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CLUSTERING FORMALISM FOR
SYNCHRONOUS DATAFLOW

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Shuvra S. Bhattacharyya

Memorandum No. UCB/ERL M92/30

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ABSTRACT

This document presents a formalization of "clustering" nodes during the scheduling of a synchronous dataflow (SDF) graph. The interpretation of clustering in this document is specific to our work in scheduling for efficient iteration, and it is possible that this formalism may not apply to clustering in other contexts. The goal of this document is to define our interpretation of a cluster; to define the process by which a cluster in a graph derives a new graph; and then to show that scheduling this new graph is equivalent to scheduling the original graph with the desired clustering effect.

The definitions and theorems presented here apply to the single-processor scheduling problem. The intent presently is to gain intuition about the iteration problem alone, before attempting to combine these considerations with parallel-processor scheduling. It is expected that this formalism will eventually be extended to the multiprocessor case.

This document expands on the discussion in chapter 3 of [1] on "consolidating a subgraph" of an SDF graph.

The following notation will be used throughout this document:

Notation

(N1) An SDF graph is denoted by an ordered pair \( (X, Y) \), where \( X \) is the set of nodes and \( Y \) is the set of arcs.

(N2) For an SDF arc \( \alpha \), \( \text{source}(\alpha) \) and \( \text{sink}(\alpha) \) denote respectively the source node and sink node of \( \alpha \). \( p(\alpha) \) denotes the number of samples produced on \( \alpha \) for every invocation of \( \text{source}(\alpha) \). Similarly, \( c(\alpha) \) denotes the number of samples consumed from \( \alpha \) by every invocation of \( \text{sink}(\alpha) \). Finally, \( \text{delay}(\alpha) \) denotes the delay on \( \alpha \).

(N3) We use the term MPASS (minimum-PASS) to denote a PASS (periodic admissible sequential schedule) of blocking factor one.

(N4) For set operands, we use "\( + \)" and "\( - \)" to denote the union and exclusion operations respectively.

Definition 1 Let \( G = (N, A) \) be an SDF graph; let \( S \) be a PASS for \( G \); and let \( V \) be a connected subset of \( N \). Then we say that \( V \) is a cluster in \( G \) for \( S \) if and only if \( S \) can be expressed as

\[
(1) \quad \Xi_0 \Pi_1 \Xi_1 \Pi_2 \cdots \Xi_{n-1} \Pi_n \Xi_n,
\]

where each \( \Xi_i \) is a (possibly empty) sequence of invocations of nodes in \( N - V \), and each \( \Pi_i \) is an MPASS for the subgraph associated with \( V \).

Observe that if the decomposition (1) exists for a schedule \( S \) and subset \( V \), then it must be unique for \((S, V)\). Thus we can define

\[
\Xi(S, V, G) = \{\Xi_0, \Xi_1, \Xi_2, \ldots, \Xi_n\}, \text{ and } \\
\Pi(S, V, G) = \{\Pi_1, \Pi_2, \ldots, \Pi_n\}
\]
whenever $V$ is a cluster in $G$ for $S$.

* * * * *

The following assumptions will be used throughout the remainder of this document.

Assumptions

(A1) Let $G = (N, A)$ be an SDF graph for which a PASS exists.

(A2) Let $V$ be a connected subset of $N$ and let $G_v = (V, A_v)$ be the subgraph associated with $V$.

(A3) Define a node $n_v$ (to represent an instance of the cluster $V$).

(A4) Let $\Gamma_v$ be the topology matrix for $G_v$ and let $q_v$ be the smallest positive integer vector in the null space of $\Gamma_v$. The existence of $q_v$ is guaranteed by lemma 1 below and theorem 4 in chapter 2 of [1].

(A5) For each arc $\alpha \in \Lambda$ which is directed from a node $n_1 \in N-V$ to a node $n_2 \in V$, define the arc $\hat{\alpha}$ to be directed from $n_1$ to $n_2$, with $p(\hat{\alpha}) = p(\alpha)$, $c(\hat{\alpha}) = q_v[n_2]c(\alpha)$, and $\text{delay}(\hat{\alpha}) = \text{delay}(\alpha)$.

(A6) Similarly, for each arc $\beta \in \Lambda$ which is directed from a node $n_1 \in V$ to a node $n_2 \in N-V$, define the arc $\hat{\beta}$ to be directed from $n_1$ to $n_2$, with $p(\hat{\beta}) = q_v[n_1]p(\beta)$, $c(\hat{\beta}) = c(\beta)$, and $\text{delay}(\hat{\beta}) = \text{delay}(\alpha)$.

(A7) Let $\Lambda_1 = \{\gamma \in \Lambda \mid \text{source}(\gamma) \in N-V, \text{and sink}(\gamma) \in V\}$;

$\Lambda_2 = \{\gamma \in \Lambda \mid \text{source}(\gamma) \in V, \text{and sink}(\gamma) \in N-V\}$;

$\hat{\Lambda}_1 = \{\gamma \mid \gamma \in \Lambda_1\}$;

$\hat{\Lambda}_2 = \{\gamma \mid \gamma \in \Lambda_2\}$; and

$\hat{\Lambda} = \hat{\Lambda}_1 + \hat{\Lambda}_2$.

Definition 2 We define $F_v(G)$, called the graph obtained by clustering $V$ in $G$, by

$$F_v(G) = (N-V + \{n_v\}, \Lambda - \Lambda_v + \hat{\Lambda} - \Lambda_1 - \Lambda_2)$$

Lemma 1 Assume A1-A2 and suppose $S$ is a PASS for $G$. Then the schedule $S'$, obtained by eliminating from $S$ the firings of nodes not in $V$, is a PASS for the subgraph associated with $V$.

Proof The subset of firings in $S$ which produce or consume data on arcs in $G_v$ is precisely $S'$. Let $x$ be a firing in $S'$ and suppose $x$ does not have enough data on input arc $\alpha \in \Lambda_v$ to fire. Then there must be some $n \in V$, which is fired before $x$ in $S$ — but not fired in $S'$ — to produce one or more of the samples which $S'$ is "missing" on $\alpha$ for $x$. But this contradicts our definition of $S'$, and it follows that $S'$ is admissible. Furthermore, since $S$ leaves no net change in the number of samples on any arc in $G$, $S'$ leaves no net change on any arc in $G_v$. We conclude that $S'$ is a PASS for $G_v$.

Definition 3 Let $G$ be an SDF graph and let $S$ be a schedule for $G$. Given an arc $\alpha$ in $G$ and a firing $f$ in $S$, we define

$$b_S(\alpha, f) =$$

$$(\text{the number of times source}(\alpha) \text{ is invoked in } S \text{ before } f) \cdot p(\alpha)$$

$$- (\text{the number of times sink}(\alpha) \text{ is invoked in } S \text{ before } f) \cdot c(\alpha)$$

$$+ \text{delay}(\alpha)$$

- 2 -
If \( S_1 \) is a nonempty subschedule (a subsequence of firings) of \( S \) then we define \( b_s(\alpha, S_1) = b_S(\alpha, f_1) \), where \( f_1 \) is the first firing in \( S_1 \).

The following fact is immediately apparent from definition 3:

**Fact 1:** If \( S \) is an admissible schedule, then \( b_s(\alpha, f) \) equals the number of samples on \( \alpha \) immediately before \( f \) is fired in \( S \).

**Lemma 2** Assume the hypotheses and notation developed in A1-A7 above. Let \( S \) be a schedule for \( F_v(G) \) and let \( S' \) be the schedule for \( G \) which results from replacing in \( S \) each firing of \( n_v \) with an MPASS for \( G_v \). Suppose \( S_i \) is the \( i \) th MPASS for \( G_v \) in \( S' \). Then:

**A** \( \alpha \in \Lambda_1 \Rightarrow b_s(\alpha, \text{the } i \text{ th invocation of } n_v) = b_S(\alpha, S_i) \); and

**B** \( \alpha \in \Lambda_2 \Rightarrow b_s(\alpha, \text{the } i \text{ th invocation of } \text{sink}(\hat{\alpha})) = b_S(\alpha, \text{the } i \text{ th invocation of } \text{sink}(\hat{\alpha})) \).

**Proof of A**

From our assumptions, \( S \) and \( S' \) can be decomposed as:

\[
(2A) \quad S = \Xi_0 \Xi_1 \Xi_2 \cdots \Xi_{M-1} \Xi_M \\
(2B) \quad S' = \Xi_0 \Pi_1 \Xi_1 \Pi_2 \cdots \Xi_{M-1} \Pi_M \Xi_M,
\]

where each \( \Xi_i \) is a subschedule involving nodes in \( N-V \) and each \( \Pi_i \) is an MPASS for \( G_v \). We wish to show that for \( 1 \leq i \leq M \), \( b_s(\alpha, S_i) = b_S(\alpha, \text{the } i \text{ th invocation of } n_v) \). Now,

\[
(3) \quad b_s(\alpha, S_i) = \\
\{\text{the number of times source}(\alpha) \text{ is invoked in } \Xi_0^- \Xi_{i-1}^+ \} \ast p(\alpha) - (i-1) \ast \{\text{the number of times sink}(\alpha) \text{ is invoked in an MPASS for } G_v \} \ast c(\alpha) + \text{delay}(\alpha)
\]

The second term in (3) is equal to

\[
(i-1) \ast q_r[\text{sink}(\alpha)] \ast c(\alpha) \\
= (i-1) \ast c(\hat{\alpha}). \quad \text{(from A5)}
\]

Also, from A5, \( p(\hat{\alpha}) = p(\alpha), \text{delay}(\hat{\alpha}) = \text{delay}(\alpha), \) and \( \text{source}(\hat{\alpha}) = \text{source}(\alpha) \), so (3) becomes

\[
\{\text{the number of times source}(\hat{\alpha}) \text{ is invoked in } \Xi_0^- \Xi_{i-1}^+ \} \ast p(\hat{\alpha}) - (i-1) \ast c(\hat{\alpha}) + \text{delay}(\hat{\alpha}),
\]

which, from definition 3, is clearly equal to \( b_s(\hat{\alpha}, \text{the } i \text{ th invocation of } n_v) \).

**Proof of B**

Suppose the \( i \) th invocation of \( \text{sink}(\alpha) \) in \( S' \) is in \( \Xi_k \) of decomposition 2B. Then, since \( \text{sink}(\alpha) \) is not in \( V \), it is clear that the \( i \) th invocation of \( \text{sink}(\hat{\alpha}) \) — which is the same as \( \text{sink}(\alpha) \) — in \( S \) is in \( \Xi_k \) of decomposition 2A. Now since subschedules \( \Pi_1, \Pi_k \) precede \( \Xi_k \) in \( S' \),

\[
b_s(\alpha, \text{the } i \text{ th invocation of } \text{sink}(\alpha)) = k q_r[\text{source}(\alpha)] \ast p(\alpha) - (i-1) \ast c(\alpha) + \text{delay}(\alpha),
\]

which from A5-A6 equals:

\[
k \ast p(\hat{\alpha}) - (i-1) \ast c(\hat{\alpha}) + \text{delay}(\hat{\alpha})
\]

\[
= b_s(\hat{\alpha}, \text{the } i \text{ th invocation of } \text{sink}(\hat{\alpha})).
\]
Lemma 3 Let $G = (N, A)$ be an SDF graph. Suppose $V$ is a connected subset of $N$ and let $G_v = (V, A_v)$ denote the subgraph associated with $V$. Let $\Gamma$, $\Gamma_v$, and $\Gamma_F$ respectively denote topology matrices for $G$, $G_v$, and $F_v(G)$. Finally, suppose that PASSes for $G$ and $G_v$ exist, and let $q$, $q_v$ and $q_F$ be the smallest positive-integer vectors in the left-side-null-spaces of $\Gamma$, $\Gamma_v$ and $\Gamma_F$, respectively.

Then, $x \in V \Rightarrow q(x) = q_F(n_v) \ast q_v(x)$.

Proof
$
\Gamma$ can be expressed as

$$
\begin{bmatrix}
\Gamma_1 & 0 \\
\Gamma_2 & \Gamma_v
\end{bmatrix},
$$

where $\Gamma_1$ denotes the submatrix associated with the interaction between nodes in $N - V$ and arcs in $A - A_v$, and $\Gamma_2$ denotes the submatrix associated with $V$ and $A - A_v$.

Also, $q$ can be decomposed as

$$
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix},
$$

where $q_1$ denotes the subvector associated with nodes in $N - V$ and $q_2$ denotes the entries associated with the elements of $V$.

With these decompositions, we have that

$$
\begin{bmatrix}
q_1 & q_2
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 & 0 \\
\Gamma_2 & \Gamma_v
\end{bmatrix} = 0,
$$

which implies that

$$
q_1\Gamma_1 + q_2\Gamma_2 = 0
$$

and

$$
q_2\Gamma_v = 0.
$$

Thus $q_2$ must be some positive integer multiple $k$ of $q_v$, i.e. $q_2 = kq_v$ for some $k \in \{1, 2, 3, \ldots\}$

Now from A5-A6, it can be deduced that we may express $\Gamma_F$ as

$$
\begin{bmatrix}
\Gamma_1 \\
q_v\Gamma_2
\end{bmatrix},
$$

where $q_v\Gamma_2$ determines the row corresponding to the node $n_v$. It follows then, that

$$
q' = \begin{bmatrix} q_1 & k \end{bmatrix}
$$

is in the null space of $\Gamma_F$, since
\[ q_1 \Gamma_1 + k q_2 \Gamma_2 = q_1 \Gamma_1 + q_2 \Gamma_2 = 0 \]

Now suppose that there was an integer \( r > 1 \) such that
\[
\frac{q'}{r}
\]
was also a positive integer vector in \( \eta(\Gamma_F) \). Then
\[
\frac{q_1}{r} \Gamma_1 + \frac{k q_2}{r} \Gamma_2 = \frac{q_1}{r} \Gamma_1 + \frac{q_2}{r} \Gamma_2 = 0
\]
and thus \( \frac{q'}{r} \) is a positive integer vector in the null space of \( \Gamma \). But this contradicts our assumption that \( q \) is the smallest such vector.

We conclude that
\[
q_F = \begin{bmatrix} q_1 & k \end{bmatrix}.
\]
Thus, if \( x \in V \) then \( q[x] = q_2[x] = k q_2[x] = q_F[n_2][q_2[x]$. 

QED.

**Theorem 1** Assume A1-A7. Let \( S_v \) be an PASS of blocking-factor \( k \) for \( F_v(G) \). Then the schedule \( S' \), obtained by substituting each occurrence of \( n_v \) in \( S_v \) with an MPASS for \( G_v \), is a PASS of blocking-factor \( k \) for \( G \), and \( V \) is a cluster in \( G \) for \( S' \).

**Proof**

From these assumptions, \( S_v \) and \( S' \) can be decomposed as \( S \) and \( S' \) are (respectively) decomposed in 2A-2B. Thus

\[
\begin{align*}
(4A) & \quad S_v = \Xi_0 n_v \Xi_1 n_v \Xi_{M-1} n_v \Xi_M \\
(4B) & \quad S' = \Xi_0 \Pi_1 \Xi_1 \Pi_2 \Xi_{M-1} \Pi_M \Xi_M,
\end{align*}
\]

and each \( \Xi_i \) is a subschedule involving nodes in \( N - V \), and each \( \Pi_i \) is an MPASS for \( G_v \). It is obvious from (4B) that \( V \) is a cluster in \( G \) for \( S' \).

Now we show by contradiction that \( S' \) is admissible. Suppose that \( x \) is the first firing in \( S' \) which does not have enough data to fire, and suppose \( x \) is "missing" samples from arc \( \alpha \).

(A) Suppose \( \alpha \in \Lambda - \Lambda_v - \Lambda_1 - \Lambda_2 \). Then \( \alpha \) is in both \( G \) and \( F_v(G) \). Since neither \( n_v \) nor any \( \Pi_i \) interacts with \( (\Lambda - \Lambda_v - \Lambda_1 - \Lambda_2) \), it follows that \( x \in \Xi_i \), and that \( x \) does not have enough samples from \( \alpha \) in \( S_v \), as well. This contradicts the assumption that \( S_v \) is admissible.

(B) Suppose \( \alpha \in \Lambda_v \). Then \( x \in \Xi_i \). Since the \( \Xi_i \)'s don't interact with \( \Lambda_v \), it follows that \( \Pi_i \) is not admissible, which is another contradiction.

(C) Suppose \( \alpha \in \Lambda_1 \). Then \( x \in \Xi_i \). Since \( \text{source}(\alpha) \) is outside \( G_v \), \( b_{\Sigma}(\alpha, \Pi_i) < q_2[sink(\alpha)] * c(\alpha) \). From Lemma 2A and assumption A5, this inequality implies that \( b_{\Sigma}(\alpha, \text{the } i \text{th invocation of } n_v) < c(\alpha) \). From fact 1, we see that this last inequality contradicts our assumption that \( S_v \) is a admissible.

(D) Finally, suppose that \( \alpha \in \Lambda_2 \). Then when \( x \) denotes some firing, say the \( i \)th one, of \( \text{sink}(\alpha) \) and \( b_{\Sigma}(\alpha, \text{the } i \text{th invocation of } \text{sink}(\alpha)) < c(\alpha) \).
From lemma 2B and assumption A5, this inequality implies that $h_{\mathcal{G}}(\alpha_i, \text{the } i\text{th invocation of sink}(\alpha)) < c(\alpha)$. From fact 1, this contradicts our assumption that $S_v$ is admissible.

(A)-(D) together prove that the assumption that $S'$ is not admissible cannot hold.

To prove that $S'$ is periodic, we must show that $S'$ produces no net change in the number of samples on any arc $\alpha$ in $G$.

(E) Suppose $\alpha \in \Lambda - \Lambda_1 - \Lambda_2 - \Lambda_v$. Then $\alpha \in F_v(G)$ and source($\alpha$) and sink($\alpha$) are the same in $G$ and $F_v(G)$. Furthermore, for both schedules, source($\alpha$) and sink($\alpha$) are fired only within the $\Sigma$ sub-schedules of decompositions 4A-4B. Thus the number of firings of source($\alpha$) and sink($\alpha$) in $S$ are the same, respectively, as the number of firings of source($\alpha$) and sink($\alpha$) in $S_v$. Since $S_v$ is periodic it follows that $S'$ can produce no net change in the number of tokens on $\alpha$.

(F) Suppose $\alpha \in \Lambda_v$. Then in $S'$, source($\alpha$) and sink($\alpha$) are fired only within $\Pi_1, \Pi_2, \ldots, \Pi_M$. Since each $\Pi_i$ is assumed to be an MPASS, it follows that there can be no net change on $\alpha$.

(G) Suppose $\alpha \in \Lambda_1$. Then from A5, each invocation of $n_v$ in $S_v$ consumes the same number of tokens from $\alpha$ as each $\Pi_i$, and thus the total number of samples consumed on $\alpha$ through $S'$ is the same as the number consumed from $\alpha$ through $S_v$. Also source($\alpha$)=source($\alpha$) is not in $V$, so it is fired the same number of times in both schedules. Since $p(\alpha) = p(\alpha)$, it follows that the net change on $\alpha$ is equal to the net change on $\alpha$, which is 0 since $S_v$ is a pass.

(H) Suppose $\alpha \in \Lambda_2$, then sink($\alpha$)=sink($\alpha$) is outside $V$, so the number of samples consumed from $\alpha$ during $S'$ is the same as the number consumed from $\alpha$ during $S_v$. Furthermore, each invocation of $n_v$ produces the same number of samples on $\alpha$ as each $\Pi_i$ produces on $\alpha$. Thus the net change on $\alpha$ equals the net change on $\alpha$, which is 0.

(E)-(H) show that $S'$ produces no net change in sample-population on any arc in $G$, and we conclude that $S'$ is periodic.

Finally, we show that $S'$ has blocking factor $k$. This is equivalent to showing that for some node $x$ in $G$, $S'$ fires $x$ $(k \cdot q(x))$ times, where $q$ is the smallest integer vector in the null space of $\Gamma$. Let our $x$ be chosen from within $V$. Also let $q_F$ and $q_v$ denote the smallest positive - integer - null - space - vectors for the topology matrices of $F_v(G)$ and $G_v$ respectively. Then $S'$ fires $k$ sets of $q_F[n_v]$ invocations of MPASSes for $F_v(G)$, each containing $q_v(x)$ invocations of $x$, for a total of $(k \cdot q_F[n_v] \cdot q_v(x))$ firings of $x$. From lemma 3, this is equal to $(k \cdot q(x))$ firings.

$QED.$

**Theorem 2** Assume A1-A7. Let $S$ be a PASS of blocking-factor $k$ for $G$, and suppose that $V$ is a cluster in $G$ for $S$. Then the schedule $S'$, obtained by substituting in $S$ each occurrence of an MPASS for $G_v$ with a firing of $n_v$, is a PASS of blocking-factor $k$ for $F_v(G)$.

The proof is analogous to that of Theorem 1, and we omit it here for brevity. Theorems 1 and 2 together indicate that scheduling the “clustered” graph $F_v(G)$ is equivalent to producing schedules in $G$ for which $V$ is cluster — every PASS for $G$ in which $V$ is a cluster can be obtained by scheduling $F_v(G)$, and from every PASS for $F_v(G)$, we can derive a schedule for $G$ in which $V$ is a cluster.

We conclude this document with a fact that is useful when considering only two nodes as a cluster. We have found that such pairwise clustering is useful for organizing looping in a schedule.

**Fact:** If $V = \{A,B\}$ is a two-element subset of $N$ then
\[ q_v[A] = \frac{q[A]}{\gcd(q[A], q[B])} ; \]

\[ q_v[B] = \frac{q[B]}{\gcd(q[A], q[B])} ; \text{ and} \]

\[ q_f[n_v] = \gcd(q[A], q[B]). \]

("gcd" denotes the greatest common divisor)

**Proof:**

From lemma 3, each of the equalities above implies the other two. Here we will directly prove the first two.

From lemma 1, we know that \( G_v \) has a PASS, and thus that \( \Gamma_v \) has rank 1. It follows that there are integers \( a \) and \( b \) such that

\[ (5) \quad q_1 a + q_2 b = 0. \]

From lemma 3, we know that \( \forall x \in V, q[x] = q_f[n_v] * q_v[x] \), and thus \( [q[A] \ q[B]] \in \eta(\Gamma_v). \) Hence:

\[ q[A]^* + q[B]^* b = 0. \]

\[ \Rightarrow \] the smallest positive integer vector that satisfies the RHS of (5) is

\[ \left[ \begin{array}{c}
\frac{q[A]}{\gcd(q[A], q[B])} \\
\frac{q[B]}{\gcd(q[A], q[B])}
\end{array} \right] \]

**QED.**

**References**