ON THE RECIPROCITY RELATIONS IN
A COMPRESSIBLE PLASMA

by

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ERL Technical Memorandum M-151
17 February 1966

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The research reported herein was supported by the Joint Services Electronics Program (Air Force Office of Scientific Research, Army Research Office, Office of Naval Research) under Grant AF-AFOSR-139-65.
ACKNOWLEDGMENT

The author wishes to thank Professor K. K. Mei for reading this communication and offering useful suggestions which make this communication possible.
In a recent paper\textsuperscript{1} H. C. Chen and D. K. Cheng proved a reciprocity relation for electromagnetic fields in a compressible plasma with high frequency sources. It was shown that the Lorentz reciprocity relation holds if $\nabla \cdot \vec{J}_e = 0$. From the well-known decomposition of the fields in a compressible plasma into electromagnetic and plasma modes, it is easy to see that such restriction on $\vec{J}_e$ results in no plasma mode, thus the fields are purely electromagnetic.\textsuperscript{2}

Therefore, the plasma only affects the dielectric constant in the Maxwell equations, the reciprocity relation of which is well known.

In this communication we wish to show that Lorentz reciprocity relation of electromagnetic fields holds in a compressible plasma in a much more general condition, namely,

(a) $\nabla \cdot \vec{J}_e \neq 0$,
(b) the ion motion is not negligible,

provided both the electromagnetic and plasma modes of the fields satisfy either the radiation condition at infinity or homogeneous boundary conditions on a closed surface.

The linearized hydrodynamic equation of motion and the equation of continuity for plasma are given by\textsuperscript{3}

$$
\omega M_e N_e \vec{V}_e = N_e \vec{E} + \nabla P_e, \quad (1)
$$

$$
\omega M_i N_i \vec{V}_i = -N_e \vec{E} + \nabla P_i, \quad (2)
$$
\[ KTN_0 \nabla \cdot \nabla V_e = i\omega P_e, \quad (3) \]
\[ KTN_0 \nabla \cdot V_i = i\omega P_i, \quad (4) \]
\[ KT = M_i U_i^2 = M_e U_e^2, \]

where the subscripts e and i denote the association of the fields with electrons and ions, respectively.

Maxwell equations are given by

\[ \nabla \times \mathbf{E} = i\omega \mu_0 \mathbf{H} - \mathbf{J}_m, \quad (5) \]
\[ \nabla \times \mathbf{H} = -i\omega \epsilon_0 \mathbf{E} + N_0 e(\nabla_i - \nabla_e) + \mathbf{J}_e \quad (6) \]

From (1) and (3), and (2) and (4), we obtain

\[ \mathbf{E} = i\omega \frac{M}{e} \mathbf{V}_e - \frac{KT}{i\omega e} \nabla \nabla \cdot \nabla_e, \quad (7) \]
\[ \mathbf{E} = -i\omega \frac{M_i}{e} \mathbf{V}_i + \frac{KT}{i\omega e} \nabla \nabla \cdot \nabla_i. \quad (8) \]

The reciprocity relations can be derived by considering two sets of sources \( \mathbf{J}_{el}, \mathbf{J}_{m1} \) and \( \mathbf{J}_{e2}, \mathbf{J}_{m2} \). Through two sets of curl equations similar to (5) and (6) we have the following:
\[ \iint_s (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{n} \, ds = \iint_v \mathbf{J}_m \cdot \mathbf{H}_2 - \mathbf{J}_m \cdot \mathbf{H}_1 + \mathbf{E}_2 \cdot \mathbf{J}_e - \mathbf{E}_1 \cdot \mathbf{J}_e \, dv \]

Let \( R_1 \) be the second integral on the right hand side of (9). Substituting (7) or (8) into \( R_1 \), we can transform it into a surface integral.

\[ \iint_s \left( \mathbf{E}_1 \cdot \mathbf{V}_e - \mathbf{E}_2 \cdot \mathbf{V}_e - \mathbf{E}_1 \cdot \mathbf{V}_i + \mathbf{E}_2 \cdot \mathbf{V}_i \right) \, dv \]  

For the unbounded plasma, we decompose the fields into E-M mode and plasma mode \(^4\) with

\[ \mathbf{E} = \mathbf{E}_o + \mathbf{E}_p, \quad \mathbf{H} = \mathbf{H}_o, \]

\[ \mathbf{V}_e = \mathbf{V}_{oe} + \mathbf{V}_{pe}, \quad \mathbf{V}_i = \mathbf{V}_{oi} + \mathbf{V}_{pi} \]

and satisfy the following sets of equations.

**E-M mode:**

\[ \nabla \times \mathbf{H}_o = -i\omega \varepsilon \mathbf{E}_o + \mathbf{J}_e, \quad \varepsilon = 1 - \frac{\omega^2_{pe}}{\omega^2} - \frac{\omega^2_{pi}}{\omega^2} \]  

\[ \nabla \times \mathbf{E}_o = i\omega \mu_0 \mathbf{H}_o - \mathbf{J}_m \]  

\[ i\omega M_e \mathbf{V}_{oe} = e\mathbf{E}_o \]
\[ i\omega M_i \nabla_{oi} = -eE_o. \]  

(14)

Plasma mode:

\[
[\nabla] = \begin{bmatrix} \nabla_{pe} \\ \nabla_{pi} \end{bmatrix} = -\frac{i\omega}{KTN_o} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} P_e \\ P_i \end{bmatrix} = -i \frac{\omega}{KTN_o} A \nabla [p],
\]

(15)

where

\[
A_{11} = \frac{\omega^2 - \omega_{pi} U^2}{\omega^4 \epsilon}, \quad A_{22} = \frac{(\omega^2 - \omega_{pe}^2) U_i^2}{\omega^4 \epsilon},
\]

\[
A_{12} = A_{21} = -\frac{\omega^2 \rho U}{\omega^4 \epsilon}.
\]

\[
\bar{E}_p = \frac{1}{\omega^2 \epsilon N_o e} \left[ \omega_{pe} \nabla P_e - \omega^2 \rho \nabla P_i \right],
\]

(16)

\[
A \nabla^2 [p] + [p] = \frac{iU^2 e N_o e \nabla \cdot \mathbf{J}_e}{\omega^3 \epsilon_0 \epsilon} \left[ \frac{1}{M_e/M_i} \right].
\]

(17)

Take the divergence of (15) and combine with (17) to eliminate the Laplacian term. Since \( \mathbf{J}_e \) is confined in a finite region, we obtain at infinity,

\[
\nabla \cdot [\nabla] = -\frac{i\omega}{KTN_o} A \nabla^2 [p] = \frac{i\omega}{KTN_o} [p].
\]

(18)
Equation (17) is a coupled wave equation of \( P_e \) and \( P_i \) through a real symmetrical matrix \( A \), which can be diagonalized by an orthonormal matrix \( T \) composed of the eigen vectors of \( A^{-1} \).

\[
T^{-1} A^{-1} T = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},
\]

where \( k_1 \) and \( k_2 \) are the eigen values of \( A^{-1} \).

We define a new vector \([ \hat{p} ]\) by

\[
\]

Thus, (17) can be written in the following form:

\[
\nabla^2 [\hat{p}] + T^{-1} A^{-1} T[\hat{p}] = \frac{i U^2 e N_0 e \nabla \cdot \bar{J}_e}{\omega^3 \varepsilon_o \varepsilon} T^{-1} A^{-1} \begin{pmatrix} -1 \\ -M_e/M_L \end{pmatrix} .
\]

Therefore,

\[
\begin{bmatrix}
\nabla^2 P_a + k_1^2 P_a \\
\nabla^2 P_b + k_2^2 P_b
\end{bmatrix} = \frac{i U^2 e N_0 e \nabla \cdot \bar{J}_e}{\omega^3 \varepsilon_o \varepsilon} T^{-1} A^{-1} \begin{pmatrix} -1 \\ -M_e/M_L \end{pmatrix} .
\]

Equation (21) gives the uncoupled equations for \( P_a \) and \( P_b \) with excitation sources of the form \( \nabla \cdot \bar{J}_e \) on the right. If \( \nabla \cdot \bar{J}_e = 0 \),
\[ d (S_{\gamma} \psi_{\lambda}) = \mathbb{T} \quad 1 \gamma \cdot \chi \]

\[ \mathcal{Q} \quad \text{for some operator } \mathcal{Q} \]

\[ \mathcal{Q} \quad \text{for some operator } \mathcal{Q} \]

\[ (\Omega) \quad \text{for some operator } \Omega \]

\[ \mathcal{P} \quad \text{for some operator } \mathcal{P} \]

\[ \mathcal{R} \quad \text{for some operator } \mathcal{R} \]

\[ \mathcal{S} \quad \text{for some operator } \mathcal{S} \]

\[ \mathcal{T} \quad \text{for some operator } \mathcal{T} \]

\[ \mathcal{U} \quad \text{for some operator } \mathcal{U} \]

\[ \mathcal{V} \quad \text{for some operator } \mathcal{V} \]

\[ \mathcal{W} \quad \text{for some operator } \mathcal{W} \]

\[ \mathcal{X} \quad \text{for some operator } \mathcal{X} \]

\[ \mathcal{Y} \quad \text{for some operator } \mathcal{Y} \]

\[ \mathcal{Z} \quad \text{for some operator } \mathcal{Z} \]
$P_a$ and $P_b$ are identically zero. It follows from (20), (15), and (16) that no plasma mode is excited. The $\overline{E}$ and $\overline{H}$ fields are completely described by (11) and (12). Reciprocity relations of the form (32) is then obvious.

If $\nabla \cdot \overline{J}_e \neq 0$, we have the following restrictions on $P_a$ and $P_b$:

$$\lim_{r \to \infty} r P_a \to \text{finite,} \quad \lim_{r \to \infty} r P_b \to \text{finite,}$$

$$\lim_{r \to \infty} r \left[\frac{\partial P_a}{\partial n} - ik_1 P_a\right] \to 0 \quad \lim_{r \to \infty} r \left[\frac{\partial P_b}{\partial n} - ik_2 P_b\right] \to 0. \quad (23)$$

The $\nabla \cdot \overline{V}_{pe_i}$ terms can be shown by (18) and (22) to behave as $\frac{1}{r}$ at infinity. Also, (16) shows that $\overline{E}_p$ is longitudinal at infinity. This implies that the transversal components diminish at infinity faster than $\frac{1}{r}$ and can be neglected in the surface integral on the left hand side of (9). We have then

$$\int \int_{s \to \infty} (\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1) \cdot \overline{nds} = \int \int_{s \to \infty} (\overline{E}_{o1} \times \overline{H}_{o2} - \overline{E}_{o2} \times \overline{H}_{o1}) \cdot \overline{nds} = 0 \quad (24)$$

From (11) we get $\nabla \cdot \overline{V}_e = \nabla \cdot \overline{J}_e$. Take the divergence of (13) and (14) and notice that the source is confined in a finite region, we have at infinity

$$\nabla \cdot \overline{V}_{oe} = \nabla \cdot \overline{V}_{oi} = 0.$$
Equation (10) can then be rewritten as

\[ R_1 = \frac{iKTN_0}{\omega} \int s \left[ (\vec{V}_{oe1} + \vec{V}_{pe1}) \nabla \cdot \vec{V}_{pe2} - (\vec{V}_{oe2} + \vec{V}_{pe2}) \nabla \cdot \vec{V}_{pel} \\
+ (\vec{V}_{oi1} + \vec{V}_{pi1}) \nabla \cdot \vec{V}_{pi2} - (\vec{V}_{oi2} + \vec{V}_{pi2}) \nabla \cdot \vec{V}_{pi1} \right] \cdot nds. \] (25)

The \( \vec{E}_o \) field is transversal at infinity which implies that \( \vec{n} \cdot \vec{E}_o \) diminishes faster than \( \frac{1}{r} \). As a consequence of (13) and (14), we conclude that \( \vec{n} \cdot \vec{V}_{oe} \) and \( \vec{n} \cdot \vec{V}_{oi} \) also behave in the same way at infinity.

The product of \( \vec{n} \cdot \vec{V}_o \) with \( \nabla \cdot \vec{V}_p \) then diminishes faster than \( \frac{1}{r^2} \), hence its surface integral vanishes as \( r \to \infty \). Then \( R_1 \) reduces to the following form:

\[ R_1 = \frac{iKTN_0}{\omega} \int s (\vec{V}_{pel} \nabla \cdot \vec{V}_{pe2} - \vec{V}_{pe2} \nabla \cdot \vec{V}_{pel} + \vec{V}_{pi1} \nabla \cdot \vec{V}_{pi2} - \\
\vec{V}_{pi2} \nabla \cdot \vec{V}_{pi1} ) \vec{n} \] (26)

The integrand in (26) can be written in matrix form

\[ R_{IN} = (\vec{V}_{pel} \vec{V}_{pi1} \nabla \left( \begin{array}{c} \vec{V}_{pe2} \\
\vec{V}_{pi2} \end{array} \right) - (\vec{V}_{pi2} \vec{V}_{pi1} \nabla \left( \begin{array}{c} \vec{V}_{pel} \\
\vec{V}_{pi2} \end{array} \right) = \\
\vec{V}_{1}^{-t} \nabla \cdot \vec{V}_{2} - \vec{V}_{2}^{-t} \nabla \cdot \vec{V}_{1} . \] (27)

Substituting (15) and (18) into (27), we have

\[ R_{IN} = \left( \frac{i\omega}{KTN_o} \right)^2 \left[ \nabla[ p_1 ]^t A[ p_2 ] - \nabla[ p_2 ]^t A[ p_1 ] \right] . \]

In terms of \( \vec{p} \), \( R_{IN} \) can be expressed as

\[ R_{IN} = -\left( \frac{i\omega}{KTN_o} \right)^2 \left[ \nabla[ \vec{p} ]^t T^t A \nabla[ \vec{p} ] - \nabla[ \vec{p} ]^t T^t A \nabla[ \vec{p} ] \right] . \] (28)
Notice that $T = T^{-1}$ and $T^{-1}AT = \begin{pmatrix} k_1^{-2} & 0 \\ 0 & k_2^{-2} \end{pmatrix}$. We obtain

$$R_{IN} = \frac{\omega^2}{(KTN_0)^2} (k_1^{-2} P_{2a} \nabla P_{1a} - k_1^{-2} P_{1a} \nabla P_{2a} + k_2^{-2} P_{2b} \nabla P_{1b} - k_2^{-2} P_{1b} \nabla P_{2b}) \quad (29)$$

Substituting (29) back into (26), we have

$$R_1 = \frac{i\omega}{KTN_0} \int_s (k_1^{-2} P_{2a} \frac{\partial P_{1a}}{\partial n} - k_1^{-2} P_{1a} \frac{\partial P_{2a}}{\partial n} + k_2^{-2} P_{2b} \frac{\partial P_{1b}}{\partial n} - k_2^{-2} P_{1b} \frac{\partial P_{2b}}{\partial n}) \, ds \quad (30)$$

Relations (22) and (23) imply

$$\frac{\partial P_a}{\partial n} = ik_1 P_a + O\left(\frac{1}{r^{1+\alpha}}\right), \quad \alpha > 0, \quad (31)$$

$P_a$ diminishes as $\frac{1}{r}$ at infinity.

Substitute (31) into (30). The integration involving a part then becomes

$$\int_s \left( P_{2a} \frac{\partial P_{1a}}{\partial n} - P_{1a} \frac{\partial P_{2a}}{\partial n} \right) \, ds = \int_s \left( P_{2a} 0 \left(\frac{1}{r^{1+\alpha}}\right) - P_{1a} 0 \left(\frac{1}{r^{1+\alpha}}\right) \right) \, ds$$

This integration is zero because $S$ is of the order $r^2$, but $P_a 0 \left(\frac{1}{r^{1+\alpha}}\right)$...
is of the order \( \frac{1}{r^{1+\alpha}} \). It follows that the integration of the b part vanishes likewise. The reciprocity relation

\[
\iiint_V (E_1 \cdot J_2 - E_2 \cdot J_1) dv = \iiint_V (H_2 \cdot \mathcal{J}_1 - H_1 \cdot \mathcal{J}_2) dv \quad (32)
\]

holds.

If the volume is bounded by a nonpenetrable perfect conductor with homogeneous boundary conditions \( \vec{n} \cdot \nabla_e = \vec{n} \cdot \nabla_i = 0 \) and \( \vec{n} \times \vec{E} = 0 \) at the surface, it is easy to show through (9) and (10) that (32) is valid.
REFERENCES


