ALGEBRAIC DESIGN OF LINEAR MULTIVARIABLE FEEDBACK SYSTEMS

by

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1. INTRODUCTION

This paper considers exclusively linear time-invariant systems with the configuration \( \Sigma(P, K) \) of Fig. 1, where the plant \( P \) has an output \( y_0 \) and a measured output \( y_m \) and the controller \( K \) has two inputs: the exogeneous input \( v \) and the feedback signal \( e_1 \). This configuration is a slight extension of the standard one considered in most textbooks and papers [Blo. 1, Cal. 1, Kai. 1, Per. 1, Ros. 1, Vid. 1, You. 1]. It is simpler than that considered in [Net. 3]. Algebraic techniques are systematically used in this paper [Des. 1, Des. 3, Des. 4, Net. 2, Vid. 1, Vid. 2]. The contribution of this work lies in its more general configuration and its standardized proofs. For previous work on decoupling, see [Ham. 1] and the references therein.

Six theorems address the crucial issues in the design of control systems: stability; achievable I/O and D/O maps; achievable decoupled I/O maps; robustness of stability; asymptotic tracking: necessary conditions; and sufficient conditions for (robust) asymptotic tracking.

The following is a list of the commonly used symbols:

- \( a := \text{means a denotes b.} \)
- \( \mathfrak{v}_n \) is the \( n \)-vector of zeros. W.l.o.g. means without loss of generality. U.t.c. means under these conditions. If \( \mathcal{G} \) is a ring, then \( \mathcal{G}(\mathcal{F}) \) denotes the set of matrices having all entries in \( \mathcal{F} \). \( \mathcal{R}_\mathcal{U} \) denotes the proper rational functions analytic in the region \( \mathcal{U} \subset \mathbb{C} \), a symmetric subset of \( \mathbb{C} \) which contains \( \mathbb{C}^+ \) and \( \mathbb{C}^+ \cup \{\infty\} \). \( \mathbb{R}(s) \) denotes the scalar rational functions in \( s \) with real coefficients, and \( \mathbb{R}[s] \) denotes the scalar polynomials in \( s \) with real coefficients.

Algebraic Structure: \([\text{Bou. 1, p. 55}], [\text{Jac. 1, p. 393}], [\text{Lang 1, p. 89}]\).

- \( \mathcal{H} \): A principal ring (principal ideal domain), i.e., an entire commutative ring in which every ideal is principal (e.g., \( \mathcal{R}_\mathcal{U} \)).
- \( \mathfrak{F} \): The field of fractions over \( \mathcal{H} \) (e.g. \( \mathbb{R}(s) \)).
- \( \mathcal{J} \): A multiplicative subset of \( \mathcal{H} \), equivalently, \( \mathcal{I} \subset \mathcal{H} \), \( 0 \notin \mathcal{I} \), \( 1 \in \mathcal{I} \) and \( x,y \in \mathcal{I} \) implies that \( xy \in \mathcal{I} \) (e.g., \( f \in \mathcal{J} \) if \( f \in \mathcal{R}_\mathcal{U} \) and \( f(\infty) = 1 \)).
- \( \mathfrak{G} : = \{ n/d : n \in \mathcal{H} \ , \ d \in \mathcal{J} \} \), a subring of \( \mathfrak{F} \) (e.g. \( \mathbb{R}_p(s) \), the ring of proper scalar rational functions).
- \( \mathcal{U}(\mathcal{H}) : = \{ m \in \mathcal{H} : m^{-1} \in \mathcal{H} \} \), the group of units in \( \mathcal{H} \) (e.g., \( f \in \mathcal{U}(\mathcal{H}) \) if \( f \in \mathcal{R}_\mathcal{U} \) and \( f(s) \neq 0 \) for all \( s \in \mathcal{U} \)).
- \( \mathfrak{G}_s : = \{ x \in \mathfrak{G} : (1+xy)^{-1} \in \mathfrak{G} \ , \ \forall \ y \in \mathfrak{G} \} \) (Jacobson radical of \( \mathfrak{G} \)).

Four examples of this algebraic structure are given in [Des. 3, Table 1].
2.1. Problem Description

We consider the multi-input-multi-output (MIMO) linear, time-invariant system \( \Sigma(P,K) \) shown in Fig. 1 (Fig. 2). Given a plant \( P \) we wish to design a controller \( K \) with two inputs and one output such that the resulting feedback system is stable and \( K \) has elements in \( \mathcal{X} \). We make the following assumptions on \( \Sigma(P,K) \):

Assumptions on the System \( \Sigma(P,K) \)

\( (P) \quad P = \begin{bmatrix} \frac{P_0}{P_m} \end{bmatrix} \in \mathcal{G}^{2n_u \times n_y} \) has a right-coprime factorization (r.c.f.) \( \begin{bmatrix} N_0 \cr D_{pr}^{-1} \end{bmatrix} N_m \) with \( D_{pr} \in \mathcal{H}^{n_u \times n_y}, \ N_0, N_m \in \mathcal{H}^{n_u \times n_k} \) and \( \det D_{pr} \in \mathcal{J} \).

\( (K) \quad K \in \mathcal{G}^{n_k \times (n_u + n_v)} \) has a left-coprime factorization (l.c.f.) \( D_{cl}^{-1}[N_m : N_f] \) with \( D_{cl} \in \mathcal{H}^{n_k \times n_k}, \ N_m \in \mathcal{H}^{n_k \times n_k}, \ N_f \in \mathcal{H}^{n_k \times n_k} \) and \( \det D_{cl} \in \mathcal{J} \) with \( \det (D_{cl}D_{pr} + N_f N_m) \in \mathcal{J} \).

It is understood that the subsystems \( P \) and \( K \), specified by their transfer functions, do not have any unstable hidden modes [Cal. 1 sec. 4.2].

Under assumptions \( (P) \) and \( (K) \) the system \( \Sigma(P,K) \) in Fig. 1 is completely described by

\[
\begin{bmatrix} I_{n_u} & -D_{pr} \cr D_{cl} & N_f N_m \end{bmatrix} \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & -I_{n_u} & 0 \\ N_m & N_f & 0 & -N_f N_m \end{bmatrix} \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \quad (2.1)
\]

\[
\begin{bmatrix} I_{n_k} & 0 \\ 0 & N_{pr} \end{bmatrix} \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_1 & y_0 & y_m \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -N_{pr} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \quad (2.2)
\]

Let \( u := (v^T, u_1^T, u_2^T, d^T)^T \), \( \xi := (y_1^T, \xi_p^T)^T \), \( y := (y_1^T, y_0^T, y_m^T)^T \). Then equations (2.1) and (2.2) are of the form

\[
D \xi = N_l u \quad (2.3)
\]

\[
N_r \xi = y + Eu \quad (2.4)
\]

where the matrices \( D, N_l, N_r, E \), defined in an obvious manner from (2.1) and (2.2), have their elements in \( \mathcal{X} \).

For any \( D_{cl} \in \mathcal{H}^{n_k \times n_k} \) and any \( N_f \in \mathcal{H}^{n_k \times n_k} \), define

\[
D_h := D_{cl}D_{pr} + N_f N_m
\]

(2.5)

Note that \( \det D = \det D_h \) and, by assumption \( (K) \), \( \det D \in \mathcal{J} \).

**Definition 2.1. (\( \mathcal{X} \)-stability):** The system \( \Sigma(P,K) \) is said to be \( \mathcal{X} \)-stable if and only if \( H_{yu} : u \mapsto y \) satisfies \( H_{yu} \in \mathcal{E} (\mathcal{X}) \).
Let assumptions (P) and (K) hold; then from equations (2.3) and (2.4) we obtain

\[ H_{yu} = N_r D^{-1} N_t + E \in \mathcal{E} (\mathcal{F}) \quad . \]  

(2.6)

**Definition 2.2 (Stabilizing Controller):** The controller \( K \) is said to stabilize \( P \) if \( K \) satisfies assumption (K) and the resulting system \( \Sigma (P,K) \) is \( \mathcal{H} \)-stable.

**Theorem 2.3 (\( \mathcal{H} \)-stability)**

Consider the system \( \Sigma (P,K) \) where \( P \) satisfies (P), and \( K \) will be specified later.

(i) Let \( K \) satisfy (K). Then \( \Sigma (P,K) \) is \( \mathcal{H} \)-stable if and only if \( \det D_h \in \mathcal{U} (\mathcal{H}) \).

(ii) Let, in addition, \( P^m \in \mathcal{G}_s^{n_x \times n_t} \), where \( \mathcal{G}_s := \text{Jacobson radical of } \mathcal{G} \). Then there is a compensator \( K \) which stabilizes \( P \) if and only if \( (N_r^m, D_{pr}) \) is a right-coprime (r.c.) pair.

**Proof:** (i) \((=>)\) To prove that \( \det D_h \in \mathcal{U} (\mathcal{H}) \), we first show that \( (D, N_t) \) is left-coprime (l.c.) and \( (N_r, D) \) is r.c., where \( N_r, D, N_t \) are defined by (2.1)-(2.4).

Let \( L, R \in \mathcal{E} (\mathcal{H}) \) be products of elementary row and column matrices, respectively. Then using Bezout identities it can be shown that \( (D, N_t) \) is l.c. \( <=> \) \( (\hat{D}, \hat{N}_t) \) is l.c. where \( \begin{bmatrix} \hat{D} & \hat{N}_t \end{bmatrix} = [D : N_t][R] \); similarly \( (N_r, D) \) is r.c. \( <=> \) \( (\hat{N}_r, \hat{D}) \) is r.c. where

\[
\begin{bmatrix}
N_r \\
\cdots \\
D
\end{bmatrix} = [L] \quad \begin{bmatrix}
N_r \\
\cdots \\
D
\end{bmatrix}
\]

By elementary column operations on \( [D : N_t] \) of equation (2.3), we obtain

\[
\hat{D} = \begin{bmatrix}
0 & \cdots & 0 \\
\cdots & \cdots & \cdots \\
D_{cl} & 0
\end{bmatrix}, \quad \hat{N}_t = \begin{bmatrix}
0 & \cdots & I_{n_t} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
N_{mt} & N_{ft} & 0 & -N_{ft}
\end{bmatrix}
\]

(2.10)

By assumption (K), \( (\hat{D}, \hat{N}_t) \) in (2.10) is l.c. By elementary row operations on \( \begin{bmatrix}
N_r \\
\cdots \\
D
\end{bmatrix} \) of equations (2.3) and (2.4), we obtain

\[
\begin{bmatrix}
I_{n_t} & 0 \\
\cdots & \cdots \\
0 & N_{pr} \\
\cdots & \cdots \\
0 & N_{pr}^m
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & D_{pr} \\
\cdots & \cdots \\
0 & 0
\end{bmatrix}
\]

(2.11)

In view of assumption (P), equation (2.11) shows that \( (N_r, D) \) is r.c.

Now for a proof by contradiction, suppose that \( \det D_h \notin \mathcal{U} (\mathcal{H}) \); then \( \det D_h = \det D \notin \mathcal{U} (\mathcal{H}) \). Hence, \( D^{-1} \notin \mathcal{E} (\mathcal{H}) \) since \( \mathcal{H} \) is a commutative ring [Jac. 1, p. 94]. Using Bezout identities it is easy to show that, since \( (N_r, D) \) are r.c., \( N_r D^{-1} \in \mathcal{E} (\mathcal{H}) \) \( <=> \) \( D^{-1} \in \mathcal{E} (\mathcal{H}) \), and that, since \( (D, N_t) \) are l.c., \( N_r D^{-1} N_t \in \mathcal{E} (\mathcal{H}) \) \( <=> \) \( N_r D^{-1} \in \mathcal{E} (\mathcal{H}) \). Therefore, \( H_{yu} = (N_r D^{-1} N_t + E) \notin \mathcal{E} (\mathcal{H}) \), which implies that \( \Sigma (P,K) \) is not \( \mathcal{H} \)-stable. Since this is a contradiction, \( \det D_h \in \mathcal{U} (\mathcal{H}) \).

(\(<=\)) Since \( \det D_h \in \mathcal{U} (\mathcal{H}) \), and \( \det D = \det D_h \), we have \( (\det D)^{-1} \in \mathcal{H} \), and \( D^{-1} \in \mathcal{E} (\mathcal{H}) \). Consequently, \( H_{yu} = (N_r D^{-1} N_t + E) \in \mathcal{E} (\mathcal{H}) \) and \( \Sigma (P,K) \) is \( \mathcal{H} \)-stable.
(ii) For a proof by contradiction, suppose that the pair \((N_{pr}^m, D_{pr})\) is not r.c. Then \((N_{pr}^m, D_{pr})\) have a greatest-common-right-divisor (gcrd) \(R\) such that \(\det R \not\in \mathcal{U}(\mathcal{H})\).\(N_{pr}^m = N_{pr}^m R, D_{pr} = D_{pr} R\) and \((N_{pr}^m, D_{pr})\) are r.c. Defining \(D_h\) is an obvious manner, we write

\[
\det D_h = \det[(D_{cl} \tilde{D}_{pr} + N_{fl} \tilde{N}_{pr}^m)R] = \det \tilde{D}_h \det R
\]

where \(\det \tilde{D}_h \in \mathcal{H}\) and \((\det R)^{-1} \not\in \mathcal{H}\). Then \(\det D_h \not\in \mathcal{U}(\mathcal{H})\), because if \(\det D_h \in \mathcal{U}(\mathcal{H})\), then from (2.18), \((\det R)^{-1} = (\det D_h) (\det R)^{-1} \in \mathcal{H}\), which is a contradiction. Therefore, for all \(D_{cl}, N_{fl}, \) det \(D_h \not\in \mathcal{U}(\mathcal{H})\), including those \(D_{cl}\) and \(N_{fl}\) for which \(K\) satisfies (K). Then by part (i), the system \(\Sigma(P, K)\) is not \(\mathcal{H}\) -stable for all \(K\) which satisfy (K). In other words, there is no such \(K\) that stabilizes \(P\).

(<=) By assumption, the pair \((N_{pr}^m, D_{pr})\) is r.c.; hence, there exist \(U_{pr}^m, V_{pr}^m \in \mathcal{E}(\mathcal{H})\) such that

\[
U_{pr}^m N_{pr}^m + V_{pr}^m D_{pr} = I_{n_k}
\]

As a compensator choose \(K := (V_{pr}^m)^{-1}[N_{fl} : U_{pr}^m]\), where \(N_{fl} \in \mathcal{E}(\mathcal{H})\) is arbitrary.

From (2.5) and (2.17), \(D_h = I\) and \(\det D_h = 1 \in \mathcal{U}(\mathcal{H})\).

It remains to show that \(\det V_{pr}^m \in \mathcal{J}\): For the chosen compensator, (2.5) implies

\[
V_{pr}^m D_{pr} = I - U_{pr}^m N_{pr}^m
\]

and taking determinants of both sides of (2.18) we obtain

\[
\det V_{pr}^m = \det(I - U_{pr}^m N_{pr}^m)(\det D_{pr})^{-1}
\]

By assumption, \(P_{pr}^m \in \mathcal{E}(\mathcal{E}_s)\); by the properties of the Jacobson radical \(\mathcal{E}_s\), we have \(P_{pr}^m D_{pr} = N_{pr}^m \in \mathcal{E}(\mathcal{E}_s)\) and \(U_{pr}^m N_{pr}^m \in \mathcal{E}^{n_k \times n_k}\) since \(D_{pr}\) and \(U_{pr}^m \in \mathcal{E}(\mathcal{H})\).

Using standard determinant expansion formulas, we see that \(\det(I - N_{fl} N_{pr}^m) \in \mathcal{J}\) and hence, \([\det(I - N_{fl} N_{pr}^m)]^{-1} \in \mathcal{E}\). Since \(\det D_{pr} \in \mathcal{J}\), equation (2.19) shows that \((\det V_{pr}^m)^{-1} \in \mathcal{E}\) and hence, \(\det V_{pr}^m \in \mathcal{J}\). Thus, the compensator \(K\) chosen above has all its elements in \(\mathcal{E}\) and for this \(K\), \(\det D_h \in \mathcal{U}(\mathcal{H})\). Therefore \(\Sigma(P, K)\) is \(\mathcal{H}\) -stable.

III. ACHIEVABLE PERFORMANCE OF \(\Sigma(P, K)\)

We now use the relationships between the stabilizing controller \(K\) and \(\det D_h\) to give global parametrizations of a) the family of all \(1/0\) maps possible for a given plant with some stabilizing controller b) the family of all disturbance-to-output (D/O) maps possible for a given plant with some stabilizing controller.

For a given system \(\Sigma(P, K)\) satisfying (P) and (K), and \(\det D_h \neq 0\), equations (2.1) and (2.2) show that the \(1/0\) map \(H_{y_{o,v}} : v \mapsto y\) and the \(D/O\) map \(H_{y_{o,d}} : d \mapsto y\) are given by:

\[
H_{y_{o,v}} = N_{pr}^m D_h^{-1} N_{fl}
\]

\[
H_{y_{o,d}} = N_{pr}^m [I - D_h^{-1} N_{fl} N_{pr}^m] = N_{pr}^m D_h^{-1} D_{cl} D_{pr}
\]

Definition 3.1 (Achievable Maps)

Let \(P\) be a given plant that satisfies assumption (P); hence the specification of the controller \(K\) determines the system \(\Sigma(P, K)\). Roughly speaking, let \(\mathcal{H}_{y_{o,v}}(P)\) denote the set of all achievable \(1/0\) maps of \(\Sigma(P, K)\), and let \(\mathcal{H}_{y_{o,d}}(P)\) denote the set of all achievable \(D/O\) maps of \(\Sigma(P, K)\); more precisely,
The following theorem characterizes all the achievable I/O maps and the achievable D/O maps for \( \Sigma(P,K) \).

**Normalization Assumption:** Since by Theorem 2.3, \( K \) stabilizes \( P \) if and only if \( \det D_h \in \mathcal{U}(H) \), we take w.l.o.g.

\[ D_h = I_n \]  
whenever \( K \) stabilizes \( P \) [Vid. 2].

**Theorem 3.2 (Achievable I/O Maps and Achievable D/O Maps)**

Consider the system \( \Sigma(P) \) of Fig. 1. Let \( P \) satisfy assumption (P) and let \( (N_{pr}, D_{pr}) \) be a r.c. pair. Let \( D_{pl}^{-1} N_{pl} \) be a l.c.f. of \( P^m \). Then

\[ (i) \mathcal{H}_{y,v} = \{ N_{pr}^o Q : Q \in \mathcal{H}_{n_x \times n_x} \} \]

equivalently, any map \( H_v \in \mathcal{H}_{n_x \times n_x} \) is an achievable I/O map of the \( \mathcal{H} \)-stable system \( \Sigma(P,K) \) if and only if \( H_v = N_{pr}^o Q \) for some \( Q \in \mathcal{H}_{n_x \times n_x} \).

\[ (ii) \mathcal{H}_{y,d} = \{ N_{pr}^o [I - (U_{pr} + RD_{pl}) N_{pr}^m] = N_{pr}^o (V_{pr}^m - RN_{pl}^m) D_{pr} : R \in \mathcal{H}_{n_x \times n_x} \text{ s.t. } \det(V_{pr}^m - RN_{pl}^m) \in \mathcal{J} \} \]

where \( V_{pr}^m, U_{pr}, N_{pr}, D_{pr} \) are as in (2.16); equivalently, any map \( H_d \in \mathcal{H}_{n_x \times n_x} \) is an achievable D/O map of the \( \mathcal{H} \)-stable system \( \Sigma(P,K) \) if and only if \( H_d = N_{pr}^o [I - (U_{pr} + RD_{pl}) N_{pr}^m] = N_{pr}^o (V_{pr}^m - RN_{pl}^m) D_{pr} \) for some \( R \in \mathcal{H}_{n_x \times n_x} \) which satisfies \( \det(V_{pr}^m - RN_{pl}^m) \in \mathcal{J} \).

**Comments:**

1) In the case that \( y_0 = y_m \) (i.e., \( N_{pr}^o = N_{pr}^m =: N_{pr} \)) the set of achievable I/O maps and the set of achievable D/O maps reduce to those in [Des. 3]:

\[ \mathcal{H}_{y,v}(P) = \{ N_{pr} Q : Q \in \mathcal{H}_{n_x \times n_x} \} \]

\[ \mathcal{H}_{y,d}(P) = \{ I - N_{pr} (U_{pr} + RD_{pl}) : R \in \mathcal{H}_{n_x \times n_x} \text{, and } R \text{ is s.t. } \det D_{cl} \in \mathcal{J} \} \]

where \( d_0 := N_{pr} d \). 2) \( H_{y,d} \) by the \( \mathcal{U} \)-zeros and the \( \mathcal{U} \)-poles of the plant when \( \mathcal{H} = \mathcal{R}_U \). If \( \Sigma(P,K) \) is \( \mathcal{H} \)-stable and if \( PF := PD_{cl}^{-1} N_{ft} \) is full normal rank in \( \mathcal{G} \), then

a) if \( z_0 \) is a \( \mathcal{U} \)-zero of \( N_{pr}^o \) (equivalently, \( \exists \alpha \neq \varphi_{n_x} \) such that \( \alpha^* N_{pr}(z_0) = \varphi_{n_i} \)) then

\[ a^* N_{pr}(I - N_{ft} N_{pr}^m)(z_0) = a^* H_{y,d}(z_0) = \alpha \]  

(3.8)

b) if \( N_{pr}^m \) has full normal rank and if \( z_m \) is a \( \mathcal{U} \)-zero of \( N_{pr} \) (equivalently, \( \exists \beta \neq \varphi_n \) such that \( N_{pr}^m(z_m) = \beta \)) then

\[ N_{pr}(I - N_{ft} N_{pr}^m)(z_m) \beta = N_{pr}(z_m) \beta = H_{y,d}(z_m) \beta \]  

(3.9)

c) if \( p_o \) is a \( \mathcal{U} \)-pole of \( P \) (equivalently, \( \exists \gamma \neq \varphi_n \) such that \( D_{pr}(p_o) \gamma = \varphi_{n_i} \)) then
Thus, whenever either \( N_{pr}^0 \) or \( N_{pr}^m \) has a \( \mathcal{U} \)-zero or when \( P \) has a \( \mathcal{U} \)-pole, the D/0 map is constrained by a vector-equality such as (3.8), (3.9) or (3.10) respectively.

**Proof of Theorem 3.2:** \((\Rightarrow)\) We are given \( P \) satisfying (P) and a controller \( K \) which stabilizes \( P \). Let \( H_u \) be the I/O map and \( H_d \) be the D/0 map of this \( \Sigma(P,K) \). We must show that \( H_u \) is of the form \( N_{pr}^0 Q \) for some \( Q \in \mathcal{H}^{n_x \times n_u} \) and \( H_d \) is of the form \( N_{pr}^0 [I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^0 (V_{pr}^m - RN_{pr}^m)D_{pr} \) for some \( R \in \mathcal{H}^{n_x \times n_u} \) satisfying \( \det(V_{pr}^m - RN_{pr}^m) \in \mathcal{J} \).

Since \( K \) satisfies (K), \( N_{nl} \in \mathcal{H}^{n_x \times n_u} \) and by Theorem 2.3, \( \det D_h \in \mathcal{U}(\mathcal{H}) \). Let 
\[ Q := D_h^{-1} N_{nl} = N_{nl} \]; then 
\[ Q \in \mathcal{H}^{n_x \times n_u} \] and by (3.1), \( H_u = N_{pr}^0 D_h^{-1} N_{nl} = N_{pr}^0 Q \).

Now from (2.5) and (3.5)

\[
N_{nl}^0 N_{pr}^m + D_{cl} D_{pr} = I
\]

Viewing (3.11) as a linear matrix equation in \( \mathcal{E}(\mathcal{H}) \), we solve for \( (D_{cl}, N_{fl}) \) subject to \( \det D_{cl} \in \mathcal{J} \) so that \( D_{cl}^{-1} N_{fl} \in \mathcal{E}^{n_x \times n_u} \); since \( (N_{pr}^m, D_{pr}) \) is a r.c. pair, from (2.17) we have

\[
U_{pr}^m N_{pr}^m + V_{pr}^m D_{pr} = I
\]

and since \( N_{pr}^m D_{pr}^{-1} = D_{pr}^{-1} N_{pr}^m = P^m \), we have

\[
D_{pl} N_{pr}^m - N_{nl} D_{pr} = 0
\]

The pair \( (U_{pr}^m, V_{pr}^m) \) is (3.12) is a particular solution to \( (N_{fl}, D_{cl}) \) in (3.11) and the pair \( (D_{pl}, -N_{nl}^m) \) is a particular solution to the homogeneous equation (3.13). Hence, any general solution of (3.11) is given by

\[
N_{fl} = U_{pr}^m + RD_{pl}
\]

\[
D_{cl} = V_{pr}^m - RN_{nl}^m
\]

We now show that \( R \in \mathcal{E}(\mathcal{H}) \). Since \( K \) satisfies (K), \( \det D_{cl} \in \mathcal{J} \); therefore \( \det(V_{pr}^m - RN_{nl}^m) \in \mathcal{J} \). Since \( (D_{pl}, N_{nl}^m) \) are l.c., there exist \( V_{pl}, U_{pl} \in \mathcal{E}(\mathcal{H}) \) such that

\[
D_{pl} V_{pl} + N_{nl}^m U_{pl} = I
\]

Thus,

\[
R = R(D_{pl} V_{pl} + N_{nl}^m U_{pl}) = (N_{fl} - U_{pr}^m) U_{pl} + (V_{pr}^m - D_{cl}) U_{pl} \in \mathcal{E}(\mathcal{H}) \text{ since } N_{fl}, D_{cl}, U_{pr}^m, V_{pr}^m, V_{pl}, U_{pl} \in \mathcal{E}(\mathcal{H})
\]

From (3.2) and (3.14a-b), \( H_d = N_{pr}^0 [I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^0 (V_{pr}^m - RN_{nl}^m)D_{pr} \). Therefore the given \( H_u \) and \( H_d \) are elements of the sets (3.6) and (3.7) respectively. \((\Leftarrow)\) For some \( Q \in \mathcal{H}^{n_x \times n_u} \), we are given \( H_u = N_{pr}^0 \), and for some \( R \in \mathcal{H}^{n_x \times n_u} \), we are given \( H_d = N_{pr}^0 [I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^0 (V_{pr}^m - RN_{nl}^m)D_{pr} \), where \( \det(V_{pr}^m - RN_{nl}^m) \in \mathcal{J} \). We must show that there exists a compensator \( K \) which stabilizes \( P \) and the \( \mathcal{H} \)-stable \( \Sigma(P,K) \) achieves the given \( H_u \) and \( H_d \).
Choose the controller \( K := D_{cl}^{-1} [N_{ml} : N_{ft}] \) with \( N_{ft} \) and \( D_{ct} \) as in (3.14a-b) and \( N_{ml} = Q \). Clearly, \( D_{ct}, N_{ml}, N_{ft} \in \mathcal{F} \). Note that \( \det D_{ct} \in \mathcal{F} \) is guaranteed by the \( R \) that was chosen. Now, by (2.5)

\[
D_h = (V_{pr}^m - RN_{pl}^m)D_{pr} + (U_{pr}^m + RD_{pl})N_{pr}^m
\]

By (3.12) and (3.13), \( D_h = I \). Rewriting (3.16) as

\[
(V_{pr}^m - RN_{pl}^m)D_{pr} + [Q : (U_{pr}^m + RD_{pl})] \begin{bmatrix} \eta_{n_y \times n_i} \\ N_{pr}^m \end{bmatrix} = I,
\]

we see that \( (D_{ct}, [N_{ml} : N_{ft}] \) are l.c., and this \( K \) satisfies (K). since \( \det D_h \in \mathcal{U}(\mathcal{F}) \), \( \Sigma(P,K) \) is \( \mathcal{F} \)-stable by Theorem 2.3(i).

By (3.1) and with \( D_h = I \), we calculate the I/O map:

\[
H_{y,u} = N_{pr}^o N_{ml} = N_{pr}^o Q = H_v.
\]

By (3.2), the D/O map is

\[
H_{y,d} = N_{pr}^o [I - N_{ft} N_{pr}^m] = N_{pr}^o [I - (U_{pr}^m + RD_{pl}) N_{pr}^m] = N_{pr}^o D_{ct} D_{pr} = N_{pr}^o (V_{pr}^m - RN_{pl}^m)D_{pr} = H_d.
\]

**Summary:** Given the setup of Theorem 3.2 and in particular the \( Q \) and the \( R \) of (3.6) and (3.7), the compensator that achieves the specified \( H_v \) and \( H_d \) and that stabilizes \( P \) is given by the coprime factorization

\[
D_{ct} = V_{pr}^m - RN_{pl}^m,
\]

\( [N_{ml} : N_{ft}] = [Q : U_{pr}^m + RD_{pl}] \).

**IV. DECOUPLING**

In this section we characterize all diagonal I/O maps which can be achieved by \( \Sigma(P,K) \) for the given plant \( P \).

Let \( P \in \mathcal{B}_{2n \times n} \); i.e., \( n_y = n_i = n \), \( K \in \mathcal{F}_{n \times 2n} \) and \( n_u = n \). Let assumption (P) and (K) hold with these new dimensions.

Let \( n_{pk} \in \mathcal{H}^{1 \times n} \) denote the \( k \)-th row of \( N_{pr}^o \in \mathcal{H}^{n \times n} \). For \( k = 1, \ldots, n \), define \( \Delta_{L_k} \) as a greatest common divisor (g.c.d.) over \( \mathcal{H} \) of the elements of \( n_{pk} \) [Lang 1, p. 71]. \( \Delta_{L_k} \) exists since \( \mathcal{H} \) is a principal ring. Then the row-vector \( n_{pk} \) is uniquely defined by

\[
\eta_{pk} = \Delta_{L_k} n_{pk} \quad \text{and} \quad \eta_{pk} \in \mathcal{H}^{1 \times n} \, \text{Let} \quad N_{pr}^o \in \mathcal{H}^{n \times n} \, \text{be defined as the matrix which has} \quad \eta_{pk} \quad \text{as its} \, k \text{-th row,} \, k = 1, \ldots, n \, \text{. Then}
\]

\[
N_{pr}^o = diag (\Delta_{L_1}, \ldots, \Delta_{L_k} \ldots, \Delta_{L_n}) \tilde{N}_{pr}^o =: \Delta_{L} \tilde{N}_{pr}^o .
\]

Note that \( \Delta_{L} \) and \( \tilde{N}_{pr}^o \) are unique within unimodular factors. (In the case that \( \mathcal{H} = \mathcal{R}_{2n} \), \( \Delta_{L_k} \) "book-keeps" the plant zeros in \( \mathcal{U} \) that are common to all elements of the \( k \)-th row of \( N_{pr}^o \).) A similar factorization is used in [Dat. 1].

The matrix \( \tilde{N}_{pr}^o \) is not necessarily invertible over \( \mathcal{H}^{n \times n} \). But by assumption (P) and since \( \det N_{pr}^o \in \mathcal{H} \), \( (N_{pr}^o)^{-1} \) has elements in the field of fractions \( [\mathcal{H}] \subseteq [\mathcal{H} \setminus 0]^{-1} \) of the entire ring \( \mathcal{H} \) [Lang 1, p. 69]. Let \( \frac{m_{ij}}{d_{ij}} \) denote the \( i,j \)-th element of \( (\tilde{N}_{pr}^o)^{-1} \), \( i,j = 1,\ldots,n \), where \( m_{ij}, d_{ij} \in \mathcal{H} \) and \( m_{ij}, d_{ij} \) are coprime; thus

\[
(\tilde{N}_{pr}^o)^{-1} = \begin{bmatrix} \frac{m_{ij}}{d_{ij}} \end{bmatrix}, \quad i,j = 1,\ldots,n \quad \text{(4.2)}
\]

For \( j = 1,\ldots,n \), let \( \Delta_{R_j} \) be a least common multiple (l.c.m.) of \( d_{1j}, d_{2j}, \ldots, d_{nj} \) of the \( j \)-th column of \( (\tilde{N}_{pr}^o)^{-1} \) [Lang 1, p. 72]. Define
Let $\Delta_R := \text{diag}(\Delta_{R1}, \ldots, \Delta_{Rj}, \ldots, \Delta_{Rn}) \in \mathbb{R}^{n \times n}$ (4.3)

$\Delta_R$ is unique within a unimodular factor.

**Lemma 4.1:** Let $\tilde{N}_p^o$ and $\Delta_R$ be defined by (4.1) and (4.3). Then $(\tilde{N}_p^o)^{-1}\Delta_R \in \mathbb{H}^{n \times n}$.

**Proof:** Since $\Delta_{Rj}$ is a l.c.m. of $(d_i)_{i=1}^n$, we have $a_{ij} \in \mathbb{H}$ such that $\Delta_{Rj} = d_i a_{ij}$ for $i = 1, \ldots, n$. Then, for $i = 1, \ldots, n$, the $ij$-th element of $(\tilde{N}_p^o)^{-1}\Delta_R = \frac{m_{ij}}{d_i} \Delta_{Rj} = m_{ij} a_{ij} \in \mathbb{H}$ by (4.2).

**Definition 4.2 (Achievable diagonal I/O map):** Let $P$ be a given plant that satisfies assumption (P). Roughly speaking, let $\mathcal{H}^d_{y,v}(P)$ denote the set of all achievable diagonal I/O maps of $Z(P,K)$; more precisely,

$$\mathcal{H}^d_{y,v}(P) := \{H^d_{y,v} : K \text{ stabilizes } P \text{ and the resulting I/O map } H^d_{y,v} \text{ is diagonal and nonsingular.}\}$$

**Theorem 4.3 (Achievable Diagonal I/O Maps)**

Consider the system $Z(P,K)$ of Fig. 1. Let $P$ satisfy assumption (P) and let $(N_{pr}, D_{pr})$ be r.c. Let $D_{pr}^{-1}N_{pr}^m$ be a l.c.f. of $P^m$. Then

$$\mathcal{H}^d_{y,v}(P) = \{\Delta_L \Delta_R Q_d : Q_d \in \mathbb{H}^{n \times n}, \text{with } Q_d \text{ diagonal and nonsingular}\}$$

(4.4)

equivalently, the map $H^d_v \in \mathbb{H}^{n \times n}$ is an achievable I/O map of the $\mathbb{H}$-stable system $Z(P,K)$ if and only if $H^d_v = \Delta_L \Delta_R Q_d$ for some nonsingular, diagonal $Q_d \in \mathbb{H}^{n \times n}$. 

**Proof:** ($\Rightarrow$) We are given $P$ satisfying (P) and $K$ which stabilizes $P$. Let $H^d_v \in \mathbb{H}^{n \times n}$ be the diagonal I/O map of this $Z(P,K)$. We must show that $H^d_v$ is of the form $\Delta_L \Delta_R Q_d$ for some diagonal, nonsingular $Q_d \in \mathbb{H}^{n \times n}$.

Since $Z(P,K)$ is $\mathbb{H}$-stable, we use (3.5). By (3.1) and (4.1), the diagonal matrix $\Delta_L$ is obviously a left-factor of $H^d_v$. It remains to show that $H^d_v$ has $\Delta_L \Delta_R$ as a left-factor. For a contradiction, suppose that $H^d_v$ is of the form

$$H^d_v = \Delta_L \tilde{\Delta}_R Q_d$$

(4.5)

where $\tilde{\Delta}_R$ is a proper factor of $\Delta_R$, and $Q_d \in \mathbb{H}^{n \times n}$ is nonsingular and diagonal. W.l.o.g. suppose, for example, that

$$\tilde{\Delta}_R = \text{diag}(\Delta_{R1}, \ldots, \Delta_{Rj-1}, \tilde{\Delta}_{Rj}, \Delta_{Rj+1}, \ldots, \Delta_{Rn})$$

(4.6)

where, for a non-unit prime element $\delta_j \in \mathbb{H}$ [Lang. 1, p. 72],

$$\Delta_{Rj} = \delta_j \tilde{\Delta}_{Rj}$$

(4.7)

Then by (3.1) and (4.5)

$$\Delta_L \tilde{N}_p^o N_{ml} = \Delta_L \tilde{\Delta}_R Q_d$$

(4.8)

Since $\mathbb{H}$ is a principal ring, we may cancel the nonsingular left-factor $\Delta_L$ and invert $\tilde{N}_p^o$ in (4.8) to obtain

$$N_{ml} = (\tilde{N}_p^o)^{-1} \tilde{\Delta}_R Q_d$$

(4.9)
By (4.2) and (4.6)

\[ N_{ml} = \begin{bmatrix} \frac{m_{ij}}{d_{ij}} \end{bmatrix} \cdot \text{diag} (\Delta_{R_1}, \ldots, \Delta_{R_j}, \ldots, \Delta_{R_n}) \cdot Q_d. \tag{4.10} \]

Recalling that \( \Delta_{R_j} \) is by definition a l.c.m. of \( (d_{ij})_{i=1}^n \) and by (4.7), for some \( i \), we have

\[ d_{ij} = \delta_j \tilde{d}_{ij} \tag{4.11} \]

where \( \tilde{d}_{ij} \in H \) is a factor of \( \Delta_{R_j} \); i.e., there is a \( \tilde{c}_{ij} \in H \), possibly a unit, such that

\[ \Delta_{R_j} = \tilde{d}_{ij} \tilde{c}_{ij}. \tag{4.12} \]

Hence, with \( g_j \in H \) denoting the \( j \)-th (non-zero) diagonal entry of some general nonsingular diagonal \( Q_d \in H^{n \times n} \), we obtain the \( ij \)-th element of \( N_{ml} \) from (4.10), (4.11) and (4.12) as

\[ \frac{m_{ij} \tilde{c}_{ij} q_j}{\delta_j}. \tag{4.13} \]

Since \( \delta_j \notin U(H) \) and in general \( \delta_j \) is not a factor of \( q_j \), (4.13) is not in \( H \). Therefore, except when the prime non-unit \( \delta_j \) is a factor of \( q_j \), \( N_{ml} \notin H^{n \times n} \), thus with \( N_{ml} \) as in (4.10), there is a diagonal, nonsingular \( Q_d \in H^{n \times n} \) such that \( K \) does not satisfy assumption (K). This contradicts the assumption that \( K \) stabilizes \( P \). Therefore, \( H^d \) must be an element of the set (4.4). (<=) For some diagonal, nonsingular \( Q_d \in H^{n \times n} \), we are given \( H^d_v = \Delta_L \Delta_R Q_d \). We must show that there exists a compensator \( K \) which stabilizes \( P \), and the \( H^d \)-stable \( \Sigma(P,K) \) achieves the given \( H^d_v \).

Choose the controller \( K := D_{cl}^{-1} [N_{ml} : N_{cl}] \) with

\[ N_{ml} := (\tilde{N}^o_{pr})^{-1} \Delta_R Q_d \tag{4.14} \]

where, by Lemma 4.1, \( N_{ml} \in H^{n \times n} \), and choose \( N_{cl}, D_{cl} \) as in (3.14a-b) with \( n_t = n_o \). To prove that this \( K \) satisfies (K) and that \( \Sigma(P,K) \) is \( H^d \)-stable, one uses the same reasoning as in Theorem 3.2 (ii). Hence, we omit the proof.

By (3.1), (3.5) and (4.14) we calculate the diagonal I/O map as

\[ H^d_{wv} = N^o_{pr}, D_{pr}^{-1} N_{ml} = \Delta_L N^o_{pr} (\tilde{N}^o_{pr})^{-1} \Delta_R Q_d = \Delta_L \Delta_R Q_d = H^d_v. \]

V. ROBUST STABILITY

The following robust stability theorem considers multiple perturbations (both plant and compensator) for the system \( \Sigma(P,K) \).

In the following, let \( \Sigma(\tilde{P}, \tilde{K}) \) denote the perturbed system where

\[ \tilde{P} = \begin{bmatrix} \cdots & \tilde{D}_{pr}^{-1} \tilde{N}^o_{pr} & \tilde{D}_{pr}^{-1} \tilde{N}^m_{pr} \end{bmatrix}, \quad \tilde{N}^o_{pr} = N^o_{pr} + \Delta N^o_{pr}, \quad \tilde{N}^m_{pr} = N^m_{pr} + \Delta N^m_{pr}, \quad \tilde{D}_{pr} = D_{pr} + \Delta D_{pr} \]

and \( \tilde{K} \) is defined similarly. Assumptions (P) and (K) become \( (\tilde{P}) \) and \( (\tilde{K}) \) with all parameters replaced by their perturbed versions.
Theorem 5.1 (Robust Stability)

Consider the system $\Sigma(P,K)$ of Fig. 1, where $P$ satisfies assumption (P) and $K$ stabilizes $P$. Let $D_{pr}, N_{pr}, N_{m}, D_{cl}, N_{fl}, N_{cl}$ be additively perturbed by, respectively, $\Delta D_{pr}, \Delta N_{pr}, \Delta N_{m}, \Delta D_{cl}, \Delta N_{fl}, \Delta N_{cl}$, with $\det D_{pr} \in \mathcal{U}$, $\det D_{cl} \in \mathcal{U}$ and $\det( D_{cl} D_{pr} + N_{fl} N_{pr} ) \in \mathcal{U}$.

(i) Let $P$ and $K$ satisfy assumptions (P) and (K). Then $\Sigma(\bar{P}, \bar{K})$ is $\mathcal{H}$-stable if and only if $\det( D_{cl} D_{pr} + N_{fl} N_{pr} ) \in \mathcal{U}(\mathcal{H})$.

(ii) Let $(\mathcal{H}, \| \cdot \|)$ be a Banach algebra and $B(0;r)$ denote the open ball in $\mathcal{H}^{p \times q}$ of radius $r$ centered at zero where $p$ and $q$ are specified by the context. Let $\rho_{dp} > 0$, $\rho_{np} > 0$, $\rho_{dc} > 0$, $\rho_{pf} > 0$ be such that

$$\| D_{cl} \| \rho_{dp} + \| N_{fl} \| \rho_{np} + \| D_{pr} \| \rho_{dc} + \| N_{pr} \| \rho_{pf} + \rho_{dp} \rho_{nc} + \rho_{np} \rho_{nf} < 1 \quad (5.1)$$

under the condition

$$\Delta D_{pr} \in B(0;\rho_{dp}), \Delta D_{cl} \in B(0;\rho_{dc}) \quad (5.2)$$

then the perturbed system $\Sigma(\bar{P}, \bar{K})$ is $\mathcal{H}$-stable.

Proof: (i) Same as the proof of Theorem 2.3(i), with all parameters replaced by the perturbed versions. (ii) The perturbed system $\Sigma(\bar{P}, \bar{K})$ is $\mathcal{H}$-stable if and only if $\det D_{h} \in \mathcal{U}(\mathcal{H})$ where $D_{h} := D_{cl} D_{pr} + N_{fl} N_{pr}$. By normalization of the unperturbed system, $D_{h} = I$. Then

$$\bar{D}_{h} = I + D_{cl} \Delta D_{pr} + N_{fl} \Delta N_{m} + \Delta D_{cl} D_{pr} + \Delta N_{fl} N_{m} + \Delta D_{cl} \Delta D_{pr} + \Delta N_{fl} \Delta N_{m}$$

$$= I + R \quad (5.3)$$

By (5.1) and (5.2), $\| R \| < 1$; hence, $(I + R)^{-1} \in \mathcal{E}(\mathcal{H})$ [Rud. 1, Theorem 18.3]. Therefore, $D_{h}^{-1} \in \mathcal{E}(\mathcal{H})$ and the conclusion follows.

Comments: 1) Similar results may be obtained for the case in which a left-coprime factorization (l.c.f.) of the plant $P$ and a right-coprime factorization (r.c.f.) of the compensator $K$ are used. 2) In the lumped case, the sufficiency result (ii) allows changes in the number and location of poles and zeros in both the stable and the unstable regions of the plane; this allows the consideration of systems of different orders having different number of unstable zeros and poles.

VI. ASYMPTOTIC TRACKING

For the tracking problem we consider the system $\Sigma(P,K)$ of Fig. 1 with $\pi_{u} = \pi_{o}$.

Definition 6.1 (Class of Inputs): The class $A$ of inputs to be tracked consists of vectors $\alpha^{-1} u$ where $\alpha \in \mathcal{I}(\mathcal{H}) \setminus \mathcal{U}(\mathcal{H})$ and $u \in \mathcal{H}^{n}$, with the property that the vector $u$ is not a multiple of $\alpha$. Consequently, the vector $\alpha^{-1} u \not\in \mathcal{E}(\mathcal{H})$; the inputs to be tracked are not stable time functions (typically steps, ramps, parabolas, sinusoids, etc.).

Definition 6.2 (Asymptotic Tracking): The closed-loop system $\Sigma(P,K)$ is said to asymptotically track the class $A$ if and only if $\nu - \nu_{o} \in \mathcal{H}^{n}$, $\forall \nu \in A$.

Comments: 1) The function $\nu - \nu_{o}$ is the tracking error; if the class $\mathcal{H}$ is suitably chosen, $\nu - \nu_{o} \in \mathcal{H}^{n}$ implies that $\nu(t) - \nu_{o}(t) \rightarrow \mathcal{V}_{n}$ as $t \rightarrow \infty$ (e.g., $\mathcal{H} = \mathcal{R}$ with...
Alternatively we could have used $D_t^{-1}$ driven by $u \in \mathcal{H}^{n_0}$ as the generator of tracked signals, where $(det D_t)^{-1} \in \mathcal{H}$. With $\alpha$ defined as the largest invariant factor of $D_t$, using the discussion in [Vid. 1] it can be shown that there is no loss of generality in adopting our definition as far as robust asymptotic tracking—to be precisely defined later—is concerned.

**Theorem 6.3 (Necessary Conditions)**

Let $P$ satisfy (P). Let $K$ stabilize $P$ and have a l.c.f. $D^{cl-1}_c [N_m : N_f] \in \mathcal{P}^{n_0 \times n_0}$, w.l.o.g. let $D_h = I_{n_0}$. U.t.c. if the system $\Sigma(P,K)$ asymptotically tracks the class $A$, then

(i) $n_i \geq n_0$  \hspace{1cm} (6.1)

(ii) $(N^0_{pr} N_{ml} , \alpha I)$ is r.c.  \hspace{1cm} (6.2)

**Comments:**

1) By calculation, $H_{y_y} = N^0_{pr} N_{ml} \in \mathcal{H}^{n_0 \times n_0}$.

2) Let $H = \mathcal{R}_U$. If $\Sigma(P,K)$ tracks the class $A$, the zeros of $N^0_{pr}$, the zeros of $N_{ml}$, and the zeros of $H_{y_y}$ are disjoint from those of $\alpha$. In particular, if $N^0_{pr}$ and $\alpha$ have some common zeros in $\mathcal{U}$, there exists no $K$ such that $\Sigma(P,K)$ tracks $A$.

**Proof:**

Let $u_4 / \alpha$ be an input to be tracked; thus, $u_4 \in \mathcal{H}^{n_0}$. The transfer matrix $H_{u_4 u_4}: u_4 \mapsto (v - y_0) =: e_t$ is given by

$$H_{u_4 u_4} = (I - N^0_{pr} N_{ml}) \alpha^{-1}$$  \hspace{1cm} (6.3)

By assumption, $H_{u_4 u_4} \in \mathcal{H}^{n_0 \times n_0}$ since asymptotic tracking is achieved.

(i) Suppose, for a proof by contradiction, that $n_i < n_0$. Then

$$rk(N^0_{pr} N_{ml}) \leq \min(rk N^0_{pr}, rk N_{ml}) \leq n_i < n_0.$$  \hspace{1cm} (6.4a)

(b) $\gamma$ is not a multiple of $\alpha$.  \hspace{1cm} (6.4b)

If $\gamma$ were a multiple of $\alpha$, say $\gamma = \alpha^k \gamma$ where $k$ is the multiplicity of $\alpha$ as a factor of $\gamma$, then $N^0_{pr} N_{ml} \gamma = \gamma_{n_0}$, and $\gamma \in \mathcal{H}^{n_0}$ and $\alpha \in \mathcal{H}$ have no (non-trivial) common factors.

Apply the input $v = \alpha^{-1} \gamma$, $\gamma^{-1} \gamma \notin \mathcal{E}(\mathcal{H})$ by (6.4b) above. Then

$$e_t = v - y_0 = (I - N^{0}_{pr} N_{ml}) \alpha^{-1} \gamma = \gamma \notin \mathcal{E}(\mathcal{H})$$;

which contradicts the assumption that $\Sigma(P,K)$ asymptotically tracks the class $A$.

(ii) Since $H_{u_4 u_4} \in \mathcal{H}^{n_0 \times n_0}$, let $(I - N^0_{pr} N_{ml}) \alpha^{-1} =: M \in \mathcal{E}(\mathcal{H})$; equivalently,

$$N^0_{pr} N_{ml} + M \alpha = I$$  \hspace{1cm} (6.5)

Hence, $(N^0_{pr} N_{ml}, \alpha I)$ is r.c. and since $N^0_{pr}, N_{ml} \in \mathcal{E}(\mathcal{H})$, $(N_{ml}, \alpha I)$ is also r.c. and $(\alpha I, N^0_{pr})$ is l.c.

**Theorem 6.4 (Robust asymptotic tracking: sufficient conditions)**

Let $P$ satisfy assumption (P) and let $N^m_{pr} D^{cl-1}_c = D^{cl-1}_c N^m_{pr}$ be a r.c.f. and a l.c.f., respectively, of $P^m$. Let $K$ stabilize $P$ and let $D^{cl-1}_c N_f = N_f D^{cl-1}_c$ be a l.c.f. and a r.c.f., respectively, of the feedback compensator.
I) (Tracking)

If (i) \( N_{pr}^m N_{fl} - N_{pr}^o N_{ml} = \alpha N_c \) for some \( N_c \in \mathcal{H}_n^{n \times n_o} \) and

(\text{ii}) \( D_{cr} = \alpha D_c \) for some \( D_c \in \mathcal{H}_n^{n \times n_o} \),

then \( \Sigma(P,K) \) tracks asymptotically the class \( A \).

II) (Robust Tracking)

Let the plant \( P \) be perturbed to \( \tilde{P} \) and let the compensator \( K \) be perturbed to \( \tilde{K} \); let \( \tilde{P} \) and \( \tilde{K} \) be described by similar coprime factorizations (i.e., all \( N \)'s become \( \tilde{N} \) and \( D \)'s becomes \( \tilde{D} \)'s) but \( \alpha \) is not perturbed. U.t.c.

If

(\text{i}) \( \tilde{K} \) stabilizes \( \tilde{P} \),

(\text{ii}) \( \tilde{N}_{pr}^m \tilde{N}_{fl} - \tilde{N}_{pr}^o \tilde{N}_{nl} = \alpha \tilde{N}_c \) for some \( \tilde{N}_c \in \mathcal{H}_n^{n \times n_o} \) and

(\text{iii}) \( \tilde{D}_{cr} = \alpha \tilde{D}_c \) for some \( \tilde{D}_c \in \mathcal{H}_n^{n \times n_o} \),

then \( \Sigma(\tilde{P},\tilde{K}) \) tracks asymptotically the class \( A \).

Comments:

1) Condition (ii) of part I requires that \( 1/\alpha \) appears in each input-channel of the compensator: the internal model must contain each unstable factor of \( \alpha \), the denominator of the signal generator. 2) Condition (i) of part I means that the difference between the closed-loop gain \( u_1 \mapsto y_m \) and the closed-loop gain \( v \mapsto y_o \) must have \( \alpha \) as a factor (if \( \mathcal{H} = \mathcal{U} \cdot N_{pr}^m N_{fl} - N_{pr}^o N_{ml} \) must have a blocking zero at each \( \mathcal{U} \)-zero of \( \alpha \) for asymptotic tracking. 3) For \( S(P,C) \), [Des. 3], condition (ii) of part I becomes a tautology: \( (N_{pr}^o = N_{pr}^m \text{ and } N_{fl} = N_{ml}) \). 4) As long as the \( \alpha \)-factor conditions (ii) and (iii) of part II are obeyed, any perturbation of the plant and of the compensator however large they may be, robust asymptotic tracking will be maintained provided that stability is maintained. 5) Condition (ii) of part I may not be minimal; i.e., some factor \( \tilde{\alpha} \) of \( \tilde{\alpha} \) may already exist in the plant and thus \( \tilde{\alpha}^{-1} D_{pl} \in E(\mathcal{H}) \). However, if (ii) is satisfied, then the internal model is present in the compensator and thus allows the plant to be arbitrarily perturbed.

Proof of I:

Since \( K \) stabilizes \( P \), \( D_h = I_{n_o} \) as in (3.5). Similarly, we can set

\[ D_{pl} D_{cr} + N_{pl}^m N_{fr} = I_{n_o} \] \hfill (6.6)

From the properties of r.c.f. and l.c.f. we have

\[
\begin{bmatrix}
D_{cl} & N_{ft} \\
-N_{pl}^m & D_{pl}
\end{bmatrix}
\begin{bmatrix}
D_{pr} & -N_{fr} \\
N_{pr}^m & D_{cr}
\end{bmatrix}
= 
\begin{bmatrix}
I_{n_o} & 0 \\
0 & I_{n_o}
\end{bmatrix}
\] \hfill (6.7)

\[
\begin{bmatrix}
D_{pr} & -N_{fr} \\
N_{pr}^m & D_{cr}
\end{bmatrix}
\begin{bmatrix}
D_{cl} & N_{ft} \\
-N_{pl}^m & D_{pl}
\end{bmatrix}
= 
\begin{bmatrix}
I_{n_o} & 0 \\
0 & I_{n_o}
\end{bmatrix}
\] \hfill (6.8)

Using (6.8), \( H_{e_1 u_1} \) of (6.3) can be written as

\[ H_{e_1 u_1} = (D_{cr} D_{pl} + N_{pr}^m N_{fl} - N_{pr}^o N_{ml}) \alpha^{-1} \] \hfill (6.9)

From (i) and (ii), we obtain \( H_{e_1 u_1} = D_c D_{pl} + N_c \in E(\mathcal{H}) \). Hence, \( \Sigma(P,K) \) asymptotically tracks the class \( A \).
Fig. 1. The System $\Sigma(P,K)$.

Fig. 2. The System $\Sigma(P,K)$. 

[- Diagram with block diagrams and equations related to systems $\Sigma(P,K)$]
Proof of II: The proof of I can be repeated word for word except that K's, P's, N's and D's are now K's, P's, N's and D's, respectively.

Conclusions: This paper presents an algebraic design theory for linear feedback systems. The results obtained rely on linearity and time-invariance, and important factors such as saturation and noise are ignored.

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References