NORMAL FORMS FOR NONLINEAR VECTOR FIELDS—PART II: APPLICATIONS

by

Leon O. Chua and Hiroshi Kokubu

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Abstract

This paper applies the normal form theory for nonlinear vector fields from Part I to several examples of vector fields whose Jacobian matrix is a typical Jordan form which gives rise to interesting bifurcation behavior. The normal forms derived from these examples are based on Ushiki's method, which is a refinement of Takens' method. A comparison of the normal forms derived by Poincare's method, Takens' method, and Ushiki's method is also given.

For vector fields imbued with some form of symmetry, we impose additional constraints in the normal form algorithm from Part I in order that the resulting normal form will inherit the same form of symmetry.

The normal forms of a given vector field is then used to derive its versal unfoldings in the form of an n-parameter family of vector fields. Such unfoldings are powerful tools for analyzing the bifurcation phenomena of vector fields when the parameter changes. Moreover, since the local bifurcation structure around a highly degenerate singularity can include some global bifurcation phenomena observed from a less degenerate family of vector fields, it follows that the concepts of normal form and versal unfolding are useful tools for analyzing such degenerate singularities.

1. INTRODUCTION

In Part I of this paper, we presented a detailed algorithm for deriving the normal forms of smooth vector fields on $\mathbb{R}^n$. In this paper, this algorithm will be used in several important applications of the normal form theory. In Section 1, we will derive the well-known normal forms associated with several typical Jordan normal forms. We will compare our results, which is based on Ushiki's method, with those derived from Poincaré and Takens' methods. In Section 2, we will investigate some important consequences of symmetry on the normal forms of vector fields. In the final Section 3, we will apply our results to the versal unfoldings of vector fields and explore its relationship to Bifurcation theory.

2. EXAMPLES OF NON-DEGENERATE NORMAL FORMS

In this section, we will derive the normal forms associated with the following 3 important Jordan normal forms:
Example 2.1. Hopf type

Consider the class of smooth vector fields on \( \mathbb{R}^2 \) which vanish at the origin where the linear part is given by (2.1); namely, those vector fields having the 1-jet

\[
v_1 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}
\]

(2.4)

on \( \mathbb{R}^2 \). To simplify our computation for this example, it is advantageous to introduce the complex coordinates

\[
\xi = x + iy
\]

(2.5)

\[
\bar{\xi} = x - iy
\]

(2.6)

where \( i = \sqrt{-1} \). The corresponding differential operators are:

\[
\frac{\partial}{\partial \xi} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)
\]

(2.7)

\[
\frac{\partial}{\partial \bar{\xi}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

(2.8)

In terms of the new complex coordinate system \( (\xi, \bar{\xi}) \), the 1-jet in (2.4) assumes the form:

\[
v_1 = i \left( \xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right)
\]

(2.9)

Similarly, the following Lie bracket formulas in terms of \( (\xi, \bar{\xi}) \) are useful in the following calculations.

\[
\left[ \xi^k \bar{\xi}^l \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} + \bar{\xi} \frac{\partial}{\partial \bar{\xi}} \right] = (1-k-l) \xi^k \bar{\xi}^l \frac{\partial}{\partial \xi}
\]

(2.10)
Observe that the set \( \{ \xi^k \frac{\partial}{\partial \xi}, \xi^l \frac{\partial}{\partial \xi} \} \) for all \( k, l \) where \( k+l = p \) forms a basis for the vector space \( H_p \) of homogeneous vector fields of degree \( p \). It follows from (2.10)-(2.13) that the linear maps

\[
Y \in H_p \mapsto \left[ Y, \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] \in H_p
\]

and

\[
Y \in H_p \mapsto \left[ Y, \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] \in H_p
\]

are both represented by diagonal matrices. For example, for \( p = 1 \), the linear map,

\[
H_1 \to H_1, Y \mapsto \left[ Y, \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right]
\]

is represented by

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Consider next the 2nd order normal form problem. The vector space \( H_2 \) in this case is spanned by

\[
\left\{ \xi^2 \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \xi^2 \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \xi^2 \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \xi^2 \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi} \right\}
\]

In terms of this basis, the linear map (2.15) is represented by the following non-singular diagonal matrix:
Since the image of (2.19) is the whole space $H_2$, dim $B_2 = 6$ where $B_2$ is defined in (5.29) of [1]. It follows that dim $G_2 = 0$ and the 2nd order components in the normal form is completely eliminated in view of Theorem 5.4 from [1].

By repeating the same procedure, we find the linear map (2.15) on $H_3$ is represented by an $8 \times 8$ diagonal matrix, whose diagonal entries are given below with its associated basis component in $H_3$:

\[
\begin{bmatrix}
-1 & 0 & 3 & 0 & -3 & -1 & 1 & 0
\end{bmatrix}
\]

It follows from (2.20) that dim $B_3 = 6$ and hence dim $G_3 = 2$. Let us choose the complementary space $G_3$ spanned by

\[
\left\{\xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi}\right\}
\]

and consider the following reduced 3rd order normal form problem on $G_3$:

\[
\frac{d}{dt} g_3(t) = -\pi_3 \left[ 2Y^2, v^2 + g_3(t) \right]
\]

where

\[
v^2 = v_1 = i \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right)
\]

The (reduced) infinitesimal generator $Y^2$ must satisfy

\[
[Y^2, v^2] = [Y^2, v_1]^2 = 0
\]

that is, $Y^2 = Y_1 + Y_2$ must satisfy

\[
[Y_1, v_1] = 0 \quad \text{and} \quad [Y_2, v_1] = 0
\]

Since the linear map $Y_2 \to [Y_2, v_1]$ is non-singular in view of (2.19), we must have $Y_2 = 0$ and hence $Y^2$ is of the form.
\[ Y^2 = Y_1 = A \left[ \ell \frac{\partial}{\partial \ell} + \ell \frac{\partial}{\partial \ell} \right] + B \left[ \ell \frac{\partial}{\partial \ell} - \ell \frac{\partial}{\partial \ell} \right] \]  

(2.26)

It follows from (2.21) that \( g_3(t) \) in (2.22) is of the form

\[ g_3(t) = \alpha(t) \ell \frac{\partial}{\partial \ell} \left[ \ell \frac{\partial}{\partial \ell} + \ell \frac{\partial}{\partial \ell} \right] + B \left[ \ell \frac{\partial}{\partial \ell} - \ell \frac{\partial}{\partial \ell} \right] \]  

(2.27)

and the right hand side of (2.22) becomes, in view of (2.26),

\[ -\pi_3 \left[ (Y^2, v^2 + g_3(t)_3 \right] = -\pi_3 \left[ [Y_1, v_1 + g_3(t)]_3 \right] \]

(2.28)

\[ = -\pi_3 \left[ A \left[ \ell \frac{\partial}{\partial \ell} + \ell \frac{\partial}{\partial \ell} \right] + B \left[ \ell \frac{\partial}{\partial \ell} - \ell \frac{\partial}{\partial \ell} \right] \right] \]

(2.29)

Equating the corresponding components from (2.27) and (2.29), we obtain the differential equations

\[ \frac{d}{dt} \alpha(t) = -2A \alpha(t) \]  

(2.30)

\[ \frac{d}{dt} \beta(t) = -i2A \beta(t) \]  

(2.31)

whose solutions are:

\[ \alpha(t) = \alpha(0) e^{-2At} \]  

(2.32)

\[ \beta(t) = \beta(0) e^{-i2At} \]  

(2.33)

If \( \alpha(0) \neq 0 \), we can choose \( 2A = \log |\alpha(0)| \) so that at \( t = 1 \), we have \( \alpha(1) = \pm 1 \). If \( \alpha(0) = 0 \) but \( \beta(0) \neq 0 \), we can choose \( 2A = \log i |\beta(0)| \) so that at \( t = 1 \), we have \( \beta(1) = \pm i \).

If \( \alpha(0) = \beta(0) = 0 \), then \( \alpha(t) = \beta(t) = 0 \).

Therefore, the 3rd order normal forms associated with the 1-jet (2.9) are as follows:

(i) non-degenerate 3rd order normal form
\[ \pm \xi \xi \left( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right) + i \left( 1 + \beta \xi \xi \right) \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \]  
\hspace{1cm} (2.34)

(ii) degenerate 3rd order normal form (case 1)

\[ i \left( 1 \pm \xi \xi \right) \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \]  
\hspace{1cm} (2.35)

(iii) degenerate 3rd order normal form (case 2)

\[ i \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \]  
\hspace{1cm} (2.36)

In terms of the polar coordinate \((r, \theta)\), where

\[ x = r \cos \theta, \quad y = r \sin \theta \]  
\hspace{1cm} (2.37)

the vector fields \( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \) and \( i \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \) can be identified with the vector fields \( r \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial \theta} \) respectively. Hence, the normal forms (2.34)-(2.36) simplify further to the following expressions in terms of \((r, \theta)\):

(i) non-degenerate 3rd order normal form: \( \pm r^3 \frac{\partial}{\partial r} + (1 + \beta r^2) \frac{\partial}{\partial \theta} \)  
\hspace{1cm} (2.38)

(ii) degenerate 3rd order normal form (case 1): \( (1 \pm r^2) \frac{\partial}{\partial \theta} \)  
\hspace{1cm} (2.39)

(iii) degenerate 3rd order normal form (case 2): \( \frac{\partial}{\partial \theta} \)  
\hspace{1cm} (2.40)

Using the same procedure, we can of course obtain higher order normal forms. It turns out that for the non-degenerate case (i), all terms in the normal form beyond the 5th order can be eliminated:

**Proposition 2.2**

The non-degenerate kth order normal form of vector fields on \( \mathbb{R}^2 \) with the 1-jet \( \frac{\partial}{\partial \theta} \) is given by

\[ (\pm r^2 + \alpha r^4) r \frac{\partial}{\partial r} + (1 + \beta r^2) \frac{\partial}{\partial \theta} \]  
\hspace{1cm} (2.41)

for any \( k \geq 5 \)

This proposition can be proved by induction, similar to that of Proposition 5.2 from Part I [1]. The proof is given in the Appendix.
Example 2.3: Hopf and Zero-interaction type

Consider the class of smooth vector fields on $\mathbb{R}^3$ which vanish at the origin where the linear part is given by (2.2); namely, those vector fields having the 1-jet

$$v_1 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

(2.42)
on $\mathbb{R}^3$. The eigenvalues associated with the Jacobian matrix are $\pm i$ and 0.

Just as Example 2.1, our calculations are greatly simplified by introducing the complex coordinate $(\xi, \zeta, z)$ in place of $(x, y, z)$. The simplification is due in part to the following simple formulas

$$\begin{align*}
\left[ \xi^k \xi^l \frac{\partial}{\partial z}, \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \zeta} \right] &= -(k+l) \xi^k \xi^l \frac{\partial}{\partial z} \\
\left[ \xi^k \xi^l \frac{\partial}{\partial z}, \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \zeta} \right] &= -(k+l) \xi^k \xi^l \frac{\partial}{\partial z}
\end{align*}$$

(2.43)-(2.44)
in addition to (2.10)-(2.13). In terms of the coordinate $(\xi, \zeta, z)$, the 1-jet $v_1$ is represented by

$$v_1 = i \left[ \xi \frac{\partial}{\partial \xi} - \zeta \frac{\partial}{\partial \zeta} \right]$$

(2.45)

The linear map

$$Y \in H_p \mapsto [Y, v_1] \in H_p$$

(2.46)
of the vector space $H_p$ of homogeneous vector fields of degree $p$ has a diagonal matrix representation with respect to the basis

$$\left\{ \xi^k \xi^l z^m \frac{\partial}{\partial \xi}, \xi^k \xi^l z^m \frac{\partial}{\partial \zeta}, \xi^k \xi^l z^m \frac{\partial}{\partial z} \right\}, k + l + m = p$$

(2.47)

For $p = 2$, the diagonal entries are given by:
Consequently, the complementary space $G_2$ can be chosen with the basis
\[
\left\{ z \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right], z \left[ \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi} \right], \frac{\partial^2}{\partial \xi^2} z - \frac{\partial}{\partial \xi} \right\}.
\]
(2.49)

The reduced 2nd order normal form problem is given by
\[
\frac{d}{dt} g_2(t) = -\pi_2 \left[ [Y_1, v_1 + g_2(t)] \right],
\]
(2.50)

where $Y_1$ satisfies
\[
[Y_1, v_1] = 0
\]
(2.51)

That is, $Y_1$ belongs to the kernel of the linear map $Y \mapsto [Y, v_1]$ on $H_1$. Since this linear map is represented by a diagonal matrix whose diagonal entries are
\[
\begin{pmatrix}
0 & 2 & 1 & -2 & 0 & -1 & -1 & 1 & 0
\end{pmatrix}
\]
(2.52)

it follows that $Y_1$ is of the form
\[
Y_1 = A \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] + B \left[ \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi} \right] + Cz \frac{\partial}{\partial z}
\]
(2.53)

On the other hand, we can write
\[
g_2(t) = \alpha(t) z \left[ \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right] + i \beta(t) z \left[ \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi} \right]
\]
\[
+ \gamma(t) \xi \frac{\partial}{\partial z} + \delta(t) z^2 \frac{\partial}{\partial z}
\]
(2.54)

Substituting (2.53) and (2.54) into the right hand side of (2.50), we obtain
\[-\pi_2 \left[ [Y, v_1 + g_2(t)] \right] = - C \alpha(t) z + iC \beta(t) z \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] + (2A - C) \gamma(t) \xi \xi \frac{\partial}{\partial z} - C \delta(t) z^2 \frac{\partial}{\partial z}\]

where we have made use of the following calculations of the Lie bracket \([Y, v]:\)

<table>
<thead>
<tr>
<th>(Y)</th>
<th>(v)</th>
<th>(z \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right])</th>
<th>(z \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right])</th>
<th>(\xi \xi \frac{\partial}{\partial z})</th>
<th>(z^2 \frac{\partial}{\partial z})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi})</td>
<td>0</td>
<td>0</td>
<td>2(\xi \xi \frac{\partial}{\partial z})</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(\xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Equating the corresponding coefficients in (2.54) and (2.55), we obtain the following 4 linear differential equations:

\[\dot{\alpha} = - C \alpha \]  
\[\dot{\beta} = - C \beta \]  
\[\dot{\gamma} = -(2A - C) \gamma \]  
\[\dot{\delta} = - C \delta \]

This implies that, generically, the coefficient \(\gamma\) and one of the coefficients \(\alpha, \beta,\) and \(\delta\) can be normalized to \(\pm 1\).

Consequently, in terms of the cylindrical coordinate \((r, \theta, z)\), the non-degenerate 2nd order normal form can be chosen to be any one of the following 3 forms:

\[(a) \ \alpha rz \frac{\partial}{\partial r} + (1 + \beta z) \frac{\partial}{\partial \theta} + (s_1 r^2 + s_2 z^2) \frac{\partial}{\partial z} \]  
\[(b) \ s_2 rz \frac{\partial}{\partial r} + (1 + \beta z) \frac{\partial}{\partial \theta} + (s_1 r^2 + s_2 z^2) \frac{\partial}{\partial z} \]  
\[(c) \ \alpha rz \frac{\partial}{\partial r} + (1 + s_2 z^2) \frac{\partial}{\partial \theta} + (s_1 r^2 + s_2 z^2) \frac{\partial}{\partial z} \]

where \(s_1, s_2 = \pm 1\).

We can repeat the same algorithm to derive higher order normal forms. Here, we will present only the result of the non-degenerate 3rd order normal form for case (a) of the preceding 2-jets:
where $s_1, s_2 = \pm 1$.

**Example 2.4: Triple-zero type**

Consider the class of smooth vector fields on $\mathbb{R}^3$ which vanish at the origin where the linear part is given by (2.3); namely, those vector fields having the 1-jet

$$v_1 = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}$$

on $\mathbb{R}^3$. The eigenvalues associated with the Jacobian matrix consist of 3 zeros. Unlike the preceding two examples, the linear map

$$L_p : Y \in H_p \mapsto [Y, v_1] \in H_p$$

for this example does not have a diagonal matrix representation. We should expect therefore much more tedious calculations in this example.

Let us examine first the kernel of the linear map

$$L_1 : H_1 \rightarrow H_1, \ Y \mapsto [Y, v_1]$$

In terms of the basis

$$\left\{ x \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, z \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, y \frac{\partial}{\partial z}, z \frac{\partial}{\partial z} \right\}$$

the map $L_1$ is represented by the matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
\end{bmatrix}
$$

Hence, the kernel of $L_1$ is spanned by
Next, let us choose a basis for $H_2$ as follows:

\[ \left\{ \frac{\partial}{\partial x}, y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right\} \]  \hspace{1cm} (2.68)

The linear map $L_2$ with respect to the above basis has the following matrix representation:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}
\]  \hspace{1cm} (2.70)

Analyzing the linear map $L_2$, we obtain the following basis for

\[ B_2 = \text{images of } L_2: \]

\[
\left\{ xy \frac{\partial}{\partial x}, xz \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, yz \frac{\partial}{\partial x}, z^2 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y}, \right. \\
\left. xz \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}, yz \frac{\partial}{\partial y}, z^2 \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial y} - 2xy \frac{\partial}{\partial z}, \right. \\
\left. xy \frac{\partial}{\partial y} - xz \frac{\partial}{\partial z} - y^2 \frac{\partial}{\partial z}, yz \frac{\partial}{\partial z}, z^2 \frac{\partial}{\partial z} \right\} \]  \hspace{1cm} (2.71)

Hence; we can choose the complementary space $G_2$ to $B_2$ in $H_2$ to be the subspace spanned by
Here we must remember that the projection \( \pi_2 : H_2 \rightarrow G_2 \) gives the following:

\[
\pi_2 \left[ xy \frac{\partial}{\partial y} \right] = \pi_2 \left[ xy \frac{\partial}{\partial y} - xz \frac{\partial}{\partial z} - y^2 \frac{\partial}{\partial z} + xz \frac{\partial}{\partial z} + y^2 \frac{\partial}{\partial z} \right]
\]

\[
= xz \frac{\partial}{\partial z} + y^2 \frac{\partial}{\partial z},
\]

\[
\pi_2 \left[ x^2 \frac{\partial}{\partial y} \right] = \pi_2 \left[ x^2 \frac{\partial}{\partial y} - 2xy \frac{\partial}{\partial z} + 2xy \frac{\partial}{\partial z} \right] = 2xy \frac{\partial}{\partial z}.
\]

\[
\pi_2 \left[ x^2 \frac{\partial}{\partial x} \right] = \pi_2 \left[ x^2 \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} + 2xy \frac{\partial}{\partial y} \right] = 2\pi_2 \left[ xy \frac{\partial}{\partial y} \right]
\]

\[
= 2xz \frac{\partial}{\partial z} + 2y^2 \frac{\partial}{\partial z}.
\]

It follows from (2.67) that a vector field \( Y_1 \in H_1 \) satisfying,

\[
[Y_1, v_1] = 0,
\]

is of the form

\[
Y_1 = Az \frac{\partial}{\partial x} + B \left( y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \right) + C \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right).
\]

Our 2nd order normal form problem is therefore given by

\[
\frac{d}{dt} g_2(t) = -\pi_2 \left[ \left[ Y_1, v_1 + g_2(t) \right] \right]
\]

\[
= -\pi_2 \left[ \left[ Y_1, g_2(t) \right] \right]
\]

where

\[
g_2(t) = \alpha(t) x^2 \frac{\partial}{\partial z} + \beta(t) xy \frac{\partial}{\partial z} + \gamma(t) xz \frac{\partial}{\partial z} + \delta(t) y^2 \frac{\partial}{\partial z},
\]

To compute the right hand side of (2.78), it is convenient to derive the following table:
Remark 2.5

Since \( y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} = v_1 \), we observe that \( y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \), \( v_1 \) \( \epsilon \) \( B_2 \) does not affect (2.78). Consequently, there is no need to evaluate any Lie bracket involving \( y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \).

Using the above table, we evaluate

\[
[Y_1, g_2] = A \alpha \left[2xz \frac{\partial}{\partial z} - x^2 \frac{\partial}{\partial x}\right] + A \beta \left[yz \frac{\partial}{\partial z} - xy \frac{\partial}{\partial x}\right]
+ A \gamma \left[z^2 \frac{\partial}{\partial z} - xz \frac{\partial}{\partial x}\right] - A \delta y^2 \frac{\partial}{\partial x} + C \alpha x^2 \frac{\partial}{\partial z}
+ C \beta xy \frac{\partial}{\partial z} + C \gamma xz \frac{\partial}{\partial z} + C \delta y^2 \frac{\partial}{\partial z}.
\]

Therefore, the right hand side of (2.78) becomes

\[
-\pi_2 \left[\left[ Y_1, g_2 \right]\right] = -C \alpha x^2 \frac{\partial}{\partial z} - C \beta xy \frac{\partial}{\partial z} - (2A \alpha + C \gamma) xz \frac{\partial}{\partial z}
- C \delta y^2 \frac{\partial}{\partial z} + A \alpha \pi_2 \left[x^2 \frac{\partial}{\partial x}\right].
\]

From (2.75), we obtain

\[
A \alpha \pi_2 \left[x^2 \frac{\partial}{\partial x}\right] = 2A \alpha xz \frac{\partial}{\partial z} + 2A \alpha y^2 \frac{\partial}{\partial z}
\]

(2.82)

Hence,

\[
-\pi_2 \left[\left[ Y_1, g_2 \right]\right] = -C \alpha x^2 \frac{\partial}{\partial z} - C \beta xy \frac{\partial}{\partial z} - C \gamma xz \frac{\partial}{\partial z}
+ (2A \alpha - C \delta) y^2 \frac{\partial}{\partial z}.
\]

Equating the corresponding coefficients of (2.79) and (2.83), we obtain the following 4 linear differential equations:
\[
\begin{aligned}
\dot{\alpha} &= -C \alpha \\
\dot{\beta} &= -C \beta \\
\dot{\gamma} &= -C \gamma \\
\dot{\delta} &= 2A \alpha - C \delta .
\end{aligned}
\]  

(2.84)

This equation generically gives,
\[\alpha = \pm 1 \quad \text{and} \quad \delta = 0 ,\]

Hence, the non-degenerate 2nd order normal form is given by:
\[
y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (sx^2 + \beta xy + \gamma xz) \frac{\partial}{\partial z} ,
\]

where \( s = \pm 1 \).

The differential operator (2.85) can be identified with the differential equation:
\[
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= sx^2 + \beta xy + \gamma xz ,
\end{aligned}
\]

(2.86)

which is equivalent to the following 3rd order differential equation:
\[
\ddot{x} = sx^2 + \beta x\dot{x} + \gamma x\dot{x} .
\]

(2.87)

Consider next the 3rd order normal form problem. The basis for the vector space \( H_3 \) can be chosen as follows:
\[
\left\{ x^k y^l z^m \frac{\partial}{\partial x} , x^k y^l z^m \frac{\partial}{\partial y} , x^k y^l z^m \frac{\partial}{\partial z} \right\} , \quad k + l + m = 3
\]

(2.88)

In terms of this basis, the linear map (lexicographic order)
\[
L_3 : H_3 \to H_3 ; \ Y_3 \to [Y_3 , v_1]
\]

(2.89)

is represented by the 30 x 30 matrix,
\[
\begin{bmatrix}
A & I & 0 \\
\vdots & \vdots & \vdots \\
0 & A & I \\
\vdots & \vdots & \vdots \\
0 & 0 & A
\end{bmatrix}
\]

where \( I \) denotes a 10 x 10 unit matrix, and
It is not difficult to obtain the image $B_3$ of the linear map $L_3$: $B_3$ is spanned by

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\]

(2.91)

Hence, we can choose the following basis for the complementary space $G_3$:

\[
\left\{ x^2y \frac{\partial}{\partial x}, xyz \frac{\partial}{\partial x}, x^2z \frac{\partial}{\partial x}, y^2 \frac{\partial}{\partial x}, y^2z \frac{\partial}{\partial x}, z^3 \frac{\partial}{\partial x},
\frac{x^2}{\partial y}, y^2 \frac{\partial}{\partial y}, yz \frac{\partial}{\partial y}, z^3 \frac{\partial}{\partial y},
\frac{x^3}{\partial z} - 3x^2 \frac{\partial}{\partial z}, x^2 \frac{\partial}{\partial z} - 2xyz \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial y} + xyz \frac{\partial}{\partial y},
3xyz \frac{\partial}{\partial z} + y^3 \frac{\partial}{\partial z}, x^3 \frac{\partial}{\partial z} - 3x^2 \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z},
\right. \\
\left. x^2y \frac{\partial}{\partial y} - x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z}, x^2z \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial x} - y^2z \frac{\partial}{\partial x},
xy \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z}, y^3 \frac{\partial}{\partial y} - 3y^2z \frac{\partial}{\partial z}, xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z}\right\}
\]

(2.92)

Hence, we can choose the following basis for the complementary space $G_3$:

\[
\left\{ x^3 \frac{\partial}{\partial z}, x^2y \frac{\partial}{\partial z}, x^2z \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial z}, xyz \frac{\partial}{\partial z}, y^2z \frac{\partial}{\partial z}\right\}
\]

(2.93)

In order to solve the 3rd order normal form problem

\[
\frac{d}{dt} g_3(t) = -\pi_3 \left[ \left[ y^2, v^2 + g_3(t) \right]_3 \right]
\]

(2.94)

we must choose the infinitesimal generator $Y^2$ such that

\[
[Y^2, v^2]^2 = 0
\]

(2.95)
that is, we must satisfy

\[ [Y_1, v_1] = 0 \]  

and

\[ [Y_1, v_2] + [Y_2, v_1] = 0 \]  

As is discussed on the 2nd order normal form problem, \( Y_1 \) is of the form,

\[ Y_1 = Az \frac{\partial}{\partial x} + Bv_1 + C \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \]  

(2.97)

Hence, if \( A \neq 0 \) or \( C \neq 0 \), then \([Y_1, v_2]\) does not belong to \( B_2 \) = Image of \( L_2 \); in view of (2.81), whereas \([Y_2, v_1] \in B_2 \). Therefore (2.96) and (2.97) implies

\[ Y_1 = \text{constant} \cdot v_1 = K \cdot v_1 \]  

(2.99)

Hence, (2.97) becomes

\[ [Y_1, v_2] + [Y_2, v_1] = [Kv_1, v_2] + [Y_2, v_1] \]  

(2.100)

It follows from (2.100) that

\[ Y_2 = Kv_2 + (\text{kernel of } L_2) \]  

(2.101)

Here, the kernel of \( L_2 \) is spanned by

\[ \begin{aligned} \left\{ z^2 \frac{\partial}{\partial x}, \right. \\
\left. yz \frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y}, \right. \\
\left. 2xz \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial x} \right. \right\} \]  

(2.102)

Thus we can choose

\[ Y^2 = K (v_1 + v_2) + A z^2 \frac{\partial}{\partial z} + B \left( yz \frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y} \right) \]  

\[ + C \left( 2xz \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial x} \right) + D \left( xz \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right) \]  

(2.103)

On the other hand, from (2.93), \( g_3(t) \) is of the form
\[ g_3(t) = a(t)x^3 \frac{\partial}{\partial z} + b(t)x^2y \frac{\partial}{\partial z} + c(t)x^2z \frac{\partial}{\partial z} + d(t)xy \frac{\partial}{\partial z} + e(t)xyz \frac{\partial}{\partial z} + f(t)y^2z \frac{\partial}{\partial z} \]  

\[ \text{(2.104)} \]

Observe that

\[ \left[ Y_2, v^2 + g_3(t) \right]_3 = \left[ Y_1, g_3(t) \right] + \left[ Y_2, v^2 \right] \]

\[ = \left[ K_{v_1}, g_3(t) \right] + \left[ K_{v_2} + (\text{kernel}), v^2 \right] \]  

\[ = \left[ K_{v_1}, g_3(t) \right] + \left[ K_{v_2}, v^2 \right] + \left( \text{kernel} \right), v^2 \].

\[ \text{(2.105)} \]

Since the first term \( \left[ K_{v_1}, g_3(t) \right] \) belongs to \( B_3 \), it vanishes under the projection \( \pi_3 : H_3 \rightarrow G_3 \). The second term vanishes by itself. Therefore only the third term \( \left( \text{kernel} \right), v^2 \) is essential: The Lie bracket \( \left( \text{kernel} \right), v^2 \) for various kernels and \( v^2 \) are summarized as follows:

<table>
<thead>
<tr>
<th>(kernel)</th>
<th>( v^2 )</th>
<th>( x^2 \frac{\partial}{\partial z} )</th>
<th>( xy \frac{\partial}{\partial z} )</th>
<th>( xz \frac{\partial}{\partial z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z^2 \frac{\partial}{\partial x} )</td>
<td>( 2xz^2 \frac{\partial}{\partial z} - 2x^2z \frac{\partial}{\partial x} )</td>
<td>( yz^2 \frac{\partial}{\partial z} - 2xyz \frac{\partial}{\partial x} )</td>
<td>( z^3 \frac{\partial}{\partial z} - 2xx^2 \frac{\partial}{\partial x} )</td>
<td></td>
</tr>
<tr>
<td>( yz \frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y} )</td>
<td>( 2xyz \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial y} - 2x^2z \frac{\partial}{\partial x} )</td>
<td>( yz^2 \frac{\partial}{\partial z} + xy^2 \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial x} )</td>
<td>( yz^2 \frac{\partial}{\partial z} - 2xyz \frac{\partial}{\partial x} )</td>
<td></td>
</tr>
<tr>
<td>( 2xz \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} )</td>
<td>( 4x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial y} - 2x^2z \frac{\partial}{\partial x} )</td>
<td>( 2xyz \frac{\partial}{\partial z} - xy^2 \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial x} )</td>
<td>( 2xz^2 \frac{\partial}{\partial z} - y^2z \frac{\partial}{\partial x} - 2x^2z \frac{\partial}{\partial x} )</td>
<td></td>
</tr>
<tr>
<td>( x^2 \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} )</td>
<td>( -x^3 \frac{\partial}{\partial z} - x^2y \frac{\partial}{\partial z} )</td>
<td>( -x^2y \frac{\partial}{\partial z} - xy^2 \frac{\partial}{\partial z} )</td>
<td>( -x^2z \frac{\partial}{\partial z} - xyz \frac{\partial}{\partial z} )</td>
<td></td>
</tr>
</tbody>
</table>

In this table, the following terms belong to \( B_3 \):

\[
\left\{ yz^2 \frac{\partial}{\partial z}, xy \frac{\partial}{\partial x}, z^3 \frac{\partial}{\partial z}, xy^2 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} \right\}
\]

\[ \text{(2.106)} \]

Moreover,

\[
2xz^2 \frac{\partial}{\partial z} - 2x^2z \frac{\partial}{\partial x} = 2 \left( xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z} \right) - 2 \left( x^2z \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z} \right),
\]

\[
2xyz \frac{\partial}{\partial z} - 2xz^2 \frac{\partial}{\partial y} = 2xyz \frac{\partial}{\partial z} - 4xyz \frac{\partial}{\partial z} - 2 \left( x^2z \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial z} \right)
\]

\[ = - 2xyz \frac{\partial}{\partial z} - 2 \left( x^2z \frac{\partial}{\partial y} - 2xyz \frac{\partial}{\partial z} \right),\]
\[ y^2z \frac{\partial}{\partial z} + xz^2 \frac{\partial}{\partial z} - xy^2 \frac{\partial}{\partial x} - 2xyz \frac{\partial}{\partial y} \]

\[ = y^2z \frac{\partial}{\partial z} - 2y^2z \frac{\partial}{\partial z} + \left[ xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z} \right] \]

\[ -y^2z \frac{\partial}{\partial z} - \left[ xy^2 \frac{\partial}{\partial x} - y^2z \frac{\partial}{\partial z} \right] + 2y^2z \frac{\partial}{\partial z} - 2 \left[ xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z} \right] \]

\[ = \left[ xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z} \right] - \left[ xy^2 \frac{\partial}{\partial x} - y^2z \frac{\partial}{\partial z} \right] - 2 \left[ xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z} \right] \]

\[ 4x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} - 2x^3 \frac{\partial}{\partial x} \]

\[ = 4x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} - 6x^2z \frac{\partial}{\partial z} - 12xy^2 \frac{\partial}{\partial z} - 2 \left[ x^3 \frac{\partial}{\partial x} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z} \right] \]

\[ = - 2x^2z \frac{\partial}{\partial z} - 14xy^2 \frac{\partial}{\partial z} - 2 \left[ x^3 \frac{\partial}{\partial x} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z} \right] \]

\[ 2xyz \frac{\partial}{\partial z} - y^3 \frac{\partial}{\partial z} = 2xyz \frac{\partial}{\partial z} + 3xyz \frac{\partial}{\partial z} - \left[ y^3 \frac{\partial}{\partial z} + 3xyz \frac{\partial}{\partial z} \right] \]

\[ = 5xyz \frac{\partial}{\partial z} - \left[ y^3 \frac{\partial}{\partial z} + 3xyz \frac{\partial}{\partial z} \right] \]

\[ 2xz^2 \frac{\partial}{\partial z} - y^2z \frac{\partial}{\partial z} - 2xz^2 \frac{\partial}{\partial z} \]

\[ = - 4y^2z \frac{\partial}{\partial z} + 2 \left[ xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z} \right] - y^2z \frac{\partial}{\partial z} + 4y^2z \frac{\partial}{\partial z} - 2 \left[ x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z} \right] \]

\[ = - y^2z \frac{\partial}{\partial z} + 2 \left[ xz^2 \frac{\partial}{\partial z} + 2y^2z \frac{\partial}{\partial z} \right] - 2 \left[ x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z} \right] \]
\[-x^3 \frac{\partial}{\partial x} - x^2 y \frac{\partial}{\partial y} \]

\[= -3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z} - \left[ x^3 \frac{\partial}{\partial x} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z} \right] \]

\[-x^2 \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} - \left[ x^2y \frac{\partial}{\partial y} - x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} \right] \]

\[= -4x^2z \frac{\partial}{\partial z} - 8xy^2 \frac{\partial}{\partial z} - \left[ x^3 \frac{\partial}{\partial x} - 3x^2z \frac{\partial}{\partial z} - 6xy^2 \frac{\partial}{\partial z} \right] - \left[ x^2y \frac{\partial}{\partial y} - x^2z \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial z} \right] \]

\[-xy^2 \frac{\partial}{\partial y} = xyz \frac{\partial}{\partial z} - \left[ xy^2 \frac{\partial}{\partial y} + xyz \frac{\partial}{\partial z} \right] \]

\[-x^2z \frac{\partial}{\partial x} - xyz \frac{\partial}{\partial y} = 2y^2z \frac{\partial}{\partial z} - \left[ x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z} \right] + y^2z \frac{\partial}{\partial z} - \left[ xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z} \right] \]

\[= 3y^2z \frac{\partial}{\partial z} - \left[ x^2z \frac{\partial}{\partial x} + 2y^2z \frac{\partial}{\partial z} \right] - \left[ xyz \frac{\partial}{\partial y} + y^2z \frac{\partial}{\partial z} \right] . \]

In all of the above expressions, the terms enclosed by parentheses belong to \( B_3 \). Hence

\[-\pi_3 \left( [Y^2, v^2 + g_3(t)]_3 \right) = -Bs \left[ -2xyz \frac{\partial}{\partial z} \right] - Cs \left[ -2x^2z \frac{\partial}{\partial z} - 14xy^2 \frac{\partial}{\partial z} \right] \]

\[-C \beta \left[ 5xyz \frac{\partial}{\partial z} \right] - C \gamma \left[ -y^2z \frac{\partial}{\partial z} \right] \]

\[-Ds \left[ -4x^2z \frac{\partial}{\partial z} - 8xy^2 \frac{\partial}{\partial z} \right] - D \beta \left[ xyz \frac{\partial}{\partial z} \right] - D \gamma \left[ 3y^2z \frac{\partial}{\partial z} \right] \]  \hspace{1cm} (2.107) \]

\[= (2Cs + 4Ds)x^2 \frac{\partial}{\partial z} + (14Cs + 8Ds)xy^2 \frac{\partial}{\partial z} \]

\[+ (2Bs - 5C \beta - D \beta)xyz \frac{\partial}{\partial z} + \left[ C \gamma - 3D \gamma \right] y^2z \frac{\partial}{\partial z} \ . \]

Finally, equating the corresponding coefficients of (2.104) and (2.107), we obtain the following system of differential equations,
\[ \begin{align*}
\dot{a} &= 0, \quad \dot{b} = 0 \\
\dot{c} &= 2Cs + 4Ds \\
\dot{d} &= 14Cs + 8Ds \\
\dot{e} &= 2Bs - 5C \beta - D \beta \\
\dot{f} &= C \gamma - 3D \gamma
\end{align*} \]

for solving the 3rd order normal form problem. By an appropriate choice of \( B, C, \) and \( D, \) we can choose
\[ d = e = f = 0 \]
at \( t = 1 \) and obtain the following non-degenerate 3rd order normal form:
\[ v_2 + (ax^3 + bx^2y + cx^2z) \frac{\partial}{\partial z}. \]  

Equation (2.109) can be identified with the differential equation
\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= sx^2 + \beta xy + \gamma xz + ax^3 + bx^2y + cx^2z
\end{align*} \]
or, equivalently,
\[ \begin{align*}
\ddot{x} &= sx^2 + \beta x\dot{x} + \gamma x\dot{z} + ax^3 + bx^2\dot{y} + cx^2\dot{z} \\
&= (sx^2 + ax^3) + (\beta x + bx^2)\dot{x} + (\gamma x + cx^2)\dot{z}
\end{align*} \]

We will conclude this section by giving a comparison of the normal forms derived via Poincaré's method, Takens' method and Ushiki's method. Since the goal for obtaining normal forms of vector fields is to eliminate as many monomials from each order as possible, we will list in the following tables the number of monomials of each degree that is still present in the normal form. For example, a number "2" under the column for degree 3 for Takens method means that there are 2 monomials of degree 3 in the Takens normal form. Similarly, the number "0" under the same column would mean that the resulting normal form has no 3rd degree terms. Hence "0" corresponds to the ideal situation where all terms of a given degree are eliminated.

It turns out that even for the remaining monomials, some coefficients can be set to \( \pm 1, \) rather than remaining arbitrary in the general case. This further refinement, due to Ushiki [2,3], is indicated by a parentheses. For example, the number 2 + (1) in the table under degree 2 means that there are 3 monomials of degree 2 present in the normal form, one of which has a coefficient equal to \( \pm 1. \)

The following tables are based on the 5 examples derived in Part I [1] as well as in this section.† They

†A similar comparison is given in Ushiki [3]. Our tables include, however, corrections to some errors in [3].
correspond to non-degenerate normal forms from each example.

**Case 1:** Simple-zero type in $\mathbb{R}^1$ (*Example 5.1, Part I*)

The Jacobian matrix in this case is a scalar: $(0)$

<table>
<thead>
<tr>
<th>degree</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
<th>7th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poincaré</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Takens</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Ushiki</td>
<td>(1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Case 2:** Hopf type in $\mathbb{R}^2$ (*Example 2.1*)

The Jacobian matrix in this case is:

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>degree</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
<th>7th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poincaré</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Takens</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Ushiki</td>
<td>0</td>
<td>1+$\text{(1)}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Case 3:** Hopf and Zero-interaction type in $\mathbb{R}^3$ (*Example 2.3*) in cylindrical coordinate

The Jacobian matrix in this case is:

\[
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>degree</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poincaré</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>Takens</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>Ushiki</td>
<td>2+$\text{ (2)}$</td>
<td>3</td>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>

**Case 4:** Double-zero type in $\mathbb{R}^2$ (*Example 5.5, Part I*)

The Jacobian matrix in this case is:

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

Since the *linear* part in this example is *not* diagonalizable, Poincaré's method is not applicable.

<table>
<thead>
<tr>
<th>degree</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Takens</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Ushiki</td>
<td>2+$\text{ (1)}$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Case 5: Triple-zero type in $\mathbb{R}^3$ (Example 2.4)

The Jacobian matrix in this case is:

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

Again, Poincaré's method is not applicable because the linear part is not diagonalizable.

<table>
<thead>
<tr>
<th>degree</th>
<th>2nd</th>
<th>3rd</th>
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<tbody>
<tr>
<td>Takens</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Ushiki</td>
<td>2 + 1</td>
<td>3</td>
</tr>
</tbody>
</table>

Remarks

1. Both Poincaré and Takens' methods give the same result for vector fields having a diagonalizable linear part.
2. Poincaré's method does not apply, in its original form, to non-diagonalizable vector fields.
3. Takens' method is applicable to all cases.
4. Ushiki's method may be considered as a refinement of Takens' method.
5. Takens' is aware that further refinements are possible and have in fact derived them for cases 1 and 2 [5,6].
6. This paper follows Ushiki's approach since his method gives an explicit algorithm for general vector fields.

3. NORMAL FORM WITH SYMMETRIES

Many vector fields of practical interest are imbued with some form of symmetry; e.g., reflection symmetry, point symmetry, rotation symmetry, etc. In such cases, it is natural that their normal forms should exhibit the same symmetries. Our goal in this section is to show how such additional constraints can be imposed upon the algorithm and in the preceding section.

Definition 3.1: Vector fields with symmetry

A symmetry for a vector field $v$ on $\mathbb{R}^n$ is a diffeomorphism $\gamma: \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying

$$
\gamma_v = v
$$

(3.1)

where

$$
\gamma_v(x) \triangleq D \gamma \left[ \gamma^{-1}(x) \right] \cdot v \left[ \gamma^{-1}(x) \right]
$$

(3.2)

Example 3.2:
Let \( v \) be a vector field on \( \mathbb{R}^2 \) defined by
\[
v(x, y) = (1 + y^2, x^2 y). \tag{3.3}
\]
Then \( v \) exhibits a symmetry via the diffeomorphism
\[
\gamma(x, y) = (x, -y), \tag{3.4}
\]
To see this, we calculate
\[
D \gamma(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^{-1}(x, y) = (x, -y) \tag{3.5}
\]
and
\[
\gamma_* v(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 + y^2 \\ -x^2 y \end{bmatrix} = v(x, y) \tag{3.6}
\]

It is easy to see that the set of all symmetries for a vector field \( v \) forms a group under composition. This group is called the symmetry group of \( v \). For the above example, the symmetry group contains at least four elements \( \{id, \gamma, \delta, \gamma \circ \delta\} \), where \( \delta: (x, y) \rightarrow (-x, y) \), since \( \gamma \circ \gamma = \delta \circ \delta = id \) and \( \gamma \circ \delta = \delta \circ \gamma \).

A group made of two elements is frequently denoted by \( \mathbb{Z}_2 \). In this section, we shall restrict our consideration to vector fields whose symmetry group contains \( \mathbb{Z}_2 = \{id, \gamma\} \) as a subgroup. To derive a normal form with symmetry \( \gamma \), we must find appropriate transformations which preserve the same symmetry \( \gamma \), after subjecting the vector field to these transformations, as in the preceding section. In other words, the resulting normal form must also exhibit the symmetry \( \gamma \). Using our notation from [1], our abstract objects in this section consists of a subspace \( \mathcal{X}_0 \) of smooth vector fields \( \mathcal{X}_0 \) which vanish at 0 and exhibit the symmetry \( \gamma \).

Our next proposition characterizes the class of transformations that preserves symmetry.

**Proposition 3.3**

Suppose a vector field \( v \) exhibits a symmetry \( \gamma \). Then a transformed vector field \( \phi_* v \) also exhibits the same symmetry \( \gamma \) if \( \phi \) commutes with \( \gamma \), i.e., \( \gamma \circ \phi = \phi \circ \gamma \), or equivalently,
\[
\gamma \circ \phi \circ \gamma^{-1} = \phi \tag{3.7}
\]

**Proof.**

Since \( \gamma_* v = v \) and \( \phi \circ \gamma = \gamma \circ \phi \),
\[
\gamma_* (\phi_* v) = (\gamma \circ \phi)_* v = (\phi \circ \gamma)_* v = \phi_* (\gamma_* v) = \phi_* v \tag{3.8}
\]

**Proposition 3.4**
If a local one-parameter group \( \{ \phi^t \} \) of transformations is generated by a vector field \( Y \), i.e.,

\[
\phi^t = \exp tY,
\]

then the local one-parameter group

\[
\tilde{\phi}^t = \gamma \circ \phi^t \circ \gamma^{-1}
\]

is generated by \( \gamma \circ Y \).

**Proof.**

Since \( \phi^t \) is the flow of the vector field \( Y \),

\[
\frac{d}{dt} \phi^t(x) = Y\left[ \phi^t(x) \right]
\]

holds. Thus,

\[
\frac{d}{dt} \tilde{\phi}^t(x) = \frac{d}{dt} (\gamma \circ \phi^t \circ \gamma^{-1})(x) = D \gamma \left[ \phi^t \circ \gamma^{-1}(x) \right] \cdot \frac{d}{dt} \phi^t \left[ \gamma^{-1}(x) \right]
\]

\[
= D \gamma \left[ \phi^t \circ \gamma^{-1}(x) \right] \cdot Y\left[ \phi^t(\gamma^{-1}(x)) \right]
\]

\[
= D \gamma \left[ \gamma^{-1}(\tilde{\phi}^t(x)) \right] \cdot Y\left[ \gamma^{-1}(\tilde{\phi}^t(x)) \right]
\]

\[
= (\gamma \circ Y) \left[ \tilde{\phi}^t(x) \right].
\]

This completes the proof.

It follows from the above propositions that the set

\[
\gamma \text{Diff}_0 = \{ \phi : \mathbb{R}^n \to \mathbb{R}^n, \text{diffeomorphism}, \phi(0) = 0, \gamma \circ \phi = \phi \circ \gamma \}
\]

can be chosen as our transformation group and \( \gamma \mathcal{X}_0 \) can be chosen as its infinitesimal generators. Once we fix this transformation group and space, the remaining steps for obtaining the normal form are identical to the preceding algorithm, as illustrated by the following example.

**Example 3.4**

Consider vector fields on \( \mathbb{R}^1 \) with a vanishing 1-jet; i.e.,

\[
v_1 = 0, \quad v(0) = v'(0) = 0.
\]

Let us impose the symmetry

\[
\gamma(x) = -x.
\]

The space \( \gamma \mathcal{X}_0 \) can be decomposed into the homogeneous parts.
\[ y X_0 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \ldots, \]  
(3.16)

where each part \( \mathcal{H}_k \) consists of homogeneous vector fields \( v_k \) of order \( k \) satisfying the symmetry constraint

\[ \gamma_* v_k = v_k. \]  
(3.17)

Hence, for even \( k \), we have

\[ \mathcal{H}_k = 0 \]  
(3.18)

and for odd \( k \), we have

\[ \mathcal{H}_k = H_k. \]  
(3.19)

In other words,

\[ \gamma_0 = H_1 \oplus H_3 \oplus H_5 \oplus \ldots. \]  
(3.20)

Since we have \( \gamma^2 = 0 \), our next step is to obtain the simplest 3rd order part; i.e., consider the 3rd order normal form problem with \( \gamma^2 = 0 \) under the symmetry \( \gamma(x) = -x \).

The 1-jet \( \nu_1 = 0 \) implies that the linear map

\[ L_k : \mathcal{H}_k \rightarrow \mathcal{H}_k, Y_k \rightarrow [Y_k, \nu_1] \]  
(3.21)

is the zero map for every \( k \). Hence, the complementary space \( \mathcal{G}_k \) to the image of \( L_k \) is simply \( \mathcal{H}_k \), and the normal form problem is already reduced:

\[ \frac{d}{dt} g_k(t) = - \left[ Y^{k-1}, \gamma^{k-1} + g_k(t) \right] \]  
(3.22)

with

\[ \left[ Y^{k-1}, \gamma^{k-1} \right]^{k-1} = 0, \gamma_* Y^{k-1} = Y^{k-1}. \]  
(3.23)

For \( k = 3 \),

\[ \frac{d}{dt} g_3(t) = - \left[ Y^2, g_3(t) \right] = - \left[ Y_1, g_3(t) \right] \]  
(3.24)

where

\[ [Y^2, \gamma^2]^2 = 0 \]  
(3.25)

automatically holds for \( \gamma^2 = 0 \). Thus, \( Y^2 \) need only satisfy \( \gamma_* Y^2 = Y^2 \); hence \( Y_2 = 0 \). Defining

\[ Y_1 = Ax \frac{\partial}{\partial x} \]  
(3.26)

and

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\[ g_3(t) = \alpha(t)x^3 \frac{\partial}{\partial x} , \]  

we obtain

\[ -\left[ Y_1, g_3(t) \right] = -2A \alpha(t)x^3 \]  

Equating the coefficients of \( 3.27 \) and \( 3.28 \), we obtain the differential equation

\[ \dot{\alpha} = -2A \alpha . \]  

It follows that we can always choose the value of \( A \) so that the non-degenerate 3rd order normal form with symmetry \( \gamma \) is given by

\[ v^3 = \pm x^3 \frac{\partial}{\partial x} . \]  

Similar to Prop. 5.2 from Part I [1], we can prove that the \( \gamma \)-symmetric kth order normal form of vector fields on \( \mathbb{R}^4 \) with a vanishing 1-jet is given by

\[ (\pm x^3 + ax^5) \frac{\partial}{\partial x} \quad a \in \mathbb{R} . \]  

for \( k \geq 5 \).

Following the same procedure, we have derived the following non-degenerate normal forms with \( \mathbb{Z}_2 \)-symmetries for several typical examples:

2-dimensional case

(i) non-degenerate 3rd order normal form with the symmetry \( (x , y) \rightarrow (-x , -y) \), whose linear part has double-zero eigenvalues:

\[ v^3 = y \frac{\partial}{\partial x} + (sx^3 + ax^2 y) \frac{\partial}{\partial y} \quad s = \pm 1 \]  

(ii) non-degenerate 3rd order normal form with the symmetry \( (x , y) \rightarrow (-x , y) \), whose linear part has double-zero eigenvalues:

\[ v^3 = s_1 xy \frac{\partial}{\partial x} + (s_2x^2 + ax^2y^2 + by^3) \frac{\partial}{\partial y} \quad s_1 = \pm 1 \quad s_2 = \pm 1 \]  

3-dimensional case

(iii) non-degenerate 3rd order normal form with the symmetry \( (x , y , z) \rightarrow (-x , -y , -z) \), whose linear part has triple-zero eigenvalues:
\[ v^3 = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (sx^3+ax^2y+bx^2z+cxyz+dxz^2) \frac{\partial}{\partial z} , s = \pm 1 \] (3.34)

(iv) non-degenerate 3rd order normal form with the symmetry \((x , y , z) \rightarrow (-x , -y , z)\), whose linear part has triple-zero eigenvalues:
\[ v^3 = y \frac{\partial}{\partial x} + (s_1xz_3 + ayz + bx^3 + cxy^2 + dyz^2) \frac{\partial}{\partial y} + (s_2x^2 + ez^2 + fyz^3) \frac{\partial}{\partial z} , s_1 = \pm 1 , s_2 = \pm 1 \] (3.35)

(v) non-degenerate 3rd order normal form with the symmetry \((x , y , z) \rightarrow (x , y , -z)\), whose linear part has triple-zero eigenvalues:
\[ v^3 = y \frac{\partial}{\partial x} + (s_1x^2 + axy + bx^3 + cxy^2 + dyz^2) \frac{\partial}{\partial y} + (exz + s_2z^2) \frac{\partial}{\partial z} , s_1 = \pm 1 , s_2 = \pm 1 \] (3.36)

4. VERSAL UNFOLDINGS

Our preceding normal form theory consists basically of methods for simplifying ordinary differential equations (ODE) in a neighborhood of their singular points. We have presented various examples of normal forms having several eigenvalues with a zero real part; namely, multiple zero eigenvalues, or pure imaginary eigenvalues, etc. All such eigenvalues which lie on the imaginary axis in the complex plane are called central eigenvalues. A singular point of a vector field is said to be hyperbolic if it does not contain any central eigenvalues. From the Hartman-Grobman theorem, we know that the local phase portraits of vector fields around hyperbolic singular points are determined by their associated 1-jets, and the center manifold theorem reduces a vector field with both central and non-central eigenvalues to that with only central eigenvalues. Hence, we will only consider vector fields around a singular point where all of its eigenvalues are central.

The above conclusion appears, at first sight, to be rather academic especially from an engineering context because almost all Jacobian matrices associated with physical systems are hyperbolic; i.e., all eigenvalues have a non-zero real part. This observation is true only for fixed vector fields. For families of vector fields, however, central eigenvalues will always be encountered. In particular, we will show that simple central eigenvalues (i.e., multiplicity one) are inevitable in one-parameter families of vector fields, while multiple central eigenvalues are inevitable in multi-parameter families of vector fields.

Example 4.1
Consider a vector field $x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}^1$. The origin is a singular point whose linear part vanishes. Hence, we have a zero central eigenvalue at the origin.

If we perturb this vector field into

$$f(x, \mu) = (\mu + x^2) \frac{\partial}{\partial x}, \quad \mu \neq 0$$  \hspace{1cm} (4.1)$$
then we have a simple singular point at $x = \pm \sqrt{-\mu}, \mu < 0$ whose eigenvalues are given by $\pm \sqrt{-\mu}$. In this example, we have succeeded in avoiding central eigenvalues by perturbing the vector field slightly. This example shows that it is possible to avoid singular points with a simple zero eigenvalue in the space of vector fields having a non-degenerate 2nd order term; namely, all vector fields whose 2-jet is given by (4.1).

Unfortunately, it can be proved that in the space of one-parameter families of vector fields, it is impossible to avoid central eigenvalues. In particular, if a one-parameter family $f(x, \mu) \frac{\partial}{\partial x}$ is sufficiently close to the one-parameter family $(\mu + x^2) \frac{\partial}{\partial x}$, as well as to its derivatives, then there exists a parameter value $\mu_0$ close to zero such that $f(x, \mu_0) \frac{\partial}{\partial x}$ has a singular point $x_0$ whose 1-jet vanishes but whose 2-jet does not vanish. In other words, there exist a nearby one-parameter vector field which has a central eigenvalues at some parameter value $\mu_0 \neq 0$.

In general, if a singular point with a central eigenvalue is avoidable in a (k-1)-parameter families of vector fields, but is inevitable in k-parameter families, then we say the singularity has a codimension $k$. Hence, it follows from the above example that a singularity with a vanishing 1-jet but non-vanishing 2-jet has a codimension 1.

A singularity of codimension $k$ should therefore be studied in the space of k-parameter families of vector fields. We are therefore concerned with the following two problems:

1. Find the codimension of singularities.
2. Study the process by which singularities are formed in a typical family of vector fields, called a versal family of the singularity, whose precise definition will be given later.

Let us begin with the versal families for matrices which was first studied by Arnold [8-9]. Let $A(\mu)$ denote a family of $n \times n$ matrices which depend smoothly on $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$.

\[\text{It is important to note that the one-parameter family of vector fields } (\mu + x^2) \frac{\partial}{\partial x} \text{ is only one point in the space of one-parameter families (plural!) of vector fields. There are infinitely many other families of vector fields, parametrized by } \mu, \text{ which are different but "close" to } (\mu + x^2) \frac{\partial}{\partial x}.\]
Definition 4.2

(i) A family \( A(\mu) \) is an unfolding of a matrix \( A_0 \) if \( A(0) = A_0 \).

(ii) Let \( A(\mu) \) and \( \tilde{A}(\tilde{\mu}) \) be unfoldings of a matrix \( A_0 \), where \( \mu \) and \( \tilde{\mu} \) need not have the same dimension. We say \( A(\mu) \) is induced from \( \tilde{A}(\tilde{\mu}) \) if there exist an unfolding \( C(\mu) \) of the identity matrix and a transformation \( \tilde{\mu} = \phi(\mu) \) satisfying \( \phi(0) = 0 \) such that

\[
A(\mu) = C(\mu) \cdot \tilde{A} \left[ \phi(\mu) \right] \cdot C(\mu)^{-1}
\]

holds for any \( \mu \) close to zero.

(iii) An unfolding \( A(\mu) \) of \( A_0 \) is versal if every unfoldings of \( A_0 \) is induced from \( A(\mu) \).

Example 4.3

(1) For any 2x2 matrix \( A_0 \), the family

\[
A(\mu) = A_0 + \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}
\]

is a versal family because any unfolding of \( A_0 \) can be written as

\[
A_0 + \begin{bmatrix} \phi_1(\lambda) & \phi_2(\lambda) \\ \phi_3(\lambda) & \phi_4(\lambda) \end{bmatrix}
\]

where \( \phi_1(0) = 0 \).

(2) Let \( A_0 \) denote the 2x2 matrix \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). Then the family

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{bmatrix}
\]

is versal because any unfolding

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \phi_1(\lambda) & \phi_2(\lambda) \\ \phi_3(\lambda) & \phi_4(\lambda) \end{bmatrix}
\]

is conjugate to the matrix

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \phi_3(1+\phi_2) - \phi_1 \phi_4 & \phi_1 + \phi_4 \end{bmatrix}
\]

under the transformation.
\[
C = \begin{bmatrix}
1 + \phi_2(\lambda) & 0 \\
-\phi_1(\lambda) & 1 
\end{bmatrix}
\]  

(4.8)

Remark 4.4

The above example shows that the matrix
\[
A_0 = \begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix}
\]

has at least 2 kinds of versal unfoldings; namely,
\[
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix} + \begin{bmatrix}
\mu_1 & \mu_2 \\
\mu_3 & \mu_4 
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
\mu_1 & \mu_2 
\end{bmatrix}
\]

(4.9)

Therefore, it is important to obtain versal unfoldings with the minimum number of parameters. Such versal unfoldings are said to be miniversal.

We will show later that the versal unfolding
\[
\begin{bmatrix}
0 & 1 \\
0 & 0 
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
\mu_1 & \mu_2 
\end{bmatrix}
\]

is miniversal.

In order to characterize versal and miniversal unfoldings, we need the following concepts from differential topology:

Definition 4.5

(1) Let \( L \) be a finite dimensional vector space over \( \mathbb{R} \) and let \( X, Y \) be its subspaces: \( X, Y \subseteq L \). We say \( X \) and \( Y \) are transversal (in \( L \)) and denote this property by the symbol

\[
X \cap Y \quad \text{(in \( L \))}
\]

(4.10)

if the sum \( X + Y \) is the whole space \( L \) (see Fig. 1). This implies that

\[
\dim L = \dim X + \dim Y.
\]

(4.11)

If the equality holds, then \( X \) and \( Y \) are said to be minitransversal.

(2) Let \( M \) and \( N \) be smooth manifolds and let \( P \) be a submanifold of \( N \). We say a smooth mapping \( f : M \to N \) is transversal to \( P \) at a point \( x \in M \) if one of the following conditions holds:

(i) \( f(x) \notin P \)

(ii) \( f(x) = p \in P \) and

\[
Df(x) \cdot T_x M \cap T_p P \quad \text{in} \quad T_p N
\]

(4.12)

where \( Df(x) \) is the tangent map (see Fig. 2)

\[
Df(x) : T_x M \to T_{f(x)} N.
\]

(4.13)

If \( Df(x) \cdot T_x M \) and \( T_p P \) are minitransversal, then we say \( f \) is minitransversal to \( P \) at \( x \in M \).
In considering an unfolding $A(\mu)$ of $A_0$, we may regard it as a smooth mapping

$$A : \mu \to A(\mu)$$

from the parameter space $\Pi$ to $M(n, \mathbb{R})$.

Our next theorem is fundamental in versal unfoldings.

**Theorem 4.6**

An unfolding $A(\mu)$ of $A_0 \in M(n, \mathbb{R})$ is (mini-)versal if and only if $A(\mu)$ is (mini-)transversal to $\mathcal{O}(A_0)$ at $\mu = 0$, where $\mathcal{O}(A_0)$ is a submanifold of $M(n, \mathbb{R})$ defined by

$$\mathcal{O}(A_0) = \left\{ C A_0 C^{-1} \mid C \in GL(n, \mathbb{R}) \right\}.$$  

**Remark 4.7**

(1) This theorem is an example of the "versality $\iff$ transversality" principle which originated from the singularity theory of smooth functions [10]. This principle works for various objects.

(2) If an unfolding $A(\mu)$ of $A_0$ is transversal to $\mathcal{O}(A_0)$ at $A_0$, then this type of singularity is inevitable in $A(\mu)$; that is, if we slightly perturb $A(\mu)$ to $A'(\mu')$, as depicted in Fig. 3, then the family $A'(\mu')$ still has a matrix $A'_0$ which is conjugate to $A_0$. If $A(\mu)$ is mini-versal, then $A(\mu)$ is an unfolding with the least number of parameters which contains $A_0$ in an inevitable manner. This shows that the number of parameters of a miniversal unfolding is equal to the codimension of $A_0$ in the space $M(n, \mathbb{R})$ of matrices.

(3) This theorem can be easily proved using the inverse function theorem. See [8,9] for the proof.

**Corollary 4.7**

(1) A miniversal unfolding of the $n \times n$ matrix

$$A_0 = \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}^n$$

is given by

$$A_0 + \begin{bmatrix} \cdots \circ \cdots \\ \mu_1 \cdots \mu_n \end{bmatrix}$$

(2) A miniversal unfolding of the matrix

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Proof

Each statement can be proved by checking the minitransversality of $A(\mu)$ to $O(A_0)$. See [8,9] for the proof.

We are now ready to consider vector fields.

Definition 4.8

1. Let $V_0$ be a vector field on $\mathbb{R}^n$ vanishing at the origin $0$. We say a family $v(\mu)$ of vector fields on $\mathbb{R}^n$ is an unfolding of $v_0$ if $v(0) = v_0$ holds. We do not assume $v(\mu)$ vanishes at $0$ for $\mu \neq 0$.

2. Let $v(\mu)$ and $w(\lambda)$ be unfoldings of $v_0$. We say $w(\lambda)$ is induced from $v(\mu)$ if there exist a family $\phi(\lambda)$ of diffeomorphisms with

$$\phi(0) = \text{identity}$$

and a transformation of parameters $\mu = \mu(\lambda)$ with

$$\mu(0) = 0$$

such that

$$w(\lambda) = \phi(\lambda) \cdot v\left[\mu(\lambda)\right]$$

(4.23)

for $\lambda$ close to zero.

3. We say an unfolding $v(\mu)$ of $v_0$ is $k$-versal if any unfolding of $v_0$ coincides with an induced unfolding up to order $k$.

Remark 4.9
Instead of coordinate transformations, we may consider the case where \( v(\mu(\lambda)) \) is locally topologically equivalent to \( w(\lambda) \). (See [8,9] for the definition.) Under this equivalence, \( v(\mu) \) is said to be topologically versal.

**Example 4.10**

An unfolding \((\mu+x^2) \frac{\partial}{\partial x} \) of the vector field \( x^2 \frac{\partial}{\partial x} \) is 2-versal. To prove this, let \( \phi(x,\lambda) \frac{\partial}{\partial x} \) be any unfolding of \( x^2 \frac{\partial}{\partial x} \). Since \( \phi(x,0) = x^2, \phi_{xx}(0,0) = 2 \neq 0 \). Therefore, by the implicit function theorem, the equation \( \phi_x(x,\lambda) = 0 \) can be solved for \( x \) as a function of \( \lambda \); namely,

\[
x = \eta(\lambda), \quad \text{with } \eta(0) = 0.
\]

By a family of coordinate transformations,

\[
X = \frac{\phi_{xx}(\eta(\lambda),\lambda)}{2} \left\{ x - \eta(\lambda) \right\},
\]

the vector fields \( \phi(x,\lambda) \frac{\partial}{\partial x} \) can be transformed into the form

\[
\left\{ \psi(\lambda) + x^2 + O(x^3) \right\} \frac{\partial}{\partial x}.
\]

Hence, upon choosing \( \mu = \psi(\lambda) \), (4.26) coincides with \((\mu+x^2) \frac{\partial}{\partial x}\) up to order two.

The general "versality \( \Leftrightarrow \) transversality" principle also works for this case. In particular, by changing \( M(n,\mathbb{R}) \) to

\[
\mathcal{X}_0^k = H_1 \oplus H_2 \oplus \cdots \oplus H_k
\]

and \( \mathcal{O}(A_0) \) to

\[
\mathcal{O}(v_0) = \{ \phi^k \cdot v_0 \mid \phi^k \in \text{Diff}_0^k \}, \quad v_0 \in \mathcal{X}_0^k,
\]

we can prove the following theorem by using the same reasoning as that of Theorem 4.6:

**Theorem 4.11**

An unfolding \( v(\mu) \) of \( v_0 \in \mathcal{X}_0^k \) is (mini-)k-versal if and only if \( v \) is (mini-)transversal to \( \mathcal{O}(v_0) \subseteq \mathcal{X}_0^k \) at \( \mu = 0 \).

**Remark 4.12**

Note that we restrict the whole space to \( \mathcal{X}_0^k \), not \( \mathcal{X}_0^k \). This means that any unfolding \( v(\mu) \) of \( v_0 \in \mathcal{X}_0^k \) should vanish at the origin; i.e., \( v(\mu)(0) = 0 \). Therefore Theorem 4.11 does not imply Example 4.10 because in that case, the unfolding \((\mu+x^2) \frac{\partial}{\partial x}\) does not vanish at \( O \). If we apply this theorem to
\( v_0 = x^2 \frac{\partial}{\partial x} \), we would obtain a 2-versal unfolding

\[
(\mu x + x^2) \frac{\partial}{\partial x}
\]

(4.27)

under the condition that the origin is always the singular point.

Nevertheless, we presented the above theorem because we can easily obtain an unfolding without the trivial singularity 0 from the k-versal unfolding in \( \mathcal{X}_0^k \), and because the following general corollary holds.

**Corollary 4.13**

Suppose \( v_0 \in \mathcal{X}_0^k \) is a non-degenerate kth order normal form with a specified 1-jet \( Ax \frac{\partial}{\partial x} \). Then a k-versal unfolding of \( v_0 \) in \( \mathcal{X}_0^k \) is obtained upon adding a linear unfolding

\[
\Phi(\mu) x \frac{\partial}{\partial x}
\]

(4.28)

where \( A + \Phi(\mu) \) is a versal unfolding of the matrix \( A \). In other words, a versal unfolding of a non-degenerate normal form can be obtained by adding a linear versal unfolding.

From this corollary, we can obtain the following versal unfoldings automatically:

(i) \( A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \ (\omega \neq 0) \) (4.29)

In polar coordinate:

\[
(\mu \pm r^2 + \omega r^4) r \frac{\partial}{\partial r} + (\omega + \beta r^2) \frac{\partial}{\partial \theta}
\]

(4.30)

(k-versal for any \( k \geq 5 \))

(ii) \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) (4.31)

\[
y \frac{\partial}{\partial x} + (\mu_1 x + \mu_2 y \pm x^2 + \beta xy + ax^2) \frac{\partial}{\partial y}
\]

(3-versal)

(iii) \( A = \begin{bmatrix} 0 & -\omega & \cdots & 0 \\ \omega & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ (\omega \neq 0) \)
\[(\mu_1 + \alpha z + \alpha z^2) r \frac{\partial}{\partial r} + (\alpha + \beta z + \beta z^2) \frac{\partial}{\partial \theta} + (\mu_2 z + s_1 r^2 + s_2 z^2 + cz^3) \frac{\partial}{\partial z} \quad (s_i = \pm 1) \quad (4.33)\]

(3-versal)

(iv) \[A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\]

\[y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (\mu_1 x + \mu_2 y + \mu_3 z + s x^2 + \beta xy + \gamma xz + ax^3 + bx^2 y + cx^2 z) \frac{\partial}{\partial z} \quad (s = \pm 1) \quad (4.34)\]

(3-versal)

(v) \[A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\] with the symmetry \((x, y) \rightarrow (-x, -y)\)

\[y \frac{\partial}{\partial x} + (\mu_1 x + \mu_2 y + s x^2 + ax^2 y) \frac{\partial}{\partial y} \quad (s = \pm 1) \quad (4.35)\]

(3-versal)

(vi) \[A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\] with the symmetry \((x, y, z) \rightarrow (-x, -y, z)\)

\[y \frac{\partial}{\partial x} + (\mu_1 x + \mu_2 y + s_1 x z + \alpha y z + bx^3 + cx^2 y + dy z^2) \frac{\partial}{\partial y} + (\mu_2 z + s_2 z^2 + ez^2 + fz^3) \frac{\partial}{\partial z} \quad (s_i = \pm 1) \quad (4.36)\]

(3-versal).

The proof of Corollary 4.13 consists of showing the addition of a linear versal unfolding gives a transversal family to \(O(v_0)\). This is assured by the non-degeneracy assumption since k-jets equivalent to non-degenerate normal form of order \(k\) forms an open set in

\[H_2 \oplus H_3 \oplus \cdots \oplus H_k\]

Due to versal unfoldings, we can study the change in the phase portraits of vector fields near a vector field with central eigenvalues in a systematic manner. This study constitutes a part of bifurcation theory. Unfortunately, since we have only k-versal unfoldings for finite \(k\), our considerations must be restricted to truncated
vector fields of finite order. Such truncation sometimes gives rise to very complicated and subtle problems which are beyond the scope of this paper. Therefore our descriptions in what follows are incomplete. The reader is referred to [7,9,11] and the references therein for more details.

Let us begin with the simplest case

\[(\mu + x^2) \frac{\partial}{\partial x} \text{ on } \mathbb{R}^1\]  

(4.37)

which is proved to be 2-versal in Example 4.10. Equation (4.37) is equivalent to the following differential equation:

\[\dot{x} = \mu + x^2\]  

(4.38)

When the parameter \(\mu\) changes from a positive to a negative value, the bifurcation behavior of this ODE is given in the \(x\)-space in Fig. 4, or in the \((\mu, x)\)-space in Fig. 5. Such a bifurcation gives birth to a pair of stable and unstable singular points, and is called a \textit{saddle-node bifurcation}.

Our next example is given by (4.1). In polar coordinate, (4.1) can be identified with the ODE

\[
\begin{cases}
\dot{r} = (\mu + r^2 + \omega r^4) r \\
\dot{\theta} = \omega + \beta r^2 , \quad \omega \neq 0
\end{cases}
\]  

(4.39)

Since \(\omega \neq 0\), say \(\omega > 0\), we have \(\dot{\theta} > 0\) for sufficiently small \(r\). Hence, the local phase portrait near the origin changes as shown in the \((x, y)\)-space in Fig. 6, and in the \((\mu, x, y)\)-space in Fig. 7, when we vary the parameter \(\mu\) from a positive to a negative value. Observe that a limit cycle is born from a singular point. This oscillation mechanism is called the \textit{Hopf Bifurcation}.

Remark 4.14

The above arguments are all based on truncated equations. But fortunately, the result is true for any ODE with a non-degenerate lower order jet. More precisely,

(i) For a one-parameter family \(v(\mu)\) of vector fields on \(\mathbb{R}^1\), if \(v(\mu_0)\) can be transformed into the \textit{non-degenerate} 2nd order normal form \(\pm x^2 \frac{\partial}{\partial x}\) for some \(\mu_0\), and if \(\left.\frac{\partial}{\partial \mu}\right|_{\mu = \mu_0} v(\mu) \neq 0\), then \(v(\mu)\) exhibits the \textit{saddle-node bifurcation} in a neighborhood of \(\mu_0\).

(ii) Similarly, for a one-parameter family \(v(\mu)\) on \(\mathbb{R}^2\), if \(v(\mu_0)\) can be transformed into the \textit{non-degenerate} 3rd order normal form

\[\pm r^3 \frac{\partial}{\partial r} + (\omega + \beta r^2) \frac{\partial}{\partial \theta} , \quad \omega \neq 0\]  

(4.40)

in polar coordinate and if \(\left.\frac{\partial}{\partial \mu}\right|_{\mu = \mu_0} v(\mu) \neq 0\), then \(v(\mu)\) exhibits the \textit{Hopf bifurcation} in a neighborhood of \(\mu_0\).
More complicated bifurcation phenomena can be observed from the family (4.2). Although this family assumes the origin is a singular point, we can eliminate this assumption by making the transformation \( x \rightarrow x - x_0 \). The resulting family, truncated up to 2nd order, is of the form

\[
y \frac{\partial}{\partial x} + \left( \mu_1 + \mu_2 x \pm x^2 + \beta xy \right) \frac{\partial}{\partial y}
\]

(4.41)

Hence, this is 2-versal. It follows from the result of Bogdanov [12-13] (see also Arnold [9]) that this family exhibits the 2-parameter bifurcation phenomena shown in Fig. 8. All of the phase portraits in Fig. 8 are shown near the origin and the parameters \( \mu_1, \mu_2 \) are also chosen close to zero. This is an example of a local 2-parameter bifurcation. It is remarkable to observe that, in this bifurcation, both the saddle-node bifurcation and the Hopf bifurcation are present, in addition to a third bifurcation phenomenon which yields a homoclinic orbit.

Now we can appreciate a fundamental observation from the above local bifurcation theory; namely, a global bifurcation can be observed from a local bifurcation of a more degenerate singularity. For example, although a limit cycle can be observed from a vector field with a hyperbolic singular point, which requires a global analysis, the Hopf bifurcation predicts a limit cycle in an arbitrarily small neighborhood of a central singular point, which requires only a local analysis. Similarly, although the bifurcation phenomena in Fig. 9 can be observed from a 1-parameter family of vector fields, which requires a global analysis, we can observe them in an arbitrarily small neighborhood of the origin in the above 2-parameter family (4.41), which requires only a local analysis.

Bifurcations of the other 2-parameter families (4.33) and (4.35) have also been studied in [5], [7], [9], [14], etc. It remains to consider the bifurcation phenomena in 3-parameter families, such as (4.34), (4.36), etc. For such families, even strange attractors can be expected to bifurcate locally. Some analytical results, as well as results based on numerical simulations [4,15], have been reported recently. However, we are very far from a complete understanding of these extremely complex phenomena. Part 3 of this paper will therefore be devoted to this subject.
APPENDIX. PROOF OF PROPOSITION 2.2

Let us begin with the 3-jet

\[ v^3 = sr^3 \frac{\partial}{\partial r} + (1+\beta r^2) \frac{\partial}{\partial \theta}, \quad s = \pm 1 \]  

(A.1)

of a vector field \( v \) on \( \mathbb{R}^2 \), expressed in polar coordinate \((r, \theta)\), corresponding to the non-degenerate 3rd order normal form (2.38). Recall the following useful formulas from (2.12) and (2.13):

\[
\begin{align*}
\xi_k \frac{\partial}{\partial \xi} & \quad \xi_k \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} = (1-k+1) \xi_k \frac{\partial}{\partial \xi} \\
\xi_k \frac{\partial}{\partial \xi} & \quad \xi_k \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} = (-1-k+1) \xi_k \frac{\partial}{\partial \xi}
\end{align*}
\]  

(A.2)  

(A.3)

Observation (a)

If \( n = k+l \) is even, then \(-k+l\) is also even and hence \( \pm 1-k+l \neq 0 \) in (A.2) and (A.3). This implies that the mapping

\[ L_k : H_k \rightarrow H_k ; \quad Y \rightarrow [Y, v] \]

is surjective for all even \( k \); i.e.,

\[ B_k = \text{Image } L_k = H_k, \quad \text{for all even } k \]  

(A.5)

Observation (b)

If \( n = k+l \) is odd, then the map \( L_k \) has a non-empty kernel only for \( k-l = \pm 1 \).

It follows from Observation (a) that all even order terms in the higher order normal forms are eliminated. It suffices therefore for us to examine only the odd order terms \( n = 2m+1, m \geq 1 \), inductively. For simplicity, we denote the non-degenerate \((2m+1)\)th order normal form by \( v_{(m)} \). Hence, \( v_{(1)} \) denotes the 3-jet \( v^3 \); i.e., \( v_{(1)} \triangleq v^3 \).

For \( m \geq 2 \), we can choose the complementary space \( G_{2m+1} \) to be the subspace spanned by

\[ \left\{ (\xi \xi)^m \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right], (\xi \xi)^m \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] \right\} \]  

(A.6)

Hence, to determine the non-degenerate 5th order normal form \( v_{(2)} \), the complementary space \( G_5 \) is spanned by

\[ \left\{ (\xi \xi)^2 \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right], (\xi \xi)^2 \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] \right\} \]  

(A.7)

The corresponding 5th order normal form problem is given by:
\[
\frac{d}{dt} g_s(t) = -\pi_s \left[ [Y^4, v^3 + g_s(t)] \right]_s
\]  
(A.8)

where \( Y^4 \) satisfies the constraint

\[
[Y^4, v^3]^4 = 0
\]  
(A.9)

This implies:

\[
[Y_1, v_1] = 0
\]  
(A.10)

\[
[Y_2, v_1] = 0
\]  
(A.11)

\[
[Y_3, v_1] + [Y_1, v_3] = 0
\]  
(A.12)

and

\[
[Y_4, v_1] + [Y_2, v_3] = 0
\]  
(A.13)

It follows from (2.19), (A.10) and (A.11) that

\[
Y_1 = A \left( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right) + B \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right)
\]  
(A.14)

and

\[
Y_2 = 0
\]  
(A.15)

Now, (2.34) and (A.14) give

\[
[Y_1, v_2] = \left[ Y_1, s \xi \xi \left( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right) + i \beta \xi \xi \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \right]
\]  
(A.16)

It follows from (A.12) and (A.16) that \( A = 0 \) and \( Y_3 \) is kernel of \( L_3 \). Hence, we can choose

\[
Y_3 = C \xi \xi \left( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right) + D \xi \xi \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right)
\]  
(A.17)

By a similar analysis, (A.13) implies \( Y_4 = 0 \). Consequently, the normal form problem (A.8) reduces to:

\[
\frac{d}{dt} g_s(t) = -\pi_s \left[ [Y_1, g_s(t)] + [Y_3, v_3] \right]
\]  
(A.18)

where
\[ g_{5}(t) = a(t)(\xi \xi^2) \left( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right) + b(t)(\xi \xi^2) \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \]  
(A.19)

Since

\[ Y_1 = B \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) = -i B v_1 \]  
(A.20)

we have

\[ [Y_1, g_5] = i B [g_5, v_1] = 0 \]  
(A.21)

On the other hand,

\[ [Y_3, v_3] = 2(C \beta - Ds)(\xi \xi^2) \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) \]  
(A.22)

Equating the corresponding coefficients of (A.19) and (A.22), we obtain the differential equations:

\[ \dot{a} = 0 \]  
(A.23)

\[ \dot{b} = 2(Ds - C \beta) , \ s = \pm 1 \]  
(A.24)

It follows from (A.23) and (A.24) that \( a(t) = a(0) \), and \( b(t) = b(0) + 2(Ds - C \beta) t \). By choosing \( C = 0 \), and an appropriate \( D \), we can always make \( b(t) = 0 \) at \( t = 1 \). Hence, the resulting 5th order normal form becomes

\[ v(2) = v(1) + a r^s \frac{\partial}{\partial r} \]  
A.25

\[ = (sr^2 + ar^4) r \frac{\partial}{\partial r} + (1 + \beta r^2) \frac{\partial}{\partial \theta} , \ s = \pm 1 \]  
(A.26)

Let us consider now the general \((2m+1)th order\) normal form problem on \( G_{2m+1} \):

\[ \frac{d}{dt} g_{2m+1}(t) = -\pi_{2m+1} \left[ [v^{2m}, v(2) + g_{2m+1}(t)]_{2m+1} \right] \]  
(A.27)

We claim that

\[ g_{2m+1} = 0 , \ m \geq 3 \]  
(A.28)

To prove (A.28), consider the following observations:

(i)
\[ Y_{2m-1} = P(\xi, \xi)^{m-1} \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] + Q(\xi, \xi)^{m-1} \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] \]  
(A.29)

satisfies
\[ [Y_{2m-1}, \nu(2)]^{2m} = 0 \]  
(A.30)

Since \( Y_{2m-1} \in \text{kernel of } L_{2m-1} \),
\[ [Y_{2m-1}, \nu(2)]^{2m} = [Y_{2m-1}, \nu_1] = 0 \]  
(A.31)

(ii) The calculation for the Lie bracket \([Y_{2m-1}, \nu_3]\) for various combinations of \( Y_{2m-1} \) and \( \nu_3 \) is summarized below:

<table>
<thead>
<tr>
<th>( Y_{2m-1} )</th>
<th>( \nu_3 )</th>
<th>( (\xi, \xi) \left( \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right) )</th>
<th>( (\xi, \xi) \left( \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\xi, \xi)^{m-1} \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] )</td>
<td>( (4-2m)(\xi, \xi)^m \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] )</td>
<td>( 2(\xi, \xi)^m \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] )</td>
<td></td>
</tr>
<tr>
<td>( (\xi, \xi)^{m-1} \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] )</td>
<td>( (2-2m)(\xi, \xi)^m \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] )</td>
<td>( 0 )</td>
<td></td>
</tr>
</tbody>
</table>

It follows from observation (i) that we can always choose \( Y_{2m-1} \) as the infinitesimal generator; namely,
\[ Y^{2m} = Y_{2m-1} \]  
(A.32)

Hence, the normal form problem (A.27) becomes
\[ -\pi_{2m+1} \left[ \left[ Y_{2m-1}, \nu(2) + g_{2m+1} \right]_{2m+1} \right] \]
\[ = -\pi_{2m+1} \left[ \left[ Y_{2m-1}, \nu_3 \right] \right] \]
\[ = (2m-4)P \left( \xi, \xi \right)^m \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] \]
\[ + \left( (2m-2)Q + 2P \beta \right) i \left( \xi, \xi \right)^m \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] \]  
(A.33)

Equating the corresponding coefficients of (A.33) and
\[ g_{2m+1}(t) = \phi(t)(\xi, \xi)^m \left[ \xi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} \right] + i \psi(t)(\xi, \xi)^m \left[ \xi \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \xi} \right] \]  
(A.34)

we obtain:
\begin{align}
\dot{\phi} &= (2m-4)Ps \\
\dot{\psi} &= 2P \beta + (2m-2)Qs
\end{align} \tag{A.35} \tag{A.36}

Since \(s \neq 0\) and \(m \geq 3\), we can always choose an appropriate \(P\) and \(Q\) at \(t = 1\) so that

\[\phi(1) = \psi(1) = 0\] \tag{A.37}

It follows that the \((2m+1)\)th order contribution to the normal form is eliminated for all \(m \geq 3\). This completes our proof of Proposition 2.2
FIGURE CAPTIONS

Fig. 1. Subspaces $X$ and $Y$ in this picture are transversal because any point in $L$ can be decomposed into a point in $X$ and a point in $Y$. Note that $X$ and $Y$ would not be transversal if they are collinear in this picture.

Fig. 2. Geometrical interpretation of the *transversality* of a smooth mapping $f$ between two manifolds $M$ and $N$. Here, the manifold $M$ on the left maps into the bold manifold $f(M)$. In particular, $x \in M$ maps to $p \in f(M)$. Here, $f$ is transversal because the tangent to $f(M)$ at $p$ intersects the tangent to the submanifold $P$ at $p$ with a finite angle.

Fig. 3. Geometrical interpretation showing the perturbation of an unfolding $A(\mu)$ into $A'(\mu')$ preserves the transversality property.

Fig. 4. Phase portraits illustrating the *saddle-node bifurcation* in the $x$-space, parametrized by $\mu$.

Fig. 5. Phase portraits illustrating the *saddle-node bifurcation* in the $(\mu, x)$-space.

Fig. 6. Phase portraits illustrating the Hopf Bifurcation on the radial coordinate (on the left) and on the $(x,y)$-plane, parametrized by $\mu$. For $\mu>0$ and $\mu=0$, the trajectory is an expanding spiral ($\dot{r}>0$ and $\dot{\theta}>0$). For $\mu<0$, a circular limit cycle with radius $r_0$ is born. All nearby trajectories inside or outside are repelled from the limit cycle.

Fig. 7. Phase portrait illustrating the Hopf Bifurcation in the $(\mu,x,y)$-space. Note that each cross section parallel to the $(x,y)$-plane on the right ($\mu>0$) or on the $(x,y)$-plane ($\mu=0$) itself consists of an expanding spiral as in Fig. 6. On the other hand, each cross section with the parabola on the left ($\mu<0$) is a circle, which corresponds to a limit cycle.

Fig. 8. Phase portraits illustrating the 2-parameter bifurcation phenomena associated with the vector field (4.41).

Fig. 9. Phase portraits exhibiting a global bifurcation producing a homoclinic orbit associated with a 1-parameter family of vector fields.
REFERENCES


Fig. 3

(i) $\mu > 0$

(ii) $\mu = 0$

(iii) $\mu < 0$

Fig. 4
$x = \sqrt{-\mu}$

UNSTABLE

$X = -\sqrt{-\mu}$

(0,0)

Fig. 5
(i) $\mu > 0$

(ii) $\mu = 0$

(iii) $\mu < 0$

Fig. 6
Fig. 7
Fig. 9