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MOMENT MATCHING IN CONGRUENCE TRANSFORM

by

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1 Introduction

This report is to prove Proposition 2 in "Multipoint multiport algorithm for passive reduced-order model of interconnect networks".

Consider a system with state equations

\[ H(s)x = bu \]  
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}_1 \), \( H(s) = Ms + N \), and \( b = b_0 + b_1s \); and output functions

\[ y = c^T x \]  
where \( y \in \mathbb{R}^m \) and \( c = c_0 + c_1s \). Note that in Proposition 2 only one input is concerned, so we use \( b \) instead of \( b_j \) for simplicity.

The input admittance function is

\[ Y(s) = c^T H(s)^{-1}b \]  
Let an orthonormal congruence transform \( V \in \mathbb{R}^{n \times q} \) with \( q \leq n \) be applied to the system. Then we have

\[ \hat{H}(s)\hat{x} = \hat{b}u \]  
where \( \hat{H} = V^T H V = s\hat{M} + \hat{N} \), \( \hat{M} = V^T M V \), \( \hat{N} = V^T N V \), and \( \hat{b} = V^T b \); and

\[ y = \hat{c}^T \hat{x} \]  
with \( \hat{c}^T = c^T V \). For the transformed system, the transfer function becomes

\[ \hat{Y}(s) = \hat{c}^T \hat{H}(s)^{-1}\hat{b} \]

For a finite matching point \( s = s_q \), let \( H(s)^{-1}b \) be expanded as \( H(s)^{-1}b = \sum_{k=0}^{\infty} r_k(s_q)(s - s_q)^k \). Let \( N(s_q) = N + s_q\hat{M} \) and \( b_0(s_q) = b_0 + s_qb_1 \). Then,

\[ r_0(s_q) = N(s_q)^{-1}b_0(s_q) \]  
\[ r_1(s_q) = A(s_q)r_0(s_q) + N(s_q)^{-1}b_1 \]
where $A(s_q) = -N(s_q)^{-1}M$, and

$$r_k(s_q) = A(s_q)r_{k-1}(s_q) \quad k > 1 \quad (9)$$

For $s_q = \infty$, let $H(s)^{-1}b$ be expanded as $H(s)^{-1}b = \sum_{k=0}^{\infty} r_k(\infty)s^{-k}$. Let $B = -M^{-1}N$. Then,

$$r_0(\infty) = M^{-1}b_1 \quad (10)$$

$$r_1(\infty) = Br_0(\infty) + M^{-1}b_0 \quad (11)$$

and

$$r_k(\infty) = Br_{k-1}(\infty) \quad k > 1 \quad (12)$$

For the reduced order model, we have similar equations as Eqs (7) - (12) with $M$, $N$, $A$, $B$, $b_0$, $b_1$ and $r$ replaced by $\hat{M}$, $\hat{N}$, $\hat{A}$, $\hat{B}$, $\hat{b}_0$, $\hat{b}_1$ and $\hat{r}$, respectively.

Now let us consider the moments of $Y(s)$. Let $c = c_0 + s_1$ where $c_0^T = [G_p, A_Lp]$ and $c_1^T = C_p$. For finite $s_q$, let $Y(s) = \sum_{k=0}^{\infty} m^k(s - s_q)^k$ where $m^k = [m_0^k, \ldots, m_m^k]^T$ is a vector of $m$ $k$-th order moments. Then,

$$m^0 = c_0^T r_0 \quad (13)$$

and

$$m^k = c_0^T r_k + c_1^T r_{k-1} \quad k > 0 \quad (14)$$

For $s_q = \infty$, when $c_1 = 0$, let $Y(s) = \sum_{k=0}^{\infty} m^k(\infty)s^{-k}$. Then

$$m^k = c_0^T r_k \quad k \geq 0 \quad (15)$$

When $c_1 \neq 0$, $Y(s) = m^{-1}s + \sum_{k=0}^{\infty} m^k(\infty)s^{-k}$. Then,

$$m^{-1} = c_1^T r_0 \quad (16)$$

and

$$m^k(\infty) = c_0^T r_k + c_1^T r_{k+1} \quad k \geq 0 \quad (17)$$

2 Lemmas

We first prove some lemmas, where matrix $V$ is supposed to be orthonormal.

Lemma 1
If vector $u \in \text{span}(V)$, then $VV^T u = u$.

Proof.
Vector $u$ can be expressed as $u = Va$. Then

$$VV^T u = VV^TVa = V(V^TV)a$$

As $V$ is orthonormal, $V^TV = I$, and $V(V^TV)a = Va = u$. □

Lemma 2
Let $K(s_q, n) = \{r_0(s_q), r_1(s_q), \ldots, r_n(s_q)\}$. If $K(s_q, n) \in \text{span}(V)$, then

$$\hat{N}(s_q)V^Tr_k = V^TN(s_q)r_k, \quad 0 \leq k \leq n \quad (18)$$
Proof.
By Lemma 1,
\[ \hat{N}(s_q)V^T r_k = V^T N(s_q)V V^T r_k = V^T N(s_q) r_k \]
\[ \square \]

Lemma 3
Under the same assumption of Lemma 2, we have
\[ \hat{r}_k(s_q) = V^T r_k(s_q) \quad 0 \leq k \leq n \]
\[
\hat{r}_k(s_q) = V^T r_k(s_q) \quad 0 \leq k \leq n
\]

Proof.
We first consider the case that \( s_q \) is finite. (\( s_q \)) will be omitted from the symbols in the following for simplicity.
The proof is done by induction. Note that if the lemma is true for some \( k \), then by Lemma 1,
\[ V \hat{r}_k = V V^T r_k = r_k \]
and
\[ M \hat{r}_k = V^T M V \hat{r}_k = V^T M r_k \]
for the same \( k \).
For \( k = 0 \),
\[ \hat{N} V^T r_0 = V^T N r_0 = V^T b_0 = \hat{b}_0 \]
So,
\[ V^T r_0 = \hat{N}^{-1} \hat{b}_0 = \hat{r}_0 \]
and Lemma 2 is true for \( k = 0 \).
For \( k = 1 \).
\[ \hat{N} V^T r_1 = V^T N r_1 = V^T (-M r_0 + b_1) = -M \hat{r}_0 + \hat{b}_1 \]
So,
\[ V^T r_1 = -\hat{N}^{-1} M \hat{r}_0 + \hat{N}^{-1} \hat{b}_1 = \hat{r}_1 \]
and Lemma is true for \( k = 1 \).
Now suppose that the lemma is true for some \( k < n \), we prove it is true for \( k + 1 \).
\[ \hat{N} V^T r_{k+1} = V^T N r_{k+1} = -V^T M r_k = -M \hat{r}_k \]
So,
\[ V^T r_{k+1} = -\hat{N}^{-1} M \hat{r}_k = \hat{r}_{k+1} \]
Thus, the Lemma is true for finite \( s_q \).
For the case that \( s_q = \infty \), Compare Eqs(10)-(12) with Eqs(7)-(9), it can be seen that if we interchange \( M \) and \( N \) and \( b_0 \) and \( b_1 \), one set of equations becomes the other. So the proof is similar to the above and is omitted.
3 Proof of Proposition 2

Now, we are ready to prove the Proposition. For either finite or infinite matching point, from Lemma 3 and Lemma 1, for $0 \leq k \leq n$,

$$
\tilde{e}_k^T \tilde{r}_k = c_0^T V V^T r_k = c_0^T r_k 
$$

Similarly,

$$
\tilde{e}_1^T \tilde{r}_k = c_1^T r_k 
$$

From Eqs(13), (14), (15) - (17), it is clear that the Proposition is true.