STABILIZATION OF MULTIPLE INPUT CHAINED FORM CONTROL SYSTEMS

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Stabilization of Multiple Input
Chained Form Control Systems\textsuperscript{1}

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Abstract

This paper gives a control law for stabilizing multiple input chained form control systems. This extends an earlier result [10] on stabilizing the above class of systems which have two inputs. Along the way, we construct a transformation from general chained form systems with multiple generators to a power form. A control law which stabilizes the origin of a three-input control system that models the kinematics of a fire truck is simulated, confirming the theoretical results.
Chapter 1

Introduction

We are interested in stabilizing control laws for control systems of the form

$$\dot{\xi} = g_0(\xi)u_0 + g_1(\xi)u_1 + \cdots + g_m(\xi)u_m$$  \hfill (1.1)

arising from systems with nonholonomic, or non-integrable, linear velocity constraints where $\xi$ in a open set $U \subset \mathbb{R}^n$ is the state of the system, $u(\xi, t) \in \mathbb{R}$ are the inputs and $g_i$ are smooth linearly independent vector fields.

A special case of the two-input problem was presented in [4]. In previous work [10] we proposed a control law motivated by [8, 11] which stabilized the control system to a point in the state space for a special class of two-input nonholonomic systems that could be transformed, by a coordinate trans-
formation and state feedback, to a two-input single-chain single-generator chained form:

\[
\begin{align*}
\dot{x}_0^0 &= v_0 \\
\dot{x}_1^0 &= v_1 \\
\dot{x}_{10}^1 &= x_1^0 v_0 \\
\dot{x}_{10}^2 &= x_{10}^1 v_0 \\
&\vdots \\
\dot{x}_{10}^{n-2} &= x_{10}^{n-3} v_0 \\
\end{align*}
\]

It was then shown that there exists a coordinate transformation to the power form

\[
\begin{align*}
\dot{y}_0 &= v_0 \\
\dot{y}_1 &= v_1 \\
\dot{x}_{10}^1 &= y_0 v_1 \\
\dot{x}_{10}^2 &= \frac{1}{2}(y_0)^2 v_1 \\
&\vdots \\
\dot{x}_{10}^{n-2} &= \frac{1}{(n-2)!}(y_0)^{n-2} v_1 \\
\end{align*}
\]
Introduction

whose origin was stabilized. This paper is motivated by the desire to extend
this work to the general \((m+1)\)-input, \(m\)-chain, single-generator case.

The extended problem can be stated as follows: given a nonlinear control
system of the form in equation (1.1), find a control law \(u(\xi, t)\) which makes
the origin globally asymptotically stable.

The major contributions of this paper are the presentation of the trans-
formation from a \((m+1)\)-input, \(m(m+1)\)-chain, \((m+1)\)-generator chained
form system into a power form and the proof of a stabilizing control law
that makes the origin of a \((m+1)\)-input, \(m\)-chain, single-generator power
form control system globally asymptotically stable.

The outline of this paper is as follows: in Section 2, the transformations
to chained form and power form are given for a \((m+1)\)-input, \(m\)-chain, single-
generator case along with the power form transformation from a general
chained form system. In Section 3, a stabilizing control law is given for
\((m+1)\)-input, \(m\)-chain, single-generator control systems in power form. In
Section 4, a three-input example, the fire truck, is used to illustrate the
theoretic results. In Section 5, conclusions are made.
Chapter 2

Transformation to Power Form

We are interested in stabilizing the origin of control systems represented in power form. To transform the original system in equation (1.1) into power form, we present a two-step algorithm. The first step is to transform the (m+1)-input original system into a m-chain single-generator chained form as described in [2]. The second step is to transform this chained form system into a power form. We choose not to prove the one-step transformation from the original system to power form since the first transformation exists and is proven in the literature and the second transformation is straightforward.
2.1 Original System to Chained Form

once the system is in chained form. We also give the transformation from a system in general chained form that has $m+1$ inputs, $m(m+1)$ chains and $m+1$ generators to power form.

We suggest Chapter 1 of [6] for reference on distributions and Lie brackets.

2.1 Original System to Chained Form

We now present sufficient conditions under which the $(m+1)$-input system in equation (1.1) may be transformed into a $m$-chain, single-generator chained form.

The general idea is to construct $m+2$ distributions using the vector fields $g_0, g_1, \ldots, g_m$ and various Lie brackets. We then check the conditions of full rank on the distribution that contains all of the vector fields

\[ g_0, g_1, \ldots, g_m, \]
\[ \text{ad}_{g_0} g_1, \text{ad}_{g_0} g_2, \ldots, \text{ad}_{g_0} g_m, \]
\[ \text{ad}^2_{g_0} g_1, \text{ad}^2_{g_0} g_2, \ldots, \text{ad}^2_{g_0} g_m, \ldots, \]
\[ \text{ad}^n_{g_0} g_1, \text{ad}^n_{g_0} g_2, \ldots, \text{ad}^n_{g_0} g_m \]
Transformation to Power Form

where \( \text{ad}_{p_0}^k g_i \) is the \( k \)-th order Lie bracket \([g_0, [g_0, \ldots k \text{ times } \ldots [g_0, g_i]]] \).

The \( m+1 \) other distributions, \( \Delta_i \), defined as in the proposition below need to be checked for involutivity, i.e., \( f, g \in \Delta_i \Rightarrow [f, g] \in \Delta_i \). The following proposition is proven in [2].

**Proposition 1** [2] Converting \((m+1)\)-input Systems to \( m \)-Chain, Single-Generator Chained Form

Given the system

\[
\dot{\xi} = g_0(\xi)u_0 + \cdots + g_m(\xi)u_m,
\]

with \( \xi \in U \subset \mathbb{R}^n \), \( u_i(\xi, t) \in \mathbb{R} \) and the \( g_i \) smooth linearly independent vector fields. Construct the following distributions:

\[
\Delta_0 := \text{span}\{g_0, \text{ad}_{p_0}^f g_j; r_j = 0, \ldots, n_j; j = 1, \ldots, m\}
\]
\[
\Delta_1 := \text{span}\{\text{ad}_{p_0}^f g_j; r_j = 0, \ldots, n_j; j = 1, \ldots, m\}
\]
\[
\Delta_2 := \text{span}\{\text{ad}_{p_0}^f g_j; r_1 = 0, \ldots, n_1-1; r_j = 0, \ldots, n_j; j = 2, \ldots, m\}
\]
\[
\Delta_3 := \text{span}\{\text{ad}_{p_0}^f g_j; r_1 = 0, \ldots, n_1-1; r_2 = 0, \ldots, n_2-1; r_j = 0, \ldots, n_j; j = 3, \ldots, m\}
\]
\[
\vdots
\]
\[
\Delta_{m+1} := \text{span}\{\text{ad}_{p_0}^f g_j; r_j = 0, \ldots, n_j-1; j = 1, \ldots, m\}
\]
where $\sum_{j=1}^{m} n_j + m + 1 = n$.

If first we have that $\Delta_0(\xi) = \mathbb{R}^n$ and second $\Delta_j$ for $1 \leq j \leq m + 1$ are involutive, then there exists a local feedback transformation

$$x = \Phi(\xi)$$
$$u = \beta(\xi)v$$

such that system is in $m$-chain, single-generator chained form:

$$\dot{x}_0^0 = v_0$$
$$\dot{x}_1^0 = v_1 \quad \dot{x}_2^0 = v_2 \quad \cdots \quad \dot{x}_m^0 = v_m$$

$$\dot{x}_{10}^1 = x_{10}^0 v_0 \quad \dot{x}_{20}^1 = x_{20}^2 v_0 \quad \cdots \quad \dot{x}_{m0}^1 = x_{m0}^0 v_0$$

$$\vdots \quad \vdots \quad \ddots \quad \vdots$$

$$\dot{x}_{10}^m = x_{10}^{n_1-1} v_0 \quad \dot{x}_{20}^m = x_{20}^{n_2-1} v_0 \quad \cdots \quad \dot{x}_{m0}^m = x_{m0}^{n_m-1} v_0.$$ 

The notation uses the convention that $x_{ji}^k$ is the state for the $k^{th}$ level of chain $j$ using state $i$ as the generator. The state $x_j^0$ is the top of chain $j$. 
2.2 Chained Form to Power Form

Given a general chained form system with \( m+1 \) inputs, \( m(m+1) \) chains and \( m+1 \) generators:

\[
\begin{align*}
\dot{x}_j^0 &= v_j \quad 0 \leq j \leq m \\
\dot{x}_j^1 &= x_j^0 v_i \quad j > i \text{ and } x_j^1 := x_j^0 x_j^0 - x_j^1 \\
\dot{x}_j^k &= x_j^{k-1} v_i \quad 1 \leq k \leq n_j, \ 0 \leq j, i \leq m; \ j \neq i,
\end{align*}
\]

the global transformation to power form is given as follows:

\[
\begin{align*}
y_j &= x_j^0 \quad 0 \leq j \leq m \\
x_j^k &= (-1)^k x_j^k + \sum_{n=0}^{k-1} (-1)^n \frac{1}{(k-n)!} (x_i^0)^{k-n} x_i^n \\
1 \leq k \leq n_j, \ 1 \leq j \leq m, \ 0 \leq i \leq m. \quad (2.1)
\end{align*}
\]

We identify \( x_j^0 = x_j^0 \) as the tops of the chains. Using this transformation, the general chained form equation is converted to the power form dynamics:

\[
\begin{align*}
\dot{y}_j &= v_j \quad 0 \leq j \leq m \\
\dot{x}_j^k &= \frac{1}{k!} (y_i)^k v_j \quad 1 \leq k \leq n_j, \ 1 \leq j \leq m, \ 0 \leq i \leq m. \quad (2.2)
\end{align*}
\]
2.2 Chained Form to Power Form

In this paper, we will be using the single-generator version of the power form. To find this form, we use the following transformation on the single-generator chained form system

\[ y_j = x_j^0 \quad 0 \leq j \leq m \]

\[ x_{j0}^k = (-1)^k x_{j0}^k + \sum_{n=0}^{k-1} \frac{1}{(k-n)!} \left( x_0^0 \right)^{k-n} x_{j0}^n \quad 1 \leq k \leq n_j, \quad 1 \leq j \leq m \] (2.3)

which yields the single-generator power form:

\[ \dot{y}_j = v_j \quad 0 \leq j \leq m \]

\[ \dot{x}_{j0}^k = \frac{1}{k!} (y_0)^k v_j \quad 1 \leq k \leq n_j, \quad 1 \leq j \leq m \] (2.4)
Chapter 3

Stabilization

We generalize the control law given in [10] to the multiple input case. The simplicity of the law is bought at the expense of the complexity of the proof.

Proposition 2 Given an \((m+1)\)-input, \(m\)-chain, single-generator control system whose evolution in time is given by:

\[
\begin{align*}
\dot{y}_j &= v_j \text{ for } 0 \leq j \leq m \\
\dot{z}_{j0}^k &= \frac{1}{k!} (y_0)^k v_j \text{ for } 1 \leq k \leq n_j \text{ and for } 0 \leq j \leq m,
\end{align*}
\]  
(3.1)
The origin \((y, z) = 0\) is locally asymptotically stable under the action of the controls:

\[
\begin{align*}
  v_0 &= -y_0 + \rho(z) (\cos(t) - \sin(t)) \\
  v_j &= -y_j + \sum_{h=1}^{n_j} c_j^h x_j^h \cos(ht) \text{ for } 1 \leq j \leq m
\end{align*}
\]

(3.2)

where \(\rho(z) = \sum_{j=1}^{m} \sum_{k=1}^{n_j} (x_j^k)^2\) and each \(c_j^h < 0\).

\textbf{Proof:} The outline of the proof is as follows: first we find an approximation to the resulting time-varying center manifold and then we examine the dynamics on this reduced space. After an averaging transformation on this reduced space we may utilize the direct method of Lyapunov to conclude that the origin of the system under the inputs (3.2) is locally asymptotically stable. We will try to follow closely [10] whenever possible.

We will now solve for the closed-loop dynamics of the system (2.4). The dynamics under the controls (3.2) are described by (for \(1 \leq j \leq m\)):

\[
\begin{align*}
  \dot{y}_0 &= -y_0 + \rho(z) (\cos(t) - \sin(t)) \\
  \dot{y}_j &= -y_j + \sum_{h=1}^{n_j} c_j^h x_j^h \cos(ht) \\
  \dot{x}_{ho}^k &= \frac{1}{k!} (y_0)^k \left( -y_j + \sum_{h=1}^{n_j} c_j^h x_j^h \cos(ht) \right) \text{ for } 1 \leq k \leq n_j
\end{align*}
\]

(3.3)
Notice that the time-varying terms in (3.3) are linear in the state; therefore we cannot directly apply center manifold theory. In order to apply this theory, we will first transform the system into one which has higher-order time-varying terms. We start with the coordinate $y_0$.

$$\dot{y}_0 = y_0 - \rho(z) \cos(t)$$

Now we solve for the dynamics of $\dot{y}_0$:

$$\dot{y}_0 = -y_0 + \rho(z)(\cos(t) - \sin(t)) + \rho(z) \sin(t) - \left( \frac{d}{dt} \rho(z) \right) \cos(t)$$

$$= -(y_0 - \rho(z) \cos(t)) - \left( \frac{d}{dt} \rho(z) \right) \cos(t)$$

$$= -\dot{y}_0 - g_0(y, z, t)$$

with the function $g_0(y, z, t)$ being

$$g_0(y, z, t) = \left( \frac{d}{dt} \rho(z) \right) \cos(t)$$

$$= 2 \sum_{j=1}^{m} \sum_{k=1}^{n_j} \frac{1}{k!} (y_0)^k \left( -y_j + \sum_{h=1}^{n_j} c_j^k z_{j0}^k \cos(ht) \right) \cos(t).$$

Note that $g_0(y, z, t)$ is at least cubic in the state $(y, z)$. This fact will aid us in estimating the time-varying center manifold. Now we will transform the
other $y_j$ coordinates.

$$\ddot{y}_j = y_j + \sum_{h=1}^{n_j} z_{j0}^h \left( \frac{-hc_j^h}{1 + h^2} \sin(ht) + \frac{-c_j^h}{1 + h^2} \cos(ht) \right) \text{ for } 1 \leq j \leq m$$

Once again, we solve for the dynamics of $\ddot{y}_j$:

$$\dot{\ddot{y}}_j = -\ddot{y}_j + \sum_{h=1}^{n_j} c_j^h z_{j0}^h \cos(ht) +$$

$$\sum_{h=1}^{n_j} z_{j0}^h \left( \frac{-hc_j^h}{1 + h^2} \cos(ht) - \frac{-c_j^h}{1 + h^2} \sin(ht) \right) + g_j(y, z, t)$$

$$= -\left( \ddot{y}_j + \sum_{h=1}^{n_j} z_{j0}^h \left( \frac{-hc_j^h}{1 + h^2} \sin(ht) - \left( \frac{-hc_j^h}{1 + h^2} + c_j^h \right) \cos(ht) \right) \right) + g_j(y, z, t)$$

$$= -\ddot{y}_j + g_j(y, z, t)$$

where the term $g_j(y, z, t)$ is given by

$$g_j(y, z, t) = \sum_{h=1}^{n_j} \frac{1}{h!} (y_0)^h \left( -\ddot{y}_j + \sum_{i=1}^{n_j} c_i^j z_{j0}^i \cos(it) \right) \left( \frac{-hc_j^h}{1 + h^2} \sin(ht) + \frac{-c_j^h}{1 + h^2} \cos(ht) \right).$$

(3.4)

Again note that $g_j(y, z, t)$ is at least quadratic in the state $(y, z)$. All of

the $g_j(y, z, t)$ may be rewritten in terms of $\ddot{y}$, with no corresponding loss of
order. Call these functions $\tilde{g}_j(\tilde{y}, z, t)$.

We will now introduce a generalization of the center manifold theorem which appears in [10]. The following statement uses the notation of [3] so that $f'(0,0,w)$ refers to the partial derivative of $f$ with respect to all variables and evaluated at $(y,z,w) = (0,0,w)$.

**Lemma 1 ("Time-varying" Center Manifold)** Consider the system:

\begin{align*}
\dot{y} &= By + g(y,z,w) \\
\dot{z} &= Az + f(y,z,w) \\
\dot{w} &= Sw
\end{align*}

(3.5)

with $y \in \mathbb{R}^{m+1}$, $z \in \mathbb{R}^{n-m-1}$, and $w \in \mathbb{R}^p$ and where the eigenvalues of $A$ and $S$ have zero real part. The functions $f$, $g$ and $h$ are $C^2$ with $f(0,0,w) = 0$, $f'(0,0,w) = 0$, $g(0,0,w) = 0$ and $g'(0,0,w) = 0$. Then, given $M > 0$, there exists a center manifold for (3.5), $y = h(z,w)$ for $\|w\| < M$, $\|z\| < \delta(M)$, for some $\delta > 0$ and dependent on $M$, where $h(z,w)$ is $C^2$ and $h(0,w) = 0$ and $h'(0,w) = 0$.

To apply this result to our system, create the vector $w$, with $w_1^j = \cos(jt)$ and $w_2^j = \sin(jt)$. Note that the dimension of $w$, given by $p$, is then twice
the maximum $n_j$ for $1 \leq j \leq m$. The differential equation describing the
evolution of this vector may be written as $\dot{w} = Sw$, with the eigenvalues of
$S$ having zero real part. We can then substitute an element of the vector
$w$ for each of the time–varying terms in the equations. In this form, we
may apply the “Time-varying” Center Manifold theorem. The time-varying
center manifold is then given by (with the time dependencies resubstituted):

$$
\tilde{y}_0 = h_0(z, t)
$$

$$
\tilde{y}_j = h_j(z, t) \text{ for } 1 \leq j \leq m .
$$

Since $g_0(y, z, t)$ is at least cubic in the state $(y, z)$ it may be verified that
by calculation that $h_0''(0, t) = 0$. We will check the stability of $z = 0$ by
solving for the dynamics of $z$ on the center manifold. It is useful to note
that

$$
v_0 = -g_0 + \rho(z)(\cos(t) - \sin(t))
$$

$$
= -\tilde{y}_0 - \rho(z) \sin(t) .
$$
And for the cases $1 \leq j \leq m$, we find

\[
v_j = -y_j + \sum_{h=1}^{n_j} c_j^h z_j^h \cos(ht)
\]

\[
= -\bar{y}_j + \sum_{h=1}^{n_j} z_j^h \left( \frac{-h c_j^h}{1 + h^2} \sin(ht) + \left( \frac{-c_j^h}{1 + h^2} + c_j^h \right) \cos(ht) \right)
\]

\[
= -\bar{y}_j + \sum_{h=1}^{n_j} \frac{-h c_j^h}{1 + h^2} z_j^h \sin(ht) - h \cos(ht))
\]

The dynamics on the reduced space are then given by:

\[
\dot{z}_{j0}^k = \frac{1}{k!} \left( h_0(x, t) + \rho(z) \cos(t) \right)^k \left( -h_j(z, t) + \sum_{h=1}^{n_j} \frac{-h c_j^h}{1 + h^2} z_j^h \sin(ht) - h \cos(ht) \right)
\]

and because of the higher order nature of the $h_i$'s, we may write that

\[
\dot{z}_{j0}^k = f_{j0}^k(z, t) = \frac{1}{k!} \rho(z) \cos^k(t) \left( \sum_{h=1}^{n_j} \frac{-h c_j^h}{1 + h^2} z_j^h \sin(ht) - h \cos(ht) \right) + O(2j + 2).
\]

Induction may be used to verify that:

\[
\cos^k(t) = \sum_{i=1}^{\ell} \alpha_{ki} \cos([k - 2(i - 1)]t)
\]
where $\alpha_{ti} > 0$ and $\ell = \frac{k}{2}$ if $k$ is even and $\frac{k+1}{2}$ if $k$ is odd. As we do not have exponential stability of the averaged system nor high frequency sinusoidal terms [7] we cannot apply averaging to the terms of equation (3.6) and conclude stability. Alternatively, one might notice that each chain (denoted by the index $j$) is decoupled from the effects of the others except through the positive semi-definite term $\rho(z)$. We would like to apply the results of for the single chain case repeatedly to argue local stability.

Lemma 2 ("Averaging" Transformation) Consider the time-varying non-linear system

$$\dot{x} = f(x, t)$$

(3.7)

where $f$ is of period $T$, $C^r$, and where the $i^{th}$ entry of the vector $f$ satisfies $f_i = O(x)^{2k+1}$, with $k \geq 1$. Then there exists a $C^r$ local change of coordinates $x = y + \Psi(y, t)$ under which (3.7) becomes

$$\dot{y} = \bar{f}(y) + \hat{f}(y, t)$$

where $\bar{f}$ is the time average of $f$ and $\hat{f}(y, t) = O(y)^{2k+2}$ and of period $T$. 

Remark: Lemma 2 closely resembles lemma 2.2 of [10], except now there is no dependence between the position and the order of an element $f_i$ of the vector $f$. Note that if we permute the vector $x$ and hence the vectors $f$ and $y$, the order of the element $f_i$ of the vector $f$ of the previous result will no longer necessarily of order $2i + 1$. In fact, if the element $f_i$ were of order $2k + 1$, the previous lemma would have asserted $\tilde{f}_i$ to be order $2k + 1$ and $\tilde{f}_i$ to be order $2k + 2$. The order of $\tilde{f}_i$ and $\tilde{f}$ is then determined by the order of $f_i$. However, a formal proof is still required.

Proof: The proof closely resembles a result described in [5]. We will divide $f(x, t)$ into its time average, given by $\bar{f}(x)$, and the remainder $\tilde{f}(x, t)$. We make the coordinate change:

$$\quad x = y + \Psi(y, t)$$

with $\Psi(y, t)$ specified later. We now solve for the dynamics of $y$.

$$\quad (I + D_y\Psi)\dot{y} + \frac{\partial \Psi}{\partial t} = \dot{x} = \bar{f}(y + \Psi) + \tilde{f}(y + \Psi, t)$$

$$\quad \dot{y} = (I + D_y\Psi)^{-1} \left( \bar{f}(y + \Psi) + \tilde{f}(y + \Psi, t) - \frac{\partial \Psi}{\partial t} \right)$$
Set $\frac{\partial \Psi}{\partial t} = \tilde{f}(y, t)$. As $\tilde{f}(y, t)$ has zero mean, $\Psi$ is a bounded function of time.

Furthermore, this implies that $\Psi$ is of higher order, thus the coordinate change is valid locally. Expanding the terms, we see:

$$
\dot{y} = (I + D_y \Psi)^{-1} \left( \tilde{f}(y) + f(y + \Psi, t) - f(y, t) \right)
$$

$$
= \left( I - D_y \Psi + O(\|D_y \Psi\|^2) \right) \left( \tilde{f} + D_y f \Psi + O(\|\Psi\|^2) \right)
$$

$$
= \tilde{f}(y) + \tilde{f}(y, t)
$$

Now we will check the order of $\tilde{f}_i$.

$$
\tilde{f} = D_y f \Psi - D_y \Psi \left( \tilde{f}(y) + f_i(y + \Psi, t) - f_i(y, t) \right) \quad (3.8)
$$

As the order of the element $f_i$ is $2k + 1$, the elements of the corresponding row of $D_y f$ are of order $2k$. The lowest order in $\Psi$, as it is the time integral of $\tilde{f}$, is 3 from the case $k = 1$. We conclude that the first term is of order $2k + 3$. The same argument holds for the second term, as the element $\Psi_i$ is of order $2k + 1$, the elements of the corresponding row of $D_y \Psi$ is of order $2k$. The product is then at least of order $2k + 3$. $\square$. 
With lemma 2 in mind we will define the vectors \( z, f(z,t) \in \mathbb{R}^{n-m-1} \).

\[
  z = \begin{pmatrix} z_{10}^1 & z_{10}^2 & \cdots & z_{10}^m \\
  \vdots & \ddots & \ddots & \vdots \\
  z_{m0}^1 & z_{m0}^2 & \cdots & z_{m0}^m \end{pmatrix}^T
\]

\[
f(z,t) = \begin{pmatrix} f_{10}^1 & f_{10}^2 & \cdots & f_{10}^m \\
  \vdots & \ddots & \ddots & \vdots \\
  f_{m0}^1 & f_{m0}^2 & \cdots & f_{m0}^m \end{pmatrix}^T
\]  

(3.9)

Note that by equation (3.6) \( z \) satisfies:

\[
\dot{z} = f(z,t)
\]

(3.10)

By lemma 2 here exists an locally valid averaging transformation. In spirit with the earlier notation, we will write the transformations maps \( z \rightarrow \tilde{z} \).

\[
\dot{\tilde{z}}_j^k = \tilde{f}_j^k(\tilde{z}) + \tilde{f}_j^k(\tilde{z}, t)
\]

Further, \( \tilde{f}_j^k(\tilde{z}) \) is the time average of \( f_j^k(\tilde{z}, t) \) and \( \tilde{f}_j^k(\tilde{z}, t) \) is of order \( 2k + 2 \).

Now we will recall the case specific Lyapunov result from [10]:

**Lemma 3 (Case Specific Lyapunov Result)** Consider the time-varying nonlinear system

\[
\dot{y} = \tilde{f}(y) + \tilde{f}(y, t)
\]

(3.11)
where \( y \in \mathbb{R}^n \). If

\[
\|\ddot{f}(y, t)\| \leq \beta_i \|y\|^{2(1+i)}
\]

for all \( y \) in some open neighborhood of the origin and

\[
\ddot{f}(y) = A \psi(y)
\]

where \( A \) is a square lower triangular matrix with \( a_{ii} < 0 \) for \( i = 1, \ldots, n \) and

\[
\psi_i(y) = y_i \|y\|^{2i}
\]

then the origin of (3.11) is locally asymptotically stable.

The same kind of permutation argument can now be applied to Lemma 3 as was applied to Lemma 2, except in this case it would be wise to preserve the order within each chain in order to retain the lower block diagonal structure of the matrix \( A \). For our case, matrix \( A \) will then have the block diagonal form (recall that the chains are decoupled except for the function
where $A_j \in \mathbb{R}^{n_j \times n_j}$ has a lower block diagonal form. The diagonal terms of the matrix $A_j$ are denoted by $a_{jk}^{kk}$ and are given by,

$$a_{jk}^{kk} = -\frac{1}{2k!} \frac{k^2}{1 + k^2 \alpha_{1k} c_j^k}$$

similar to those in [10]. As $\alpha_{1k} c_j^k < 0$, we may conclude local asymptotic stability.

**Remark 1:** Corollary 2.1 of [10] may be easily extended to the multiple input case. We may therefore conclude that the controls (3.2) will locally asymptotically stabilize the chained form system as well as the power form.

**Remark 2:** As the chains are decoupled, we may can make the stability global by using saturation functions as described in [10]. The example will incorporate this strategy. The controls with saturation functions incorpo-
rated are given by:

\[
\begin{align*}
    v_0 &= -y_0 + \sigma(\rho(z))(\cos(t) - \sin(t)) \\
    v_j &= -y_j + \sum_{k=1}^{n_h} c_j^k \sigma(z_{j0}^k) \cos(ht) \text{ for } 1 \leq j \leq m
\end{align*}
\]

with \( c_j^k < 0 \) and with \( \sigma : \mathbb{R} \to \mathbb{R} C^3 \) and nondecreasing. The function is the identity map near the origin but never greater in magnitude than some \( \epsilon > 0 \). Provided this \( \epsilon \) is small enough, the origin will be globally asymptotically stable.

**Remark 3:** These control laws may be adapted in order to regulate a class of mechanical systems. It has been shown in the literature [1, 9] that a large number of mechanical systems with nonholonomic velocity contraints may be written as:

\[
\begin{align*}
    \dot{\xi} &= \sum_{i=1}^{m} g_i(\xi)u_i \\
    \dot{u} &= v
\end{align*}
\]

The first equation describes what is sometimes referred to as the *kinematic* system. We have developed controls laws \( u_d(\xi, t) \) which render \( \xi = 0 \) globally asymptotically stable, solving the *kinematic* point stability problem for a
class of mechanical systems. Since $u_d(\xi, t)$ is a $C^1$ function of $\xi$ and $t$ we may apply the following control law in order to render the point $(\xi, u) = (0, 0)$ globally asymptotically stable,

$$v(\xi, u, t) = -(u - u_d(\xi, t)) + \left( \frac{\partial u(\xi, t)}{\partial \xi} g(\xi) u + \frac{\partial u(\xi, t)}{\partial t} \right)$$

To demonstrate that this control law stabilizes $(0, 0)$, examine the dynamics of the error of the input $e = u - u_d(\xi, t)$.

$$\dot{e} = \dot{u} - \dot{u}_d(\xi, t)$$

$$= -e$$

Note that if we view the state of the system as $(e, \xi)$, the equations will look like $\dot{\xi} = g(\xi)u_d(\xi, t) + g(\xi)e$. The linearly stable coordinates will then contain not only the terms due to $u_d(\xi, t)$ as before but also the entire vector $e$. In the center manifold, $e = 0$. Consequently the the equations for the dynamics on the center manifold are exactly the same as in the kinematic case thus we may conclude stability.
Chapter 4

An Example

The stabilization method presented in this paper will be illustrated by a simple three-input example. The example we use is the fire truck, or tiller truck shown in Figure (4.1). The kinematic equations were derived in [2] to have the following form:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi}_0 \\
\dot{\theta}_0 \\
\dot{\phi}_1 \\
\dot{\theta}_1
\end{pmatrix}
= \begin{pmatrix}
1 & \tan \theta_0 \\
0 & \tan \phi_0 \\
0 & \frac{\tan \theta_0}{l_0 \cos \theta_0} \\
0 & \frac{\tan \phi_0}{l_0 \cos \theta_0} \\
0 & \frac{-\sin(\phi_1 - \theta_0 + \theta_1)}{l_1 \cos \phi_1 \cos \theta_0}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
The system has six states which are the Cartesian location of the center of the rear axle of the cab, \((x, y)\); the steering angle of the front wheels relative to the cab's orientation, \(\phi_0\); the absolute cab orientation with respect to the horizontal axis of the inertial frame, \(\theta_0\); the steering angle of the rear wheels with respect to the trailer body, \(\phi_1\); and the absolute trailer orientation, \(\theta_1\). The constants \(l_0\) and \(l_1\) correspond to physical parameters of the system. The three inputs \(u_1, u_2\) and \(u_3\) correspond to the driving velocity, the cab's steering velocity and the trailer's steering velocity, respectively.

The coordinate transformation [2],

\[
x_0^0 = x
\]
An Example

\[ x_1^0 = \frac{\tan \phi_0}{l_0 \cos^3 \theta_0} \]
\[ x_2^0 = -\sin(\phi_1 - \theta_0 + \theta_1) \]
\[ x_{10}^1 = \tan \theta_0 \]
\[ x_{20}^1 = \theta_1 \]
\[ x_{10}^2 = y \]

is used to put the system into a two-chained single-generator form:

\[ \dot{x}_0^0 = v_0 \quad \dot{x}_1^0 = v_1 \quad \dot{x}_2^0 = v_2 \]
\[ \dot{x}_{10}^1 = x_{10}^1 v_0 \quad \dot{x}_{20}^1 = x_{20}^2 v_0 \]
\[ \dot{x}_{10}^2 = x_{10}^1 v_0 \] \hspace{1cm} (4.1)

Using equation (2.3), we calculate the coordinate transformation

\[ y_0 = x_0^0 \]
\[ y_1 = x_1^0 \]
\[ y_2 = x_2^0 \]
\[ x_{10}^1 = -x_{10}^1 + x_0^0 x_1^0 \]
\[ x_{10}^2 = x_{10}^2 - x_0^0 x_2^0 + \frac{1}{2} (x_0^0)^2 x_1^0 \]
\[ x_{20}^1 = -x_{20}^1 + x_0^0 x_2^0 \]

to give us the power form

\[ \begin{align*}
\dot{y}_0 &= v_0 \\
\dot{y}_1 &= v_1 \\
\dot{y}_2 &= v_2 \\
\dot{z}_{10}^1 &= y_0 v_1 \\
\dot{z}_{20}^1 &= y_0 v_2 \\
\dot{z}_{10}^2 &= \frac{1}{2} (y_0)^2 v_1.
\end{align*} \] (4.2)

We now present the simulation of the fire truck system. The simulation was performed on the system in power form with the states being stabilized to the origin from a given initial point by using the following control laws:

\[ \begin{align*}
v_0 &= -y_0 + \rho_0 (\cos t - \sin t) \\
v_1 &= -y_1 + c_1 \rho_1 \cos t + c_2 \rho_2 \cos 2t \\
v_2 &= -y_2 + c_3 \rho_3 \cos t. \end{align*} \] (4.3)

The \( c_1, c_2, c_3 \) are constants and \( \rho_1, \rho_2, \rho_3, \rho_4 \) are saturation functions designed to yield global stabilization results and defined as follows:

\[ \rho_0 = \left( \sigma \left( \sqrt{(z_{10}^1)^2 + (z_{10}^2)^2 + (z_{10}^2)^2} \right) \right)^2 \]
\begin{align*}
\rho_1 &= \sigma(z_{10}^1) \\
\rho_2 &= \sigma(z_{10}^2) \\
\rho_3 &= \sigma(z_{20}^1).
\end{align*}

where \(\sigma(z)\) is a saturation function linear between \((-\epsilon, \epsilon)\).

The coordinates in power form were then transformed back into the original coordinates for analysis and a movie animation. Then the inverse coordinate transformation from power form to the original coordinates is as follows.

\begin{align*}
x &= y_0 \\
y &= z_{10}^2 + y_0 y_1 - \frac{1}{2} (y_0)^2 y_1 \\
\phi_0 &= \tan^{-1} \left( l_0 y_1 \cos^3(\tan^{-1}(y_0 y_1 - z_{10}^1)) \right) \\
\theta_0 &= \tan^{-1} \left( y_0 y_1 - z_{10}^1 \right) \\
\phi_1 &= \frac{1}{\cos (y_0 y_2 - z_{10}^1 - \tan^{-1}(y_0 y_1 - z_{10}^1))} \\
&\quad \times \tan^{-1} \left( l_1 y_2 \cos(\tan^{-1}(y_0 y_1 - z_{10}^1)) + \sin \left( y_0 y_2 - z_{10}^1 - \tan^{-1}(y_0 y_1 - z_{10}^1) \right) \right) \\
\theta_1 &= y_0 y_2 - z_{10}^1
\end{align*}

Figure 4.2 shows nine frames from a movie of the simulation results for
Figure 4.2: Movie of Parallel Parking Trajectory
An Example

a parallel parking maneuver, starting from an arbitrary initial point that illustrates the control law. The fire truck is being stabilized to the origin. The $y = 0$ line is shown in the plots and we see that after 8000 iterations, the fire truck is heading towards the origin.

Figure 4.3 shows the x-y plot for the same maneuver. In this plot we see the Lissajous figures nicely.

Figure 4.4 shows the trajectories of the body angles and $y$ position of the center of the rear axle of the cab for the same parallel parking maneuver. We can see the success of the stabilization after 8000 iterations in the $y$ position plot.

Figure 4.5 shows the saturation functions used in equation (4.3). For the parallel parking trajectory, we chose $\epsilon = 0.5$. 
Figure 4.3: X-Y Trajectory of the Parallel Parking Maneuver
Figure 4.4: Trajectories of the Parallel Parking Maneuver
Figure 4.5: Saturation Functions for Parallel Parking Maneuver
Chapter 5

Conclusion

In summary, our contributions are the presentation of a transformation from most general chained form to power form and a control law for a class of multiple input nonholonomic control systems without drift. In addition, we illustrated our results with an example from the literature, solving incidentally the parallel parking problem.

It is important to notice that the control laws presented here are smooth and, thus, are easily adapted to dynamical control systems. Furthermore, the convergence in the coordinate directions can be modified by changing the coefficients of the control law.

Possible future work includes finding a control law that asymptotically
stabilizes to the origin an \((m+1)\)-generator control system. We are also looking to enlarge the class of nonholonomic control systems which we may transform into chained form.

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Bibliography


