RESISTANCE TRIANGLE INEQUALITY

by

Leon O. Chua

Memorandum No. UCB/ERL M99/21

6 April 1999
RESISTANCE TRIANGLE INEQUALITY

by

Leon O. Chua

Memorandum No. UCB/ERL M99/21

6 April 1999

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720
Abstract

This note proves that the 3 input resistances measured across any 3 nodes of a connected circuit made of linear positive resistors satisfy the triangle inequality.

1 Formal statement of triangle inequality

Let \( N \) be any connected circuit made of 2-terminal linear positive resistors, and choose any 3 nodes \( \{1\}, \{2\}, \) and \( \{3\} \), as depicted in Fig.1(a). Connect 3 current sources \( I_i, i = 1, 2, 3 \), as shown in Fig.1(b). Let

\[
R_{ii} = \frac{V_i}{I_i \bigg|_{I_j=0, j\neq i}}, \quad i = 1, 2, 3
\]

(1)

denote the input resistance across the driving-point terminals formed by the node-pairs \( \{1\}, \{2\}, \{2\}, \{3\} \), and \( \{3\}, \{1\} \), respectively.

Theorem: Resistance Triangle Inequality

The resistances \( R_{ii}, i = 1, 2, 3 \), satisfy the following triangle inequality:
\[ R_{ii} + R_{i+1,i+1} \geq R_{i+2,i+2} \]  

\( i \in \{ \text{positive integers mod } 3 \} \).

**Proof.** Since \( N \) is connected and contains only 2-terminal linear positive resistors (\( R > 0 \)), the 3-port in Fig.1(b) can be characterized uniquely by the following resistance matrix representation [1]:

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{12} & R_{22} & R_{23} \\
R_{13} & R_{23} & R_{33}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
\]

(3)

where the 3 \( \times \) 3 open-circuit resistance matrix \( R \) is *symmetric and positive semi-definite* [2]. Since nodes 1, 2, and 3 form a loop,

\[ V_1 + V_2 + V_3 = 0 \]  

(4)

Substituting \( V_i \) from Eq.(3) into Eq.(4) and assigning \((1,0,0)\), \((0,1,0)\), and \((0,0,1)\) respectively to \((I_1, I_2, I_3)\), we obtain

\[ R_{11} + R_{12} + R_{13} = 0 \]  

(5)

\[ R_{12} + R_{22} + R_{23} = 0 \]  

(6)

\[ R_{13} + R_{23} + R_{33} = 0 \]  

(7)

Subtracting Eq.(7) from the sum of Eqs.(5) and (6), we obtain

\[ R_{11} + R_{22} = R_{33} - 2R_{12} \]  

(8)
Now since

$$R_{12} = V_1|_{I_1=I_5=0,I_7=1}$$  \hspace{1cm} (9)

it follows from the methods presented in [1] that:

$$R_{12} \leq 0$$  \hspace{1cm} (10)

Applying inequality (10) to Eq.(8), we obtain the triangle inequality (2). □

2 Remark

1. By duality, the above triangle inequality also holds for the input conductances of connected circuits made of 2-terminal linear positive resistors.

2. The resistance triangle inequality (2) is a special case of a more general results presented in [2].

Acknowledgment

I would like to thank Professor David Gale for first posing to me the “Resistance triangle inequality” as a conjecture.

References


\(^1\)The inequality (10) can also be proved directly by the same procedure as in the proof the maximum node-voltage property on page 272-273 of [1].
Figure Caption

Fig.1: (a) A connected resistor circuit $N$ with 3 arbitrarily chosen nodes $1$, $2$, and $3$. (b) Driving $N$ with 3 current sources across the node-pairs $1-2$, $2-3$, and $3-1$. 
Fig. 1 (a) A connected linear resistor circuit $N$ with 3 arbitrarily chosen nodes $\bullet$, $\bullet$, and $\bullet$.
(b) Driving $N$ with 3 current sources across the node-pairs $\bullet-\bullet$, $\bullet-\bullet$, and $\bullet-\bullet$. 

\[ R > 0 \]