CYCLES OF CHAOTIC INTERVALS IN A 1-D PIECEWISE-LINEAR MAP

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Cycles of Chaotic Intervals in a 1-D Piecewise-Linear Map

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Abstract

We study the bifurcations of attractors of a one-dimensional 2-segment piecewise linear map. We prove that the parameter regions of existence of stable point cycles $\gamma$ are separated by regions of existence of stable interval cycles $\Gamma$ containing chaotic trajectories. Moreover, we show that the period-doubling phenomenon for stable interval cycles is characterized by two universal constants $\alpha$ and $\delta$, whose values are calculated from explicit formulas.
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Introduction.
In this work we consider the endomorphisms of the interval $I = [0, 1]$:

$$f_{l,p}: x \mapsto f_{l,p}(x), \quad x \in I,$$

where $f_{l,p}$ denotes a 2-segment piecewise-linear function with one extremum and having slopes $l$, $p$ as parameters. These maps arise in the consideration of the time-delayed Chua’s circuits modeled by a difference equations with a continuous argument

$$x(t + 1) = f_{l,p}(x(t)), \quad t \in \mathbb{R}^+.$$  

Since the dynamics of this difference equation is governed by the dynamics of the trajectories of the 1-D map $f_{l,p}$, we will consider only the 1-D map $f_{l,p} : I \mapsto I$ in this paper.

There are many publications dealing with one-dimensional piecewise-linear maps. In particular the kneading theory is developed in [Misiurewicz & Visinescu, 1988] and [Marcuard & Visinescu, 1989]. The paper [Sharkovsky et al., 1993] considered an ideal model of Chua’s circuits containing a time delay and proved the existence of stable point cycles $\gamma_n$ of all periods $n$. Moreover, the conditions for the existence of stable interval cycles $\Gamma$ and some results for a two-dimensional generalization of this one-dimensional model are given in [Maistrenko et al., 1992].

The order of the bifurcation sequence in piecewise-linear maps $f_{l,p}$ is different from that of smooth maps. In the case of our piecewise-linear maps, when a period-$n$ point cycle $\gamma_n$ loses its stability, a "rigid" period-doubling bifurcation occurs which leads to the emergence of not point cycles but interval cycles $\Gamma_{n,2n}$ of double period having chaotic trajectories. This is followed by an inverse period-doubling bifurcation; i.e., interval cycles $\Gamma_{n,2n}$ of period $2n$ are merged pairwise, giving birth to a period-$n$ interval cycle $\Gamma_{n,n}$. Finally, in the next bifurcation all intervals of interval cycles $\Gamma_{n,n}$ will merge into an interval cycle $\Gamma_{n,1} = I$. In this case, there are no subintervals of $I$ which recur periodically under the map $f$.

The bifurcation of a period-2 point cycle ($n = 2$) is different from the above scenario and is therefore somewhat special. When a period-2 point cycle $\gamma_2$ loses its stability, an interval cycle $\Gamma_{2,2^k}$ of period-$2^k$ occurs, where $k$ is any integer, depending on the values of the parameters $l$, $p$. In this case, the next bifurcation consists of a pairwise merging of period-$2^k$ interval cycles, giving birth to an interval cycle $\Gamma_{2,2^k-1}$ of period $2^{k-1}$.

At the point $(l,p) = (1,-1)$ two universal constants associated with period-doubling interval cycles ($\delta = 2$ and $\alpha = \infty$) are obtained which are analogous to the "point cycle" period-doubling Feigenbaum’s universal constants.

Therefore, for general one-dimensional piecewise-linear maps with one extremum, the following ordering of attractor bifurcations must occur:

\[
\gamma_1 \Rightarrow \gamma_2 \Rightarrow \Gamma_{2,2^k} \Rightarrow \Gamma_{2,2^{k-1}} \Rightarrow \cdots \Rightarrow \Gamma_{2,2} \Rightarrow I \Rightarrow \gamma_3 \Rightarrow (\Gamma_{3,6} \Rightarrow \Gamma_{3,3} \Rightarrow I) \Rightarrow \gamma_4 \Rightarrow \\
\Rightarrow (\Gamma_{4,8} \Rightarrow \Gamma_{4,4} \Rightarrow I) \Rightarrow \cdots \Rightarrow \gamma_n \Rightarrow (\Gamma_{n,2n} \Rightarrow \Gamma_{n,n} \Rightarrow I) \Rightarrow \gamma_{n+1} \Rightarrow \cdots
\]

This result is similar to the well-known "period-adding" phenomenon [Pei et al., 1986], [Kennedy & Chua, 1986], [Chua, 1986], observed in non-autonomous circuits where the period increases consecutively: i.e., by "addition" of the unit integer, i.e.,

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow \cdots \Rightarrow n \Rightarrow n + 1 \Rightarrow \cdots,$$

and not by multiplication, as in the period-doubling route to chaos. Here, every two consecutive stable periodic orbits are separated by a chaotic region.
1 Stable point cycles \( \gamma_n \) in natural ordering

In this paper, we will consider a continuous, 2-parameter piecewise-linear map \( f : [0, 1] \mapsto [0, 1] \) with one extremum (maximum) point defined by:

\[
f = f_{l,p} : x \mapsto f_{l,p}(x) = \begin{cases} f_1(x) &\text{def } l x + a, \quad x \in [0, b], \\ f_2(x) &\text{def } p x - p, \quad x \in (b, 1]. \end{cases}
\]  

We assume that the parameters \( l, p \) belong to the region

\[
\Pi = \{(l, p) : 0 \leq l \leq \frac{p}{p+1}, \quad p \in (-\infty, -1)\}. \tag{2}
\]

Since \( f_{l,p} \) in (1) is assumed to be continuous, the constants \( a \) and \( b \) are defined by the formulas

\[
a = 1 - l(1 + \frac{1}{p}), \quad b = 1 + \frac{1}{p}. \tag{3}
\]

It should be noted, that any continuous piecewise-linear 1-D map with one breakpoint, having a nontrivial invariant interval, can be reduced to the map (1) by a linear transformation of the real line (see Appendix).

The graphs of the map \( f_{l,p} \) and its next two iterations are shown at Fig. 1(a)-1(c).

Let \( \gamma_n = \{x_1, \ldots, x_n\}, n = 2, 3, \ldots \) denote a period-n cycle, i.e,

\[
x_i < x_{i+1}, \quad f(x_i) = x_{i+1}, \quad i = 1, \ldots, n-1, \quad f(x_n) = x_1. \tag{4}
\]

Let us denote by

\[
L_n \overset{\text{def}}{=} 1 + l + l^2 + \cdots + l^n = \frac{1 - l^{n+1}}{1 - l}.
\]

We need later on the following basic theorem which was proved in [Sharkovsky et al., 1993].

**Theorem 1** A point cycle \( \gamma_n \) of the 1-D map \( f_{l,p} \) in (1) exists if, and only if,

\[
p \leq -\frac{L_{n-2}}{L_{n-1}}; \tag{5}
\]

and is attracting if, and only if,

\[
p > -\frac{1}{L_{n-1}}. \tag{6}
\]

It follows from Theorem 1 that for each \( n \), the existence and stability region of the point cycles \( \gamma_n \) in the \((l, p)\)-parameter space is defined by

\[
\Pi_n = \{(l, p) : -\frac{1}{L_{n-1}} \leq p \leq -\frac{L_{n-2}}{L_{n-1}}, \quad n = 2, 3, \ldots \} \tag{7}
\]

To avoid clutter, the regions \( \Pi_n \) are plotted in the \((l, p^*)\)-parameter plane in Fig. 2, where \( p^* = \log_2(-p) \).

Each region \( \Pi_n \) is bounded from below by an "existence curve", denoted by \([E, n]\), and from above by a "stability curve", denoted by \([S, n]\), as shown in Fig. 2. These two curves
intersect at a point $O_n = (l_n, p_n)$, $n = 2, 3, \ldots$, which defines the end point (apex) of the stability region $\Pi_n$, where the first coordinate $l = l_n$ is the root of the algebraic equation

$$ll_{n-2} = 1, \quad (l^n - 2l + 1 = 0) \tag{8}$$

in the interval $(1/2, 1)$, The second coordinate of the point $O_n$ is located at $p_n = -l_n^{(n-1)}$. The apex points $O_n$, $n = 2, 3, \ldots$, are situated on a branch of the hyperbola

$$p = -\frac{1}{2} - \frac{1}{4(l-1/2)} = \frac{l}{1-2l}. \tag{9}$$

The coordinate $p_n$ has the asymptotic property

$$p_n \sim -2^{n-1} + 1, \quad n \to \infty. \tag{10}$$

The formula (10) is derived from the properties that the curve $[E, n]$ passes through the point $(1/2, 1 - 2^{n-1})$ and the curve $[S, n]$ passes through the point $(1/2, -2^{n-1})$. In particular, it follows from (9) and (10) that

$$\lim_{n \to \infty} l_n = \frac{1}{2}, \quad \lim_{n \to \infty} p_n = -\infty.$$ 

Therefore, if we fix some parameter value $l \in (0, 1/2)$ and vary the parameter $p$ from $-1$ to $-\infty$ then the stable point cycles (separated by chaotic regions) of all integer periods will be observed for the map $f_{l,p}$:

$$2 \Rightarrow 3 \Rightarrow 4 \Rightarrow \ldots \Rightarrow n \Rightarrow n+1 \Rightarrow \ldots \tag{11}$$

These cycles arise as the parameter $(l, p)$ passes through the regions $\Pi_2, \Pi_3, \ldots, \Pi_n, \Pi_{n+1}, \ldots$. This phenomenon is known in electronic circuits as the "period adding" phenomenon, which consists of the appearance of periodic oscillations whose period increases consecutively through all integers as a system parameter is tuned continuously. Observe that the period increases according to a natural ordering. In particular, as $p \to -\infty$ and $l \in (0, 1/2)$, the period of the cycle must tend to infinity.

On the other hand, if we fix some parameter value $p \in (-\infty, -1]$ and increase the parameter $l$ from 0 to 1, then the period-adding phenomenon will also be observed, however, in this case, the period will increase only up to the some finite integer, depending on the value of $p$.

It should be noted, that the Schwarzian derivative $Sf = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2$ is equal to zero everywhere except at extremum point in which case it is not defined. This is one reason which leads to the period-adding bifurcation (11). It is known, that if the Schwarzian derivative of a one-dimensional map is not equal to zero, then a period-doubling point cycle bifurcation must occur as a parameter changes.

2 Stable interval cycles $\Gamma_{n,2n}$, $\Gamma_{n,n}$ for $n \geq 3$

The map $f_{l,p}$ does not have attracting point cycle for $(l, p) \in \Pi \setminus \bigcup_{n=2}^{\infty} \Pi_n$. However, in this case, it has attracting cycles of intervals with chaotic dynamics, i.e. an invariant measure exists; it is concentrated on intervals and is absolutely continuous with respect to the Lebesgue measure.
We will show that the stability regions of interval cycles of periods $2n$ and $n$, respectively, exist in the parameter space $\Pi$ for all $n \geq 2$ (see fig. 3). These regions are denoted by $\Pi_{n,2}$ and $\Pi_{n,1}$ respectively. The bifurcation curve which separates the regions $\Pi_{n,2}$ and $\Pi_{n,1}$ is denoted by $[D, n]$. The curve which bounds the region $\Pi_{n,1}$ from above is denoted by $[C, n]$. The equations of the curves $[D, n]$ and $[C, n]$ will be obtained in the proof of the following theorem.

**Theorem 2.** Let $n \geq 3$.

1) If $(l, p) \in \Pi_{n,1}$, then the map $f_{l,p}$ in the form of (1) will have a stable interval cycle $\Gamma_{n,n}$ of period $n$.

2) If $(l, p) \in \Pi_{n,2}$, then the map $f_{l,p}$ in the form of (1) will have a stable interval cycle $\Gamma_{n,2n}$ of period $2n$.

3) If $(l, p) \in \Pi \setminus (\bigcup_{n=3}^{\infty} (\Pi_n \cup \Pi_{n,1} \cup \Pi_{n,2}))$, then the map $f_{l,p}$ will have a stable interval cycle $\Gamma_{n,1} = [0, 1]$ of period 1.

**Proof.** Consider a parameter point $(l, p) \in \Pi$. Let this point cross the curve $[E, n]$ and enter the region $\Pi_n$. It is easy to see that at the moment (critical bifurcation parameter) where one crosses the curve $[E, n]$, two period-$n$ cycles $\gamma_n = \{x_1, \ldots, x_n\}$ and $\tilde{\gamma}_n = \{\tilde{x}_1, \ldots, \tilde{x}_n\}$ emerged.

These cycles satisfy the following condition:

$$x_i \leq \tilde{x}_i, \quad i = 1, \ldots, n - 1, \quad x_n \leq \tilde{x}_n. \quad (12)$$

At the above critical bifurcation point, these two cycles coincide with each other, and then split off into two distinct cycles (see fig.4). The cycle $\tilde{\gamma}_n$ is always unstable, but the cycle $\gamma_n$ is stable for $(l, p) \in \Pi_n$. Consider next the case where the parameter point $(l, p)$ leave the region $\Pi_n$ and cross the stability curve $[S, n]$. It is easy to see that an interval cycle of double period, i.e., $2n$, is born at this bifurcation point.

Indeed, let us consider the rightmost upper angle of the graph of the function $f^n$ shown in Fig.4 and expanded in Fig.5 over the subinterval $[\tilde{x}_n, 1]$ at the moment when the point $(l, p)$ crossed the curve $[S, n]$. The slope of the right segment of the function $f^n$, denoted by $l'$, is slightly less than $-1$, and the slope of the left segment is equal to $l''$. Obviously, $f^{2n}(1) > x_n$ in some neighborhood of the curve $[S, n]$. Therefore the map $f^n$ has an interval cycle of period 2:

$$\Gamma_{n,2n} \overset{\text{def}}{=} \{[f^n(1), f^{3n}(1)], \quad [f^{2n}(1), 1]\}.$$ 

This interval cycle is attracting as soon as it is born, but at the precise bifurcation point $(l, p) \in [S, n]$ it coincides with the point cycle $\gamma = \{f^n(1), 1\}$ of period 2. The interval cycle $\Gamma_{n,2n}$ of period $2n$ is obtained by iterating the interval $[f^{2n}(1), 1]$ under the action of the map $f$.

If we continue to vary the parameter values so that the slopes of $f^n$ increases then at some bifurcation parameter, the intervals $[f^n(1), f^{2n}(1)]$ and $[f^{2n}(1), 1]$ of cycle $\Gamma_{n,2n}$ touched each other and merged into one, as shown in Fig. 6. This bifurcation parameter defines the bifurcation curve $[D, n]$ and the onset of an inverse period-doubling bifurcation of interval cycles: $\Gamma_{n,2n} \implies \Gamma_{n,n}$. The period-$n$ interval cycle $\Gamma_{n,n}$ is obtained by iterating the interval $[f^n(1), 1]$ under the action of $f$.

It is easy to see that the bifurcation phenomenon $\Gamma_{n,2n} \implies \Gamma_{n,n}$ occurs when the $2n$-th iteration of the point $x = 1$ maps into the point $x_n$ of the cycle $\gamma_n$. Figures 6 and 7 illustrate this situation for $f$ in the case of $n = 4$. The analytical expression defining this condition,
shown in Fig. 7, is

\[ f_2 f_1^{n-2} f_2 f_1^{n-1}(0) = x_n, \quad (13) \]

where \( f_1 \) and \( f_2 \) denote the linear parts of the map \( f_{1,p} \) (see Fig. 1). Here we used the property \( f_2(1) = 0 \).

We will derive formula (13) in term of the parameters \( l \) and \( p \) later, but for now let us continue to vary the values of the parameters \( l, p \) further. As the magnitude of the slope \( l \) and the magnitude of the slope \( p \) increases (in general, this involves a decrease of the value of \( a \) and an increase of the value of \( b \) as shown in Fig. 1(a)), we come to a situation when \( f^n(1) = \bar{x}_n \) (see Fig. 8).

At this moment the bifurcation phenomenon \( \Gamma_{1,1} \Rightarrow \Gamma_{1,0} = [0,1] \) occurs (curve \([C,n]\))

The stable interval cycle of period \( n \) bifurcates into a stable interval cycle of period 1. It is easy to see that the condition for this bifurcation is

\[ f^{n-1}(0) = \bar{x}_n. \quad (14) \]

Figure 9 shows this situation for a cycle of period 4.

To derive conditions (13) and (14) in terms of the parameters \((l, p)\), we must first derive the formulas for the points \( x_n \) and \( \bar{x}_n \) belonging to the cycles \( \gamma_n \) and \( \bar{\gamma}_n \). The point \( x_n \) is defined by the equation

\[ f_1^{n-1} f_2(x) = x; \quad (15) \]

The point \( \bar{x}_n \) is defined by the equation

\[ f_2 f_1^{n-2} f_2(x) = x. \quad (16) \]

Since these equations are linear, we can solve them for \( x_n \) and \( \bar{x}_n \) as follow:

\[ x_n = 1 + \frac{1}{p} + \frac{L_{n-1}}{(l^{n-1}p - 1)p}, \quad (17) \]

\[ \bar{x}_n = 1 + \frac{1}{p} + \frac{1 + pL_{n-2}}{(l^{n-2}p^2 - 1)p}. \quad (18) \]

Substituting (17) into (13) and using the expression for \( f_1 \) and \( f_2 \) (see (1)), we obtain the following relation between \( l \) and \( p \), which defines the bifurcation \( \Gamma_{1,2n} \Rightarrow \Gamma_{n,n} \):

\[ l^{2n-4}p^4 + l^{2(n-1)}L_{n-2}p^3 - l^{n-2}p^2 + (l^{n-1} - L_{n-2})p + lL_{n-2} = 0. \quad (19) \]

It should be noted that the bifurcation curve \([E,n]\) satisfies the relation (19) (the relation (13) is satisfied upon the birth of the cycles \( \gamma_n \) and \( \bar{\gamma}_n \)). Therefore, if we eliminate the factor \( l^{n-2} + L_{n-2} \), we will obtain

\[ l^{2(n-1)}p^3 - p + l = 0. \quad (20) \]

Equation (20) defines the bifurcation curve \([D,n]\). As an example, for \( n = 2 \) we obtain the curve \([D,2]\)

\[ l^2p^3 - p + l = 0, \quad (21) \]
which can be solved explicitly for $l$:

$$l = \frac{-1 - \sqrt{1 + 4p^4}}{2p^3}.$$ 

The equation for the curve $[D, 3]$ is given by

$$l^4p^3 - p + l = 0. \tag{22}$$

Substituting (18) into (14), we obtain a relation between $l$ and $p$, which defines the bifurcation phenomenon $\Gamma_{n,n} \Rightarrow \Gamma_{n,1}$:

$$l^{2n-3}p^3 + l^{n-2}L_{n-1}p^2 + (L_{n-2} - l^{n-1})p - lL_{n-2} = 0. \tag{23}$$

The curve $[E, n]$ satisfies the relations (23) and (19). Therefore, if we divide the left side of the relation by $l^{n-2}p + L_{n-2}$ we will obtain

$$l^{n-1}p^2 + p - l = 0. \tag{24}$$

Equation (24) defines the bifurcation curve $[C, n]$. In particular, we have the curve $[C, 2]$

$$lp^2 + p - l = 0, \quad \left( l = \frac{p}{1 - p^2} \right) \tag{25}$$

for $n = 2$, and the curve $[C, 3]$

$$l^2p^2 + p - l = 0 \tag{26}$$

for $n = 3$.

Therefore, the map $f_{i,p}$ has a stable interval cycle of period $2n$ in the regions $\Pi_{n,2}$, bounded by the curves $[S, n]$, $[E, n]$, $[D, n]$, and a period-$n$ stable interval cycle for the region $\Pi_{n,1}$, bounded by the curves $[D, n]$, $[E, n]$, $[C, n]$, for all $n = 2, 3, ...$.

*This completes our proof of theorem 2.*

**Remark:** Although theorem 2 was formulated for $n > 2$, it is also true for $n = 2$ except for some neighborhood of the point

$$(l, p) = (1, -1). \tag{27}$$

The curves $[E, n]$, $[S, n]$, $[D, n]$, $[C, n]$ for the first 3 values of $n$ ($n = 2, 3, 4$) and the regions $\Pi_n, \Pi_{n,2}, \Pi_{n,1}$ are shown in the Fig. 10.

### 3 Period-doubling bifurcation of interval cycles ($n = 2$)

In this section we consider in detail the case $n = 2$. We will study the bifurcations phenomena which are observed when a period-2 point cycle $\gamma_2$ loses its stability. We will show that this case is different from the cases $n > 2$, which were described by theorem 2. The difference is in the appearance of an attracting interval cycle of period $2^m$ for all integers $m$. This bifurcation sequence occurs when the point $(l, p)$ passes through the curve $[S, 2]$. Moreover, if the curve $[S, 2]$ is crossed by varying the parameter $(l, p)$ though the point $(l, p) = (1, -1)$,
then an interval cycle of period \(2^\infty\) appears. Subsequent parameter variations lead to an inverse period-doubling bifurcation of interval cycles.

In section 2 we have given the formulas for the bifurcation curves \([D,n]\) and \([C,n]\), \(n = 2, 3, \ldots\). As Fig. 10 shows, the curve \([D,n]\) separates regions of stable interval cycles \(\Gamma_{n,2^n}\) and \(\Gamma_{n,n}\) of periods \(2^n\) and \(n\), respectively. Analogously, the curve \([C,n]\) separates regions of stable interval cycles \(\Gamma_{n,n}\) and \(\Gamma_{n,1}\) of periods \(n\) and 1, respectively.

Let us consider in detail a parameter point on the curve \([S,n]\) where a period-\(n\) cycle \(\gamma_n\) loses its stability. Figure 5 shows a part of the graph of the map \(f^n\) at this parameter point; namely, the "tent-like" map from the extreme right position in Fig. 4. Let us examine the \(f^{2n}\) graph (see Fig. 11) at once after crossing this parameter bifurcation point. Here \(\{x_{n,1}, x_{n,2}\}\) is a point cycle of period 2 for the map \(f^n\). Does there exist a stable interval cycle of period 2 for the map \(f^{2n}\)? It follows from the arguments in the preceding section that this interval cycle exists if, and only if, the value of the second iteration of the point \(x = 1\) under the action of \(f^{2n}\) is greater than \(x_{n,2}\); i.e., \(f^{4n}(1) > x_{n,2}\). This inequality must be fulfilled at the bifurcation point \((l,p) \in [S,n]\), when the slope of the extreme right segment of the graph, shown in Fig. 12, is equal to \(p/l\), but the slope of the second rightmost segment is equal to 1.

Let us consider an auxiliary map \(g\) in the form of (1) with slopes 1 and \(p' = p/l\) (Fig. 12), respectively. If the map \(g\) has an interval cycle of period 2, then the original map \(f\) will have an interval cycle of period \(4n\) as \((l,p)\) crosses the curve \([S,n]\). It follows from (25) that the condition for the existence of an interval cycle of period 2 is \((p')^2 + p' - 1 < 0\), i.e.

\[
\left(\frac{p}{l}\right)^2 + \frac{p}{l} - 1 < 0, \quad (28)
\]

or:

\[
p > -\frac{1 + \sqrt{5}}{2}. \quad (29)
\]

Therefore, if at the parameter point where the cycle \(\gamma_n\) loses its stability (i.e. for \((l,p) \in [S,n]\)) the condition (29) is violated, then, the interval cycle \(\Gamma_{n,4n}\) of period \(4n\) will not occur. Instead, we will have an interval cycle \(\Gamma_{n,2n}\) of period \(2n\). It is easy to see (Fig. 10) that in region \(\Pi\) the straight line

\[
p = -\frac{1 + \sqrt{5}}{2} \quad (30)
\]

is situated above the regions \(\Pi_3, \Pi_4, \ldots\). Therefore the loss of stability of the cycle \(\gamma_n\) leads to the birth of a stable interval cycle \(\Gamma_{n,2n}\) of period \(2n\), for any \(n = 3, 4, \ldots\).

Let us consider the cycle of period 2. The straight line (30) passes through the curve \((S,2)\) at the point

\[
(l,p) = \left(\sqrt{\frac{2}{1+\sqrt{5}}}, \frac{-\sqrt{1+\sqrt{5}}}{2}\right). \quad (31)
\]

Consequently, if \(l < \sqrt{2}/(1+\sqrt{5})\), when this cycle loses its stability at \((l,p) \in [S,2]\), then a period-doubling bifurcation of the interval cycle \(\Gamma_{2,4}\) will occur. Otherwise, a stable interval cycle of some periods \(2^3, 2^4, \ldots\) will occur.
Let us consider the map \( f \) and its iterated maps \( f^{2m}, m = 1,2, \ldots \). The slope of the rightmost segment of the graph \( f^{2m} \) is equal to

\[
p^{(2m)} = p^{2m} - p^{2m+1}, \quad m = 0,1,\ldots, \tag{32}
\]

where \( \alpha_m \) is a solution of the difference equation

\[
\alpha_{m+1} = \alpha_m + 2\alpha_{m-1}, \quad m = 1,2,\ldots, \tag{33}
\]

with initial conditions \( \alpha_0 = 0, \) and \( \alpha_1 = 1. \) This solution is equal to

\[
\alpha_m = \frac{1}{3} \left( 2^m + (-1)^m \right), \quad m = 0,1,\ldots, \tag{34}
\]

The slope of the second rightmost segment of the graph \( f^{2m} \) is equal to

\[
l^{(2m)} = l^{2m-1} - l^{2m+1}, \quad m = 0,1,\ldots. \tag{35}
\]

As an example, the slopes for \( m = 1,2,\ldots, 6 \) are given below:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( l^{(2m)} )</th>
<th>( p^{(2m)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>( p )</td>
</tr>
<tr>
<td>1</td>
<td>( l^2 )</td>
<td>( lp )</td>
</tr>
<tr>
<td>2</td>
<td>( l^2p^2 )</td>
<td>( lp^3 )</td>
</tr>
<tr>
<td>3</td>
<td>( l^2p^6 )</td>
<td>( lp^8 )</td>
</tr>
<tr>
<td>4</td>
<td>( l^5p^{10} )</td>
<td>( lp^{11} )</td>
</tr>
<tr>
<td>5</td>
<td>( l^5p^{22} )</td>
<td>( lp^{21} )</td>
</tr>
<tr>
<td>6</td>
<td>( l^2p^{42} )</td>
<td>( lp^{43} )</td>
</tr>
</tbody>
</table>

In order that the original map \( f_{i,p} \) has an interval cycle of period \( 2^{m+1} \), it is necessary and sufficient that \( f^{2m} \) has an interval cycle of period 2. Granting this and using formulas (32), (35) and (25) we obtain the following equation of the curve for the bifurcation phenomenon \( \Gamma_{2,2m} \Rightarrow \Gamma_{2,2m+1} \):

\[
p^{m+1}l^{m} + (-1)^m(p - l) = 0 \quad m = 0,1,\ldots, \tag{36}
\]

where \( \delta_m, \quad m = 0,1,\ldots, \) is the solution of the inhomogeneous difference equation

\[
\delta_{m+1} = 2\delta_m + \frac{1}{2}(1 + (-1)^m), \quad m = 1,2,\ldots, \tag{37}
\]

with initial condition \( \delta_0 = 1. \)

The bifurcation curves defined by equations (36) are denoted by \( [D,2,2^m] \), for any \( m = 0,1,\ldots. \) It should be noted that \( [D,2,2^0] = [C,2], [D,2,2^1] = [D,2] \). The regions bounded by the curves \( [D,2,2^{m-1}], [D,2,2^m], [S,2] \) and \( [E,2], \) are denoted by \( \Pi_{2,2m}. \)

It follows that the following theorem is true for any \( m = 1,2,\ldots. \)

**Theorem 3** Let \( (l,p) \in \Pi_{2,2m}. \) Then the map \( f_{i,p} \) has a stable interval cycle of period \( 2^m. \)
Figure 13 shows the bifurcation curves \([D,2,2^m]\) converge to the point \((l, p) = (1, -1)\). As an example, the equations of these curves for \(m = 0, 1, 2, \ldots, 6\) are as follow:

\[
\begin{align*}
p^2l + p - l &= 0, \quad [D, 2, 1]; \\
p^3l^2 - p + l &= 0, \quad [D, 2, 2]; \\
p^6l^3 + p - l &= 0, \quad [D, 2, 2^2]; \\
p^{11}l^4 - p + l &= 0, \quad [D, 2, 2^3]; \\
p^{22}l^5 + p - l &= 0, \quad [D, 2, 2^4]; \\
p^{43}l^6 - p + l &= 0, \quad [D, 2, 2^5]; \\
p^{86}l^7 + p - l &= 0, \quad [D, 2, 2^6].
\end{align*}
\tag{38}
\]

Theorems 1-3 allow us to conclude that in the general case of a one-dimensional piecewise-linear map with one extremum, the following ordering of attractor bifurcations must occur:

\[
\gamma_1 \Rightarrow \gamma_2 \Rightarrow (\Gamma_{2,2^1} \Rightarrow \Gamma_{2,2^{k-1}} \Rightarrow \cdots \Rightarrow \Gamma_{2,2} \Rightarrow I) \Rightarrow \gamma_3 \Rightarrow \cdots \Rightarrow (\Gamma_{n,2} \Rightarrow \Gamma_{n,n} \Rightarrow I) \Rightarrow \gamma_{n+1} \Rightarrow \cdots.
\]

4 Universal constants of period-doubling bifurcation of interval cycles

Since the period-doubling bifurcation curves have been found in explicit forms (see (36), (37)), we can derive two universal constants \(\delta\) and \(\alpha\) for period-doubling bifurcations of interval cycles, just like the Feigenbaum's constants, for period-doubling point cycles. To define the constants \(\delta\) and \(\alpha\) we consider in the \((l, p)\) parameter space any straight line \(p = k(l-1) + 1\), which passes through the point \((l, p) = (1, -1)\). Let \((l^{(m)}, p^{(m)})\), \(m = 0, 1, ...,\) be the intersection point of this straight line with the bifurcation curve in the form of (37) for some given fixed \(m\). The distance between the points \((l^{(m)}, p^{(m)})\) and \((l^{(m+1)}, p^{(m+1)})\) is denoted by \(d_m\) for any \(m = 0, 1, ...\). Then the constant \(\delta\) is defined as

\[
\delta = \lim_{m \to \infty} \frac{d_m}{d_{m+1}}.
\tag{39}
\]

Analogously the constant \(\alpha\) is defined as

\[
\alpha = \lim_{m \to \infty} \frac{1 - x_m}{1 - x_{m+1}},
\tag{40}
\]

where \(x_m = x_m(l^{(m)}, p^{(m)})\) and \(x_{m+1} = x_{m+1}(l^{(m+1)}, p^{(m+1)})\) are point cycles of periods \(2^m\) and \(2^{m+1}\), defined by formulas (32) and (35), respectively. These points are calculated with the following bifurcation conditions: \(x_m\) at \((l, p) = (l^{(m)}, p^{(m)})\) and \(x_{m+1}\) at \((l, p) = (l^{(m+1)}, p^{(m+1)})\).

We will say that the family of maps \(f_{l,p}\) at the point \((l, p) = (1, -1)\) is characterized by an universal behavior with constant \(\delta\) and \(\alpha\), if the limits in (39) and (40) exist and do not depend on choice of the straight line through the point \((l, p) = (1, -1)\).
Theorem 4 The family of maps $f_{l,p}$ is characterized by an universal behavior at the point $(l, p) = (1, -1)$ with universal constants $\delta = 2$ and $\alpha = \infty$.

Proof. Let us first prove the existence of the universal constant $\delta = 2$. The proof will be carried out for the case $l = 1$, i.e. when the slope of the straight line is equal to $\infty$ (see fig. 13).

The intersection point of the straight line $l = 1$ and the bifurcation curve $[D, 2 \cdot 2^{n-1}]$ is denoted by $p_m$ for all $m = 1, 2, \ldots$. This bifurcation curve is the curve of the interval cycle of period $2^m$. Then

$$\delta = \lim_{m \to \infty} \left| \frac{p_{m-1} - p_m}{p_m - p_{m+1}} \right|.$$ 

We will prove, that this limit exists and is equal to 2.

Let us consider the family of functions $y_n(x) = x^n - 1, x > 1, n = 1, 2, \ldots$. Let $x_n$ be the root of the equation $x_n = x_n^n - 1$, which is nearest to $x = 1$ with $x_n \geq 1$. Graphically, $x_n$ is the abscissa of the intersection point of the graph $y = y_n(x)$ and the bisectrix $y = x$ (Fig. 14).

Lemma 1. The sequence $x_n, n = 1, 2, \ldots$, has the property

$$\lim_{n \to \infty} \left| \frac{x_{2n} - x_n}{x_{2n} - x_{4n}} \right| = 2.$$ 

Proof. Let us estimate the distance between the points $x_n$ and $\sqrt{2}$. Using the boundary condition $y_n(\sqrt{2}) = 1$, we find the derivative

$$y' = n x^{n-1} \big|_{x = \sqrt{2}} = n 2^{n-1} = \frac{2n}{\sqrt{2}}.$$ 

Then the equation of the tangent at the point $(\sqrt{2}, 1)$ has the form $y = (2n/ (\sqrt{2}))x - 2n + 1$. The tangent crosses the bisectrix at the point $x = x_n^*$, where

$$x_n^* = (2n - 1)/(\sqrt{2} - 1).$$ 

Assuming $x_n^* > x_n$, then

$$x_n^* - \sqrt{2} = \sqrt{2} \frac{\sqrt{2} - 1}{2n - \sqrt{2}} \equiv \varepsilon_n.$$ 

Let us prove that $\varepsilon_n$ is a higher-degree infinitesimal than $\sqrt{2} - x_n^*$. Indeed we have

$$\lim_{n \to \infty} \frac{\sqrt{2}(\sqrt{2} - 1)}{(2n - \sqrt{2})(\sqrt{2} - 2\sqrt{2})} = \lim_{n \to \infty} \frac{(\sqrt{2} - 1)(3\sqrt{2} + 1)}{(2n - \sqrt{2})(\sqrt{2} - 1)} = 0.$$ 

Moreover, it is easy to see that $\varepsilon_n \sim 1/n$, as $n \to \infty$. Then

$$\lim_{n \to \infty} \left| \frac{x_n - x_{2n}}{x_{2n} - x_{4n}} \right| \leq \lim_{n \to \infty} \frac{\sqrt{2} - 2\sqrt{2} + (\varepsilon_n + \varepsilon_{2n})}{2\sqrt{2} - 4\sqrt{2} - (\varepsilon_{2n} + \varepsilon_{4n})} = 2.$$ 

This completes our proof of lemma 1.
Our calculations give the following results for \( p_i, i = 1, 2, \ldots, 10 \):

\[
\begin{align*}
p_1 &= -1.618022, & p_2 &= -1.324698, \\
p_3 &= -1.134732, & p_4 &= -1.068296, \\
p_5 &= -1.032771, & p_6 &= -1.016444, \\
p_7 &= -1.008140, & p_8 &= -1.004074, \\
p_9 &= -1.002032, & p_{10} &= -1.001017.
\end{align*}
\]

Using these numbers, we obtain the following approximations for \( \delta \):

\[
\begin{align*}
\delta_1 &= 1.544, & \delta_2 &= 2.860, \\
\delta_3 &= 1.87, & \delta_4 &= 2.18, \\
\delta_5 &= 1.97, & \delta_6 &= 2.04, \\
\delta_7 &= 1.99, & \delta_8 &= 2.02.
\end{align*}
\]

To obtain the universal constant \( \alpha \) we consider point cycles \( x_{2m} \) of period \( 2^m \) on the bifurcation curves \([D, 2, 2^{m-1}]\). Then

\[
\alpha = \lim_{m \to \infty} \frac{x_{2m-1} - x_{2m}}{x_{2m} - x_{2m+1}}.
\]

The constant \( \alpha \) was obtained by using \( x_{2m} \) in the following algorithm. Let \( x_{2m} \) be a root of the linear equation \( a_m x + b_m = x \), where \((a_m, b_m)\) is the result obtained after \( m \) iterations of the map

\[
G_n : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} p^{(-1)^{m+1}} a^2 \\ p^{(-1)^{m+1}} (b - ab - 1) + 1 \end{pmatrix},
\]

where \( n = 1, 2, \ldots, m \). The map \( G_n \) is employed at the point \((a, b) = (p, -p - 1/p)\) for the value \( p = p_n \), on the bifurcation curve \([D, 2, 2^{m-1}]\). That is

\[
\begin{pmatrix} a_m \\ b_m \end{pmatrix} = G_m \cdots G_2 G_1 \begin{pmatrix} p_m \\ -p_m - 1/p_m \end{pmatrix},
\]

Then we find \( x_{2m} = -b_m/(a_m - 1) \) for all \( m = 1, 2, \ldots \). It should be noted that the initial condition \((p_m, -p_m - 1/p_m)\) varies with \( m \).

Using this algorithm the following results are obtained

\[
\begin{align*}
\alpha_1 &= 2.820, & \alpha_2 &= 14.058, & \alpha_3 &= 17.777, \\
\alpha_4 &= 50.462, & \alpha_5 &= 84.501, & \alpha_6 &= 190.896, \\
\alpha_7 &= 358.839, & \alpha_8 &= 672.111.
\end{align*}
\]

It follows from the above result that

\[
\alpha = \infty,
\]

where

\[
\alpha_n \sim \alpha_0 \cdot 2^n, \quad n \to \infty, \quad \alpha_0 \approx \sqrt{2}.
\]

This completes our proof of theorem 4.

Four one-dimensional bifurcation diagrams for \( l = 1 \) in successively enlarged scale are shown in the Fig. 15(a-d).
5 References


6 Appendix

There are two cases where a continuous piecewise linear 1D-map $g$ with one breakpoint has a nontrivial invariant interval. Both are for the slopes $l$ and $p$ such as:

$$(l, p) \in \Pi = \{0 \leq l \leq \frac{p}{p+1}, \quad p < -1\}.$$ 

In the first case

$$g : x \mapsto g_{l,p}(x) = \begin{cases} 
lx + A, & x \leq \frac{B-A}{l-p}, \\
px + B, & x > \frac{B-A}{l-p},
\end{cases}$$

where $A$ and $B$ satisfy

$$A > \frac{1 - l}{1 - p} B.$$ 

In the second case

$$g : x \mapsto g_{l,p}(x) = \begin{cases} 
px + B, & x \leq \frac{B-A}{l-p}, \\
lx + A, & x > \frac{B-A}{l-p},
\end{cases}$$

where $A$ and $B$ satisfy

$$A < \frac{1 - l}{1 - p} B.$$ 

It is easy to see that in both cases the map $g$ can be reduced by the linear transformation

$$\sigma : x \mapsto \sigma(x) = 1 + \frac{(1-2p)(l-p)}{[A(1-p) + B(l-1)]p} \left[ x - \frac{1B - pA}{l-p} \right]$$

to obtain a map $f$ in the form (1) with an invariant interval $[0, 1]$:

$$f = \sigma \circ g \circ \sigma^{-1}.$$
7 Figure captions

Fig. 1(a). Graph of piecewise-linear function $f_{l,p}(x)$, with two slopes $l$, and $p$.

Fig. 1(b),(c). Graphs of iterations $f^2_{l,p} = f(f(x))$ and $f^3_{l,p} = f(f(f(x)))$ of the piecewise-linear map $f : x \mapsto f_{l,p}$.

Fig. 2. The existence and stability regions $\Pi_n$ of the point cycles $\gamma_n$ in the parameter space $(p^*, l)$, where $p^* = \log_2(-p)$. Each region $\Pi_n$ is bounded from below by an existence curve $[E, n]$ and from above by a stability curve $[S, n]$.

Fig. 3. The stability regions $\Pi_n$ of the point cycle $\gamma_n$ and $\Pi_{n,2}$, $\Pi_{n,1}$ of the interval cycles $\Gamma_{n,2n}$ and $\Gamma_{n,n}$ of periods $2n$ and $n$ respectively, in the parameter space $(p^*, l)$ for all $n > 2$. The regions $\Pi_{n,2}$ and $\Pi_{n,1}$ are separated by the bifurcation curve $[D, n]$. The curve $[C, n]$ bounds the region $\Pi_{n,1}$ from above.

Fig. 4. The graph of the function $f^n_{l,p}$ when the point $(l, p)$ crosses the curve $[E, n]$ and enters the region $\Pi_n$. The points of the period-$n$ stable cycle are given by $\gamma_n = \{x_1, \ldots, x_n\}$. Those for a period-$n$ unstable cycle are given by $\tilde{\gamma}_n = \{\tilde{x}_1, \ldots, \tilde{x}_n\}$.

Fig. 5. The rightmost upper angle of the graph of the function $f^n_{l,p}$, from Fig. 4 at the moment when the point $(l, p)$ crosses the curve $[S, n]$ and enters the region $\Pi_{n,2}$. At this moment each point $x_n$ of a stable period-$n$ cycle $\gamma_n = \{x_1, \ldots, x_n\}$ creates the interval cycle $\Gamma_{n,2n}$ of period $2n$.

Fig. 6. The rightmost upper angle of the graph of the function $f^n_{l,p}$ from Fig. 4 at the moment when interval cycle $\Gamma_{n,n}$ of period-$n$ is born.

Fig. 7. The graph of the function $f^n_{l,p}$ ($n = 4$) when the point $(l, p)$ crosses the curve $[D, n]$ and an interval cycle $\Gamma_{n,n}$ of period-$n$ was born. At this moment each pair of intervals $[f^n(1), f^{2n}(1)]$ and $[f^{2n}(1), 1]$ of the cycle $\Gamma_{n,2n}$ touched each other and merged into one interval.

Fig. 8. The rightmost upper angle of the graph of the function $f^n_{l,p}$ from Fig. 4 at the moment when interval cycle $\Gamma_{n,1} = [0, 1]$ is born.

Fig. 9. The graph of the function $f^n_{l,p}$ ($n = 4$) when the point $(l, p)$ crosses the curve $[C, n]$ and all intervals of the interval cycle $\Gamma_{n,n}$ merged into one interval $[0, 1]$.

Fig. 10. The stability region $\Pi_n$ of the point cycles $\gamma_n$ and the stability regions $\Pi_{n,2}$ and $\Pi_{n,1}$ of the interval cycles $\Gamma_{n,2n}$, $\Gamma_{n,n}$ of periods $2n$ and $n$, respectively in the parameter space $(p^*, l)$ for $n = 2, 3, 4$. The regions $\Pi_{n,2}$ and $\Pi_{n,1}$ are separated by the bifurcation curve $[D, n]$. The curve $[C, n]$ bounds the region $\Pi_{n,1}$ from above.

Fig. 11. The graph of the function $f^n_{l,p}$ at the moment when the point $(l, p)$ crossed the curve $[S, n]$ and the cycle $\gamma_n$ lost its stability. Here $\{x_{n,1}, x_{n,2}\}$ is a period-2 cycle of $f^n$. 

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Fig. 12. The rightmost upper angle of the graph of the function \( f_{i,p}^{2n} \) from Fig. 11.

Fig. 13. The stability regions \( \Pi_2 \) of the point cycles \( \gamma_2 \) and \( \Pi_{n,2m} \) of the interval cycles \( \Gamma_{2,2m} \) in the parameter space \( (p^*, l) \) for \( m = 0, 1, \ldots \). The regions \( \Pi_{n,2m} \) and \( \Pi_{n,2m+1} \) are bounded by the bifurcation curves \([D, 2, 2^m]\).

Fig. 14. The graph of the functions \( y_n(x) = x^n - 1 \). The point \( x_n \) is the abscissa of the intersection point of the graph \( y = y_n(x) \) and the bisectrix \( y = x \).

Fig. 15(a-d). Four parameter bifurcation diagrams in successively enlarged scale, which illustrate the cascade of period-doubling bifurcations of interval cycles for \( l = 1 \). The bifurcation points \( p_m^* = \log_2(-p_m), m = 1, 2, 3 \ldots \) belong to the curves \([D, 2, 2^m]\).
Fig. 1(a)
Fig. 1(b)
Fig. 1(c)
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 9
Fig. 11
Fig. 12
Fig. 13
Fig. 14
Fig. 15(c)
Fig. 15(d)