Rational Curves with Polynomial Parametrization

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Abstract: Rational curves and splines are one of the building blocks of computer graphics and geometric modeling. Although a rational curve is more flexible than its polynomial counterpart, many properties of polynomial curves are not applicable to it. For this reason it is very useful to know if a curve presented as a rational space curve has a polynomial parametrization. In this paper, we present an algorithm to decide if a polynomial parametrization exists, and to compute the parametrization.

In algebraic geometry it is known that a rational algebraic curve is polynomially parametrizable iff it has one *place* at infinity. This criterion has been used in earlier methods to test polynomial parametrizability of space curves. These methods project the curve into the plane and test parametrizability there. But this gives only a sufficient condition for the original curve. In this paper we give a simple condition which is both necessary and sufficient for polynomial parametrizability. The calculation of the polynomial parametrization is simple, and involves only a rational reparametrization of the curve.

[1]This research was supported in part by David and Lucile Packard Fellowship and in part by National Science Foundation Presidential Young Investigator Award (number IRI-8958577).

1 Introduction

Rational curves are a central tool in graphics and modeling. The rational formulation has the ability to represent conics (in fact all *genus 0* algebraic curves) as well as free-form (controlled) curves. The coordinates for each point on the curve can be expressed as:

$$\mathbf{Q}(t) = (x, y, z) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)}, \frac{z(t)}{w(t)}) \quad t \in [a, b]$$

where $x(t), y(t), z(t)$ and $w(t)$ are polynomials in $\mathcal{R}[t]$, the ring of all polynomials in t, whose coefficients are real numbers. A curve given in this representation is said to be *polynomial* if $w(t) = 1$, otherwise it is a *rational curve.*

The rational formulation offers more flexibility than its polynomial counterpart, but many properties of polynomial curves are not applicable to it. The rational curves of a fixed degree (defined in terms of the homogeneous representation) do not constitute a vector space. Moreover, there can be more than one homogeneous representation for a rational curve. These different representations are obtained in many applications, *e.g.* degree elevation algorithms.

At times a rational curve has a polynomial representation. Some examples of such rational curves are:

- If $w(t)$ divides all the three polynomials, $x(t), y(t)$ and $z(t)$, then $\mathbf{Q}(t)$ has a polynomial representation of the form $(\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)}, \frac{z(t)}{w(t)})$.
- Consider a polynomial curve $\mathbf{q}(s)$. A rational reparametrization obtained by substituting $s = f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials in t, results in a rational curve of the form $\mathbf{Q}(t)$. That is

  $$\mathbf{q}(\frac{f(t)}{g(t)}) = \mathbf{Q}(t).$$

  In this case $\mathbf{Q}(t)$, a rational curve, has a polynomial parametrization, $\mathbf{q}(s)$.

Not only does a polynomial representation possess extra properties, it is also more efficient to evaluate. This raises an important question: given a rational curve, does it have a polynomial representation? It is known from algebraic geometry that among the conics, the circle, ellipse and hyperbola have a rational representation only (*i.e.* no polynomial parametrization), whereas a parabola has a polynomial parametrization. The criterion for determining whether a rational algebraic curve has a polynomial parametrization, namely that it has only one place at infinity, is well known in algebraic geometry. It has been used on rational plane algebraic curves [Abhyankar '86]. The algorithm makes use of g-adic expansions and Newton polygons [Walker '50] and is quite complicated. Moreover, it is applied to rational plane curves represented as $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y, and requires that we implicitize a plane rational parametric curve before applying it.

In this paper we give a simple criterion for deciding if a properly parametrized rational curve has a polynomial parametrization. This criterion is constructive, and we derive the polynomial representation by doing a rational reparametrization of the curve. Since a proper parametrization can be computed from an arbitrary parametrization, this gives us an algorithm that works on arbitrary rational curves. The degree of the polynomial parametrization is not more than the rational one.

The paper is organized as follows: In Section 2 we specify the notation and define the terms which are used in the rest of the paper. We also state the criterion for polynomial parametrizability from algebraic geometry there. In Section 3, we present our new criterion and show that if it holds, the curve can be polynomial parametrized. Furthermore the parametrization, if it exists, is obtained by a simple rational reparametrization of the curve (as opposed to a more general analytic reparametrization). This section also contains the algorithm for making the test, and computing the polynomial representation. In Section 4, we show that if the criterion from algebraic geometry holds, then so does our new criterion. This together with the results from Section 3 shows that our criterion is equivalent to parametrizability (and to the algebraic geometry criterion).

All our algorithms are for space curves. Their restriction to plane curves is obvious.

2 Rational Curves

In our applications a rational space curve is a vector valued function of the type

$$\mathbf{Q}(t) = (X(t), Y(t), Z(t)), \quad t \in [a, b]$$

where $X(t), Y(t)$ and $Z(t)$ are polynomial or rational functions and $[a, b]$ denotes a real interval. The set of all rational functions includes the polynomial functions. We restrict the use of the word *rational* in the following manner: a given function of the form $u(t)/w(t)$, where $u(t)$ and $w(t)$ are polynomial functions, is not a rational function if $w(t)$ divides $u(t)$. Such functions are also referred to as *integral* functions to distinguish them from rational functions. We will use lower case letters to denote polynomial functions like $x(t), y(t), z(t)$ and upper case letters to denote rational functions like $X(t), Y(t), Z(t)$. The boldface letters are used to represent vector valued functions like $\mathbf{q}(\mathrm{t})$ or $\mathbf{Q}(\mathrm{t})$.

A rational curve also has a homogeneous representation of the type

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$$

associated with it, where $X(t) = \frac{x(t)}{w(t)}, Y(t) = \frac{y(t)}{w(t)}, Z(t) = \frac{z(t)}{w(t)}$. The homogeneous representation corresponds to a polynomial curve in a higher dimensional space. In Fig. I, we have shown how one can view a 2-dimensional rational curve as the projection of a 3-dimensional polynomial curve on a plane ($w = 1$). The homogeneous representation of a rational curve is not unique. Our problem of determining whether a rational curve has a polynomial representation is equivalent to that of determining whether the rational curve has a homogeneous representation in which the last term, corresponding to $w(t)$ is a constant ($x(t), y(t)$ and $z(t)$ are still polynomials).

Each polynomial $x(t), y(t), z(t)$ or $w(t)$ is assumed to have *power basis* representation. All Bézier, B-spline or Beta-spline curves can be converted into power basis representation. The degree of $\mathbf{q}(t)$ is the maximum of degrees of $x(t), y(t)$ and $z(t)$ and the degree of $\mathbf{Q}(t)$ is the maximum of the degrees of $x(t), y(t), z(t)$ and $w(t)$.

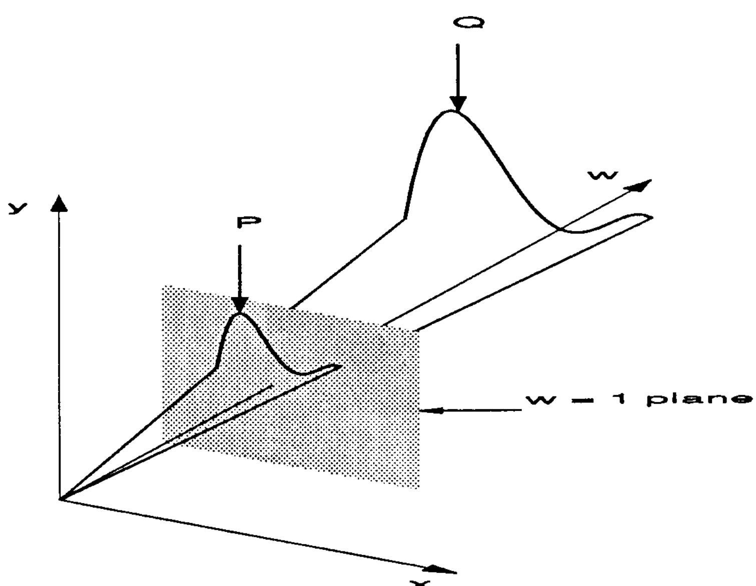

Fig. I
A 2-D rational curve $\mathbf{P}(\mathrm{u})$ and its associated 3-D homogeneous curve $\mathbf{Q}(\mathrm{u})$.

2.1 Affine and Projective Spaces

In computer graphics and geometric modeling a curve parametrization, $\mathbf{Q}(t)$, denotes a mapping of the form

$$\mathbf{Q}: \ \mathcal{R} \rightarrow \mathcal{R}^3,$$

where $\mathcal{R}$ denotes the set of real numbers. The domain is often restricted to a finite interval, of the form $[a, b]$. Since the field of real numbers is not an algebraically closed field, we extend this definition to its algebraic closure, $\mathcal{C}$, the set of complex numbers. This will be useful later on for defining proper and improper parametrizations. Hence we consider the parametrization as a mapping of the form

$$\mathbf{Q}: \ \mathcal{C} \rightarrow \mathcal{C}^3.$$

Till now we have viewed our curve as a geometric object in the *affine* space. Affine n-dimensional over the complexes is the familiar n-space. Using Cartesian coordinates, a point in this space has coordinates

$$(x_1, x_2, \ldots, x_n)$$

where $x_i \in \mathcal{C}$ (and is therefore finite). However, there are a lot of advantages in considering the object in *projective* space. Projective n-dimensional space consists of the affine n-dimensional space plus the points at *infinity.* Using Cartesian coordinates, a point in this space is represented as

$$(x_1, x_2, \ldots, x_n, x_{n+1})$$

where not all x_i are zero and $x_i \in \mathcal{C}$. Moreover, for any nonzero complex number, s, $(x_1, x_2, \ldots, x_n, x_{n+1})$ and $(sx_1, sx_2, \ldots, sx_n, sx_{n+1})$ denote the same point in the space. The variable x_{n+1} is considered a *homogenizing* variable. If $x_{n+1} = 0$ then $(x_1, \ldots, x_n, x_{n+1})$ is a point at infinity. More on affine and projective spaces is given in [Abhyankar '86; Hoffmann '89; Walker '50].

In computer graphics and geometric modeling the homogeneous representation of a rational curve

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)) \tag{1}$$

indicates a mapping of the form

$$\mathbf{Q}: \ \mathcal{C} \rightarrow \mathcal{P}^3$$

where $\mathcal{P}$ denotes the complex projective space (affine complex space plus the points at infinity). It is reasonable to assume that $GCD(x(t), y(t), z(t), w(t)) = 1$, where GCD denotes the greatest common factor of the given polynomials. If $t = t_0$ is a root of $w(t)$ then $\mathbf{Q}(t_0)$ corresponds to a point at infinity.

A rational curve (1) should be regarded as a mapping of the form

$$\mathbf{Q}: \ \mathcal{P} \rightarrow \mathcal{P}^3.$$

That is, the domain of the parameter t consists of values at infinity as well [Semple and Kneebone '59]. A parameter value in the domain, $\mathcal{P}$, is represented by the pair (t, u) and $u = 0$ corresponds to the parameter values at infinity. The rational curve $\mathbf{Q}(t)$ should be interpreted as a representation of the form

$$\overline{\mathbf{Q}}(t, u) = (\overline{x}(t, u), \overline{y}(t, u), \overline{z}(t, u), \overline{w}(t, u)) \tag{2}$$

where $\overline{x}(t, u), \overline{y}(t, u), \overline{z}(t, u)$ and $\overline{w}(t, u)$ are homogeneous polynomials in t and u and the degree of each polynomial is d, the degree of the rational curve, $\mathbf{Q}(t)$. Moreover,

$$GCD(\overline{x}(t, u), \overline{y}(t, u), \overline{z}(t, u), \overline{w}(t, u)) = 1.$$

Many applications assume that the domain is a finite interval on the real line and the image is contained in $\mathcal{R}^3$. Using a representation of the form (2) may not offer any advantages as far as the geometric operations are concerned. However, the representation in terms of projective coordinates helps us in determining certain properties of rational curves. For example, the criterion in algebraic geometry about determining whether a rational curve has a polynomial parametrization (which we will mention in the next section) uses that fact.

In the rest of the paper we continue with a rational representation of the form (1) and will explicitly mention a representation of the form (2) whenever needed.

Definition: A point $\mathbf{P}(t_0)$ on the curve $\mathbf{P}(t)$ is called a *regular point* if the curve permits a regular parametrization $x = \beta_1(t), y = \beta_2(t), z = \beta_3(t)$, where $\beta_i(t)$ are analytic functions, in a neighborhood of t_0 satisfying the condition $x'^2 + y'^2 + z'^2 \neq 0$ at the point $\mathbf{P}(t_0)$.

2.2 Rational Algebraic Curves

An irreducible plane algebraic curve in $\mathcal{P}^2$ is the aggregate of all the points of $\mathcal{P}^2$, whose coordinates are zeros of some irreducible homogeneous polynomial $f(x, y, w)$. The representation of a curve as the zero set of a polynomial is its *implicit* representation. Every planar algebraic curve, $\mathbf{P}(t) = (x(t), y(t), w(t))$ corresponds to an irreducible plane algebraic curve. Algorithms to implicitize a plane rational parametric curve are given in [Hoffmann '89].

An algebraic surface in $\mathcal{P}^3$ is defined as the zero set of an irreducible homogeneous polynomial of the form $g(x, y, z, w)$. An algebraic space curve is the common intersection of two or more algebraic surfaces. Each algebraic space curve is in *birational* correspondence with an algebraic plane curve [Semple and Kneebone '59; Walker '50]. All rational parametric space curves can be represented as the intersection of two or more algebraic surfaces. Many algorithms in geometric modeling are restricted to algebraic space curves, which can be represented as the intersection of two surfaces. It is not known at the moment whether all space curves can be represented as the intersection of two surfaces [Abhyankar '86]. Consider a non-planar, properly parametrized *cubic* rational parametric curves of the form, $\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$. Using *Bezout's* theorem [Walker '50] one can show that the algebraic degree of $\mathbf{Q}(t)$ is three. The implicit representation of $\mathbf{Q}(t)$ is obtained by representing it as the intersection of three algebraic surfaces. It is possible that there are two algebraic surfaces, say $g_1(x, y, z, w)$ and $g_2(x, y, z, w)$ such that their intersection contains $\mathbf{Q}(t)$ as one of the many irreducible components or the intersection multiplicity of the surfaces along $\mathbf{Q}(t)$ is greater than one.

Only a few algebraic curves have rational parametrization. In particular, an algebraic curve has a rational parametrization if and only if its *genus* is zero [Semple and Kneebone '59; Walker '50]. Such algebraic curves are categorized as *rational algebraic curves.* Algorithms to parametrize rational algebraic curves are given in [Hoffmann '89].

2.3 Properly Parametrized Rational Curves

In many applications a rational curve can be identically described by a polynomial or rational parametrization of lower degree. Such a curve is *improperly* parametrized, which means to every point on the curve there corresponds more than one parameter value. The domain of the parameter is not restricted to the finite real interval and the curve is considered as a mapping

$$\mathbf{Q}: \mathcal{P} \rightarrow \mathcal{P}^3.$$

Curves which have a one-to-one relationship between the parameter values and the points on the curve (except for a finite number of points) are called *properly* parametrized curves. Consider the polynomial curve

$$\mathbf{q}(t) = (t, t^2, t^3).$$

$\mathbf{q}(t)$ is properly parametrized and has no multiple points. Hence there is a unique parameter value corresponding to each point on the curve [Semple and Kneebone '59]. If we reparametrize the curve by substituting $t = \frac{2s}{s^2+1}$, we get a rational curve of the form

$$\mathbf{Q}(s) = (2s(s^2+1)^2, 4s^2(s^2+1), 8s^3, (s^2+1)^3).$$

$\mathbf{Q}(t)$ is improperly parametrized, because for each value of the parameter $t = t_0$ there are two values of the parameter s corresponding to it and they are given by the roots of the quadratic equation $t_0(s^2-1) - 2s = 0$. Another popular terminology for the proper and improper parametrizations are *faithful* and *unfaithful* parametrizations, respectively.

Every rational algebraic curve has a rational parametrization. According to *Lüroth's theorem*, given a rational algebraic curve $f(x, y) = 0$, then there exist two rational[2] functions $\phi(t)$ and $\psi(t)$, where $t \in \mathcal{C}$, such that:

- For all but a finite set of $t \in \mathcal{C}$, $f(\phi(t), \psi(t)) = 0$.
- With a finite number of exceptions, for every x_0,y_0 for which $f(x_0, y_0) = 0$, there is a unique $t \in \mathcal{C}$ such that $x_0 = \phi(t)$ and $y_0 = \psi(t)$.

More details on Lüroth's theorem are given in [Semple and Kneebone '59; Walker '50]. Thus, corresponding to every improperly parametrized planar rational curve there is a properly parametrized planar rational[2] curve.

Given an improperly parametrized rational space curve $\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$, we can consider the plane curves $\mathbf{Q}_1(t) = (x(t), y(t), w(t))$ and $\mathbf{Q}_2(t) = (x(t), z(t), w(t)$ as two plane curves. The fact that $\mathbf{Q}(t)$ is improperly parametrized implies that $\mathbf{Q}_1(t)$ and $\mathbf{Q}_2(t)$ are improperly parametrized, too. According to Lüroth's theorem each plane rational curve has a proper rational parametrization and it implies that the space curve $\mathbf{Q}(t)$ has a proper rational parametrization as well. More on proper and improper parametrizations and algorithms to compute the proper rational parametrizations for improperly parametrized rational space curves are given in [Manocha and Canny '89; Sederberg '84; Sederberg '86].

[2] Here the set of rational functions includes the polynomial functions as well.

2.4 Place of a Curve

Let $\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$ be a parametric curve and define $\mathbf{P} = \mathbf{Q}(0) = (x_0, y_0, z_0, w_0)$. In the neighborhood of $\mathbf{P}$, the curve can always be defined by a formal power series. If $w_0 \neq 0$, then $(\overline{x}(t), \overline{y}(t), \overline{z}(t), 1)$, where

$$\overline{x}(t) = \frac{x(t)}{w(t)} \quad \overline{y}(t) = \frac{y(t)}{w(t)} \quad \overline{z}(t) = \frac{z(t)}{w(t)},$$

is a power series representation in the neighborhood of $\mathbf{P}$. The formal power series or the local parametrization is called a *place* of $\mathbf{Q}(t)$ at $\mathbf{P}$ and exists because of *Newton's* theorem [Semple and Kneebone '59]. The notion of a place is more specific than that of a curve point. Corresponding to every curve point the curve has a place. The curve may have more than one place at a singular point and has one place at every non-singular point. In particular, the curve has two or more places at a node or loop and one place at a cusp. More on places and their representation as branches is given in [Abhyankar '86; Hoffmann '89; Semple and Kneebone '59].

A rational parametric space curve $\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$ is a properly parametrized curve if it has one-to-one relationship between the parameter t and points on the curve, except for a finite number of exceptional points. Let $\mathbf{R}$ be one of these exceptional points (not every curve consists of such points). In other words, there is more than one value of the parameter t, which gives rise to the point $\mathbf{R}$. At such points, the curve has more than one place. The exact relationship between the number of parameter values corresponding to a point and the number of places at the same point is given by the following theorem [Semple and Kneebone '59]:

Theorem I *The number of values of t that give rise to a point $\mathbf{P}$ on a properly parametrized curve $\mathbf{Q}(t)$ is the number of places on the curve at $\mathbf{P}$.*

Consider the cubic plane curve

$$\mathbf{Q}(t) = (x(t), y(t), w(t)) = (t(1-t^2), 1-t^2, t^3)$$

which is a parametrization of $f(x,y,w) = x^2w - y^2w + x^3 = 0$. Since the degree of $\mathbf{Q}(t)$ is equal to the degree of $f(x,y,w)$ (which is three), $\mathbf{Q}(t)$ is properly parametrized. The curve has one place at every point except at the origin, $(0,0,1)$, which has two places corresponding to $t = 1$ and $t = -1$.

In algebraic geometry the criterion for determining whether a rational algebraic curve has a polynomial parametrization has been defined in terms of the number of places the curve has at infinity. The following theorem specifies the criterion for a curve to have polynomial parametrization [Abhyankar '86]:

Theorem II *A curve has a polynomial parametrization if and only if the curve has one place at infinity and it can be parametrized by rational functions.*

The curve has at least one place corresponding to every point. Thus, the number of places at infinity is greater than or equal to the number of points at infinity. A circle, ellipse or a hyperbola have two points at infinity and therefore, they do not have a polynomial parametrization.

2.5 Multiplicity of a Polynomial

A polynomial $f(t)$ has multiplicity n at $t = t_0$, iff

$$f(t_0) = 0,$$

$$f^1(t_0) = 0,$$

$$\vdots$$

$$f^{m-1}(t_0) = 0,$$

where $f^i(c)$ denotes the i^{th} derivative of $f(t)$ at $t = t_0$. The algorithm to decide for the existence of a polynomial parametrization and its computation needs to verify whether $w(t)$ is a polynomial of the form $\alpha(t-\beta)^n$. That is, it has a single root of multiplicity n, the degree of $w(t)$.

Lemma I: *Given a polynomial $w(t)$ of degree n, the necessary and sufficient condition that it has a single root of multiplicity n is*

$$degree\ of\ GCD(w(t), w'(t)) = n - 1,$$

where $w'(t)$ is the first derivative of $w(t)$.

Proof: *Necessity*

If $w(t)$ has a single root of multiplicity n, it is of the form:

$$w(t) = \alpha(t-\beta)^n,$$

$$\Rightarrow w'(t) = n\alpha(t-\beta)^{n-1},$$

$$\Rightarrow GCD(w(t), w'(t)) = \alpha(t-\beta)^{n-1},$$

$$\Rightarrow degree\ of\ GCD(w(t), w'(t)) = n - 1.$$

Sufficiency

Let us assume that $w(t)$ has a root of multiplicity less than n, say β_1. Without loss of generality we can assume that $w(t)$ is a polynomial of the form:

$$w(t) = (t-\beta_1)^k p(t),$$

where $k < n$, degree $p(t) = n - k$ and $p(\beta_1) \neq 0$. Therefore,

$$w'(t) = k(t - \beta_1)^{k-1}p(t) + (t - \beta_1)^k p'(t),$$

$$\Rightarrow w'(t) = (t - \beta_1)^{k-1}(p(t) + (t - \beta_1)p'(t)).$$

Thus,

$$GCD(w(t), w'(t) = (t - \beta_1)^{k-1}GCD(p(t), p'(t)).$$

Since degree $p'(t) = n - k - 1$,

$$\Rightarrow \textit{degree of } (GCD(p(t), p'(t))) \leq n - k - 1,$$

$$\Rightarrow \textit{degree of } GCD(w(t), w'(t)) \leq n - 2.$$

Q.E.D.

We already know that

$$\textit{degree of } w'(t) = n - 1.$$

Therefore, the lemma implies that $w(t)$ has a single root of multiplicity n iff

$$w'(t) \textit{ divides } w(t).$$

Based on these results a simple and robust algorithm to decide whether $w(t)$ has a single root of multiplicty n is:

1. Compute $w'(t)$.
2. Let

   $$p(t) = w(t) - tw'(t).$$

   $w(t)$ has a single root of multiplicty n iff

   $$p(t) = k\ w'(t),$$

   where k is a constant. This can be verified by comparing the coefficients of t^i, $0 \leq i < n$.

This algorithm does not involve polynomial division. It is impossible using finite precision arithmetic to decide whether a polynomial with floating point coefficients divides the other. Hence, we expect our algorithm to be robust even if the coefficients of the given polynomials are floating point numbers.

3 Reparametrizing Rational Curves

In this section we present a criterion for determining whether a rational curve can be reparametrized into a polynomial curve. We make use of this criterion to present an algorithm for determining whether a given rational curve has a corresponding polynomial parametrization. We only consider rational reparametrizations as opposed to a more general analytic reparametrization.

Given a rational curve

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)),$$

where $GCD(x(t), y(t), z(t), w(t)) = 1$. Without loss of generality we can assume that $GCD(x(t), w(t)) = 1$. Let us analyze the conditions under which such a function can be reparametrized into a polynomial function.

Theorem III: *Given a rational function*

$$X(t) = \frac{x(t)}{w(t)},$$

where

$$x(t) = a_0 + a_1 t + \ldots + a_m t^m,$$
$$w(t) = b_0 + b_1 t + \ldots + b_n t^n,$$
$$GCD(x(t), w(t)) = 1$$

$n \geq 1$ *and* $b_n \neq 0$. *The necessary and sufficient conditions that it can be reparametrized into a polynomial function are:*

1. $m \leq n$.

2. $w(t)$ *is of the form* $b_n(t-\beta)^n$. *That is, it has a single root of multiplicity* n.

Proof: *Necessity*

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the m roots (over the field of complex numbers) of $x(t)$. They may not all be distinct. Similarly $w(t)$ has n roots $\beta_1, \beta_2, \ldots, \beta_n$. Thus,

$$X(t) = \frac{a_m(t-\alpha_1)(t-\alpha_2)\ldots(t-\alpha_m)}{b_n(t-\beta_1)(t-\beta_2)\ldots(t-\beta_n)}$$

The fact that $GCD(x(t), w(t)) = 1$ implies that

$$\alpha_i \neq \beta_j, \quad i = 1, m; \quad j = 1, n.$$

To reparametrize $X(t)$, we can substitute $t = f(s)$. There are two possible choices for $f(s)$. It can either be a polynomial or a rational function.

Case: $f(s)$ is a polynomial

Let

$$\overline{X}(s) = X(f(s)),$$

$$\Rightarrow \overline{X}(s) = \frac{a_m(f(s)-\alpha_1)\dots(f(s)-\alpha_m)}{b_n(f(s)-\beta_1)\dots(f(s)-\beta_n))}.$$

Let

$$g(s) = \prod_{i=1}^{m}(f(s)-\alpha_i).$$

A necessary condition for $\overline{X}(s)$ to be a polynomial is that $(f(s)-\beta_j)$ divides $g(s)$. Moreover this should hold for all j's. But $(f(s)-\beta_j)$ divides $g(s)$, if and only if every root of $(f(s)-\beta_j)$ is a root of $g(s)$. Let s_0 be a root of $(f(s)-\beta_j)$. Thus,

$$f(s_0) = \beta_j.$$

$$\Rightarrow g(s_0) = \prod_{i=1}^{m}(\beta_j-\alpha_i) \neq 0$$

since

$$\beta_j \neq \alpha_i$$

for all i.

Thus, the necessary condition for $\overline{X}(s)$ to be a polynomial is not satisfied. Hence, reparametrizing a rational function by a polynomial substitution always results in a rational function (i.e. it is not polynomial).

Case: Rational reparametrization

Let us substitute $t = \frac{p(s)}{q(s)}$. Without loss of generality we can assume that $GCD(p(s), q(s)) = 1$ and $q(s)$ is not a constant. Thus,

$$\overline{X}(s) = \frac{a_m(q(s))^n(p(s)-\alpha_1 q(s))\dots(p(s)-\alpha_m q(s))}{b_n(q(s))^m(p(s)-\beta_1 q(s))(p(s)-\beta_2 q(s))\dots(p(s)-\beta_n q(s))}$$

Let

$$g(s) = \prod_{i=1}^{m}(p(s)-\alpha_i q(s)).$$

and

$$h(s) = \prod_{i=j}^{n}(p-\beta_j q(s)).$$

Depending on the relative values of m and n there are 2 possible cases:

Case $n < m$: The denominator of $\overline{X}(s)$ contains a term of the form $(q(s))^{m-n}$. The fact that $p(s)$ and $q(s)$ are relatively prime implies

$$GCD(g(s), q(s)) = 1.$$

Else let

$$GCD(g(s), q(s)) = r(s),$$

where $r(s)$ is not a constant. Let s_0 be a root of $h(s)$. Therefore

$$g(s_0) = 0; \qquad q(s_0) = 0,$$

$$\Rightarrow g(s_0) = \prod_{i=1}^{m} (p(s_0) - \alpha_i q(s_0)) = 0,$$

$$\Rightarrow (p(s_0) - \alpha_k q(s_0)) = 0 \quad 1 \leq k \leq m,$$

$$\Rightarrow p(s_0) = 0.$$

This contradicts our assumption that $GCD(p(s), q(s)) = 1$. Thus, $\overline{X}(s)$ cannot be an integral function.

Case: $m \leq n$: The argument used in previous case also implies that

$$GCD(h(s), q(s)) = 1.$$

Thus, $\overline{X}(s)$ is an integral function if and only if $h(s)$ divides $g(s)$. A necessary condition for the division is obtained if $(p - \beta_j q)$ divides $g(s)$, for all j. There are two possible cases

- $(p(s) - \beta_j q(s))$ is a constant. In this case the necessary condition is satisfied.
- $(p(s) - \beta_j q(s))$ is not a constant. Thus, it divides $g(s)$ if and only if every root of $(p(s) - \beta_j q(s))$ is a root of $g(s)$. Let s_0 be a root of $(p(s) - \beta_j q(s))$. There are two possibilities:

  1. $\beta_j = 0$.

  $$\Rightarrow p(s_0) = 0$$

  $$\Rightarrow g(s_0) = \prod_{i=1}^{m} (-\alpha_i q(s_0)) \neq 0$$

  This is because each $\alpha_i \neq 0$ (since $\beta_j = 0$) and $q(s_0) \neq 0$ (since $p(s_0) = 0$). Thus, s_0 cannot be a root of $g(s)$.

  2. $\beta_j \neq 0$.
  This implies that $p(s_0) \neq 0$ and $q(s_0) \neq 0$. Otherwise, if

  $$p(s_0) = 0$$

  $$\Rightarrow q(s_0) = 0$$

  which is contrary to the assumption that $p(s)$ and $q(s)$ are relatively prime and vice versa.

The fact

$$\begin{aligned} p(s_0) - \beta_j q(s_0) &= 0 \\ \Rightarrow p(s_0) &= \beta_j q(s_0) \\ \Rightarrow \frac{p(s_0)}{q(s_0)} &= \beta_j \end{aligned}$$

Thus,

$$\begin{aligned} g(s_0) &= \prod_{i=1}^{m} (p(s_0) - \alpha_i q(s_0)) \\ &= \frac{1}{(q(s_0))^m} \prod_{i=1}^{m} (\beta_j - \alpha_i) \neq 0 \end{aligned}$$

because $\beta_j \neq \alpha_i$ for all i. Thus, s_0, a root of $h(s)$, is not a root of $g(s)$. Hence $h(s)$ does not divide $g(s)$ and reparametrization results in a rational function rather than an integral function.

Thus, we have shown that a necessary condition for $\overline{X}(s)$ to be a polynomial is that $(p(s) - \beta_j q(s))$ is a constant. However, $p(s)$ and $q(s)$ are polynomials and this relation has to hold for all β_j's. Thus,

$$\beta_1 = \beta_2 = \beta_3 = \ldots = \beta_n.$$

This implies that $w(t)$ is a polynomial of the type

$$b_n(t - \beta)^n.$$

Sufficiency:

Given a rational function of type

$$X(t) = \frac{x(t)}{b_n(t - \beta)^n},$$

where (degree of $x(t)$) $\leq n$. Substitute

$$t = \frac{1 + s\beta}{s}, \qquad (s = \frac{1}{t - \alpha}).$$

Thus, $\overline{X}(s)$ is a polynomial.

Q.E.D.

3.1 Algorithm

Given a rational curve

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t))$$

the theorem places constraints on the relative degrees of various polynomials and on $w(t)$. These constraints along with the algorithms to compute the GCD of a set of polynomials and proper parametrization of an improperly parametrized curve gives us an algorithm to decide whether a given rational curve has a corresponding polynomial parametrization. Algorithm to compute the proper parametrization of an improperly parametrized curve is given in [Sederberg '86]. It is based on Lüroth's theorem and *involutions* [Walker '50] and makes use of the GCD operation.

Given $\mathbf{Q}(t)$, an algorithm to determine whether it has a corresponding polynomial parametrization is given below. If the curve has a polynomial parametrization, the algorithm computes that.

1. Let
$$h(t) = GCD(x(t), y(t), z(t), w(t))$$
and
$$\overline{\mathbf{Q}}(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t), \overline{w}(t)) = (\frac{x(t)}{h(t)}, \frac{y(t)}{h(t)}, \frac{z(t)}{h(t)}, \frac{w(t)}{h(t)}).$$
If $\overline{w}(t)$ is a constant, then $\overline{\mathbf{Q}}(t)$ is the polynomial parametrization.

2. Let $\mathbf{P}(t) = (x_1(t), y_1(t), z_1(t), w_1(t))$ be the proper parametrization corresponding to $\overline{\mathbf{Q}}(t)$. If $w_1(t)$ is a constant, then $\mathbf{P}(t)$ is the polynomial parametrization.

3. $\mathbf{P}(t)$ has a polynomial parametrization if

   (a) degree of $w_1(t)$ = maximum (degree of $x_1(t)$, degree of $y_1(t)$, degree of $z_1(t)$, degree of $w_1(t)$).

   (b) $w_1(t)$ is a polynomial of the form $\alpha(t-\beta)^n$. That is, it has a single root of multiplicity n, the degree of $w_1(t)$. An algorithm to verify whether $w_1(t)$ is of this form has been given in the previous section.

   In case $\mathbf{P}(t)$ satisfies the two constraints given above, then substitute
$$t = \frac{1+s\beta}{s}$$
   and $s^n\overline{\mathbf{P}}(s)$, where
$$\overline{\mathbf{P}}(s) = \mathbf{P}(t)$$
   is its polynomial parametrization. $\overline{\mathbf{P}}(s)$ is a rational function and after multiplying by s^n, the last term of $\overline{\mathbf{P}}(s)$, i.e. $w_1(s)$, turns into a constant. Moreover
$$\textit{degree of } s^n\overline{\mathbf{P}}(s) = \textit{degree of } \mathbf{P}(t) \leq \textit{degree of } \mathbf{Q}(t).$$

3.2 Examples

A unit circle has the following rational parametrization

$$\mathbf{Q}(t) = (x(t), y(t), w(t)) = (2t, 1-t^2, 1+t^2).$$

We know that $GCD(2t, 1-t^2, 1+t^2) = 1$ and $\mathbf{Q}(t)$ is a proper parametrization of the unit circle centered at the origin. The circle does not have a polynomial parametrization as

$$w(t) = 1 + t^2 = (1+it)(1-it), \quad \textit{where} \quad i = \sqrt{-1},$$

and it has two distinct roots rather than a single root as required by the second condition of theorem III. Similarly we can show that an ellipse cannot have a polynomial parametrization.

Consider the rectangular hyperbola

$$\mathbf{P}(t) = (x(t), y(t), w(t)) = (t^2, 1, t).$$

$\mathbf{P}(t)$ is a proper parametrization of a hyperbola and $GCD(t^2, 1, t) = 1$. In this case

$$\textit{degree of } x(t) > \textit{degree of } w(t)$$

and therefore, the first condition of theorem III does not hold. Thus, a hyperbola does not have a polynomial parametrization.

4 Interpretation of Constraints

In this section we show the equivalence between our procedure based on theorem III and the criterion in algebraic geometry, mentioned in theorem I and II. Without loss of generality we assume that all rational curves of the form

$$\mathbf{Q}(t) = (x(t), y(t), z(t), w(t)), \tag{3}$$

considered in this section are properly parametrized and $GCD(x(t), y(t), z(t), w(t)) = 1$.

Theorem IV: *Given a properly parametrized rational curve of the form (3). If the curve has one place at infinity then*

1. *degree of $w(t)$ = maximum (degree of $x(t)$, degree of $y(t)$, degree of $z(t)$, degree of $w(t)$).*
2. *$w(t)$ is a polynomial of the form $\alpha(t-\beta)^n$. That is, it has a single root of multiplicity n, the degree of $w(t)$.*

Proof: Let us assume that the curve has one place at infinity. We initially prove the second condition and use it for the proof of first condition. It is required that $w(t)$ has a single root. Let us assume that it has two distinct roots, say β_1 and β_2. $\mathbf{Q}(\beta_1)$ and $\mathbf{Q}(\beta_2)$ are the two corresponding points at infinity. There are two possibilities:

1. $\mathbf{Q}(\beta_1) \neq \mathbf{Q}(\beta_2)$. In this case we have two distinct points at infinity. The number of places at infinity is greater than or equal to the number of points at infinity. Thus, the curve has at least two places at infinity, which is contrary to our assumptions.
2. $\mathbf{Q}(\beta_1) = \mathbf{Q}(\beta_2)$. If $w(t)$ has some other root, say β_3, and $\mathbf{Q}(\beta_3) \neq \mathbf{Q}(beta_1)$, we are done (as shown above). From now onwards we assume that the curve has a unique point at infinity. Since the curve is properly parametrized and there are at least two distinct values of the parameter t, β_1 and β_2, which correspond to the point at infinity, the curve has at least two places at infinity, according to theorem I, and is therefore contrary to our assumptions.

To analyze the first condition of the theorem, we need to consider the homogeneous representation (2) of a curve:

$$\overline{\mathbf{Q}}(t,u) = (\overline{x}(t,u), \overline{y}(t,u), \overline{z}(t,u), \overline{w}(t,u))$$

where $\overline{x}(t,u)$, $\overline{y}(t,u)$, $\overline{z}(t,u)$ and $\overline{w}(t,u)$ are homogeneous polynomials of same degree and

$$GCD(\overline{x}(t,u), \overline{y}(t,u), \overline{z}(t,u), \overline{w}(t,u)) = 1.$$

Without loss of generality we may assume that

$$degree\ of\ x(t) > degree\ of\ w(t).$$

Since $w(t)$ is not a constant, it has a root, say $t = t_0$. We use u as a homogenizing variable and obtain an expression of the form $\overline{\mathbf{Q}}(t,u)$. Since

$$degree\ of\ \overline{x}(t,u) = degree\ of\ \overline{w}(t,u)$$

and

$$degree\ of\ x(t) > degree\ of\ w(t)$$
$$\Rightarrow u \mid \overline{w}(t,u), \qquad (i.e., u\ divides\ \overline{w}(t,u)).$$

Thus, $\overline{w}(t,u)$ has at least two roots

$$(t,u) = (t_0, 1)$$

and

$$(t,u) = (1,0).$$

In this case $\overline{\mathbf{Q}}(t_0, 1)$ and $\overline{\mathbf{Q}}(1, 0)$ are either two distinct points at infinity or they are two parameter values corresponding to the point at infinity. In either case the curve has two places at infinity. Q.E.D.

Main Theorem: *Given a properly parametrized rational curve, (3), the following three statements are equivalent:*

1. *The parametrization has the following properties:*
   (a) *degree of $w(t)$ = maximum (degree of $x(t)$, degree of $y(t)$, degree of $z(t)$, degree of $w(t)$).*
   (b) *$w(t)$ is a polynomial of the form $\alpha(t-\beta)^n$. That is, it has a single root of multiplicity n, the degree of $w(t)$.*
2. *The curve has a corresponding polynomial parametrization. The polynomial parametrization can be obtained after rational reparametrization.*
3. *The curve has one place at infinity.*

Proof: According to theorem III, (1) $\Rightarrow$ (2) and according to theorem II, (2) $\Rightarrow$ (3). The fact, (3) $\Rightarrow$ (1) follows from theorem IV.

Q.E.D.

4.1 Example

Let us again consider the rectangular hyperbola

$$\mathbf{P}(t) = (x(t), y(t), w(t)) = (t^2, 1, t).$$

$t = 0$ is a root of $w(t)$. After homogenizing we obtain

$$\overline{\mathbf{P}}(t, u) = (\overline{x}(t, u), \overline{y}(t, u), \overline{w}(t, u)) = (t^2, u^2, tu).$$

The curve has two points at infinity given by the corresponding parameter values. They are

$$\overline{\mathbf{P}}(0, 1) = (0, 1, 0)$$

and

$$\overline{\mathbf{P}}(1, 0) = (1, 0, 0).$$

Thus we have shown that the hyperbola has two points at infinity and therefore, cannot have a polynomial parametrization.

5 Conclusion

In this paper we presented a simple algorithm to decide if a given rational plane or space curve has a polynomial parametrization. If the polynomial parametrization exists, our algorithm computes that by a rational reparametrization of the curve. The criterion for determining whether a curve has a polynomial parametrization has been known in algebraic geometry and we showed it equivalent to our procedure. It is of great interest to decide whether a given rational parametric surface has a polynomial parametrization. We also need algorithms for computing the proper parametrization of improperly parametrized rational surfaces.

6 Acknowledgements

We are grateful to Ron Goldman and Ray Sarraga for productive discussions.

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