TWO-VIEW MULTIBODY STRUCTURE FROM MOTION

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Two-View Multibody Structure from Motion *

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Abstract. In this paper we present a complete solution to 3-D motion estimation and segmentation of arbitrarily many moving objects seen in two perspective views. We adopt a geometric approach to the problem that exploits the algebraic and geometric properties of the so-called *multibody epipolar constraint* and its associated *multibody fundamental matrix*, which are natural generalizations of the epipolar constraint and of the fundamental matrix to multiple moving objects. We derive a rank condition on the data that allows to estimate the number of independently moving objects as well as linearly solve for the multibody fundamental matrix. We prove that the epipoles of each independent motion lie exactly in the intersection of the left null space of the multibody fundamental matrix with the so-called Veronese surface. We then show that individual epipoles and epipolar lines can be uniformly and efficiently computed by using a novel polynomial factorization technique. Given the epipoles and epipolar lines, the estimation of individual fundamental matrices becomes a linear problem. Then, motion and feature point segmentation is automatically obtained from either the epipoles and epipolar lines or the individual fundamental matrices. We demonstrate the proposed approach by segmenting a real image sequence.

Keywords: Multibody structure from motion, motion segmentation, multibody epipolar constraint, multibody fundamental matrix, polynomial factorization.

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1. Introduction

Motion is one of the most important cues for segmenting an image sequence into different objects. Classical approaches to 2-D motion segmentation try to separate the image flow into different regions either by looking for flow discontinuities [18], while imposing some regularity conditions [2], or by fitting a mixture of probabilistic models [11, 22]. The latter is usually done using an iterative process that alternates between segmentation and motion estimates using the Expectation-Maximization (EM) algorithm [5]. Alternative approaches are based on local features that incorporate spatial and temporal motion information. Similar features are grouped together using, for example, normalized cuts [16] or the eigenvectors of a similarity matrix [23].

3-D motion segmentation and estimation based on 2-D imagery is a more recent problem and various special cases have been analyzed using a geometric approach: multiple points moving linearly with constant speed [8, 15] or in a conic section [1], multiple moving objects seen by an orthographic camera [4, 12], self-calibration from multiple motions [7, 9], or two-object segmentation from two perspective views [24]. Alternative probabilistic approaches to 3-D motion segmentation are based on model selection techniques [19, 12] or combine normalized cuts with a mixture of probabilistic models [6].

In this paper we consider the problem of estimating and segmenting the motion of an unknown number of rigidly moving objects from a set of feature points seen in two perspective views. Due to the generality of the problem, we develop new algebraic and geometric techniques that generalize the well-known epipolar geometry to multiple motions.

In section 2 we introduce the *multibody epipolar constraint* as a geometric relationship between the motion of the objects and the image points that is satisfied by all the image points, regardless of the body to which they belong. The multibody epipolar constraint defines the so-called *multibody fundamental matrix*, which is a generalization of the fundamental matrix to multiple bodies. Section 3 derives a rank condition on the image measurements that allows to estimate the number of motions as well as linearly solve for the multibody fundamental matrix, after embedding all the image points in a higher-dimensional space. In Section 4 we prove that the epipoles of each independent motion lie exactly in the intersection of the left null space of the multibody fundamental matrix with the so-called Veronese surface.

A complete solution and an algorithm for segmentation and estimation of multiple motions is presented in Section 5, where we show that individual epipoles and epipolar lines can be uniformly and efficiently computed using a novel polynomial factorization technique introduced in this paper. Given the epipoles and the epipolar lines, the estimation of the individual fundamental matrices becomes a simple linear problem. Then, motion and feature point segmentation is automatically obtained from either the epipoles and epipolar lines or the individual fundamental matrices. We demonstrate the proposed approach to segment a real image sequence in Section 6.

Even though the polynomial factorization technique introduced in this paper is algebraically equivalent to the factorization of symmetric tensors, we avoid the use of tensorial notation throughout the paper, because algorithms based on polynomial factorization are computationally more straightforward and better established. As a consequence, this paper requires little background beyond conventional linear algebra and polynomial algebra.

2. Two-view multibody structure from motion, multibody epipolar constraint and multibody fundamental matrix

2.1. Two-view multibody structure from motion problem

Consider two images of a scene containing an *unknown* number n of independent and rigidly moving objects. The motion of each object relative to the camera between the two frames is described by the fundamental matrix $F_i \in \mathbb{R}^{3\times 3}$ associated to the motion of object $i = 1, \ldots, n$. We assume that the motions of the objects are such that all the fundamental matrices are distinct and different from zero, and hence the relative translation between the two image frames is non-zero.

The image of a point $q^j \in \mathbb{R}^3$ with respect to image frame I_k is denoted as $\boldsymbol{x}_k^j \in \mathbb{P}^2$, for $j = 1, \ldots, N$ and $k = 1, 2$. In order to avoid degenerate cases, we will assume that the image points are in general position in 3-D space, i.e. they do not all lie in any critical surface, for example. We will drop the superscript when we refer to a generic image pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$. Also, we will always use the homogeneous representation[1] $\boldsymbol{x} = [x, y, z]^T \in \mathbb{R}^3$ to refer to an arbitrary (image) point in $\mathbb{P}^2$.

We define the *multibody structure from motion problem* as follows:

Problem 1 (Multibody structure from motion problem).

Given a collection of image pairs $\{(\boldsymbol{x}_1^j, \boldsymbol{x}_2^j)\}_{j=1}^N$ corresponding to an unknown number of independent and rigidly moving objects, estimate the number of independent motions n, the fundamental matrices $\{F_i\}_{i=1}^n$, and the object to which each image pair belongs.

[1] In this paper we will treat $\mathbb{P}^2$ as $\mathbb{R}^3$ since we will not use any projective entity.

2.2. THE MULTIBODY EPIPOLAR CONSTRAINT

Given an image pair (x_1, x_2) corresponding to the i^{th} moving object, we know that the image pair and the associated fundamental matrix $F_i \in \mathbb{R}^{3\times 3}$ satisfy the so-called epipolar constraint [14]

$$x_2^T F_i x_1 = 0. \tag{1}$$

If we do not know the motion associated to the image pair (x_1, x_2), then we know that there exists an object i such that $x_2^T F_i x_1 = 0$. Thus, the following constraint must be satisfied by the number of objects, the relative motions and the image pair, regardless of the object to which the image pair belongs

$$\boxed{L(x_1, x_2) \doteq \prod_{k=1}^{n} \left(x_2^T F_i x_1\right) = 0.} \tag{2}$$

We call this constraint the *multibody epipolar constraint*, since it is a natural generalization of the epipolar constraint valid for $n = 1$. The main difference is that the multibody epipolar constraint is defined for an arbitrary number of objects, which is typically unknown (e.g., traffic surveillance). Furthermore, even if n is known, the algebraic structure of the constraint is neither bilinear in the image points nor linear in the fundamental matrices as illustrated in the following example.

Example 1 (Two rigid body motions). Imagine the simplest scenario of a scene containing only two independently moving objects as shown in Figure 1.

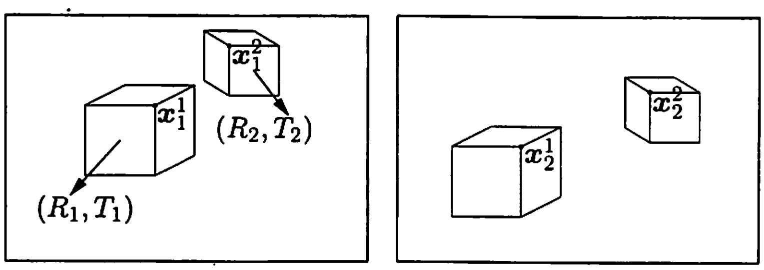


Figure 1. Two views of two independently moving objects, with two different rotations and translations: (R_1, T_1) and (R_2, T_2) relative to the camera frame.

In this case, both image pairs (x_1^1, x_2^1) and (x_1^2, x_2^2) satisfy the equation

$$\left(x_2^T F_1 x_1\right)\left(x_2^T F_2 x_1\right) = 0$$

for $F_1 = \widehat{T_1} R_1$ and $F_2 = \widehat{T_2} R_2$.[2] This equation is no longer bilinear but rather bi-quadratic in the two images x_1 and x_2 of any point q on one of

[2] In this paper, we use $\widehat{u}$ to denote the 3×3 skew symmetric matrix associated to a vector $u \in \mathbb{R}^3$ such that $\widehat{u}v = u \times v$ for all $v \in \mathbb{R}^3$.

these objects. Furthermore, the equation is no longer linear in F_1 or F_2 but rather bilinear in (F_1, F_2). However, if enough number of image pairs $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ are given, we can still recover some information about the two fundamental matrices F_1 and F_2 from such equations, in spite of the fact that we do not know the object or motion to which each image pair belongs. This special case $(n = 2)$ has been studied in [24]. In this paper we provide a general solution for an arbitrary number of motions n. ∎

2.3. THE MULTIBODY FUNDAMENTAL MATRIX

The multibody epipolar constraint allows to convert the multibody structure from motion problem (Problem 1) into that of solving for the number of independent motions n and the fundamental matrices $\{F_i\}_{i=1}^n$ from the *nonlinear* equation (2). A standard technique used in algebra to render a nonlinear problem into a linear one is to find an "embedding" that lifts the problem into a higher-dimensional space. In this case, we notice that the multibody epipolar constraint defines a homogeneous polynomial of degree n in either $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$. For example, if we let $\boldsymbol{x}_1 = [x_1, y_1, z_1]^T$, then equation (2) viewed as a function of $\boldsymbol{x}_1$ can be written as a linear combination of the following monomials $\{x_1^n, x_1^{n-1}y_1, x_1^{n-1}z_1, \ldots, z_1^n\}$. It is readily seen that there are a total of

$$M_n \doteq \binom{n+2}{2} = \frac{(n+1)(n+2)}{2} \tag{3}$$

different monomials, thus the dimension of the space of homogeneous polynomials in 3 variables with real coefficients, R_n, is M_n. Therefore, we can define the following embedding (or lifting) from $\mathbb{P}^2$ into $\mathbb{P}^{M_n-1}$:

DEFINITION 1 (Veronese map). *The* Veronese map *of degree n is defined as $\nu_n : \mathbb{P}^2 \rightarrow \mathbb{P}^{M_n-1}$*

$$\nu_n : \; [x, y, z]^T \mapsto [\ldots, \boldsymbol{x}^I, \ldots]^T, \tag{4}$$

where $\boldsymbol{x}^I$ is a monomial of the form $x^{n_1}y^{n_2}z^{n_3}$, with $0 \leq n_1, n_2, n_3 \leq n$, and $n_1+n_2+n_3 = n$, and the $\boldsymbol{x}^I$'s are ordered in the degree-lexicographic order.

Thanks to the Veronese map [10], we can convert the multibody epipolar constraint (2) into a bilinear expression in $\nu_n(\boldsymbol{x}_1)$ and $\nu_n(\boldsymbol{x}_2)$ as stated by the following proposition.

PROPOSITION 1 (The bilinear multibody epipolar constraint). *The multibody epipolar constraint (2) can be written in bilinear form as*

$$\boxed{\nu_n(\boldsymbol{x}_2)^T F \nu_n(\boldsymbol{x}_1) = 0,} \tag{5}$$

where $F \in \mathbb{R}^{M_n \times M_n}$ is a matrix whose entries are symmetric multilinear functions of degree n on the entries of the fundamental matrices $\{F_i\}_{i=1}^n$.

Proof. Let $\boldsymbol{\ell}_i = F_i \boldsymbol{x}_1 \in \mathbb{R}^3$, for $i = 1, \ldots, n$. Then, the multibody epipolar constraint $L(\boldsymbol{x}_1, \boldsymbol{x}_2) = \prod_{i=1}^n \boldsymbol{x}_2^T \boldsymbol{\ell}_i$ is a homogeneous polynomial of degree n in $\boldsymbol{x}_2 = [x_2, y_2, z_2]^T$, i.e.

$$L(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum a_{n_1,n_2,n_3} x_2^{n_1} y_2^{n_2} z_2^{n_3} \doteq \sum a_I \boldsymbol{x}_2^I \doteq \nu_n(\boldsymbol{x}_2)^T \boldsymbol{a},$$

where $\boldsymbol{a} \in \mathbb{R}^{M_n}$ is the vector of coefficients. From the properties of polynomial multiplication, each a_I is a symmetric multilinear function of $(\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_n)$, i.e. it is linear in each $\boldsymbol{\ell}_i$ and $a_I(\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_n) = a_I(\boldsymbol{\ell}_{\sigma(1)}, \ldots, \boldsymbol{\ell}_{\sigma(n)})$ for all $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the permutation group of n elements. Since each $\boldsymbol{\ell}_i$ is linear in $\boldsymbol{x}_1$, each a_I is indeed a homogeneous polynomial of degree n in $\boldsymbol{x}_1$, i.e. $a_I = \boldsymbol{f}_I^T \nu_n(\boldsymbol{x}_1)$, where each entry of $\boldsymbol{f}_I \in \mathbb{R}^{M_n}$ is a symmetric multilinear function of the entries of the F_i's. Letting

$$F \doteq [\boldsymbol{f}_{n,0,0}\ , \ \boldsymbol{f}_{n-1,1,0}\ , \ \cdots \ , \ \boldsymbol{f}_{0,0,n}]^T \ \in \mathbb{R}^{M_n \times M_n},$$

we obtain

$$L(\boldsymbol{x}_1, \boldsymbol{x}_2) = \nu_n(\boldsymbol{x}_2)^T F \nu_n(\boldsymbol{x}_1) = 0.$$

∎

We call the matrix F the *multibody fundamental matrix* since it is a natural generalization of the fundamental matrix to the case of multiple moving objects. Since equation (5) clearly resembles the bilinear form of the epipolar constraint for a single rigid body motion, we will refer to both equations (2) and (5) as the *multibody epipolar constraint.*

REMARK 1 (Multibody fundamental tensor). *The multibody fundamental matrix is a matrix representation of the symmetric tensor product of all the fundamental matrices*

$$\sum_{\sigma \in \mathfrak{S}_n} F_{\sigma(1)} \otimes F_{\sigma(2)} \otimes \cdots \otimes F_{\sigma(n)}, \tag{6}$$

where $\mathfrak{S}_n$ is the permutation group of n elements and $\otimes$ represents the tensor product.

Although the multibody fundamental matrix F seems a complicated mixture of all the individual fundamental matrices $F_1, \ldots, F_n$, it is still possible to recover all the individual fundamental matrices from F (alone), under some mild conditions (e.g., the F_i's are distinct). The rest of this paper is devoted to providing a constructive proof for this. We will show how to recover n and F from data, and $\{F_i\}_{i=1}^n$ from F.

3. Estimation of the number of independent motions n and of the multibody fundamental matrix F

Notice that, by definition, the multibody fundamental matrix F depends explicitly on the number of independent motions n. Therefore, even though the multibody epipolar constraint (5) is *linear* in F, we cannot use it to estimate F without knowing n in advance. It turns out that one can use the multibody epipolar constraint to derive a rank constraint on the image measurements that allows to compute n explicitly. Given n, the estimation of F becomes a *linear* problem.

Let us first rewrite the multibody epipolar constraint (5) as

$$(\nu_n(\boldsymbol{x}_2) \otimes \nu_n(\boldsymbol{x}_1))^T \boldsymbol{f} = 0, \tag{7}$$

where $\boldsymbol{f} \in \mathbb{R}^{M_n^2}$ is the stack of the columns of F and $\otimes$ represents the Kronecker product. Then, given a collection of image pairs $\{(\boldsymbol{x}_1^j, \boldsymbol{x}_2^j)\}_{j=1}^N$, the vector $\boldsymbol{f}$ satisfies the system of linear equations

$$A_n \boldsymbol{f} \doteq \begin{bmatrix} (\nu_n(\boldsymbol{x}_2^1) \otimes \nu_n(\boldsymbol{x}_1^1))^T \\ (\nu_n(\boldsymbol{x}_2^2) \otimes \nu_n(\boldsymbol{x}_1^2))^T \\ \vdots \\ \left(\nu_n(\boldsymbol{x}_2^N) \otimes \nu_n(\boldsymbol{x}_1^N)\right)^T \end{bmatrix} \boldsymbol{f} = 0. \tag{8}$$

In order to determine $\boldsymbol{f}$ uniquely (up to scale) from (8), we must have that

$$\operatorname{rank}(A_n) = M_n^2 - 1.$$

The above rank condition on the matrix A_n provides an effective criterion to determine the number of independent motions n from the given image pairs, as stated by the following Theorem.

THEOREM 1 (Number of independent motions). *Let $\{(\boldsymbol{x}_1^j, \boldsymbol{x}_2^j\}_{j=1}^N$ be a collection of image pairs corresponding to 3-D points in general configuration and undergoing an unknown number n of distinct rigid body motions with nonzero translation. Let $A_i \in \mathbb{R}^{N \times M_i^2}$ be the matrix defined in (8), but computed using the Veronese map ν_i of degree $i \geq 1$. Then, if the number of image pairs is big enough ($N \geq M_n^2 - 1$ when n is known) and at least* 8 *points correspond to each motion, we have*

$$\operatorname{rank}(A_i) \begin{cases} > M_i^2 - 1, & \text{if } i < n, \\ = M_i^2 - 1, & \text{if } i = n, \\ < M_i^2 - 1, & \text{if } i > n. \end{cases} \tag{9}$$

Therefore, the number of independent motions n is given by

$$\boxed{n \doteq \min\{i : \operatorname{rank}(A_i) = M_i^2 - 1\}.} \tag{10}$$

Proof. Since each fundamental matrix F_i has rank 2, the polynomial $p_i = \boldsymbol{x}_2^T F_i \boldsymbol{x}_1$ is irreducible over the real field $\mathbb{R}$. Let Z_i be the set of $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ that satisfy $\boldsymbol{x}_2^T F_i \boldsymbol{x}_1 = 0$. Then due to the irreducibility of p_i, any polynomial p in $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ that vanishes on the entire set Z_i must be of the form $p = p_i h$, where h is some polynomial. Hence if $F_1, \ldots, F_n$ are distinct, a polynomial which vanishes on the set $\cup_{i=1}^n Z_i$ must be of the form $p = p_1 p_2 \cdots p_n h$ for some h. Therefore, the only polynomial of *minimal* degree that vanishes on the same set is

$$p = p_1 p_2 \cdots p_n = \left(\boldsymbol{x}_2^T F_1 \boldsymbol{x}_1\right)\left(\boldsymbol{x}_2^T F_2 \boldsymbol{x}_1\right) \cdots \left(\boldsymbol{x}_2^T F_n \boldsymbol{x}_1\right). \tag{11}$$

Since the entries of $\nu_n(\boldsymbol{x}_2) \otimes \nu_n(\boldsymbol{x}_1)$ are exactly the independent monomials of p (as we will show below), this implies that if the number of data points per motion is at least 8 and $N \geq M_n^2 - 1$, then:

1. There is no polynomial of degree $i < n$ whose coefficients are in the null space of A_i, i.e. $\text{rank}(A_i) = M_i^2 > M_i^2 - 1$ for $i < n$.

2. There is a unique polynomial of degree n, namely p, with coefficients in the null space of A_n, i.e. $\text{rank}(A_n) = M_n^2 - 1$.

3. There is more than one polynomial of degree $i > n$ (one for each independent choice of the $(i\text{–}n)$-degree polynomial h) with coefficients in the null space of A_i, i.e. $\text{rank}(A_i) < M_i^2 - 1$ for $i > n$.

The rest of the proof is to show that the entries of $\nu_n(\boldsymbol{x}_2) \otimes \nu_n(\boldsymbol{x}_1)$ are exactly the independent monomials in the polynomial p, which we do by induction. Since the claim is obvious for $n = 1$, we assume that it is true for n and prove it for $n + 1$. Let $\boldsymbol{x}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{x}_2 = (x_2, y_2, z_2)$. Then the entries of $\nu_n(\boldsymbol{x}_2) \otimes \nu_n(\boldsymbol{x}_1)$ are of the form $(x_2^{m_1} y_2^{m_2} z_1^{m_3})(x_1^{m_1} y_1^{m_2} z_1^{m_3})$ with $m_1 + m_2 + m_3 = n_1 + n_2 + n_3 = n$, while the entries of $\boldsymbol{x}_2 \otimes \boldsymbol{x}_1$ are of the form $(x_2^{i_1} y_2^{i_2} z_2^{i_3})(x_1^{j_1} y_1^{j_2} z_1^{j_3})$ with $i_1 + i_2 + i_3 = j_1 + j_2 + j_3 = 1$. Thus a basis for the product of these monomials is given by the entries of $\nu_{n+1}(\boldsymbol{x}_2) \otimes \nu_{n+1}(\boldsymbol{x}_1)$. ∎

The significance of Theorem 1 is that the number of independent motions can now be determined incrementally using equation (10). Once the number n of motions is found, the multibody fundamental matrix $\mathcal{F}$ is simply the 1-D null space of the corresponding matrix A_n, which can be linearly obtained. Nevertheless, in order for this scheme to work, the minimum number of image pairs needed is $N \geq M_n^2 - 1$. For $n = 1, 2, 3, 4$, the minimum N is $8, 35, 99, 225$, respectively. If n is large, N grows approximately in the order of $O(n^4)$ – a price to pay for working with a linear representation of Problem 1. In Section 5.5 we will discuss many variations to the general scheme that will dramatically reduce the number of data points required, especially for large n.

4. Multibody epipolar geometry

In this section, we study the relationships between the multibody fundamental matrix F and the epipoles $e_1, \ldots, e_n$ associated to the fundamental matrices $F_1, \ldots, F_n$. The relationships between epipoles and epipolar lines will be studied in the next section, where we will show how they can be computed from the multibody fundamental matrix F.

First of all, recall that the epipole e_i associated to the i^{th} motion in the second image is defined as the left kernel of the rank-2 fundamental matrix F_i, that is

$$e_i^T F_i \doteq 0. \tag{12}$$

Hence, the following polynomial (in x) is zero for any e_i, $i = 1, \ldots, n$

$$\left(e_i^T F_1 x\right)\left(e_i^T F_2 x\right) \cdots \left(e_i^T F_n x\right) = \nu_n(e_i)^T F \nu_n(x) = 0. \tag{13}$$

We call the vector $\nu_n(e_i)$ the *embedded epipole* associated to the i^{th} motion. Since $\nu_n(x)$ as a vector spans the entire $\mathbb{R}^{M_n}$ when x ranges over $\mathbb{P}^2$ (or $\mathbb{R}^3$),[3] we have

$$\nu_n(e_i)^T F = 0. \tag{14}$$

Therefore, the embedded epipoles $\{\nu_n(e_i)\}_{i=1}^n$ lie on the left null space of F while the epipoles $\{e_i\}_{i=1}^n$ lie on the left null space of $\{F_i\}_{i=1}^n$. Hence, the rank of F is bounded depending on the number of *distinct* (pairwise linearly independent) epipoles as stated by Lemmas 1 and 2.

LEMMA 1 (Null space of F when the epipoles are distinct). *Let F be the multibody fundamental matrix generated by the fundamental matrices $F_1, \ldots, F_n$ with pairwise linearly independent epipoles $e_1, \ldots, e_n$. Then the (left) null space of $F \in \mathbb{R}^{M_n \times M_n}$ contains at least the n linearly independent vectors*

$$\nu_n(e_i) \in \mathbb{R}^{M_n}, \quad i = 1, \ldots, n. \tag{15}$$

Therefore the rank of the multibody fundamental matrix F is bounded by

$$\boxed{rank(F) \leq (M_n - n).} \tag{16}$$

Proof. We only need to show that if the e_i's are distinct, then the $\nu_n(e_i)$'s are linearly independent. If we let $e_i = [x_i, y_i, z_i]^T, i = 1, \ldots, n$, then we only need to prove the rank of the following matrix

$$U \doteq \begin{bmatrix} \nu_n(e_1)^T \\ \nu_n(e_2)^T \\ \vdots \\ \nu_n(e_n)^T \end{bmatrix} = \begin{bmatrix} x_1^n & x_1^{n-1}y_1 & x_1^{n-1}z_1 & \cdots & z_1^n \\ x_2^n & x_2^{n-1}y_2 & x_2^{n-1}z_2 & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^n & x_n^{n-1}y_n & x_n^{n-1}z_n & \cdots & z_n^n \end{bmatrix} \in \mathbb{R}^{n \times M_n} \tag{17}$$

[3] This is simply because the M_n monomials in $\nu_n(x)$ are linearly independent.

is exactly n. Since the e_i's are distinct, we can assume without loss of generality that $\{[x_i, z_i]\}_{i=1}^n$ are already distinct and that $z_i \neq 0$.[4] Then, after dividing the i^{th} row of U by z_i^n and letting $t_i = x_i/z_i$, we can extract the following Van Der Monde sub-matrix of U

$$V \doteq \begin{bmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ t_n^{n-1} & t_n^{n-2} & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{18}$$

Since $\det(V) = \prod_{i<j}(t_i - t_j)$, the Van Der Monde matrix V has rank n if and only if $t_1, \ldots, t_n$ are distinct. Hence $\mathrm{rank}(U) = \mathrm{rank}(V) = n$. ∎

Even though we know that the linearly independent vectors $\nu_n(e_i)$'s lie on the left null space of F, we do not know if the n-dimensional subspace spanned by them will be exactly the left null space of F, i.e. we do not know if $\mathrm{rank}(F) = M_n - n$. Simulations confirm that this is true when all the epipoles are distinct.

Now, if one of the epipoles is repeated, then one would expect that the dimension of the null space of F decreases. However, this is *not* the case: the null space of F is actually enlarged by higher-order derivatives of the Veronese map as stated by the following Lemma.

LEMMA 2 (Null space of F when one epipole is repeated). *Let F be the multibody fundamental matrix generated by the fundamental matrices $F_1, \ldots, F_n$ with epipoles $e_1, \ldots, e_n$. Let e_1 be repeated k times, i.e. $e_1 = \cdots = e_k$, and let the other $n - k$ epipoles be distinct. Then the rank of the multibody fundamental matrix F is bounded by*

$$\mathit{rank}(F) \leq M_n - M_{k-1} - (n-k). \tag{19}$$

Proof. When $k = 2$, $e_1 = e_2$ is a "repeated root" of $\nu_n(x)^T F$ as a polynomial (matrix) in $x = [x, y, z]^T$. Hence we have

$$\frac{\partial \nu_n(x)^T}{\partial x} F\Big|_{x=e_1} = 0, \quad \frac{\partial \nu_n(x)^T}{\partial y} F\Big|_{x=e_1} = 0, \quad \frac{\partial \nu_n(x)^T}{\partial z} F\Big|_{x=e_1} = 0.$$

Notice that the Jacobian of the Veronese map $D\nu_n(x)$ is full rank for all $x \in \mathbb{P}^2$, because $D\nu_n(x)^T D\nu_n(x) \succeq x^T x I_{3\times 3}$. Thus, the vectors $\frac{\partial \nu_n(e_1)}{\partial x}, \frac{\partial \nu_n(e_1)}{\partial y}, \frac{\partial \nu_n(e_1)}{\partial z}$ are linearly independent, because they are the

[4] This assumption is not always satisfied, e.g., for $n = 3$ motions with epipoles along the X, Y and Z axis. However, as long as the e_i's are distinct, one can always find a non-singular linear transformation $e_i \mapsto Le_i$ on $\mathbb{R}^3$ that makes the assumption true. Furthermore, this linear transformation induces a linear transformation on the lifted space $\mathbb{R}^{M_n}$ that preserves the rank of the matrix U.

columns of $D\nu_n(e_1)$ and $e_1 \neq 0$. In addition, their span contains $\nu_n(e_1)$, because

$$n\nu_n(x) = D\nu_n(x)x \doteq \left[\frac{\partial \nu_n(x)}{\partial x}, \frac{\partial \nu_n(x)}{\partial y}, \frac{\partial \nu_n(x)}{\partial z}\right] x, \quad \forall x \in \mathbb{R}^3. \quad (20)$$

Hence $\text{rank}(F) \leq M_n - M_1 - (n-1) = M_n - 3 - (n-1)$. Now if $k > 2$, one should consider the $(k-1)^{th}$ order partial derivatives of $\nu_n(x)$ evaluated at e_1. There is a total of M_{k-1} such partial derivatives, which give rise to M_{k-1} linearly independent vectors in the (left) null space of F. Similar to the case $k = 2$, one can show that the embedded epipole $\nu_n(e_1)$ is in the span of these higher-order partials. ∎

Example 2 (Two repeated epipoles). In the two-body problem, if F_1 and F_2 have the same (left) epipole, i.e. $F_1 = \widehat{T}R_1$ and $F_2 = \widehat{T}R_2$, then the rank of the two-body fundamental matrix F is $M_2 - M_1 - (2-2) = 6 - 3 = 3$ instead of $M_2 - 2 = 4$. ∎

Since the null space of F is enlarged by higher-order derivatives of the Veronese map evaluated at repeated epipoles, in order to identify the embedded epipoles $\nu_n(e_i)$ from the left null space of F we will need to exploit the algebraic structure of the Veronese map. Let us denote the image of the real projective space $\mathbb{P}^2$ under the Veronese map of degree n as $\nu_n(\mathbb{P}^2)$.[5] The following theorem establishes a key relationship between the null space of F and the epipoles of each fundamental matrix.

THEOREM 2 (Veronese null space of multibody fundamental matrix). *The intersection of the left null space of the multibody fundamental matrix F, Null(F), with the Veronese surface $\nu_n(\mathbb{P}^2)$ is exactly*

$$\boxed{Null(F) \cap \nu_n(\mathbb{P}^2) = \{\nu_n(e_i)\}_{i=1}^n.} \quad (21)$$

Proof. Let $x \in \mathbb{P}^2$ be a vector whose Veronese map is in the left null space of F. We then have

$$\nu_n(x)^T F = 0 \quad \Leftrightarrow \quad \nu_n(x)^T F \nu_n(y) = 0, \forall y \in \mathbb{P}^2. \quad (22)$$

Since F is a multibody fundamental matrix,

$$\nu_n(x)^T F \nu_n(y) = \prod_{i=1}^n \left(x^T F_i y\right).$$

This means for this x,

$$\prod_{i=1}^n (x^T F_i y) = 0, \quad \forall y \in \mathbb{P}^2. \quad (23)$$

[5] This is the so-called (real) Veronese surface in Algebraic Geometry [10].

If $x^T F_i \neq 0$ for all $i = 1, \ldots, n$, then the set of y that satisfy the above equation is simply the union of n 2-dimensional subspaces in $\mathbb{P}^2$, which will never fill the entire space $\mathbb{P}^2$. Hence we must have $x^T F_i = 0$ for some i. Therefore x is one of the epipoles. ■

The significance of Theorem 2 is that, in spite of the fact that repeated epipoles may enlarge the null space of F, and that we do not know if the dimension of the null space equals n for distinct epipoles, one may always find the epipoles exactly by intersecting the left null space of F with the Veronese surface $\nu_n(\mathbb{P}^2)$, as illustrated in Figure 2.

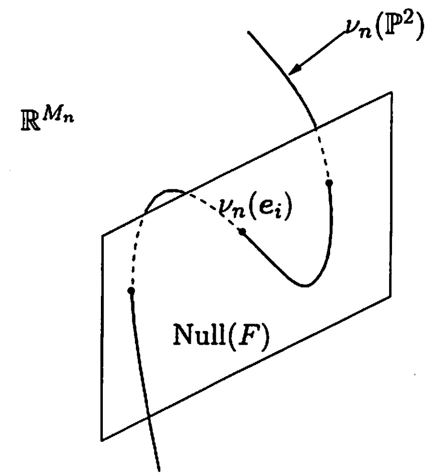


Figure 2. The intersection of $\nu_n(\mathbb{P}^2)$ and Null(F) is exactly n points representing the Veronese map of the n epipoles, repeated or not.

The question is now how to compute the intersection of Null(F) with $\nu_n(\mathbb{P}^2)$ in practice. One possible approach, explored in [24] for $n = 2$ and generalized in [21] to $n \geq 2$, consists of determining a vector $v \in \mathbb{R}^n$ such that $Bv \in \nu_n(\mathbb{P}^2)$, where B is a matrix whose columns form a basis for the (left) null space of F. Finding v, hence the epipoles, is equivalent to solving for the roots of polynomials of degree n in $n-1$ variables. Although this is feasible for $n = 2$ and even for $n = 3$, it is computationally formidable for $n > 3$.

In the next section, we take a completely different approach that combines the multibody epipolar geometry developed so far with a novel polynomial factorization technique. In essence, we will show that the epipoles (and also the epipolar lines) can be computed by solving a polynomial of degree n in *one* variable plus one linear system in n variables. Given the epipoles and the epipolar lines, the computation of individual fundamental matrices becomes a *linear* problem. Therefore, there exists a *closed form solution* to Problem 1 for $n \leq 4$.

5. Multibody motion estimation and segmentation

Given the multibody fundamental matrix F and the number of independent motions n, we are now interested in recovering the motion parameters (or fundamental matrices) and the segmentation of the image points. In this section we show how to solve these two problems from the epipoles of each fundamental matrix and the epipolar lines associated to each image point. The estimation of epipoles and epipolar lines will be based on the factorization of a given homogeneous polynomial of degree n in 3 variables with real coefficients into n distinct polynomials of degree 1 also with real coefficients. Once the epipoles and the epipolar lines have been estimated, the estimation of individual fundamental matrices becomes a simple *linear* problem from which the segmentation of the image points is automatically obtained.

We will first describe how to solve the polynomial factorization problem in Section 5.1, and then dive into the details of motion estimation (Sections 5.2 and 5.3) and segmentation (Section 5.4). We conclude with an algorithm for multibody structure from motion in Section 5.5.

5.1. Homogeneous polynomial factorization

Let $\{\boldsymbol{\ell}_i\}_{i=1}^n$ be a collection of n *distinct* vectors in $\mathbb{R}^3$ and let $p_n(\boldsymbol{x})$ be the homogeneous polynomial of degree n in $\boldsymbol{x}=[x,y,z]^T \in \mathbb{R}^3$ given by

$$\begin{aligned} p_n(\boldsymbol{x}) &= (\boldsymbol{\ell}_1^T\boldsymbol{x})(\boldsymbol{\ell}_2^T\boldsymbol{x})\cdots(\boldsymbol{\ell}_n^T\boldsymbol{x}) \\ &= (\ell_{11}x+\ell_{12}y+\ell_{13}z)(\ell_{21}x+\ell_{22}y+\ell_{23}z)\cdots(\ell_{n1}x+\ell_{n2}y+\ell_{n3}z) \\ &= \sum a_{n_1,n_2,n_3}\, x^{n_1}y^{n_2}z^{n_3} \doteq \boldsymbol{a}^T\,\nu_n(\boldsymbol{x}). \end{aligned} \tag{24}$$

where $\boldsymbol{a} \in \mathbb{R}^{M_n}$ is the vector of coefficients of the polynomial $p_n(\boldsymbol{x})$.

REMARK 2 (Symmetric multilinear tensor). *The vector $\boldsymbol{a} \in \mathbb{R}^{M_n}$ is a vector representation for the symmetric tensor product of all the vectors $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \ldots, \boldsymbol{\ell}_n \in \mathbb{R}^3$,*

$$\sum_{\sigma\in\mathfrak{S}_n} \boldsymbol{\ell}_{\sigma(1)} \otimes \boldsymbol{\ell}_{\sigma(2)} \otimes \cdots \otimes \boldsymbol{\ell}_{\sigma(n)}, \tag{25}$$

where $\mathfrak{S}_n$ is the permutation group of n elements and $\otimes$ represents the tensor product of vectors.

Given the vector of coefficients $\boldsymbol{a} \in \mathbb{R}^{M_n}$ of the polynomial $p_n(\boldsymbol{x})$, we would like to compute the set of vectors $\{\boldsymbol{\ell}_i\}_{i=1}^n$ up to scale. To this end, we consider the last $n+1$ coefficients of $p_n(\boldsymbol{x})$, which define the following homogeneous polynomial of degree n in y and z

$$\sum a_{0,n_2,n_3} y^{n_2} z^{n_3} = \prod_{i=1}^{n} (\ell_{i2}y + \ell_{i3}z). \tag{26}$$

Letting $w = y/z$ we have that

$$\prod_{i=1}^{n} (\ell_{i2}y + \ell_{i3}z) = 0 \quad \Leftrightarrow \quad \prod_{i=1}^{n} (\ell_{i2}w + \ell_{i3}) = 0,$$

hence the n roots of the univariate polynomial

$$q_n(w) = a_{0,n,0}w^n + a_{0,n-1,1}w^{n-1} + \cdots + a_{0,0,n} \tag{27}$$

are exactly $w_i = -\ell_{i3}/\ell_{i2}$, for $i = 1, \ldots, n$. Therefore, after dividing $\boldsymbol{a}$ by $a_{0,n,0}$ (if nonzero), we obtain the last two entries of each $\boldsymbol{\ell}_i$ as

$$(\ell_{i2} \ , \ \ell_{i3}) = (1 \ , \ -w_i), \quad i = 1, \ldots, n. \tag{28}$$

If $\ell_{i2} = 0$ for some i, then some of the leading coefficients of $q_n(w)$ are zero and we cannot proceed as before, because $q_n(w)$ has less than n roots. More specifically, assume that the first $r \leq n$ coefficients of $q_n(w)$ are zero and divide $\boldsymbol{a}$ by the $(r+1)$-st coefficient. In this case, we can choose $(\ell_{i2}, \ell_{i3}) = (0, 1)$, for $i = 1, \ldots, r$, and obtain $\{(\ell_{i2}, \ell_{i3})\}_{i=n-r+1}^{n}$ from the $n - r$ roots of $q_n(w)$ by using equation (28). Finally, if all the coefficients of $q_n(w)$ are equal to zero, we set $(\ell_{i2}, \ell_{i3}) = (0, 0)$, for all $i = 1, \ldots, n$.

REMARK 3 (Solvability of roots of univariate polynomial). *It is well-known from Abstract Algebra (in particular from Galois's theory) [13] that there is no closed-form solution for the roots of univariate polynomials of degree $n \geq 5$. Hence, there is no closed-form solution to homogeneous polynomial factorization for $n \geq 5$ either. Since one can always find the roots of a univariate polynomial numerically using efficient polynomial time algorithms [17], we will consider this problem as "solved".*

We are left with the computation of the coefficients of the variable x of each factor of $p_n(\boldsymbol{x})$, i.e. $\{\ell_{i1}\}_{i=1}^{n}$. For that, we consider the n coefficients a_{1,n_2,n_3} of $p_n(\boldsymbol{x})$. We notice that these coefficients are *linear* functions of the unknowns $\{\ell_{i1}\}_{i=1}^{n}$, given that we already know $\{(\ell_{i2}, \ell_{i3}\}_{i=1}^{n}$. Therefore, we can solve for ℓ_{i1} from the linear system

$$\begin{bmatrix} \mathcal{V}_1 & \mathcal{V}_2 & \cdots & \mathcal{V}_n \end{bmatrix} \begin{bmatrix} \ell_{11} \\ \ell_{21} \\ \vdots \\ \ell_{n1} \end{bmatrix} = \begin{bmatrix} a_{1,n-1,0} \\ a_{1,n-2,1} \\ \vdots \\ a_{1,0,n-1} \end{bmatrix}, \tag{29}$$

where $\mathcal{V}_i \in \mathbb{R}^n$ are the coefficients of the following homogeneous polynomial of degree $n - 1$ in y and z

$$g_i(\boldsymbol{x}) = \prod_{k=1}^{i-1} (\ell_{k2}y + \ell_{k3}z) \prod_{k=i+1}^{n} (\ell_{k2}y + \ell_{k3}z). \tag{30}$$

In order for the linear system in (29) to have a unique solution, the column vectors $\{\mathcal{V}_i\}_{i=1}^n$ (in the matrix on the left hand side) must be linearly independent. It is shown in [20] that this is indeed the case if and only if the vectors $\{(\ell_{i2}, \ell_{i3})\}_{i=1}^n$ are pairwise linearly independent. This latter condition is always satisfied, except for some degenerate cases described in Remark 4 below. In those degenerate cases, as long as the original polynomial $p_n(\boldsymbol{x})$ has n distinct factors, one can always perform an invertible linear transformation

$$\boldsymbol{x} \mapsto L\boldsymbol{x}, \quad L \in \mathbb{R}^{3\times 3} \tag{31}$$

that induces a linear transformation on the vector of coefficients $\boldsymbol{a} \mapsto T\boldsymbol{a}$, $T \in \mathbb{R}^{M_n \times M_n}$, such that the new vectors $\{(\ell_{i2}, \ell_{i3})\}_{i=1}^n$ are pairwise linearly independent. A typical choice for such L is of the form

$$L \doteq \begin{bmatrix} 1 & t & t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $t \in \mathbb{R}$ can always be chosen so that the new polynomial $q_n(w)$ in (27) has distinct roots. We refer interested readers to [20] for further details on the solution of these degenerate cases.

REMARK 4 (Degenerate cases). *There are essentially three cases in which the vectors* $\{(\ell_{i2}, \ell_{i3})\}_{i=1}^n$ *are not pairwise linearly independent:*

1. *The original polynomial* $p_n(\boldsymbol{x})$ *is such that the polynomial* $q_n(w)$ *has repeated roots, e.g.,* $p_n(\boldsymbol{x}) = (2x + y + 3z)(x + y + 3z)$.

2. *The polynomial* $q_n(w)$ *associated to some factorable* $p_n(\boldsymbol{x})$*, e.g.,* $p_n(\boldsymbol{x}) = (x + z)z$*, has more than one zero leading coefficients. In this case we have* $(\ell_{i2}, \ell_{i3}) = (0, 1)$ *for more than one* i.

3. *The original polynomial* $p_n(\boldsymbol{x})$ *is not factorable. This happens, for example, when the vector of coefficients* $\boldsymbol{a}$ *is corrupted by noise. In this case the polynomial* $q_n(w)$ *may have complex roots, e.g.,* $p_n(\boldsymbol{x}) = x^2 + y^2 + yz + z^2$*, and one could "project" these complex roots onto their real parts. This typically introduces repeated real roots in the resulting polynomial, e.g., after "projection" the above polynomial* $p_n(\boldsymbol{x})$ *is effectively converted to* $x^2 + y^2 + yz + \frac{1}{4}z^2$.

We conclude that the homogeneous polynomial factorization problem can be completely solved from the roots of a univariate polynomial of degree n and the solution to a linear system in n variables.[6] This factorization technique will be used repeatedly in the following sections in the computation of the epipoles and epipolar lines associated to the multibody structure from motion problem.

[6] In fact, the problem admits a *unique* up to $n-1$ scales as demonstrated in [20].

5.2. Estimation of epipolar lines and epipoles

Given a point $\boldsymbol{x}_1$ in the first image frame, the epipolar lines associated to it are defined as $\boldsymbol{\ell}_i \doteq F_i \boldsymbol{x}_1 \in \mathbb{R}^3$, $i = 1, \dots, n$. From the epipolar constraint, we know that one of such lines passes through the corresponding point in the second frame $\boldsymbol{x}_2$, i.e. there exists an i such that $\boldsymbol{x}_2^T \boldsymbol{\ell}_i = 0$. Let $\mathcal{F}$ be the multibody fundamental matrix. We have that

$$L(\boldsymbol{x}_1, \boldsymbol{x}_2) = \nu_n(\boldsymbol{x}_2)^T \mathcal{F} \nu_n(\boldsymbol{x}_1) = \prod_{i=1}^{n} \left(\boldsymbol{x}_2^T F_i \boldsymbol{x}_1 \right) = \prod_{i=1}^{n} (\boldsymbol{x}_2^T \boldsymbol{\ell}_i), \qquad (32)$$

from which we conclude that the vector $\tilde{\boldsymbol{\ell}} \doteq \mathcal{F} \nu_n(\boldsymbol{x}_1) \in \mathbb{R}^{M_n}$ represents the coefficients of the homogeneous polynomial in $\boldsymbol{x}$

$$\boxed{g(\boldsymbol{x}) \doteq (\boldsymbol{x}^T \boldsymbol{\ell}_1)(\boldsymbol{x}^T \boldsymbol{\ell}_2) \cdots (\boldsymbol{x}^T \boldsymbol{\ell}_n) = \nu_n(\boldsymbol{x})^T \tilde{\boldsymbol{\ell}}.} \qquad (33)$$

We call the vector $\tilde{\boldsymbol{\ell}}$ the *multibody epipolar line* associated to $\boldsymbol{x}_1$. Notice that $\tilde{\boldsymbol{\ell}}$ is a vector representation of the symmetric tensor product of all the epipolar lines $\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_n$ and it is in general *not* the Veronese map (or lifting) $\nu_n(\boldsymbol{\ell}_i)$ of any particular epipolar line $\boldsymbol{\ell}_i, i = 1, \dots, n$.

From $\tilde{\boldsymbol{\ell}}$, we can compute the individual epipolar lines $\{\boldsymbol{\ell}_i\}_{i=1}^n$ associated to any image point $\boldsymbol{x}_1$ using the polynomial factorization technique given in Section 5.1. In essence, the multibody fundamental matrix $\mathcal{F}$ allows us to "transfer" a point $\boldsymbol{x}_1$ in the first image to a set of epipolar lines in the second image. This is exactly the multibody version of the conventional "epipolar transfer" that maps a point in the first image to an epipolar line in the second image. The *multibody epipolar transfer* process can be described by the sequence of maps

$$\boldsymbol{x}_1 \overset{Veronese}{\longmapsto} \nu_n(\boldsymbol{x}_1) \overset{Epipolar\ Transfer}{\longmapsto} \mathcal{F}\nu_n(\boldsymbol{x}_1) \overset{Polynomial\ Factorization}{\longmapsto} \{\boldsymbol{\ell}_i\}_{i=1}^n,$$

which is illustrated geometrically in Figure 3.

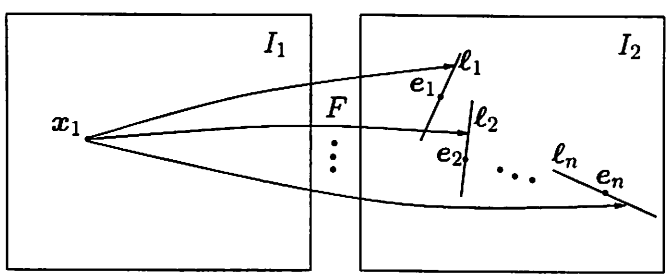


Figure 3. The multibody fundamental matrix $\mathcal{F}$ maps each point $\boldsymbol{x}_1$ in the first image to n epipolar lines $\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_n$ which pass through the n epipoles $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ respectively. Furthermore, one of these epipolar lines passes through $\boldsymbol{x}_2$

Given a set of epipolar lines, we now describe how to compute the epipoles. Recall that the (left) epipole associated to each rank-2 fundamental matrix $F_i \in \mathbb{R}^{3\times3}$ is defined as the vector $e_i \in \mathbb{R}^3$ lying in the (left) null space of F_i, that is e_i satisfies that $e_i^T F_i = 0$. Now let $\boldsymbol{\ell} \in \mathbb{R}^3$ be an arbitrary epipolar line associated to some image point in the first frame. Then there exists an i such that $e_i^T \boldsymbol{\ell} = 0$. Therefore, every epipolar line $\boldsymbol{\ell}$ has to satisfy the following polynomial constraint

$$\boxed{h(\boldsymbol{\ell}) \doteq (e_1^T \boldsymbol{\ell})(e_2^T \boldsymbol{\ell}) \cdots (e_n^T \boldsymbol{\ell}) = \tilde{e}^T \nu_n(\boldsymbol{\ell}) = 0,} \tag{34}$$

regardless of the motion to which it is associated. We call the vector $\tilde{e} \in \mathbb{R}^{M_n}$ the *multibody epipole* associated to the n motions. As before, $\tilde{e}$ is a vector representation of the symmetric tensor product of the individual epipoles $e_1, \ldots, e_n$ and it is in general different from any of the embedded epipoles $\nu_n(e_i)$, $i = 1, \ldots, n$.

Given a collection $\{\boldsymbol{\ell}^j\}_{j=1}^m$ of $m \geq M_n - 1$ epipolar lines (which can be computed from the multibody epipolar transfer described before), we can obtain the multibody epipole $\tilde{e} \in \mathbb{R}^{M_n}$ as the solution of the linear system

$$B_n \tilde{e} \doteq \begin{bmatrix} \nu_n(\boldsymbol{\ell}^1)^T \\ \nu_n(\boldsymbol{\ell}^2)^T \\ \vdots \\ \nu_n(\boldsymbol{\ell}^m)^T \end{bmatrix} \tilde{e} = 0. \tag{35}$$

In order for equation (35) to have a unique solution (up to scale), we will need to replace n by the number of distinct epipoles n_e, as stated by the following proposition:

PROPOSITION 2 (Number of distinct epipoles). *Assume that we are given a collection of epipolar lines $\{\boldsymbol{\ell}^j\}_{j=1}^m$ corresponding to 3-D points in general configuration and undergoing n distinct rigid body motions with nonzero translation. Then, if the number of epipolar lines m is at least $M_n - 1$, then we have*

$$rank(B_i) \begin{cases} > M_i - 1, & \text{if } i < n_e, \\ = M_i - 1, & \text{if } i = n_e, \\ < M_i - 1, & \text{if } i > n_e. \end{cases} \tag{36}$$

Therefore, the number of distinct epipoles $n_e \leq n$ is given by

$$\boxed{n_e \doteq \min\{i : rank(B_i) = M_i - 1\}.} \tag{37}$$

Proof. Similar to the proof of Theorem 1. ∎

Once the number of distinct epipoles, n_e, has been computed, the vector $\tilde{e} \in M_{n_e}$ can be obtained from the linear system $B_{n_e}\tilde{e} = 0$. Once $\tilde{e}$ has been computed, the individual epipoles $\{e_i\}_{i=1}^{n_e}$ can be computed from $\tilde{e}$ using the factorization technique of Section 5.1. We illustrate the computation of the epipoles in Figure 4. Each epipole e_i corresponds to the intersection of the epipolar lines associated to the i^{th} motion. The polynomial factorization process performs all such intersections simultaneously *without* knowing the segmentation of the epipolar lines.

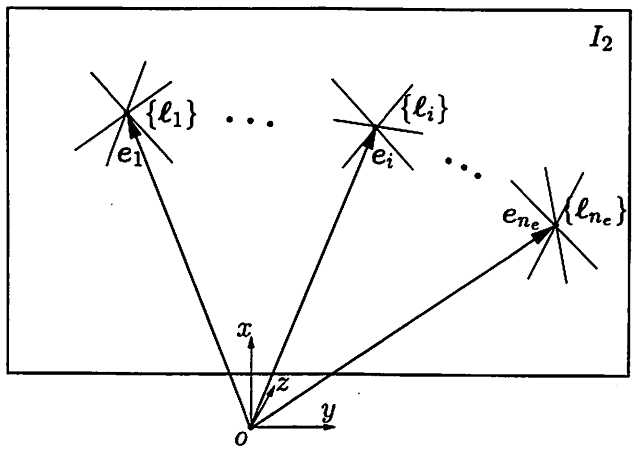


Figure 4. When n objects move independently in front of a fixed camera, the epipolar lines associated to image pairs form n_e groups and intersect respectively at n_e distinct epipoles. Here epipolar lines and epipoles are drawn in the second image I_2.

5.3. Recovery of individual fundamental matrices

Given the epipolar lines and the epipoles, we show now how to recover each one of the individual fundamental matrices $\{F_i\}_{i=1}^n$. To avoid degenerate cases, we assume that all the epipoles are distinct, i.e.

$$n_e = n.$$

Let $F_i = [f_i^1\ f_i^2\ f_i^3] \in \mathbb{R}^{3\times 3}$ the fundamental matrix associated to motion i, with columns $f_i^1, f_i^2, f_i^3 \in \mathbb{R}^3$. We know from Section 5.2 that, given $x_1 = [x_1, y_1, z_1]^T \in \mathbb{R}^3$, the vector $F\nu_n(x_1) \in \mathbb{R}^{M_n}$ represents the coefficients of the following homogeneous polynomial in x

$$g(x) = \left(x^T(f_1^1 x_1 + f_1^2 y_1 + f_1^3 z_1)\right) \cdots \left(x^T(f_n^1 x_1 + f_n^2 y_1 + f_n^3 z_1)\right).$$

Therefore, given the multibody fundamental matrix F, one can estimate any linear combination of the columns of the fundamental matrix

F_i up to a scale, i.e. we can get vectors $\boldsymbol{\ell}_i \in \mathbb{R}^3$ satisfying

$$\lambda_i \boldsymbol{\ell}_i \doteq (\boldsymbol{f}_i^1 x_1 + \boldsymbol{f}_i^2 y_1 + \boldsymbol{f}_i^3 z_1), \quad \lambda_i \in \mathbb{R},\ i = 1, \ldots, n.$$

These vectors are nothing but the epipolar lines associated to the multibody epipolar line $\mathcal{F}\nu_n(\boldsymbol{x}_1)$, which can be computed using the polynomial factorization technique of Section 5.1 as described in Section 5.2. Notice that, in particular, we can obtain the three columns of F_i up to scale by choosing $\boldsymbol{x}_1 = [1,0,0]^T$, $\boldsymbol{x}_1 = [0,1,0]^T$ and $\boldsymbol{x}_1 = [0,0,1]^T$, respectively. However:

1. We do not know the fundamental matrix to which the recovered epipolar lines belong;
2. The recovered epipolar lines, hence the columns of each F_i, are obtained up to scale only. Hence, we do not know the relative scales between the columns of the same fundamental matrix.

The first problem is easily solvable: if a recovered epipolar line $\boldsymbol{\ell} \in \mathbb{R}^3$ corresponds to a linear combination of columns of the fundamental matrix F_i, then it must be perpendicular to the previously computed epipolar line $\boldsymbol{e}_i$, i.e. we must have $\boldsymbol{e}_i^T \boldsymbol{\ell} = 0$. As for the second problem, for each i let $\boldsymbol{\ell}_i^j$ be the epipolar line associated to $\boldsymbol{x}_1^j$ that is perpendicular to $\boldsymbol{e}_i$, for $j = 1, \ldots, m$. Since the $\boldsymbol{x}_1^j$'s can be chosen arbitrarily, we choose the first three to be $\boldsymbol{x}_1^1 = [1,0,0]^T$, $\boldsymbol{x}_1^2 = [0,1,0]^T$ and $\boldsymbol{x}_1^3 = [0,0,1]^T$ to form a simple basis. Then for every $\boldsymbol{x}_1^j = [x_1^j, y_1^j, z_1^j]^T, j \geq 1$, there exist unknown scales $\lambda_i^j \in \mathbb{R}$ such that

$$\begin{aligned} \lambda_i^j \boldsymbol{\ell}_i^j &= \boldsymbol{f}_i^1 x_1^j \quad + \boldsymbol{f}_i^2 y_1^j \quad + \boldsymbol{f}_i^3 z_1^j \qquad j \geq 4, \\ &= (\lambda_i^1 \boldsymbol{\ell}_i^1) x_1^j + (\lambda_i^2 \boldsymbol{\ell}_i^2) y_1^j + (\lambda_i^3 \boldsymbol{\ell}_i^3) z_1^j, \quad j \geq 4. \end{aligned}$$

Multiplying both sides by $\widehat{\boldsymbol{\ell}_i^j}$, we obtain

$$0 = \widehat{\boldsymbol{\ell}_i^j} \left((\lambda_i^1 \boldsymbol{\ell}_i^1) x_1^j + (\lambda_i^2 \boldsymbol{\ell}_i^2) y_1^j + (\lambda_i^3 \boldsymbol{\ell}_i^3) z_1^j \right), \quad j \geq 4 \tag{38}$$

where $\lambda_i^1, \lambda_i^2, \lambda_i^3$ are the only unknowns. Therefore, the fundamental matrices are given by

$$\boxed{F_i = [\boldsymbol{f}_i^1 \ \boldsymbol{f}_i^2 \ \boldsymbol{f}_i^3] = [\lambda_i^1 \boldsymbol{\ell}_i^1 \ \lambda_i^2 \boldsymbol{\ell}_i^2 \ \lambda_i^3 \boldsymbol{\ell}_i^3],} \tag{39}$$

where λ_i^1, λ_i^2 and λ_i^3 can be obtained as the solution to the linear system

$$\begin{bmatrix} \widehat{\boldsymbol{\ell}_i^4}[x_1^4 \boldsymbol{\ell}_i^1 \ y_1^4 \boldsymbol{\ell}_i^2 \ z_1^4 \boldsymbol{\ell}_i^3] \\ \widehat{\boldsymbol{\ell}_i^5}[x_1^5 \boldsymbol{\ell}_i^1 \ y_1^5 \boldsymbol{\ell}_i^2 \ z_1^5 \boldsymbol{\ell}_i^3] \\ \vdots \\ \widehat{\boldsymbol{\ell}_i^m}[x_1^m \boldsymbol{\ell}_i^1 \ y_1^m \boldsymbol{\ell}_i^2 \ z_1^m \boldsymbol{\ell}_i^3] \end{bmatrix} \begin{bmatrix} \lambda_i^1 \\ \lambda_i^2 \\ \lambda_i^3 \end{bmatrix} = 0. \tag{40}$$

We have given a constructive proof for the following statement:

THEOREM 3 (Factorization of the multibody fundamental matrix). *Let $F \in \mathbb{R}^{M_n \times M_n}$ be the multibody fundamental matrix associated to fundamental matrices $\{F_i \in \mathbb{R}^{3\times3}\}_{i=1}^n$. If the n epipoles are distinct, then the matrices $\{F_i\}_{i=1}^n$ can be uniquely determined (up to a scale).*

5.4. 3-D Motion segmentation

The 3-D motion segmentation problem refers to the problem of assigning each image pair $\{(\boldsymbol{x}_1^j, \boldsymbol{x}_2^j)\}_{j=1}^N$, to the motion it corresponds. This can be easily done from either the epipoles $\{e_i\}_{i=1}^n$ and epipolar lines $\{\boldsymbol{\ell}^j\}_{j=1}^m$, or from the fundamental matrices $\{F_i\}_{i=1}^n$, as follows.

1. *Motion segmentation from the epipoles and epipolar lines*: Given an image pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$, the factorization of $\tilde{\boldsymbol{\ell}} = F\nu_n(\boldsymbol{x}_1)$ gives n epipolar lines. One of these lines, say $\boldsymbol{\ell}$, passes through $\boldsymbol{x}_2$, i.e. $\boldsymbol{\ell}^T \boldsymbol{x}_2 = 0$. The pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is assigned to the i^{th} motion if $\boldsymbol{\ell}^T \boldsymbol{e}_i = 0$.

2. *Motion segmentation from the fundamental matrices*: The image pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is assigned to the i^{th} motion if $\boldsymbol{x}_2^T F_i \boldsymbol{x}_1 = 0$.

Figure 5 illustrates how a particular image pair, say $(\boldsymbol{x}_1, \boldsymbol{x}_2)$, which belongs to the i^{th} motion, $i = 1, \ldots, n$ is successfully segmented.

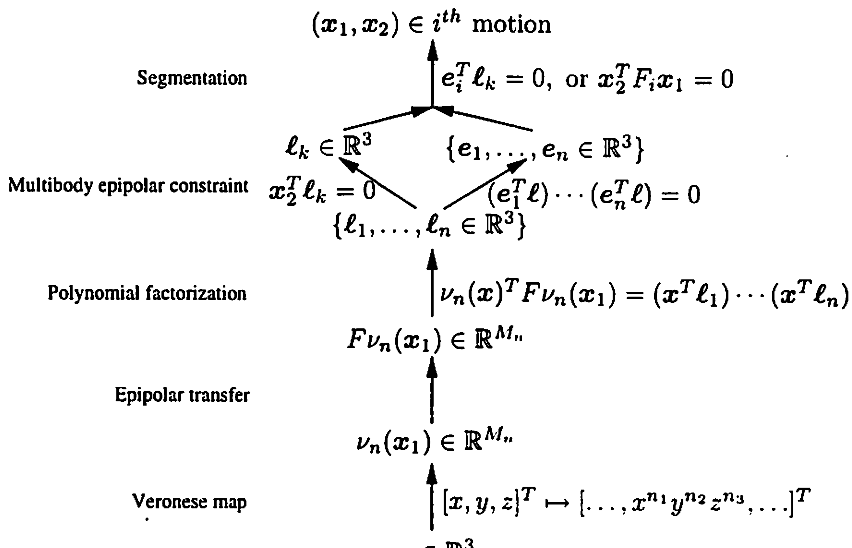


Figure 5. Transformation diagram associated to the segmentation of an image pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in the presence of n motions.

In the presence of noise, $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is assigned to the motion i that minimizes $(e_i^T \boldsymbol{\ell})^2$ or $(\boldsymbol{x}_2^T F_i \boldsymbol{x}_1)^2$, respectively.

5.5. MULTIBODY STRUCTURE FROM MOTION ALGORITHM

We are now ready to present a complete algorithm for multibody motion estimation and segmentation from two perspective views.

Algorithm 1 (Multibody structure from motion algorithm).

Given a collection of image pairs $\{(x_1^j, x_2^j\}_{j=1}^N$ of points undergoing n different motions, recover the number of independent motions n and the fundamental matrix F_i associated to motion i as follows:

1. **Number of motions.** Compute the number of independent motions n from the rank connstraint in (10), using the Veronese map of degree $i = 1, 2, \ldots, n$ applied to the image points $\{(x_1^j, x_2^j\}_{j=1}^N$.
2. **Multibody fundamental matrix.** Compute the multibody fundamental matrix F as the solution of the linear system $A_n f = 0$ in (8), using the Veronese map of degree n.
3. **Epipolar transfer.** Pick $N \geq M_n - 1$ vectors $\{x_1^j \in \mathbb{R}^3\}_{j=1}^N$, with $x_1^1 = [1,0,0]^T$, $x_1^2 = [0,1,0]^T$ and $x_1^3 = [0,0,1]^T$, and compute their corresponding epipolar lines $\{\ell_k^j\}_{k=1,\ldots,n}^{j=1,\ldots,N}$ using the factorization algorithm of Section 5.1 applied to the vectors $F\nu_n(x_1^j) \in \mathbb{R}^{M_n}$.
4. **Multibody epipole.** Use the epipolar lines $\{\ell_k^j\}_{k=1,\ldots,n}^{j=1,\ldots,N}$ to estimate the multibody epipole $\tilde{e}$ as coefficients of the polynomial $h(\ell)$ in (34) by solving the system $B_n \tilde{e} = 0$ in (35).
5. **Individual epipoles.** Use the polynomial factorization algorithm of Section 5.1 to compute the individual epipoles $\{e_i\}_{i=1}^n$ from the multibody epipole $\tilde{e} \in \mathbb{R}^{M_n}$.
6. **Individual fundamental matrices.** For each j, choose $k(i)$ such that $e_i^T \ell_{k(i)}^j = 0$, i.e. assign each epipolar line to its motion. Then use equations (39) and (40) to obtain each fundamental matrix F_i from the epipolar lines assigned to epipole i.
7. **Features segmentation by motion.** Assign image pair (x_1^j, x_2^j) to motion i if $e_i^T \ell_{k(i)}^j = 0$ or if $x_2^T F_i x_1 = 0$.

One of the main drawbacks of Algorithm 1 is that it needs a lot of image pairs in order to compute the multibody fundamental matrix, which often makes it impractical for large n (See Remark 5 below). In practice, one can significantly reduce the data requirements by incorporating partial knowledge about the motion or segmentation of the objects with minor changes in the general algorithm. We discuss a few of such possible variations to Algorithm 1 below.

Multiple linearly moving objects. In many practical situations, the motion of the objects can be well approximated by a linear motion, i.e. there is only translation but no rotation. In this case, the epipolar constraint reduces to $\boldsymbol{x}_2^T \widehat{\boldsymbol{e}_i} \boldsymbol{x}_1 = 0$ or $\boldsymbol{e}_i^T \widehat{\boldsymbol{x}_2} \boldsymbol{x}_1 = 0$, where $\boldsymbol{e}_i \in \mathbb{R}^3$ represents the epipole associated to the i^{th} motion, $i = 1, \ldots, n$. Therefore, the vector $\boldsymbol{\ell} = \widehat{\boldsymbol{x}_2} \boldsymbol{x}_1 \in \mathbb{R}^3$ is an epipolar line satisfying the equation

$$g(\boldsymbol{\ell}) = (\boldsymbol{e}_1^T \boldsymbol{\ell})(\boldsymbol{e}_2^T \boldsymbol{\ell}) \cdots (\boldsymbol{e}_n^T \boldsymbol{\ell}) = 0. \tag{41}$$

Therefore, given a set of image pairs $\{(\boldsymbol{x}_2^j, \boldsymbol{x}_1^j)\}_{j=1}^N$ of points undergoing n *distinct* linear motions $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \in \mathbb{R}^3$, one can use the set of epipolar lines $\boldsymbol{\ell}^j = \widehat{\boldsymbol{x}_2^j} \boldsymbol{x}_1^j, j = 1, \ldots, N$ to estimate the epipoles $\boldsymbol{e}_i$ using Steps 4 and 5 of Algorithm 1. Notice that the epipoles are recovered directly using polynomial factorization *without* estimating the multibody fundamental matrix $\mathcal{F}$ first. Furthermore, given the epipoles the fundamental matrix is trivially obtained as $F_i = \widehat{\boldsymbol{e}_i}$. The segmentation of the image points is then obtained from Step 7 of Algorithm 1. We conclude that if the motions are linear, we only need $N = M_n - 1$ image pairs versus $N = M_n^2 - 1$ needed in the general case. So when n is large, the number of image pairs needed grows as $O(n^2)$ for the linear motion case versus $O(n^4)$ for the general case. In other words, the number of feature points that need to be tracked on each object grows linearly in the number of independent motions. For instance, when $n = 10$, one only needs to track 7 points on each object, which is a mild requirement given that the case $n = 10$ occurs rather rarely in most applications.

Constant motions. In many vision and control applications, the motion of the objects in the scene changes slowly relative to the sampling rate. Thus, if the image sampling rate is even, we may assume that for a number of image frames, say m, the motion of each object between consecutive pairs of images is the same. Hence *all* the feature points corresponding to the $m-1$ image pairs in between can be used to estimate the *same* multibody fundamental matrix. For example, when $m = 5$ and $n = 4$, we only need to track $(M_4^2 - 1)/4 = 225/4 \approx 57$ image points between each of the 4 consecutive pairs of images instead of 255. That is about $57/4 \approx 15$ features on each object on each image frame, which is rather feasible to do in practice. In general if $m = O(n)$, $O(n^2)$ feature points per object need to be tracked in each image. For example, when $m = n + 1 = 6$, one needs to track about 18 points on each object, which is not so demanding given the nature of the problem.

Internal structure of the multibody fundamental matrix. The only step of Algorithm 1 that requires $O(n^4)$ image pairs is the estimation of the multibody fundamental matrix F. Step 2 requires a lot of data points, because F is estimated linearly without taking into account the rich internal (algebraic) structure of F (e.g., $\text{rank}(F) \leq M_n - n$). In the future, we expect to be able to reduce the number of image pairs needed by considering constraints among entries of F, in the same spirit that the well-known 8-point algorithm for $n = 1$ can be reduced to 7 points if the algebraic property $\det(F) = 0$ is used.

REMARK 5 (Comments about the algorithm).

1. **Repeated epipoles.** *If two individual fundamental matrices share the same (left) epipoles, we cannot segment the epipolar lines as described in Step 6 of Algorithm 1. In this case, one can consider the right epipoles (in the first image frame) instead, since it is extremely rare that two motions give rise to the same left and right epipoles.*[7]

2. **Repeated roots.** *If the polynomial $q_n(w)$ in (27) has repeated roots or more than one of its leading coefficient is zero, then a linear transformation (31) must be pre-applied to the polynomial $p_n(\boldsymbol{x})$ before factoring it in Steps 3 and 5 of Algorithm 1.*

3. **Algebraic solvability.** *The only nonlinear part of Algorithm 1 is to solve for the roots of univariate polynomials of degree n in Steps 3 and 5. Therefore, the multibody structure from motion problem is algebraically solvable (i.e. there is closed-form solution) if and only if the number of motions is $n \leq 4$ (see [13]). When $n \geq 5$, the above algorithm must rely on a numerical solution for the roots of those polynomials.*

4. **Computational complexity.** *In terms of data, Algorithm 1 requires $O(n^4)$ image pairs to estimate the multibody fundamental matrix F associated to the n motions. In terms of numerical computation, it needs to factor $O(n)$ polynomials*[8] *and hence solve for the roots of $O(n)$ univariate polynomials of degree n.*[9] *As for the rest of computation, which can be well approximated by the most costly Steps 1 and 2, the complexity is about $O(n^6)$.*

[7] This happens only when the rotation axes of the two motions are equal to each other and parallel to the translation direction.

[8] One needs about $M_n - 1 \approx O(n^2)$ epipolar lines to compute the epipoles and fundamental matrices, which can be obtained from $O(n)$ polynomial factorizations since each one generates n epipolar lines. Hence it is not necessary to compute the epipolar lines for all $N = M_n^2 - 1 \approx O(n^4)$ image pairs in Step 3.

[9] The numerical complexity of solving for the roots for an n^{th} order polynomial in one variable is polynomial in n for a given error bound, see [17].

5. **Special motions.** *Algorithm 1 works for distinct motions with nonzero translation. Future research is needed for special motions, e.g., pure rotation or repeated epipoles parallel to the rotation axis.*

6. **Noise sensitivity.** *Algorithm 1 gives a purely algebraic solution to the multibody structure from motion problem. Future research will need to address the sensitivity of the algorithm to noise in the image measurements. Since the polynomial factorization in Steps 3 and 5 is very robust to noise [20], one should pay attention to Step 2, which is sensitive to noise, because it does not exploit the algebraic structure of the multibody fundamental matrix $\mathcal{F}$.*

7. **Optimality.** *Notice that linearly solving for the multibody fundamental matrix through the Veronese embedding is sub-optimal from a statistic point of view. We refer interested readers to [21] for a derivation of the optimal function for the estimation of $\mathcal{F}$.*

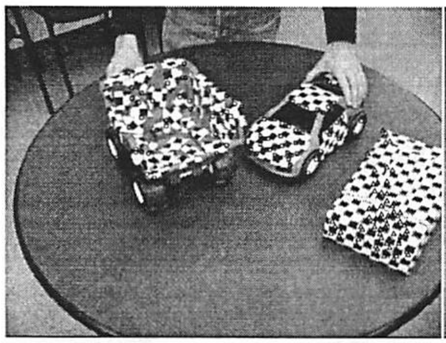
At the end of our theoretical development, Table I summarizes our results with a comparison of the geometric entities associated to two views of 1 rigid body motion and two views of n rigid body motions.

Table I. Comparison between the geometry for two views of 1 rigid body motion and that for n rigid body motions.

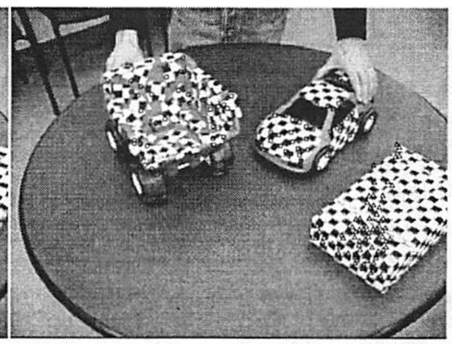
Comparison of	2 views of 1 body	2 views of n bodies
An image pair	$x_1, x_2 \in \mathbb{R}^3$	$\nu_n(x_1), \nu_n(x_2) \in \mathbb{R}^{M_n}$
Epipolar constraint	$x_2^T F x_1 = 0$	$\nu_n(x_2)^T \mathcal{F} \nu_n(x_1) = 0$
Fundamental matrix	$F \in \mathbb{R}^{3\times 3}$	$\mathcal{F} \in \mathbb{R}^{M_n \times M_n}$
Linear estimation from N image pairs	$\begin{bmatrix} x_2^1 \otimes x_1^1 \\ x_2^2 \otimes x_1^2 \\ \vdots \\ x_2^N \otimes x_1^N \end{bmatrix} f = 0$	$\begin{bmatrix} \nu_n(x_2^1) \otimes \nu_n(x_1^1) \\ \nu_n(x_2^2) \otimes \nu_n(x_1^2) \\ \vdots \\ \nu_n(x_2^N) \otimes \nu_n(x_1^N) \end{bmatrix} \mathbf{f} = 0$
Epipole	$e^T F = 0$	$\nu_n(e)^T \mathcal{F} = 0$
Epipolar lines	$\ell = F x_1 \in \mathbb{R}^3$	$\tilde{\ell} = \mathcal{F} \nu_n(x_1) \in \mathbb{R}^{M_n}$
Epipolar line & point	$x_2^T \ell = 0$	$\nu_n(x_2)^T \tilde{\ell} = 0$
Epipolar line & epipole	$e^T \ell = 0$	$\tilde{e}^T \nu_n(\ell) = 0$

6. Segmentation results

We tested the proposed approach by segmenting a real image sequence with $n = 3$ moving objects: a truck, a car and a box. Figure 6 shows two frames of the sequence with the tracked features superimposed. We used the algorithm in [3] to track a total of $N = 173$ point features: 44 for the truck, 48 for the car and 81 for the box. Figure 6 plots the segmentation of the image points obtained using Algorithm 1. Notice that the obtained segmentation has no mismatches.



(a) First image frame



(b) Second image frame

Figure 6. A motion sequence with a truck, a car and a box. Tracked features are marked as follows: "○" for the truck, "□" for the car and "△" for the box.

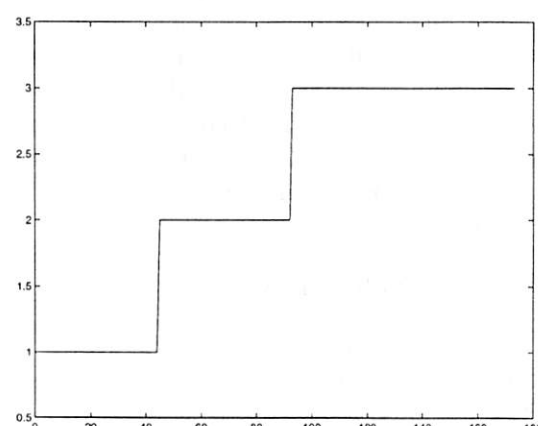


Figure 7. Motion segmentation results. Each image pair is assigned to the fundamental matrix for which the algebraic error is minimized. The first 44 points correspond to the truck, the next 48 to the car, and the last 81 to the box. The correct segmentation is obtained.

7. Discussions, conclusions and future work

This paper has presented a complete algebraic characterization of the multibody structure from motion problem from two perspective views. We provided a general solution based on a clear geometric interpretation of the algebraic properties of the so-called multibody fundamental matrix. We have proven that the multibody structure from motion problem is algebraically equivalent to the factorization of homogeneous polynomials, and provided a novel solution to the latter problem with polynomial time complexity. The rest of the algorithm is based mostly on linear techniques and hence it is also polynomial time. The algorithm proposed here provides a principled solution to the problem and paves the way to a more systematic study of its many variations, which account for different practical scenarios and conditions.

The internal algebraic structure of the multibody fundamental is not taken into account in the current approach. Besides, our discussion has also suggested that the use of *multiple* images may also reduce the amount of feature points needed from each image (pair). We expect to investigate these issues further in the future and the outcome could likely be an algorithm which requires much less image data than the current linear one.

On the other hand, the paper only focuses on the algebraic and geometric aspects of the multibody structure from motion problem, thus the current solution is purely algebraic. Issues such as the effect of noise and numerical errors have not been systematically studied. Due to the large size of matrices and data associated, algorithms for the multibody structure from motion problem become more sensitive to particular numerical implementations, especially in the estimation of the multibody fundamental matrix. Although preliminary simulations and experimental results are encouraging, we currently conducting more tests on different synthetic data sets and real images. A more complete report on such issues and proposals for better numerical approaches will be presented in future papers.

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