_n\}$$

and

$$\{\lambda_1', \ldots, \lambda_n'\} \quad .$$

These eigenvalues are obtained from the equations

$$\det\left[\lambda S - \frac{1}{2}\dot{S}\right] = 0$$

and

$$\det\left[\lambda' C + \frac{1}{2}\dot{C}\right] = 0 .$$

For a fixed L-C network,

$$\dot{S} = \dot{C} = 0 ;$$

hence,

$$\lambda_k = \lambda_k' = 0 \text{ , for all k.}$$

This is the well-known property of no damping for L-C networks.

V. R-L-C networks.

The problem here is that the resistive effects are only associated with one kind of energy storage element. On the node basis

$$I = (S + Gp + pCp)\,\Phi$$

leads to

$$\det\left[\lambda S - \frac{1}{2}\dot{S}\right] = 0$$

and

$$\det\left[\lambda' C + \frac{1}{2}\dot{C} + G\right] = 0.$$

A similar relation arises on the mesh basis.

MEVV:mt
5-15-63

UNIVERSITY OF CALIFORNIA
Electrical Engineering

Dr. S. Darlington Spring 1963

Lecture 5: Time-Varying LC Networks.

I. A property of the basis functions of LC networks.

If

$$H_\sigma e^{(s_\sigma + j\omega_\sigma)t} \quad \text{is a basis function,}$$

then

$$\check{H}_\sigma e^{-(s_\sigma + j\omega_\sigma)t} \quad \text{is a basis function;}$$

where

H_σ and $\check{H}_\sigma$ are different periodic functions.

Proof:

The solution to the driven equation in Green's function form is

$$\Phi(t) = \int_{t_1}^{t} \phi_1(t)\, W_1(\tau)\, I(\tau)\, d\tau + \int_{t_1}^{t} \phi_2(t)\, W_2(\tau)\, I(\tau)\, d\tau .$$

$\phi_i(t)$ are basis functions.

$$\Phi(t) = \int_{t_1}^{t} \phi(t)\, W(\tau)\, I(\tau)\, d\tau$$

$$\phi = \underbrace{(\phi_1, \phi_2)}_{\leftarrow 2n \rightarrow} \; \overset{\uparrow}{\underset{\downarrow}{n}}$$

$$W = \underbrace{\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}}_{\leftarrow n \rightarrow} \; \overset{\uparrow}{\underset{\downarrow}{2n}}$$

Must show that $\Phi(t)$ is a solution to the differential equation

$$(S + pCp)\,\Phi = I .$$

$$\Phi(t) = \phi(t) \int_{t_1}^{t} W(\tau)\, I(\tau)\, d\tau$$

$$p\Phi = \left[p\phi\right] \int_{t_1}^{t} W(\tau)\, I(\tau)\, d\tau + \phi(t)\, W(t)\, I(t) .$$

But physical realizability dictates that

$$\phi(t)\, W(t) = 0 ;$$

$$\therefore\ p\Phi = \left[p\phi\right] \int_{t_1}^{t} W(\tau)\, I(\tau)\, d\tau .$$

$$pCp\Phi = pC \left\{ \left[p\phi\right] \int_{t_1}^{t} W(\tau)\, I(\tau)\, d\tau \right\}$$

$$= \left[pCp\phi\right] \int_{t_1}^{t} W(\tau)\, I(\tau)\, d\tau + C\dot{\phi}WI.$$

Hence,

$$(S + pCp)\Phi = \left[(S + pCp)\phi\right] \int_{t_1}^{t} W(\tau)\, I(\tau)\, d\tau + C\dot{\phi}WI.$$

But

$$(S + pCp)\phi = 0 ,$$

as ϕ is the solution of the homogeneous equation.

$$\therefore (S + pCp)\Phi = C\dot{\phi}WI .$$

But

$$(S + pCp)\Phi = I$$

is given; therefore it must be true that

$$C\dot{\phi}WI = I,$$

or

$$C\dot{\phi}W = U \quad \text{(the identity matrix).}$$

Succinctly stated, the conditions have become

$$\phi(t)\, W(t) = 0 ,$$

$$C(t)\, \dot{\phi}(t)\, W(t) = U ,$$

$$(S + pCp)\,\phi(t) = 0 \ .$$

From the second condition, one obtains

$$\dot{\phi}W = C^{-1} \ ,$$

or, from the first and second conditions,

$$\begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix} W = \begin{pmatrix} 0 \\ C^{-1} \end{pmatrix} \ .$$

But

$$\begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}$$

is a 2n x 2n nonsingular matrix

$$\begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 & \dot{\phi}_2 \end{pmatrix}}_{\longleftarrow\ 2n\ \longrightarrow} \begin{matrix} \uparrow \\ 2n \\ \downarrow \end{matrix} \ ;$$

therefore,

$$W = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C^{-1} \end{pmatrix}$$

We wish to examine column σ of the basis function matrix

$$(\phi)_\sigma = H_\sigma e^{p_\sigma t}$$

(where $p_\sigma = s_\sigma + j\omega_\sigma$), and column σ of

$$\dot{\phi} = H_\sigma' \, e^{p_\sigma t} \ ;$$

$$H_\sigma' = H_\sigma + p_\sigma H_\sigma \ .$$

Column σ of

$$\begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}$$

is

$$\begin{pmatrix} H_\sigma \\ H_\sigma' \end{pmatrix} e^{p_\sigma t} \ .$$

Thus,

$$\begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} H \\ H' \end{pmatrix} \begin{pmatrix} e^{p_1 t} & & 0 \\ & e^{p_2 t} & \\ 0 & & \ddots \; e^{p_n t} \end{pmatrix}$$

$$= \begin{pmatrix} H \\ H' \end{pmatrix} D_e ,$$

where $\begin{pmatrix} H \\ H' \end{pmatrix}$ is periodic and D_e is the diagonal matrix of exponentials.

Hence,

$$W = \left[\begin{pmatrix} H \\ H' \end{pmatrix} D_e \right]^{-1} \begin{pmatrix} 0 \\ C^{-1} \end{pmatrix}$$

$$W = D_e^{-1} \begin{pmatrix} H \\ H' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C^{-1} \end{pmatrix} ,$$

and

$$W^t = \left[\begin{pmatrix} H \\ H' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C^{-1} \end{pmatrix} \right]^t D_e^{-1} ,$$

$$W^t = \left[\begin{pmatrix} H \\ H' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C^{-1} \end{pmatrix} \right]^t \begin{pmatrix} e^{-p_1 t} & & 0 \\ & e^{-p_2 t} & \\ 0 & & \ddots \; e^{-p_n t} \end{pmatrix}$$

But W^t is also a solution to the given differential equation:

$$\phi W = 0 \tag{1}$$

$$C\dot{\phi} W = U \tag{2}$$

$$(S + pCp)\phi = 0 . \tag{3}$$

The second condition leads to

$$p \left[(Cp\phi) W \right] = 0 ,$$

or

$$(pCp\phi) W + (Cp\phi) \dot{W} = 0 .$$

Adding $S\phi W = 0$ to both sides of this equation yields

$$(S\phi + pCp\phi) W + (Cp\phi) \dot{W} = S\phi W = 0 .$$

But, from the third condition, this equation reduces to

$$C\dot{\phi}\dot{W} = 0 \ .$$

From the first condition, one obtains

$$\dot{\phi}W + \phi\dot{W} = 0 \ .$$

$$\therefore \ C\dot{\phi}W = - C\phi\dot{W} \ ;$$

hence, from the second condition

$$C\dot{\phi}\dot{W} = - U$$

or

$$\phi\dot{W}C = - U.$$

Differentiating the last result, one obtains

$$\dot{\phi}\dot{W}C + \phi p(\dot{W}C) = 0 \ .$$

Now let

$$\dot{\phi}\dot{W}C = \phi WF \ ;$$

therefore,

$$\phi \left[WF + p(\dot{W}C) \right] = 0 \ ,$$

which expands to

$$\phi_1 \left[W_1F + p(\dot{W}_1C) \right] + \phi_2 \left[W_2F + p(\dot{W}_2C) \right] = 0 \ .$$

Choose F such that

$$W_1F + p(\dot{W}_1C) = 0 \ ;$$

but

$$\phi_2 \neq 0 \ ;$$

therefore,

$$W_2F + p(\dot{W}_2C) = 0$$

as well. Hence,

$$WF + p\dot{W}C = 0 \ .$$

But

$$C\dot{\phi}\dot{W}C = SWC$$

$$(C\dot{\phi})(\dot{W}C) = SWC \ .$$

Moreover,

$$p\left[(Cp\phi)\dot{W}C\right] = (pCp\phi)(\dot{W}C) + (C\dot{\phi})\ p(\dot{W}C) \ .$$

Thus, premultiplying

$$WF + p(\dot{W}C) = 0$$

by $(C\dot{\phi})$ yields

$$(C\dot{\phi})(WF) + (C\dot{\phi})\ p(\dot{W}C) = 0 \ ,$$

or

$$F + p\left[(Cp\phi)\dot{W}C\right] - (pCp\phi)(\dot{W}C) = 0,$$

or

$$F + p\left[(Cp\phi)\dot{W}C\right] + (S\phi)(\dot{W}C) = 0,$$

or

$$F + p(C\dot{\phi}\dot{W}C) - S = 0 \ .$$

But, from above,

$$C\dot{\phi}\dot{W} = 0;$$

$$\therefore\ F - S = 0 \ ,$$

or

$$F = S \ .$$

Therefore,

$$WS + p(pW)C = 0 \ .$$

Or, taking the transpose, one obtains

$$SW^t + pCpW^t = 0,$$

or

$$(S + pCp)W^t = 0.$$

Thus, since W^t is a solution of the equation, the basis functions have the exponents

$$e^{-p_\sigma t}$$

as was shown above. This result could also be obtained from the classical

theory of Green's functions for a self-adjoint differential operator (ref: Courant and Hilbert):

$$S + pCp \text{ is self-adjoint .}$$

The counterpart of this theorem in fixed networks follows from

$$\frac{E}{I} = S \left[\text{even function of } S\right] .$$

$$\therefore \frac{\Phi}{I} = \left[\text{even function of } S\right] ;$$

thus, the poles occur in $\pm$ pairs (quadrantal symmetry).

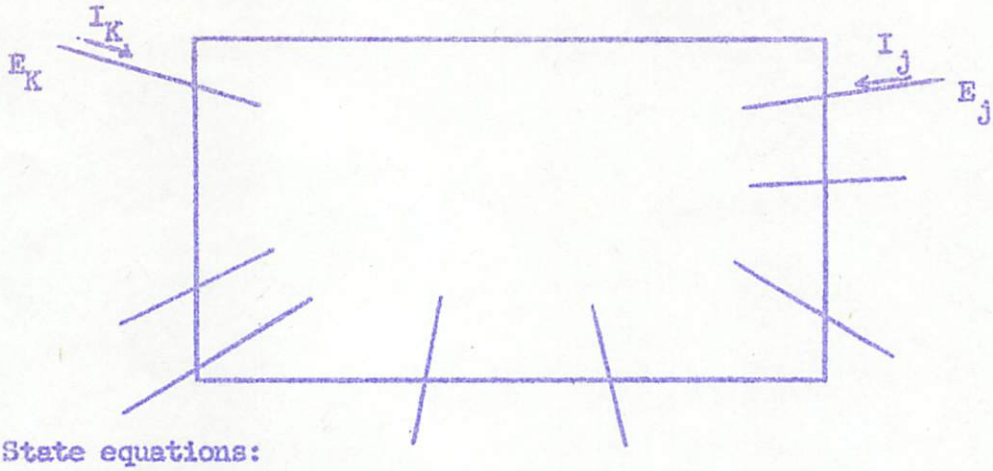
MEVV:mt
5-16-63

Dr. S. Darlington

Spring 1963

Lecture 6: Transformations, Equivalent Circuits, and Synthesis.

I. New Interpretation of Synthesis Problem.



A. State equations:

$$E(t) = \int_{-\infty}^{t} \phi(t)\, W(\tau)\, I(\tau)\, d\tau .$$

B. But the actual equation for a single transducer might specify the output $E_j(t)$ for an input $I_K(\tau)$:

$$E_j(t) = \int_{-\infty}^{t} \left[\text{row } j \text{ of } \phi(t)\right] \times \begin{bmatrix} \text{column } K \\ \text{of} \\ W(\ \ ) \end{bmatrix} I_K(\tau)\, d\tau .$$

C. The problem of filling in the entire ϕ and W matrices given but these parts is an alternate statement the classical network synthesis problem.

D. The network constraints (e.g., reciprocity) become constraints on the matrices ϕ and W.

II. Equivalent Circuits by Transformation.

A. Problem: Given a complete circuit, find its equivalents.

This can be restated as in the R-C example where

$$I = (G + pC)E ;$$

when G and C are given, one seeks new matrices G and C via a transformation

which leaves certain desired components of I and E invariant. The most general transformation might have the form

$$E = N\hat{E} ,$$

$$\hat{I} = M^t I ;$$

then

$$\hat{I} = M^t(G + pC)N\hat{E} .$$

Recall

$$M^t p = - \dot{M}^t + pM^t ;$$

then

$$\hat{I} = (\hat{G} + p\hat{C})\hat{E} ,$$

where

$$\hat{G} = M^t GN - \dot{M}^t CN$$

and

$$\hat{C} = M^t CN .$$

A restriction (for reciprocal networks) is of course that

$$\hat{G}^t = \hat{G}$$

and

$$\hat{C}^t = \hat{C} ;$$

M and N must be found which preserve this property. N might be chosen arbitrarily, and then a compatible M could be found.

B. State equation approach.

$$\hat{E} = N^{-1}E$$

$$I = (M^t)^{-1} \hat{I}$$

But the state equation is

$$E(t) = \int_{-\infty}^{t} \phi(t)\, W(\tau)\, I(\tau)\, d\tau .$$

In view of the above transformations, this relation becomes

$$\hat{E}(t) = N^{-1}(t)\, E(t)$$

$$= \int_{-\infty}^{t} \left[N^{-1}(t)\, \phi(t) \right] \left[W(\tau)(M^{t}(\tau))^{-1} \right] \hat{I}(\tau)\, d\tau \, .$$

Consequently, in the new $(\hat{E} - \hat{I})$ coordinate system the matrices

$$N^{-1}(t)\, \phi(t)$$

and

$$W(\tau) \left[M^{t}(\tau) \right]^{-1}$$

play the roles of ϕ and W in the old, respectively.

C. Special case.

Assume that the element values have periodic variations, and that all coefficients of the exponentials in the fundamental matrix are periodic:

$$\therefore\ \phi(t) = H(t)\, D_e(t)$$

where H(t) is periodic and

$$D_e(t) = \begin{pmatrix} e^{s_1 t} & & & 0 \\ & e^{s_2 t} & & \\ & & \ddots & \\ 0 & & & e^{s_n t} \end{pmatrix}$$

Moreover, as was shown last time,

$$W^{t}(\tau) = Q(\tau)\, D_e^{-1}(\tau) \, .$$

Hence, an acceptable transformation is to let

$$N(t) = H(t)$$

and

$$M(\tau) = Q(\tau) \, ;$$

then

$$\hat{E}(t) = \int_{-\infty}^{t} D_e(t)\, D_e^{-1}(\tau)\, \hat{I}(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} \begin{pmatrix} e^{s_1(t-\tau)} & & & 0 \\ & e^{s_2(t-\tau)} & & \\ & & \ddots & \\ 0 & & & e^{s_n(t-\tau)} \end{pmatrix} \times I(\tau)\, d\tau \, .$$

This equation is equivalent to the canonic differential equation

$$\hat{I} = \left(- \begin{pmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{pmatrix} + pU \right) \hat{E}.$$

For a fixed network, one diagonalizes a matrix to find the basis functions; for a time-varying network, one needs the basis functions to find the diagonalization.

D. Howitt (K-affine) transformations.

Suppose one is interested in I_1 and E_1 and nothing else (one port network). To keep I_1 and E_1 invariant under the transformation N and M must be partially specified:

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \cancel{\quad} & & & & & \\ \cancel{\quad} & & & & & \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \cancel{\quad} & & & & & \\ \cancel{\quad} & & & & & \end{pmatrix}.$$

Then

$$E_1 = \hat{E}_1,$$
$$I_1 = \hat{I}_1.$$

For two-port equivalence,

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \cancel{\quad} & & & & & \\ \cancel{\quad} & & & & & \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \cancel{\quad} & & & & & \\ \cancel{\quad} & & & & & \end{pmatrix}$$

An open question: are there enough coefficients left to meet symmetry requirements?

E. L-C case.

$$I = (S + pCp)E$$

is self-adjoint. Similarly,

$$\hat{I} = M^t(S + pCp)NE$$

must remain self-adjoint. A sufficient condition (which is probably necessary as well) is

$$M = N .$$

The most general self-adjoint, second-order differential system is

$$\hat{I} = (\hat{S} + pJ - Jp + p\hat{C}p)\hat{E} ,$$

where

$$\hat{S}^t = \hat{S} ,$$

$$\hat{C}^t = \hat{C} ,$$

$$J^t = - J ;$$

the L-C circuit gives a special subclass of this general self-adjoint differential operator where

$$J \equiv 0 .$$

III. Synthesis of Network from Basis Functions.

$$0 = (G + pC)E_\sigma$$

$$G = \begin{pmatrix} G_{11} & -G_{12} \\ -G_{12} & G_{22} \end{pmatrix}$$

$$C = U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_\sigma = \begin{pmatrix} x_1 + jy_1 \\ x_2 + jy_2 \end{pmatrix} = \begin{pmatrix} \rho_1 e^{j(\omega t + \theta_1)} \\ \rho_2 e^{j(\omega t + \theta_2)} \end{pmatrix}$$

Placing E_σ, G, and C in the above equation (and separating real and imaginary parts), one obtains

$$\begin{aligned} x_1 G_{11} \qquad - x_2 G_{12} + \dot{x}_1 &= 0 \\ y_1 G_{11} \qquad - y_2 G_{12} + \dot{y}_1 &= 0 \\ x_2 G_{22} - x_1 G_{12} + \dot{x}_2 &= 0 \\ y_2 G_{22} - y_1 G_{12} + \dot{y}_2 &= 0 \ . \end{aligned}$$

Because of the assumed symmetry of G, we have four equations in three unknowns: G_{11}, G_{12}, and G_{22}. For compatability, the determinant of the coefficients must be zero, thus restricting the possible basis functions:

$$\det \begin{pmatrix} x_1 & 0 & -x_2 & \dot{x}_1 \\ y_1 & 0 & -y_2 & \dot{y}_1 \\ 0 & x_2 & -x_1 & \dot{x}_2 \\ 0 & y_2 & -y_1 & \dot{y}_2 \end{pmatrix} = 0 \ .$$

Upon expansion of the determinant this condition becomes

$$(x_1 y_2 - y_1 x_2) \left[(x_1 \dot{y}_1 + y_1 \dot{x}_1) + (x_2 \dot{y}_2 - y_2 \dot{x}_2) \right] = 0 \ ;$$

or, in polar form,

$$\rho_1 \rho_2 \sin(\theta_2 - \theta_1) \left[\rho_1^2 (\omega + \dot{\theta}_1) + \rho_2^2 (\omega + \dot{\theta}_2) \right] = 0 \ .$$

This condition can be met if either

$$(\theta_2 - \theta_1) = 0 \text{ or } \pi \quad \text{(not interesting)}$$

or

$$\rho_1^2 (\omega + \dot{\theta}_1) + \rho_2^2 (\omega + \dot{\theta}_2) = 0 \ .$$

Define

$$\rho_\sigma^2 = \rho_1^2 + \rho_2^2 ;$$

then

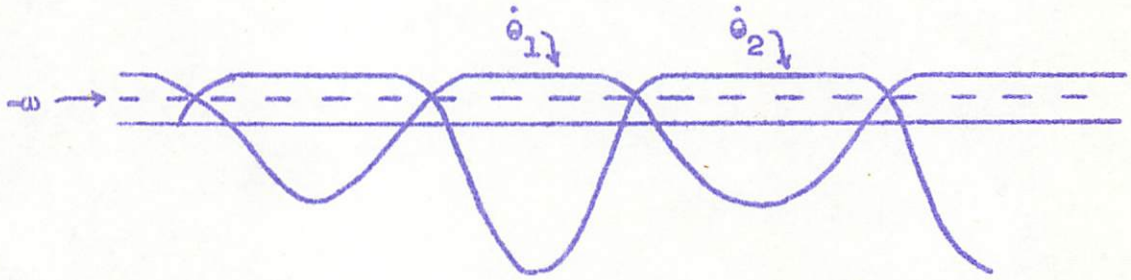
$$\dot{\theta}_1 \rho_1^2 + \dot{\theta}_2 \rho_2^2 = - \rho_\sigma^2 \omega ,$$

and

$$\rho_1^2 = \rho_\sigma^2 \frac{\dot{\theta}_2 + \omega}{\theta_2 - \theta_1} ,$$

$$\rho_2^2 = \rho_\sigma^2 \frac{\dot{\theta}_1 + \omega}{\theta_1 - \theta_2} .$$

But $(\dot{\theta}_1 - \dot{\theta}_2)$ has zero average value; therefore, $(\dot{\theta}_1 - \dot{\theta}_2)$ must have zeros. Since ρ_1^2, ρ_2^2, and ρ_σ^2 are all positive, $(\dot{\theta}_2 + \omega)$ and $(\dot{\theta}_1 + \omega)$ must both have zeros at the zeros of $(\dot{\theta}_1 - \dot{\theta}_2)$, and, furthermore, they must be of opposite sign. Hence, a typical plot of $\dot{\theta}_1$ and $\dot{\theta}_2$ might be as follows:



Note: all intersections at same level, where

$$\dot{\theta}_1 = \dot{\theta}_2 = -\omega .$$

One can also obtain damping and complex modes from time-varying R-C networks.

$$0 = (G + pU)E_\sigma$$

Consider the transformation N so that

$$0 = (N^t G N + N^t p N)\hat{E}_\sigma .$$

Fix N so that

$$N^t N = \begin{pmatrix} 1 & -K \\ -K & 1 \end{pmatrix} , \; K < 1 .$$

Then

$$0 = (N^t G N + \alpha N^t N + p N^t N)\hat{E}_\sigma \, e^{-\alpha t} .$$

MEVV:mt
5-21-63

UNIVERSITY OF CALIFORNIA
Electrical Engineering

Dr. S. Darlington

Spring, 1963

Lecture 7: Properties of Multiterminal Networks without Transformers.

References:

a) IRE Trans. on Circuit Theory, Dec. 1955; papers by
   1. Lucal
   2. Darlington

b) S. Darlington, "Some properties of multiterminal RC Networks," IRE Int. Conv. Record, Part 2, 1962.

I. Some general concepts of three terminal networks.

A three terminal network is commonly viewed as a two-port network in which the two ports share a common terminal. As in Fig. 1 the two ports of the network share terminal 3. Given the network with no further restrictions, we can form two other two-ports by using terminal 1 and 2 as a common terminal.

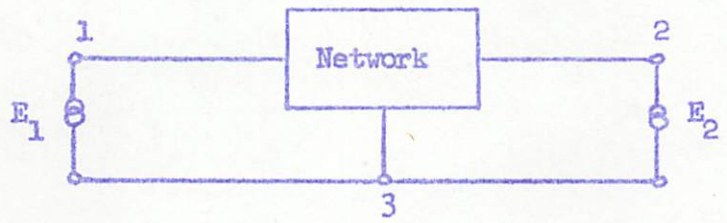


Fig. 1. A three-terminal network as a two-port.

Thus a three-terminal RC network can be represented by the block diagram in Fig. 2 below.

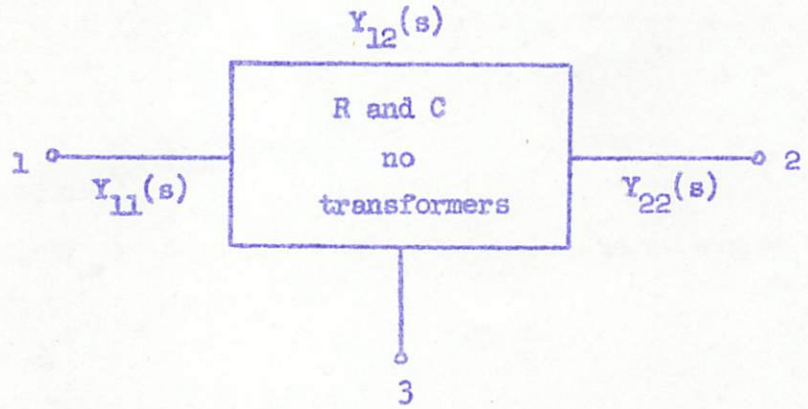


Fig. 2. A three-terminal RC network.

A matrix represention which retains the three-way symmetry is the so-called indefinite matrix Y:

$$Y = \begin{bmatrix} Y_{11} & -Y_{12} & -Y_{13} \\ -Y_{12} & Y_{22} & -Y_{23} \\ -Y_{13} & -Y_{23} & Y_{33} \end{bmatrix} .$$

Corresponding to the general three-terminal network there is an equivalent π network as shown in Fig. 3.

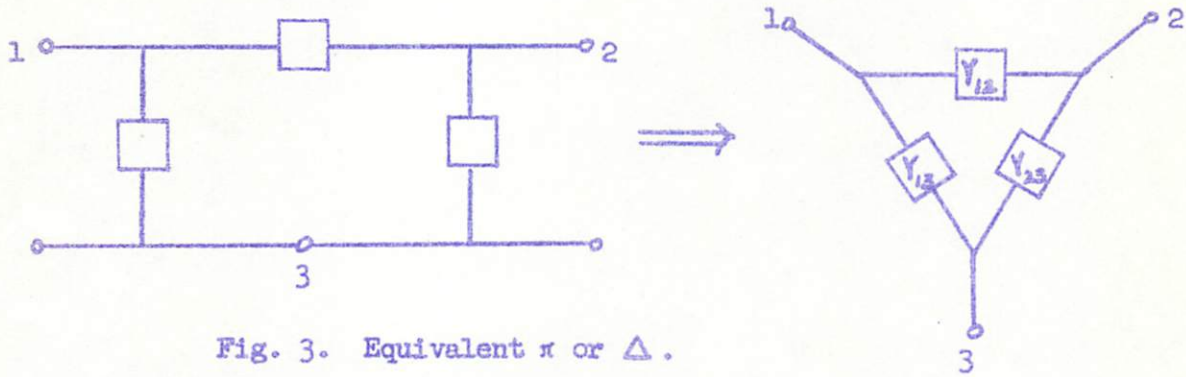


Fig. 3. Equivalent π or Δ.

It can be shown that

$$Y_{11} = Y_{12} + Y_{13}$$

$$Y_{22} = Y_{12} + Y_{23}$$

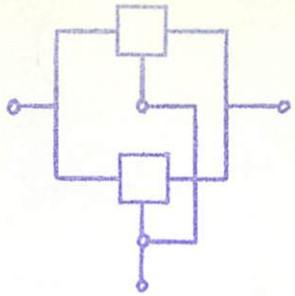
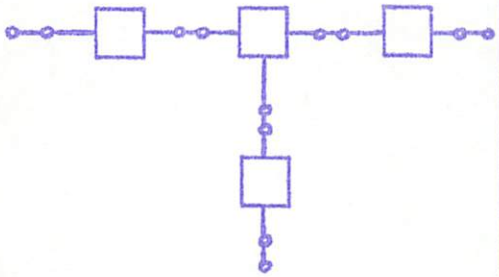
$$Y_{33} = Y_{13} + Y_{23} .$$

Thus, the external behavior of the three-terminal network may be specified in terms of Y_{12}, Y_{13}, Y_{23} (as a function of frequency) rather than Y_{11}, Y_{22}, Y_{12}.

II. Decompositions

There are two decomposition techniques, namely series and parallel as given in Table 1.

Table 1. Parallel and Series Connection of Subnetworks.

Parallel Connection of Subnetworks		The short circuit admittance of the network is equal to the sum of the short circuit admittances of subnetworks.
Series Connection of Subnetworks		The open circuit impedance of the network is equal to the sum of the open circuit impedances of subnetworks.

For a series-parallel decomposition, we separate the admittance functions or the impedance functions into parts appropriate for parallel connected or series connected subnetworks. Then we decompose each subnetwork in a similar way until the subnetworks are single branches. Here we come up to Darlington's conjecture:

<u>Darlington's conjecture:</u> We can realize any 3-set as series-parallel form (two internal nodes cases only).

A typical realization of such form is shown in Fig. 4.

In every branch we can move out one of the two components by general transformation schemes. After tranformation we have the network as in Fig. 5.

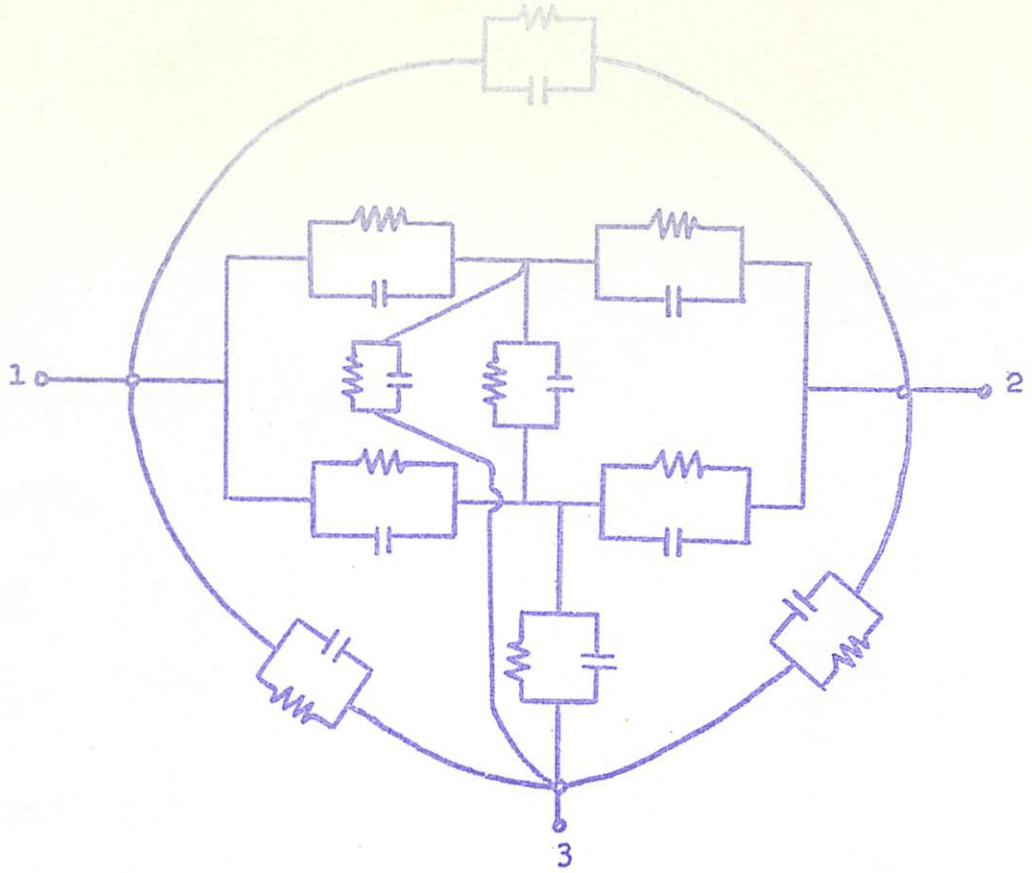
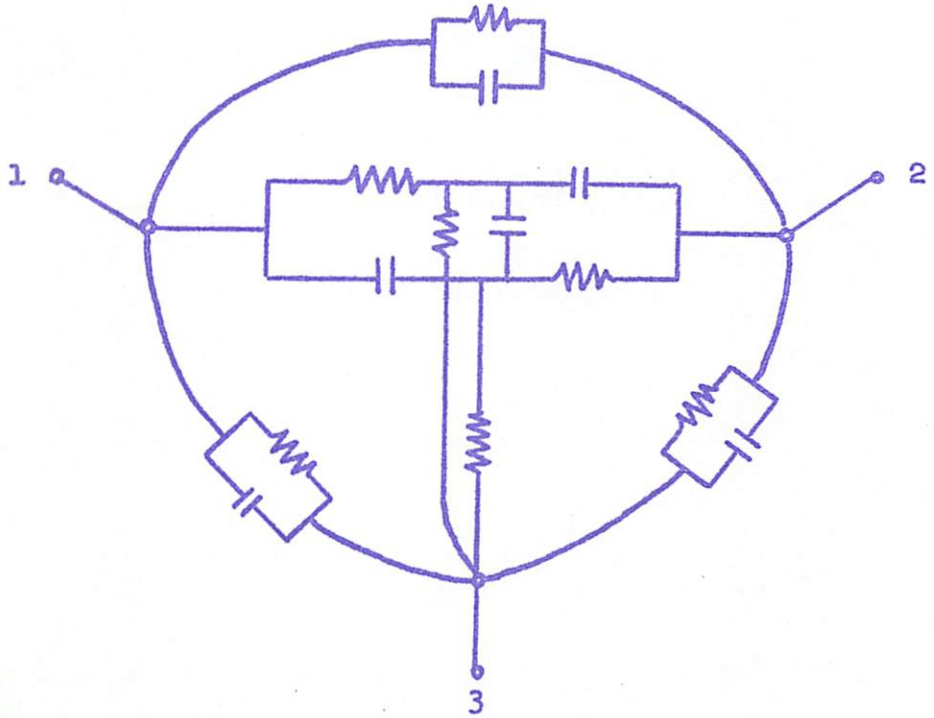
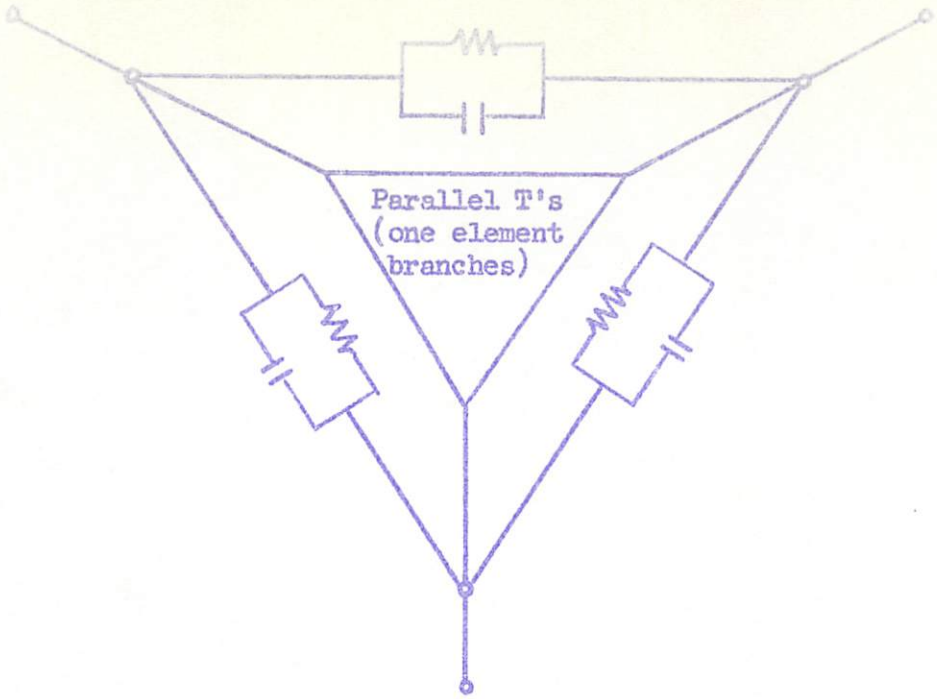


Fig. 4. A typical realization of series-parallel form.



5(a)



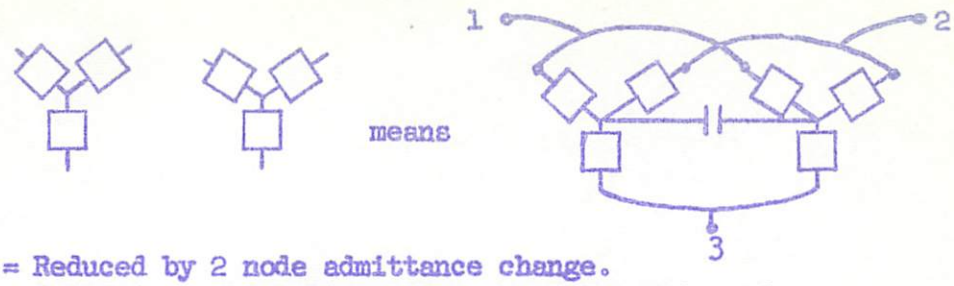
5(b)

Fig. 5. A typical network synthesis in terms of parallel-T subnetworks.

Since the outside elements of the network in Fig. 5(b) have the general nature of the inside elements, so we omit them from consideration. A complete set of thirteen RC two-element kind, 3T, 5 node networks is given in Table 2.

Table 2. RC, 3T, 5 node networks, y_{45} = capacitance
— Topologically distinct configurations.

Notation:



Symbols: A-C = Reduced by 2 node admittance change.
C-N = Reduced by elimination of a capacitor-only node.
Δ-Y = Reduced by a Δ to Y transformation.

(1)

(A-C)

(C-N)

(2)

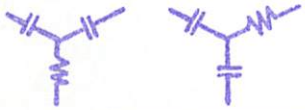
(Δ-Y)

(Δ-Y)

(C-N)

(3)

(Δ-Y)



(Δ-Y)

(C-N)

(4)

(C-N)

III. A transformation technique.

Let us consider a typical RC, 3T, 5 node network as in Fig. 6.

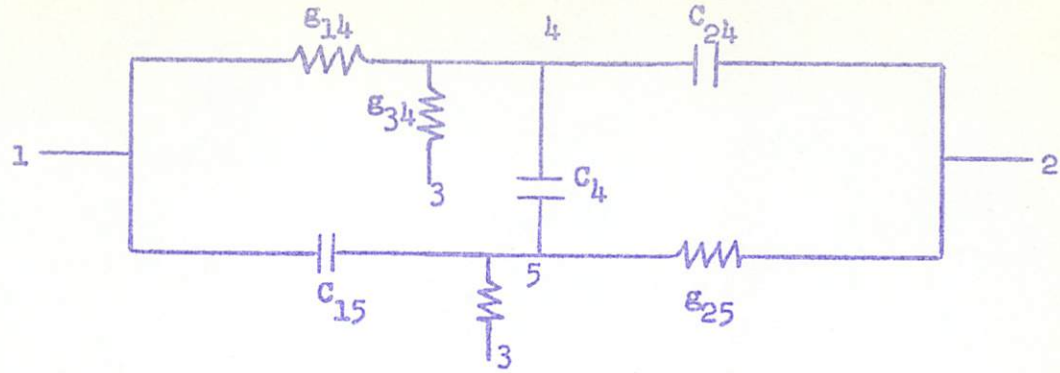


Fig. 6. A typical RC, 3T, 5 node network.

The admittance matrix of the network in Fig. 6 is

$$Y = \left[\begin{array}{ccc|cc} \epsilon_1 & 0 & 0 & -g_{14} & -C_{15} \\ 0 & \epsilon_2 & 0 & -C_{24} & -g_{25} \\ 0 & 0 & \epsilon_3 & -g_{34} & -g_{35} \\ \hline -g_{14} & -C_{24} & -g_{34} & \epsilon_4 & -C_4 \\ -C_{15} & -g_{25} & -g_{35} & -C_4 & \epsilon_5 \end{array}\right]$$

where

$$\epsilon_1 = g_{14} + C_{15}$$

$$\epsilon_2 = g_{25} + C_{24}$$

$$\epsilon_3 = g_{34} + g_{35}$$

$$\epsilon_4 = g_{44} + C_{44}$$

$$g_{44} = g_{14} + g_{34}$$

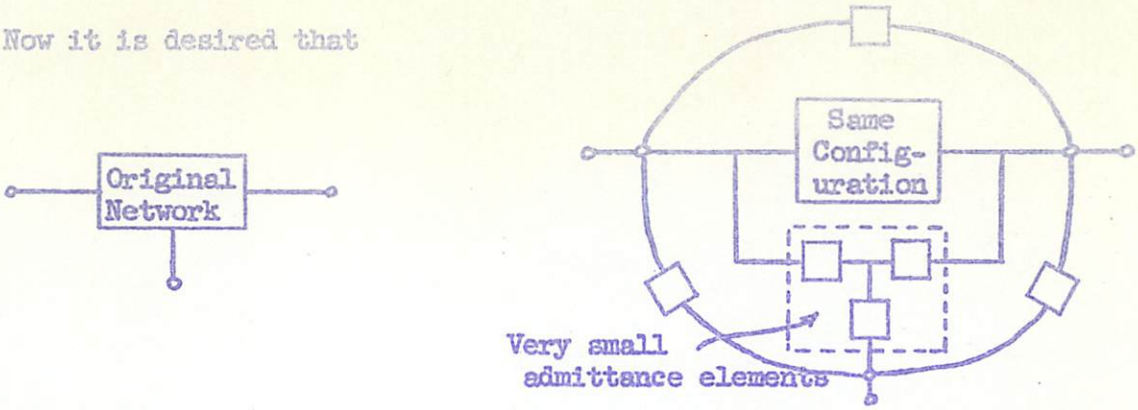
$$C_{44} = C_{24} + C_4$$

$$\epsilon_5 = g_{55} + C_{55}$$

$$g_{55} = g_{25} + g_{35}$$

$$C_{55} = C_{15} + C_4 .$$

Now it is desired that



This transformation can be obtained by adding an extra node and then making transformation which is available.

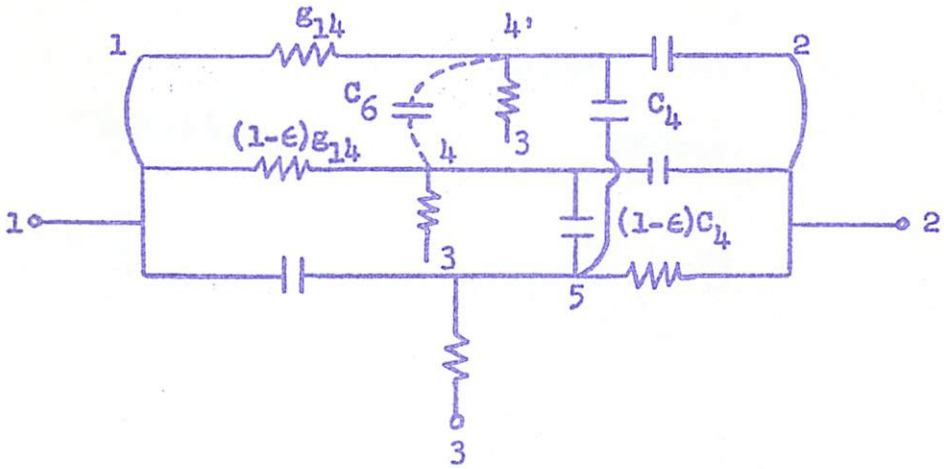


Fig. 7. The network after an extra node (node 4') is added to the original network of Fig. 6.

In Fig. 7, because the voltages at the same nodes are the same, i.e., $V_j - V_j' = 0$, so we can add extra elements between these pairs of internal nodes. The admittance matrix of the network in Fig. 7 is

$$Y = \left[\begin{array}{ccc|ccc} \epsilon_1 & 0 & 0 & -g_{14} & -C_{15} & -\epsilon g_{14} \\ 0 & \epsilon_2 & 0 & -C_{24} & -g_{25} & -\epsilon g_{24} \\ 0 & 0 & \epsilon_3 & -g_{34} & -g_{35} & -\epsilon g_{34} \\ \hline -g_{14} & -C_{24} & -g_{34} & \epsilon_4+\epsilon C_6 & -C_4 & -\epsilon C_6 \\ -C_{15} & -g_{25} & -g_{35} & -C_4 & \epsilon_5 & -\epsilon C_4 \\ -\epsilon g_{14} & -\epsilon C_{24} & -\epsilon g_{34} & -\epsilon C_6 & -\epsilon C_4 & \epsilon(\epsilon_4+C_6) \end{array}\right] .$$

Let us concentrate our attention on the submatrix (in the dotted frame) of matrix Y.

$$\begin{array}{c} \\ -r\epsilon \end{array}\overset{-r\epsilon}{\begin{vmatrix} \epsilon_4+\epsilon C_6 & -C_4 & -\epsilon C_6 \\ -C_4 & \epsilon_5 & -\epsilon C_4 \\ -\epsilon C_6 & -\epsilon C_4 & \epsilon(C_4+C_6) \end{vmatrix}}$$

$$\begin{matrix} \epsilon_4+\epsilon C_6 & -C_4 & \epsilon(rC_4-C_6) \\ -C_4 & \epsilon_5 & -\epsilon(r\epsilon_5+C_4) \\ \epsilon(rC_4-C_6) & -\epsilon(r\ {}_5+C_4) & \epsilon(\epsilon_4+rC_4)+\epsilon^2 r(r\epsilon_5+C_4) \end{matrix}$$

Letting $\epsilon(rC_4 - C_6) = 0$, we have

$$\boxed{C_6 = rC_4}.$$

Then

$$\begin{array}{c} \\ -\rho \end{array}\overset{-\rho}{\begin{vmatrix} \epsilon_4+\epsilon C_6 & -C_4 & 0 \\ -C_4 & \epsilon_5 & -\epsilon(r\epsilon_5+C_4) \\ 0 & -\epsilon(r\epsilon_5+C_4) & \epsilon(\epsilon_4+rC_4) \end{vmatrix}}$$

Similarly,

$$\begin{vmatrix} \epsilon_4+ C_6 & -C_4 & 0 \\ -C_4 & \epsilon_5+\rho\epsilon\,(r\epsilon_5+C_4) & 0 \\ 0 & 0 & \epsilon(\epsilon_4+rC_4) \end{vmatrix}$$

where $$r\epsilon_5 + C_4 + \rho(\epsilon_4 + rC_4) = 0 \qquad \text{(A)}$$

we find that

$$\rho = -r\,\frac{g_{55}}{g_{44}}. \qquad \text{(B)}$$

Substituting (B) into (A) we get

$$\boxed{C_4 + r\,\epsilon_5 = +\,r\,\frac{g_{55}}{g_{44}}(\epsilon_4 + rC_4)}$$

The resultant:

$$Y = \left[\begin{array}{ccc|ccc}
\epsilon_1 & 0 & 0 & -g_{14} & -C_{15}+\epsilon\rho(g_{14}-rC_{15}) & \epsilon M(rC_{15}-g_{14}) \\
0 & \epsilon_1 & 0 & -C_{24} & -g_{25}+\epsilon\rho(g_{24}-rg_{25}) & \epsilon M(rg_{25}-g_{24}) \\
0 & 0 & \epsilon_1 & -g_{34} & -g_{35} & \epsilon M(rg_{35}-g_{34}) \\
\hline
-g_{14} & -C_{24} & -g_{34} & \epsilon_4+\epsilon C_6 & -C_4 & 0 \\
-(C_{15}-\epsilon\rho g_{14}) & -(g_{25}-\epsilon\rho C_{24}) & -g_{35} & -C_4 & \epsilon_5 & 0 \\
-\epsilon M(g_{14}-rC_{15}) & -\epsilon M(C_{24}-rg_{25}) & -\epsilon M(g_{34}-rg_{35}) & 0 & 0 & \epsilon M^2(\epsilon_4+rC_4)
\end{array}\right]$$

Remarks:

1. r condition gives single M for g and C parts.
2. r condition gives T with a job of original network.

MEVV:mt
6-13-63