A STUDY OF LIMITATIONS ON THE CHARACTERISTICS OF NETWORK FUNCTIONS

by

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ABSTRACT

In this report the general form of the limitations imposed on driving point and transfer immittances when some part of the input circuit of a network is fixed is investigated. The limitations are found to be bounds on the real and imaginary parts of the functions at points in the right half plane. At certain specific frequencies determined by the fixed part of the network, the values of the immittance function and some of its derivatives are fixed.

These limitations on the values of immittance functions at points in the right half plane are then expressed in terms of integrals of the function along the imaginary axis by means of Cauchy's integral formula.

The limitations on integrals are utilized to find limitations on the characteristics of ideal immittance functions. Several examples are given to illustrate the use of these limitations.
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I. INTRODUCTION

Electronic circuits such as filters or amplifiers invariably contain parasitic elements such as interelectrode capacitance or lead inductance. It is well known that these parasitic elements impose certain limitations on the overall characteristics of the circuit. Bode and others have investigated the limitations imposed by a single shunt capacitance on network response functions. Fano has considered limitations on the reflection coefficient of a given passive impedance. However the general form of network limitations has not been investigated. In this report we consider the general limitations imposed on the response functions of linear, lumped, time-invariant networks by fixed elements which the network is required to contain. We first consider limitations on driving point and transfer immittance functions which are imposed by fixing part of the input circuit of the network. Next limitations on integrals of these immittance functions are considered. Finally some examples are given in which limitations on the bandwidth and magnitude of ideal transfer functions are found.

II. LIMITATIONS ON DRIVING POINT AND TRANSFER IMMITTANCE FUNCTIONS

Previous investigations of this subject have been concerned almost entirely with the effects of a single shunt capacitance. In this section the general restrictions imposed on driving point and transfer immittance functions by fixed input circuits of any degree of complexity have been considered.

The term "immittance" is used to indicate either impedance or admittance. A driving point immittance is an immittance function for
which the input and output are measured at the same terminal pair
while for a transfer immittance the input and output may be measured
at different terminal pairs.

The general notation \( K_{ij} \) rather than \( Z_{ij} \) or \( Y_{ij} \) has been
used to indicate an immittance function with excitation at the \( j \)
terminal pair and response at the \( i \) terminal pair whenever it was
not important to specify either impedance or admittance.

A driving point immittance function is said to be passive at a
point \( s_0 \) in the right half plane of the complex frequency variable
\( s \) if

\[
| \text{Arg} K(s_0) | \leq | \text{Arg} s_0 | \quad \ldots \quad (2:1)
\]

for

\[
| \text{Arg} s_0 | \leq \frac{\pi}{2}
\]

If this condition is satisfied in the entire right half plane then
\( K \) is said to be a positive real or "p. r." function. We shall refer
to an RLC network whose driving point immittances are p. r. as a
passive network. The transfer immittance of a passive network has
no poles in the right half plane but may have zeros anywhere in the
plane.

A driving point immittance function is said to be active at a
point \( s_0 \) in the right half plane if its value at \( s_0 \) is the negative
of some passive driving point immittance. There are no restric-
tions on the transfer function of an active network. Note that a
driving point immittance may be neither active nor passive at a
point \( s_0 \).

General Form of the Limitations on Driving Point Immittance Functions

The general limitations imposed on driving point immittance func-
tions by fixing part of the input circuit of the network take the form of
bounds on the real and imaginary parts of the immittance at points in the right half plane of the complex frequency variable.

Referring to Figure 1, \( N_a \) represents a fixed two-port network while \( N_b \) represents an arbitrary passive network. \( N \) denotes the overall network consisting of the fixed two-port \( N_a \) terminated in the passive one-port \( N_b \). \( K_1 \) is the driving point immittance of \( N \) and \( K_2 \) is the (p.r.) driving point immittance of the passive network \( N_b \). The fixed network \( N_a \) is described by a set of two-port parameters denoted by \( k_{ij}, \) \( i, j = 1, 2 \). The \( k_{ij} \)'s are used here as a general notation to indicate one of the two-port parameter sets: open circuit impedance parameters, short circuit admittance parameters or either of the hybrid parameter sets. Now for the circuit of Figure 1, the limitations imposed on \( K_1 \) by the fixed network \( N_a \) are considered.

The admittance \( K_1 \) may be expressed in terms of \( K_2 \) and one of the sets of \( k_{ij} \) parameters as

\[
K_1 = k_{11} - \frac{k_{21} k_{12}}{k_{22} + K_2}
\]  \hspace{1cm} (2.2)

or, more convenient for our purposes as

\[
K_1 = (k_{11}) \left\{ \frac{k_{11} k_{22} - k_{21} k_{12}}{k_{11}} + K_2 \right\}
\]  \hspace{1cm} (2.3)

The \( k_{ij} \) are all known functions of the complex frequency variable \( s \) since the network \( N_a \) is fixed. Then it follows that for a particular value \( s_0 \) of \( s \) the expression for \( K_1 \) is a function of \( K_2 \) only:
\[ K_1 = (a) \left\{ \frac{b + K_2}{c + K_2} \right\} \]  

(2.4)

where a, b, and c are complex constants. This equation has the form of the well known linear fractional transformation.

\( K_2 \) is the driving point immittance of a passive network and is therefore a positive real function. Then from the properties of immittances stated previously we have for \( s_o \) in the right half plane

\[ | \text{Arg } K_2 | \leq | \text{Arg } s_o | \]  

(2.5)

This represents a wedge shaped region in the right half of the \( K_2 \) plane, bounded by the lines

\[ \text{Arg } K_2 = \pm \text{Arg } s_o \]  

(2.6)

These lines are then mapped into arcs of circles in the \( K_1 \) plane by the transformation (2.4). The region in the \( K_1 \) plane bounded by these arcs then represents the allowable values of \( K_1 (s_o) \). This mapping is illustrated in Figure 2 for a particular choice of \( N_a \). Referring to Figure 2(a), \( N_a \) is conveniently represented by its open circuit impedance parameters:

\[ z_{11} = \frac{1}{s + 1} = z_{21} = z_{12} \]

\[ z_{22} = \frac{s + 2}{s + 1} \]

(2.7)

Consider the limitations imposed on \( Z_1 \) by \( N_a \) at a frequency \( s_o = (1 + j) \) in the right half plane as represented in Figure 2(b). \( Z_2 (s_o) \) must fall in the wedge shaped region illustrated in Figure 2(c) which is bounded by the lines

\[ \text{Arg } Z_2(s_o) = \pm \text{Arg } s_o = \pm \frac{\pi}{4} \]  

(2.8)
The allowable values of \( Z_1(s_o) \) are obtained by mapping this region onto the \( Z_1 \) plane via the transformation

\[
Z_1(s_o) = \frac{1}{(s_o + 1)} \left\{ \frac{1 + Z_2(s_o)}{\left(\frac{s_o + 2}{s_o + 1} + Z_2(s_o)\right)} \right\}
\] (2.9)

This mapping is illustrated in Figure 2(d).

Thus it has been shown that the limitations imposed on \( K_1 \) by \( N_a \) take the form of bounds on the possible values of \( K_1 \) at points in the right half of the complex frequency plane. Note that if \( k_{22}(s_o) \) is active as defined previously, then \( K_1 \) may have a pole at \( s_o \) and is therefore unbounded. Also, if \( N_a \) is lossless and passive then \( K_1 \) may have poles on the imaginary axis and is thus not bounded there.

The Invariants of Driving Point and Transfer Immittance

For certain frequencies \( s_o \) in the right half plane, the value of \( K_1 \) may be completely independent of the choice of \( N_b \). At these frequencies the input terminals of \( N_a \) are "isolated" from the output terminals of \( N_a \) and there is no signal transmitted through the fixed network to \( N_b \). This may be demonstrated as follows where \( N_a \) is assumed to be passive.

In equation 2.4 it may happen that \( b = c \), in which case the transformation becomes degenerate. In other words the entire wedge shaped region of allowable values of \( K_2(s_o) \) is mapped into a point in the \( K_1 \) plane. It is clear that such a value of \( K_1 \) is completely independent of the passive network \( N_b \). Such values of \( K_1 \) are referred to as invariants of the driving point immittance. In order to
investigate such points in more detail it is convenient to write the transformation in the form of equation 2. 2. Then note that \( b = c \) implies either

\[
(1) \quad k_{12}(s_0) k_{21}(s_0) = 0
\]

or

\[
(2) \quad k_{22}(s_0) = \infty, \quad k_{12}(s_0) k_{21}(s_0) < \infty
\]

In any case, the second term on the right hand side of equation (2. 2) vanishes at these frequencies and we have

\[
K_1(s_0) = k_{11}(s_0)
\]

Now suppose that \( k_{12} k_{21} \) has a zero of order \( m \) at \( s_0 \). Then it follows that in addition to the value of \( K_1 \) at \( s_0 \), the values of the first \((m - 1)\) derivatives of \( K_1 \) at \( s_0 \) are also independent of the passive network \( N_b \) and are given by

\[
K_1^{(n)}(s_0) = k_{11}^{(n)}(s_0), \quad n \leq (m - 1)
\]

Thus the first \( m \) terms of the Taylor series expansion of \( K_1(s) \) about the point \( s_0 \) are invariant quantities and are equal to the first \( m \) terms of the Taylor series expansion of \( k_{11}(s) \) about \( s_0 \).

If \( N_a \) is a lossless network and \( s_0 \) lies on the \( j\omega \) axis then \( k_{11} \) may have a private pole at \( s_0 \), that is, a pole which does not appear in \( k_{12} k_{21} \). In this case the invariant quantities are the coefficients of the Laurent series of \( K_1 \) about \( s_0 \) rather than the Taylor series. In the same way \( k_{22} \) may have a private pole at \( s_0 = j\omega_0 \) as indicated by equation (2. 11) in which case the first \( (m + 1) \)
terms of the series are invariant. If \( k_{22} \) does not have a private pole at \( s = j\omega_o \) then \( (k_{22} + K_2) \) may have a simple zero there, in which case only \( (m - 1) \) terms of the series are invariant.

Consider now the transfer immittance \( K_{21} \) of the overall network \( N \) of Figure 1 consisting of \( N_b \) connected in cascade with \( N_a \). \( K_{21} \) can be expressed as

\[
K_{21} = \frac{k_{21} T_b}{k_{22} + K_2}
\]  

(2.14)

\( T_b \) in this expression represents some transfer function of the network \( N_b \). This may be a transfer immittance or a dimensionless transfer ratio. Now suppose that at a point \( s_o \) in the right half plane \( k_{21} \) has a zero of order \( m \). Then it can be seen from inspection of equation 2.14 that \( K_{21} \) must likewise have a zero of at least order \( m \). The order of the zero can be increased arbitrarily since \( T_b \) can have zeros of any order anywhere in the plane but it can only be decreased by one and this only when the zero lies on the \( j\omega \) axis.

Thus the first \( m \) (or \((m-1)\) ) terms of the Taylor series expansion of \( K_{21} \) about \( s_o \) are invariant and are in fact all identically zero.

The previous statements concerning invariant quantities are true for analytic functions of immittances as well as immittances. Thus, for example, a reflection coefficient is also subject to limitations of this type. The invariant properties of reflection coefficients have been considered in detail by Fano.

III. INTEGRALS OF DRIVING POINT AND TRANSFER IMMITTANCE FUNCTIONS

It has been shown that fixing part of the input circuit of a network imposes certain restrictions on the driving point and transfer immittances of the network at specific points in the complex frequency
plane. From the point of view of the circuit designer however, it is of more interest to know how the behavior of the function along the real frequency (imaginary) axis is restricted. To this end we now consider the limitations on integrals of network functions evaluated along the imaginary axis.

Application of the Cauchy Integral Formula to Network Limitations

Cauchy's integral formula states that if $F(s)$ is analytic within and on a closed curve $C$ and if $s_o$ is any point interior to $C$ then

$$F(s_o) = \frac{1}{2\pi j} \oint_C \frac{F(s)}{s - s_o} \, ds$$  \hspace{1cm} (3.1)$$

where the integral is taken in the positive sense around $C$. This formula expresses the value of $F(s)$ at any point inside $C$ in terms of its values on $C$.

The most useful curve $C_R$ for our purposes is illustrated in Figure 3. This curve consists of a portion of the imaginary axis from $-jR$ to $+jR$ and a semicircle in the right half plane of radius $R$. The small indentations in the path along the imaginary axis are included to avoid any singularities of the integrand which may occur there.

With this choice of $C$ the value of an immittance function at any point in the right half plane can be expressed in terms of an integral along the imaginary axis. If the discussion is restricted to integrands which are such that the integral over the semicircle exists and can be evaluated as $R$ becomes indefinitely large then we
have an equation of the form

\[
\int_{-\infty}^{\infty} \frac{F(j\omega)}{j\omega - s_o} \, dj\omega + \lim_{R \to \infty} \int_{C_R} \frac{F(s)}{s - s_o} \, ds = -2\pi j F(s_o) \quad \text{(*)} \tag{3.2}
\]

Integrals involving only the real part or imaginary part of a network function are most useful. In order to obtain such integrals we can make use of the fact that on the imaginary axis, the real part of a physically realizable network function is an even function of frequency while the imaginary part is an odd function of frequency. Since the integral is over an interval of the imaginary axis which is symmetric about the origin, the integral of the odd part of the integrand vanishes. Then a function such as \( W(s)K(s) \) can be chosen as our integrand where \( W(s) \) is either an even function or an odd function which introduces a right half plane singularity at \( s_o \) and \( K(s) \) is the immittance under consideration. If \( W(s) \) is an even function, an integral of the real part of \( K(s) \) over the imaginary axis is obtained and if \( W(s) \) is odd, an integral of the imaginary part results. This may be illustrated for the example shown in Figure 2. A real part integral will be derived in terms of the possible values of \( Z_1(1+j) \). The function

\[
W(s) = \frac{1}{s^4 + 4} \quad \text{(3.3)}
\]

is chosen as an even function which introduces right half plane singularities at \( s_o = (1+j) \) and \( (1-j) \). Then integrating yields the result

\[
\int_{0}^{\infty} \frac{R_1(\omega)}{\omega^4 + 4} \, d\omega = \frac{\pi}{16} \text{ Re} \left\{ (1+j) Z_1(1+j) \right\} \quad \text{(*)} \tag{3.4}
\]

*The terms due to the paths around the j\omega axis poles go to zero.*
In this case no singularities of the integrand can occur on the imaginary axis and the value of the integral around the large semi-circle is zero. If the possible values of $Z_1(l + j)$, illustrated in Figure 2 (d), are now examined for maxima and minima, the following numerical bounds for the integral are obtained.

$$\frac{2\pi}{80} \leq \int_{0}^{\infty} \frac{R_1(\omega)}{\omega^4 + 4} d\omega \leq \frac{3\pi}{80} \quad (3.5)$$

Thus the bounds on $Z_1(l + j)$ have been expressed in terms of restrictions on the values of the real part of $Z_1(j\omega)$.

A general form for such integrals for imittance functions analytic at infinity can be obtained by choosing for the weighting function $W(s)$, the even function

$$W_1(s) = \frac{\left[ s^2 - \left( \sigma_o^2 - \omega_o^2 \right) \right]}{s^4 - 2 \left( \sigma_o^2 - \omega_o^2 \right) s^2 + \left( \sigma_o^2 + \omega_o^2 \right)^2} \quad (3.6)$$

for real part integrals and the odd function

$$W_2(s) = \frac{s \left[ s^2 - \left( \sigma_o^2 - \omega_o^2 \right) \right]}{s^4 - 2 \left( \sigma_o^2 - \omega_o^2 \right) s^2 + \left( \sigma_o^2 + \omega_o^2 \right)^2} \quad (3.7)$$

for imaginary part integrals. The resulting integral equations are

$$\int_{0}^{\infty} W_1(j\omega) \text{Re}K(j\omega)d\omega = \frac{\pi}{2} \text{Re} \left\{ \frac{K(s_o)}{s_o} \right\} - \pi \sum_{\nu} k_{\nu} W_1(j\omega_{\nu}) \quad (3.8)$$

and
where the $k_2$'s are residues at possible imaginary axis poles at $j\omega_2$ of the integrand. These equations relate the values of the real and imaginary parts of an immittance function along the imaginary axis to the value of the function at any point $s_o$ in the right half plane. It should be noted that these equations have the form of the familiar Hilbert transforms if $s_o$ is on the imaginary axis.

**Resistance and Reactance Integral Theorems**

Bode and others have investigated the limitations imposed on integrals of driving point immittance functions by a single shunt capacitance across the input terminals of the network. More generally their results have been stated for functions whose power series expansions at zero and infinite frequency are known. These results are the resistance and reactance integral theorems. If the immittance function has power series expansions at zero and infinite frequency given by

\[ K(s) = A_\infty + B_\infty s - A_1 s^2 - B_1 s^3 + \]  
\[ K(s) = A_\infty - B_\infty s - A_1 s^2 - B_1 s^3 + \]

then the integrals of the real and imaginary parts of $K(s)$ on the imaginary axis are given by

\[
\int_0^\infty [\text{Re } K(j\omega) - A_\infty] \, d\omega = -\frac{\pi}{2} B_\infty - \frac{\pi}{2} \sum \kappa_v
\]
and

\[ \int_0^\infty \frac{\text{Im} K(j\omega)}{\omega} \, d\omega = \frac{\pi}{2} \left[ A_\infty - A_0 \right] \]  

(3.13)

where the \( k_v \)'s are the residues at any possible poles of the integrand on the \( j\omega \) axis. For passive networks the residues are real and non-negative. Bode has also pointed out that integral equations involving higher ordered coefficients of the two series expansions can be derived.

In section II it was shown that fixing part of the input circuit of a network was in general equivalent to fixing the first part of the Taylor or Laurent series expansion of the function about specific points in the right half plane. These points are the right half plane transmission zeros and the private poles of the fixed part of the network. Integrals of the real or imaginary parts of the immittance function along the imaginary axis can be expressed in terms of these fixed Taylor series coefficients by means of Cauchy's integral formula. The remainder of this section is devoted to consideration of some specific real part integral formulas which may be derived. Similar formulas for the imaginary part of an immittance can be derived if desired. The functions under consideration are assumed to be analytic in the right half plane but logarithmic singularities and simple poles with positive real residues are permitted on the imaginary axis. Thus immittances and analytic functions of immittance are included.

**Integral Formulas for Complex Points**

Suppose the first \( n \) terms of the Taylor series expansion of an analytic function about a complex point \( s_o = \sigma_o + j\omega_o \) in the right
half plane are given as

$$K(s) + A_0 + A_1 (s-s_0^1) + \ldots$$  \hfill (3.14)

where the $A_k$'s are complex numbers with real and imaginary parts denoted by

$$A_k = A_{rk} + jA_{ik}$$  \hfill (3.15)

so that $2n$ quantities are specified. For each specified quantity, a real part integral can be written. The appropriate weighting functions for the $k^{th}$ real part integrals are

$$W_{1k}(s) = Ev\left\{ \frac{1}{(s-s_0)^{k+1}} + \frac{1}{(s-s_0)^{k+1}} \right\}$$  \hfill (3.16)

for $A_{rk}$ and

$$W_{2k}(s) = Ev\left\{ \frac{1}{(s-s_0)^{k+1}} - \frac{1}{(s-s_0)^{k+1}} \right\}$$  \hfill (3.17)

for $A_{ik}$ where Ev denotes "even part of". The resulting real part integral formulas are, for $k=0$.

$$\int_0^\infty W_{10}(j\omega) \text{Re}K(j\omega) \, d\omega = -2\pi A_{ro} + \pi\sigma \int_0^\infty - \frac{\pi}{2} \sum_k W_{10}(j\omega)$$  \hfill (3.18)

and
\[ \int_{0}^{\infty} \omega_{20}(j\omega) \text{Re} K(j\omega) \, d\omega = -2\pi jA_{i0} + \pi j\omega k_{\infty} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_{20}(j\omega_{\nu}) \quad (3.19) \]

where \( k_{\infty} \), and the \( k_{\nu} \)'s represent the residues in possible poles of \( K(s) \) on the imaginary axis at infinity and \( j\omega_{\nu} \) respectively. The summation is over all singularities on the finite imaginary axis.

For \( k = 1 \) the integral formulas are

\[ \int_{0}^{\infty} \omega_{11}(j\omega) \text{Re} K(j\omega) \, d\omega = -2\pi A_{r1} + \pi k_{\infty} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_{11}(j\omega_{\nu}) \quad (3.20) \]

and

\[ \int_{0}^{\infty} \omega_{21}(j\omega) \text{Re} K(j\omega) \, d\omega = -2\pi jA_{i1} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_{21}(j\omega_{\nu}) \quad (3.21) \]

For \( k \geq 2 \) the general formulas are

\[ \int_{0}^{\infty} \omega_{1k}(j\omega) \text{Re} K(j\omega) \, d\omega = -2\pi A_{rk} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_{1k}(j\omega_{\nu}) \quad (3.22) \]

and

\[ \int_{0}^{\infty} \omega_{2k}(j\omega) \text{Re} K(j\omega) \, d\omega = -2\pi jA_{i1} - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_{2k}(j\omega_{\nu}) \quad (3.23) \]

**Integral Formulas for Points on the Real Axis**

Suppose the first \( n \) terms of the Taylor series expansion of an immittance function about a point \( \sigma_0 \) on the positive real axis
are given as

\[ K(s) = A_0 + A_1 (s - \sigma_0) + \ldots \]  
(3.24)

where the \( A_k \)'s are real numbers so that for this case only \( n \) quantities are specified. Again a real part integral formula can be derived for each specified quantity. For the \( k^{th} \) coefficient the appropriate weighting function is given by

\[ W_k(s) = Ev \left\{ \frac{1}{(s - \sigma_0)^{k+1}} \right\} \]  
(3.25)

The resulting integral formulas are, for \( k = 0 \)

\[ \int_{-\infty}^{\infty} W_0(j\omega) \Re K(j\omega) d\omega = -\pi A_0 + \frac{\pi}{2} k_{\infty} \sigma_0 - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_0(j\omega_{\nu}) \]  
(3.26)

for \( k = 1 \)

\[ \int_{-\infty}^{\infty} W_1(j\omega) \Re K(j\omega) d\omega = -\pi A_1 + \frac{\pi}{2} k_{\infty} \sigma_0 - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_1(j\omega_{\nu}) \]  
(3.27)

and for \( k \geq 2 \)

\[ \int_{-\infty}^{\infty} W_k(j\omega) \Re K(j\omega) d\omega = -\pi A_k - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_k(j\omega_{\nu}) \]  
(3.28)

where as before \( k_{\infty} \) and \( k_{\nu} \) represent the residues in possible imaginary axis poles at infinity and \( j\omega_{\nu} \) respectively.
Integral Formulas for Points on the Imaginary Axis

Points on the imaginary axis cannot be dealt with in the same way as points in the right half plane since the weighting functions introduce singularities in the path of integration. This leads in general to non-convergent integrals. However if that part of the input circuit which is fixed is a lossless network then a set of real part integrals can be derived. This follows from the fact that in this case, imaginary axis transmission zeros are also zeros of the real part of the input immittance. Thus the even ordered terms among the fixed Taylor series coefficients are all imaginary while the odd ordered terms are all real. Under these conditions suppose that the first n terms of the Taylor series expansion of an immittance function about a point \( j\omega_0 \) on the imaginary axis are given as

\[
K(s) = A_0 + A_1 (s-j\omega_0) + \ldots
\]  

(3.29)

where the \( A_k \)'s are either real or imaginary numbers so that \( n \) quantities are specified. Then the appropriate weighting function for the \( k^{th} \) coefficient is

\[
W_k(s) = Ev \left\{ \frac{1}{(s-j\omega_0)^{k+1}} \right\}
\]  

(3.30)

The resulting integral formulas are, for \( k=0 \)

\[
\int_{-\infty}^{\infty} W_0(j\omega) \Re K(j\omega) \, d\omega = -\pi A_0 - j\omega_0 \frac{\pi}{2} k_\infty - \frac{\pi}{2} k_{y\nu} W_0(j\omega_\nu)
\]  

(3.31)

For \( k=1 \) the integral formula is
For $k > 2$ the general formula is

$$
\int_{0}^{\infty} W_k(j\omega) \Re K(j\omega) \, d\omega = -\pi A_k - \frac{\pi}{2} \sum_{\nu} k_{\nu} W_{k\nu}(j\omega_{\nu})
$$

Again $k_{\infty}$ is the residue at a possible pole of $K(s)$ at infinity and the $k_{\nu}$'s are the residues at any possible poles on the finite imaginary axis.

**Integral Formulas for the Points at Zero and Infinity**

Zero and infinity are points on the imaginary axis so that transmission zeros here of the fixed lossless network are also zeros of the real part of the input immittance. Thus the even ordered terms among the fixed Taylor series coefficients are all zero and the odd ordered terms are all real numbers. Then suppose the first $2n$ coefficients of the Taylor series expansion about zero are given by

$$
K(s) = A_1 s + A_3 s^3 + \ldots
$$

where the $A_k$'s are all real numbers. A real part integral formula can be derived for each non-zero coefficient. The appropriate weighting function for the $k^{th}$ coefficient is given by

$$
W_k(s) = \frac{1}{s^{k+1}}
$$
The resulting integral formula is for \( k = 1 \)

\[
\int_0^\infty W_1(j\omega) \text{Re} \ K(j\omega) \, d\omega = -\frac{\pi}{2} A_1 + \frac{\pi}{2} k_\infty - \frac{\pi}{2} \sum \kappa \nu W_1(j\nu) \tag{3.36}
\]

and for \( k>1 \) the general equation is

\[
\int_0^\infty W_k(j\omega) \text{Re} \ K(j\omega) \, d\omega = -\frac{\pi}{2} A_k - \frac{\pi}{2} \sum \kappa \nu W_k(j\nu) \tag{3.37}
\]

Suppose that the first \( 2n \) terms of the power series expansion at infinity are given as

\[
K(s) = \frac{A_1}{s} + \frac{A_3}{s^3} + \;
\tag{3.38}
\]

Then again a real part integral formula can be derived for each non-zero coefficient. The appropriate weighting function for the \( k^{th} \) coefficient is

\[
W_k(s) = s^{k - 1} \tag{3.39}
\]

and the resulting integral formulas are for all values of \( k \)

\[
\int_0^\infty W_k(j\omega) \text{Re} \ K(j\omega) \, d\omega = -\frac{\pi}{2} A_k - \frac{\pi}{2} \sum \kappa \nu W_k(j\nu) \tag{3.40}
\]

Note that for \( k=0 \) this is just Bode's resistance integral theorem with \( A_\infty \) equal to zero.
Integrals Over Paths of Finite Length

All of the integral formulas which have been derived up to this point have been over paths of infinite length. This is a consequence of utilizing Cauchy's integral formula, which requires a closed path of integration, in the derivation. In this section however, limitations on integrals of passive driving point immittance functions over paths of finite length are considered. The driving point immittance is limited, as before, by fixing part of the input circuit of the network. This problem has been considered by Spilker for the case of a single shunt capacitance.

The path of integration is restricted to be in the right half plane so that singularities of the integrand on the path are avoided. The integration is performed between conjugate points in the right half plane as illustrated in Figure 4(a) so that the value of the integral is not a complex number. Of course the value of the integral is independent of the particular right half plane path as long as the end points are fixed.

The circuit to be considered is illustrated in Figure 4(b). Here $N_a$ is a fixed passive two-port network and $N_b$ is an arbitrary passive one-port with p. r. driving point impedance $Z_2'$. $N_b$ is connected to the output of $N_a$ through an ideal transformer of turns ratio $n$. Suppose $N_a$ is described by its open circuit impedance parameters $z_{ij}$, and consider the integral of $Z_1$ given by

$$I(n) = \int_{\gamma_0}^{\gamma_0} Z_1(s) \, ds = \int_{\gamma_0}^{\gamma_0} \left\{ z_{11} - \frac{z_2 Z_{12}}{z_{22} + Z_2 n^2} \right\} \, ds$$

(3.41)
It was shown in section II that the integrand, \( Z_1 \), is bounded at each point of the path of integration and hence the integral must have both upper and lower bounds. Also it is clear that for a particular choice of \( Z_2 \) the integral is a function of \( n \) only. Therefore for each \( Z_2 \) it is reasonable to ask for what values of \( n \) the integral attains its extreme values. Differentiating yields

\[
I'(n) = \int_{s_o}^{s_0} \frac{2z_2 z_1 n Z_2}{\left[ z_2^2 + n^2 Z_2^2 \right]^2} \, ds \tag{3.42}
\]

Setting the derivative equal to zero gives \( n=0 \) and \( n=\infty \) as solutions for extreme values of the integral. This corresponds physically to terminating \( N_a \) in a short circuit or an open circuit. For any other value of \( n \), \( Z_2 \) can be chosen such that the integral is not zero, for example \( Z_2 = z_{22} \). Thus the extreme values of the integral are given by

\[
\int_{s_o}^{s_0} z_{11}(s) \, ds , \quad \int_{s_o}^{s_0} \frac{1}{y_{11}(s)} \, ds \tag{3.43}
\]

To illustrate this result consider again the example of Figure 2. For this example the extreme values of the integral of \( Z_1 \) are found to be

\[
\int_{s_o}^{s_0} \frac{ds}{y_{11}(s)} = j \ 2 \ \tan^{-1} \frac{\omega_0}{2 + \sigma_o}
\]

and
\[ \int_{s_0}^{s_0} z_{11}(s) \, ds = j2 \tan^{-1} \frac{\omega_0}{1 + \sigma_0} \]

Thus if \( s_0 \) is chosen on the imaginary axis, the integral of \( Z_1 \) is bounded by

\[ \tan^{-1} \frac{\omega_0}{2} \leq \int_{0}^{\omega} R_1(\omega) \, d\omega \leq \tan^{-1} \omega_0 \]

IV. LIMITATIONS ON THE TRANSFER FUNCTIONS OF PASSIVE TWO-PORT NETWORKS

This section is devoted to consideration of the limitations imposed by lossless two-port networks on passive transfer functions. The lossless character of the networks allows us to make use of certain simple relationships between driving point and transfer immittances which are based on considerations of power flow in the network.

Limitations on the Transfer Function of Lossless Two-Ports Terminated in a Resistance

In this section we consider the case illustrated in Figure 5 (a) in which \( N_a \) is a fixed lossless two-port network. \( N_a \) is terminated in a second lossless network \( N_b \) which is in turn terminated in a resistance \( R_0 \). Thus \( Z_2 \) may represent any p.r. driving point impedance realized in the Darlington form. The input to \( N_a \) is an ideal current source \( I_1 \) and the output of the system...
is the voltage $V_2$ measured across $R_o$. The transfer impedance $Z_{21}$ is the function whose limitations are of interest.

Since $N_a$ and $N_b$ are lossless we can equate the power delivered to the input of $N_a$ to the power dissipated in the load resistance $R_o$.

$$\text{[Power in]} = |I_1|^2 \Re Z_1 = \frac{|V_2|^2}{R_o} = \text{[Power out]}$$

Then upon dividing through by $|I_1|^2$ we obtain

$$\Re Z_1 = \frac{1}{R_o} \left| \frac{V_2}{I_1} \right|^2 = \frac{|Z_{21}|^2}{R_o}$$

Now it has been shown in section III that from the known properties of the fixed network $N_a$ we can write a set of integral equations of the form

$$\int_0^\infty W_k(j\omega) \Re Z(j\omega) \, d\omega = A_k$$

or using (4.2)

$$\int_0^\infty W_k(j\omega) \left| \frac{Z_{21}(j\omega)}{R_o} \right|^2 \, d\omega = A_k$$

If the desired form of $|Z_{21}|$ is specified then we can immediately obtain a set of equations giving the limitations on this form of magnitude function. Of course the form specified must contain the transmission zeros of $N_a$. In order that a solution of the set of equations so obtained exist, it will in general be necessary to allow the driving point immittance to be a non-minimum function, that is, to have
imaginary axis poles. The following simple example illustrates the technique outlined here.

Referring to Figure 5 (b), the fixed network $N_a$ is an LC ladder with three transmission zeros at infinity. This particular example has been chosen in order that the result might be compared with that obtained by Bode for the case of a single shunt capacitance. For this particular network, according to the results obtained in section III, we can write three equations involving integrals of the real part of the driving point impedance $Z_1$. These are

$$\int_0^\infty R_1(\omega) \, d\omega = \frac{\pi}{2C_1} - \frac{\pi}{2} \sum \nu k_\nu$$ \hspace{1cm} (4.5)$$

$$\int_0^\infty \omega^2 R_1(\omega) \, d\omega = \left(\frac{\pi}{2C_1}\right) \left(\frac{1}{LC_1}\right) - \frac{\pi}{2} \sum \nu k_\nu \omega_\nu^2$$ \hspace{1cm} (4.6)$$

$$\int_0^\infty \omega^4 R_1(\omega) \, d\omega = \left(\frac{\pi}{2C_1}\right) \left(\frac{1}{LC_1}\right)^2 \left(1 + \frac{C_1}{C_2}\right) - \frac{\pi}{2} \sum \nu k_\nu \omega_\nu^4$$ \hspace{1cm} (4.7)$$

The $k_\nu$'s represent the residues of any imaginary axis poles of $Z_1$ which may be present and the $\omega_\nu$'s represent the locations of such poles. The form of $|Z_{21}|$ can be specified as long as the transmission zeros at infinity are included. The simplest choice for $|Z_{21}|$ is a lowpass function of bandwidth $\omega_\nu$, equal to a constant $K$, in the passband and zero outside the passband. With this choice the integrals can be evaluated to give
The first of these equations is the same as the result obtained by Bode for the case of a single shunt capacitance. The additional equations represent constraints imposed by the inductance and second shunt capacitance. In order that these equations have a solution it is in general necessary that \( Z \) have at least one pole on the imaginary axis. Such a pole must lie outside the passband and since the network is passive the residue at the pole must be positive. For a pole on the imaginary axis at \( \omega_1 \), these last two conditions lead to the following inequalities:

\[
\omega_1 > \omega_0 \Rightarrow \frac{K^2}{R_o} \left[ \frac{\sqrt{C_1}}{L} \right] > \frac{3\pi}{4} \frac{\left( \frac{\omega_0 \sqrt{LC_1}}{L} \right)^2 - 1}{\left( \frac{\omega_0 \sqrt{LC_1}}{L} \right)^3} \tag{4.11}
\]

\[
k_1 > 0 \Rightarrow \frac{K^2}{R_o} \left[ \frac{\sqrt{C_1}}{L} \right] < \frac{\pi}{2} \frac{1}{\left( \frac{\omega_0 \sqrt{LC_1}}{L} \right)} \tag{4.12}
\]

If \( k_1 \) and \( \omega_1 \) are eliminated from (4.8), (4.9), and (4.10) then an equation in the following normalized variables is obtained.
This equation, depending on a parameter \( a = \frac{C_1}{C_2} \) is

\[
\left( \frac{X}{\pi} \right)^2 \left( \frac{4Y^6}{45} \right) - \left( \frac{X}{\pi} \right) \left[ \frac{Y^5}{10} - \frac{Y^3}{3} + \frac{(1 + a)}{2} \frac{Y}{4} \right] + \frac{a}{4} = 0
\]  

(4.14)

Solutions of this equation for various values of the parameter \( a \) are plotted in Figure 6 along with the constraints represented by 4.11 and 4.12.

**Limitations on the Transfer Functions of Coupling Networks with Fixed Input and Output Circuits**

In this section we consider limitations imposed on the transfer function of a passive network by fixed lossless input and output circuits. The problem is illustrated in Figure 7 (a) where \( N_a \) and \( N_b \) are fixed lossless two-port networks and \( N_c \) is an arbitrary passive coupling network. It is assumed that \( N_a \) and \( N_b \) have the same transmission zeros and private poles so that equivalent sets of real part integrals can be written for \( Z_{11} \) and \( Z_{22} \). Limitations on the transfer impedance \( Z_{21} \) are to be considered. The solution to this problem for the case of shunt capacitance at input and output has been given by Bode. This development is intended to show that Bode's technique may be extended to problems involving more complex fixed lossless networks.

Since the three networks are connected in cascade, any transmission zeros of \( N_a \) or \( N_b \) are zeros of \( Z_{21} \) as shown in section II. Thus, because of the assumed equivalence of \( N_a \) and \( N_b \), the first \( m \) coefficients of the Taylor series expansions of \( Z_{11} \), \( Z_{22} \) and \( Z_{21} \)
are fixed at a transmission zero of order \( m \) of \( N_a \) or \( N_b \). These corresponding Taylor series coefficients can be expressed as corresponding integrals of the real parts of the three functions as discussed in section III.

\[
\int_0^\infty W(\omega) R_{11}(\omega) \, d\omega = A_{11} \tag{4.15}
\]

\[
\int_0^\infty W(\omega) R_{22}(\omega) \, d\omega = A_{22} \tag{4.16}
\]

\[
\int_0^\infty W(\omega) R_{21}(\omega) \, d\omega = 0 \tag{4.17}
\]

Now the overall network is passive and this condition can be expressed as

\[
R_{11}(\omega) > 0, \quad R_{22}(\omega) > 0, \quad \sqrt{R_{11}(\omega) R_{22}(\omega)} \geq |R_{21}(\omega)| \tag{4.18}
\]

This can be multiplied by a weighting function and integrated to give

\[
\int_0^\infty |W(\omega) R_{21}(\omega)| \, d\omega \leq \int_0^\infty \sqrt{W^2(\omega) R_{11}(\omega) R_{22}(\omega)} \, d\omega \tag{4.19}
\]

The Schwarz inequality can be applied to the integral on the right to give
Combining this with the inequality (4.19) we have

\[
\int_{0}^{\infty} |W(\omega)R_{21}(\omega)| d\omega \leq \sqrt{\int_{0}^{\infty} |W(\omega)R_{11}(\omega)| d\omega - \int_{0}^{\infty} |W(\omega)R_{22}(\omega)| d\omega} \quad (4.20)
\]

If \(W(\omega)R_{11}(\omega)\) and \(W(\omega)R_{22}(\omega)\) are non-negative then the two integrals on the right can be evaluated explicitly as \(A_{11}\) and \(A_{22}\) to give

\[
\int_{0}^{\infty} |W(\omega)R_{21}(\omega)| d\omega \leq \sqrt{A_{11}A_{22}} \quad (4.22)
\]

The integral on the left can be evaluated for the desired form of transfer function in terms of the pertinent parameters. The result is the desired limitations. The technique outlined here is illustrated by the following example.

Referring to Figure 7 (b), \(N_a\) and \(N_b\) are specified as simple L networks. This configuration corresponds to the problem considered by Bode with the added constraint of a series inductance. Because of the fixed transmission zeros at infinity a lowpass transfer function, constant in the passband, is considered. The given networks \(N_a\) and \(N_b\) lead to six integral equations.

\[
\int_{0}^{\infty} R_{11}(\omega) d\omega = \frac{\pi}{2C_1} - \frac{\pi}{2} \sum_{\nu} k_{\nu 11} \leq \frac{\pi}{2C_1} \quad (4.23)
\]
The ideal form of lowpass transfer function which is constant in the passband and which satisfies (4.27) and (4.28) is given by

\[ Z_{21}(s) = \frac{K}{\left[ 1 + \left( \frac{s}{\omega_0} \right)^2 + \frac{s}{\omega_0} \right]^n} \]  

(4.29)

where \( n \) is an odd integer greater than three and \( K \) is a real constant.

Then it is easily found that for this transfer function
From these expressions the limitations on bandwidth and impedance level $K$ are found to be

$$K^o < \frac{n^2 - 1}{n} \tan \frac{\pi}{2n} \cdot \frac{\pi}{\sqrt{2} C_1 C_2}$$  \hspace{1cm} (4.32)$$

$$K^o < \frac{2\pi}{\left[ \left( \frac{n}{n^2 - 1} \right) \cot \frac{\pi}{2n} - \left( \frac{n}{n^2 - 9} \right) \cot \frac{3\pi}{2n} \right] C_1 C_2 \sqrt{L_1 L_2}}$$  \hspace{1cm} (4.33)$$

These limitations for the normalized bandwidth and impedance level with $L_1 = L_2 = L$ and $C_1 = C_2 = C$ are plotted in Figure 8. These equations depend on the parameter $n$ but the variation with $n$ is so slight that it may be ignored for purposes of calculating the curves of Figure 8.

**Lossless Two-port Networks With Input and Output Terminated in Resistance**

In this section the problem of transmission through a fixed lossless two-port network with arbitrary passive terminations is considered. The problem is illustrated in Figure 9 (a). Here $N_a$
is the fixed lossless two-port and \( N_b \) and \( N_c \) are arbitrary lossless two-ports terminated in resistance. It is required to find the limitations on the transducer power gain of the filter. A solution to this problem for the case of a simple capacitive coupler has been given by Baerwald.\(^7\) In this development a different approach to the problem is used and the technique developed can be applied to problems involving any lossless two-port network.

It is clear that since neither \( N_b \) nor \( N_c \) is required to be finite it will be possible to obtain perfect power transmission, that is, the transducer power gain is unity, over some band(s) of frequencies. This is accomplished by constructing an image matched filter which uses \( N_a \) as the basic section. However the bandwidth of perfect power transmission and the location of the passband may be changed to some extent by adding additional elements to the basic section. Of course, the transmission zeros of \( N_a \) must always appear as transmission zeros of the overall network. Thus the pertinent problem to consider is that of limitations on the bandwidth of perfect power transmission.

The general procedure to be used is to first bisect \( N_a \) thus obtaining two networks with fixed input circuits. Since the transducer power gain is unity in the passband, the driving point immittances \( K_1 \) and \( K_2 \) (both impedances or both admittances) of the two networks must be conjugates of one another in the passband. A necessary condition for the conjugateness of \( K_1 \) and \( K_2 \) over the passband is the equality of corresponding integrals over the passband of the real parts of the two functions. A second necessary condition is that corresponding integrals over the imaginary parts of the two functions must be negatives of one another. The solution of the set of equations obtained from the application of these necessary conditions leads to limitations on the bandwidth.
The location of the bisection of \( N_a \) is only important in that a judicious choice may simplify the manipulations involved in obtaining the result. If \( N_a \) is represented in the Darlington form the bisection should be made so that equal numbers of sections representing a particular transmission zero remain on each side of the cut. In case there is an odd number of such sections then the middle section can be bisected or the cut can be made on one side of the middle section. An example which follows will illustrate the technique outlined here.

Referring to Figure 9(b), \( N_a \) is given as an LC pi network. This network has three transmission zeros at infinity. In view of the location of the fixed transmission zeros a lowpass transfer function is considered. The center section of the three is bisected by cutting the inductance as indicated in Figure 9(c). Then each half of \( N_a \) has one transmission zero and a private pole at infinity so that two integrals of the real and two integrals of the imaginary part may be written for \( Y_1 \) and the same for \( Y_2 \). The simplest solution which utilizes the available real and imaginary part areas most efficiently is obtained by letting \( Y_1 \) and \( Y_2 \) be band-limited minimum susceptive functions, that is, the real parts of the two admittances are identically zero outside the passband. Then for \( Y_1 \) the following integral equations can be written, where the irrational factors are used as described by Bode to obtain integrals of the imaginary part of the admittance over the passband

\[
\int_{\omega_0}^{\omega} G_1(\omega) \, d\omega = \frac{\pi}{L} \tag{4.34}
\]

\[
\int_{\omega_0}^{\omega} \omega^2 G_1(\omega) \, d\omega = \frac{2\pi}{L^2 (C_1 + C_x)} \tag{4.35}
\]
\[
\int_{0}^{\omega_0} \frac{\sqrt{\omega^2 - \omega_0^2}}{\omega} \ B_1(\omega) \, d\omega = \frac{\pi}{2} \left[ \frac{2}{L} - \omega_0 G_{01} \right] \quad (4.36)
\]

\[
\int_{0}^{\omega_0} \frac{\sqrt{\omega^2 - \omega_0^2}}{\omega} \ B_1(\omega) \, d\omega = \frac{\pi}{2} \left[ \frac{4}{L^2(C_1 + C_x)} - \frac{\omega_0^2}{L} \right] \quad (4.37)
\]

and similarly for \( Y_2 \):

\[
\int_{0}^{\omega_0} G_2(\omega) \, d\omega = \frac{\pi}{L} \quad (4.38)
\]

\[
\int_{0}^{\omega_0} \omega^2 G_2(\omega) \, d\omega = \frac{2\pi}{L^2(C_2 + C_y)} \quad (4.39)
\]

\[
\int_{0}^{\omega_0} \frac{\sqrt{\omega^2 - \omega_0^2}}{\omega} \ B_2(\omega) \, d\omega = \frac{\pi}{2} \left[ \frac{2}{L} - \omega_0 G_{02} \right] \quad (4.40)
\]

\[
\int_{0}^{\omega_0} \omega \sqrt{\omega^2 - \omega_0^2} \ B_2(\omega) \, d\omega = \frac{\pi}{2} \left[ \frac{4}{L^2(C_2 + C_y)} - \frac{\omega_0^2}{L} \right] \quad (4.41)
\]

Here \( C_x \) and \( C_y \) represent any additional capacitance that may be added to \( C_1 \) and \( C_2 \) as the first elements of \( N_1 \) and \( N_2 \), and \( G_{01} \) and \( G_{02} \) represent the values of the real parts of \( Y_1 \) and \( Y_2 \) at zero frequency. The solution of the four equations obtained by application of the above mentioned necessary conditions leads to the result.
where $C$ is the larger of $C_1$ and $C_2$.

V. SUMMARY AND CONCLUSIONS

It has been shown that the driving point immittance of a passive network is subject to certain definite limitations when some part of the input circuit of the network is fixed. These limitations take the form of bounds on the real and imaginary parts of the function at points in the right half of the complex frequency plane. At right half plane frequencies where the input terminals of the network are "isolated" from the variable part of the circuit, the value of the input immittances and the values of some of its derivatives are invariant. These frequencies are the "transmission zeros" of the fixed part of the network. These limitations apply to driving point and transfer immittances and to analytic functions of immittance such as reflection coefficients.

The limitations on the values of the functions and its derivatives at points in the right half plane can be related to limitations on the behavior of the function along the imaginary axis by means of Cauchy's integral formula. A number of specific formulas have been given which correspond to the various types of transmission zeros which may arise. It is shown that Bode's resistance and reactance integral theorems are special cases of such integral formulas.

Upper and lower bounds are derived for integrals of passive driving point immittance over paths of finite length in the right half plane.
The limitations on immittance functions are applied to derive formulas for restrictions on the magnitude and bandwidth of the transfer functions of passive networks when the fixed part of the network is lossless.

This study has shown that requiring part of a network to be in a particular form places definite restrictions on response functions of the network. These restrictions manifest themselves as limitations on characteristics of the network functions such as magnitude of real or imaginary parts of the function, magnitude of the function, and bandwidth of constant magnitude.
CONSTRAINTS ON DRIVING POINT IMMITTANCE

FIGURE 1
EXAMPLE OF LIMITATIONS ON $Z_1$

FIGURE 2
CURVE: $C_R$ FOR CONTOUR INTEGRATION

FIGURE 3
INTEGRATION BETWEEN CONJUGATE POINTS
(a)

TRANSFORMER COUPLING $N_a$ AND $N_b$
(b)

FIGURE 4
RESISTANCE TERMINATED LOSSLESS 2-PORT
(a)

LC Pi INPUT CIRCUIT
(b)

FIGURE 5
FIGURE 6 PLOTS OF EQUATIONS (4.11) AND (4.12)
FIXED INPUT AND OUTPUT CIRCUITS

(a)

LC INPUT AND OUTPUT CIRCUITS

(b)

FIGURE 7
FIGURE 8  PLOTS OF EQUATIONS (4.32) AND (4.33)
FIGURE 9

LOSSLESS NETWORKS WITH FIXED INPUT AND OUTPUT CIRCUITS
REFERENCES


