Representation Theorems for Symmetric and Intuitionistic Algebras by Classical Sets

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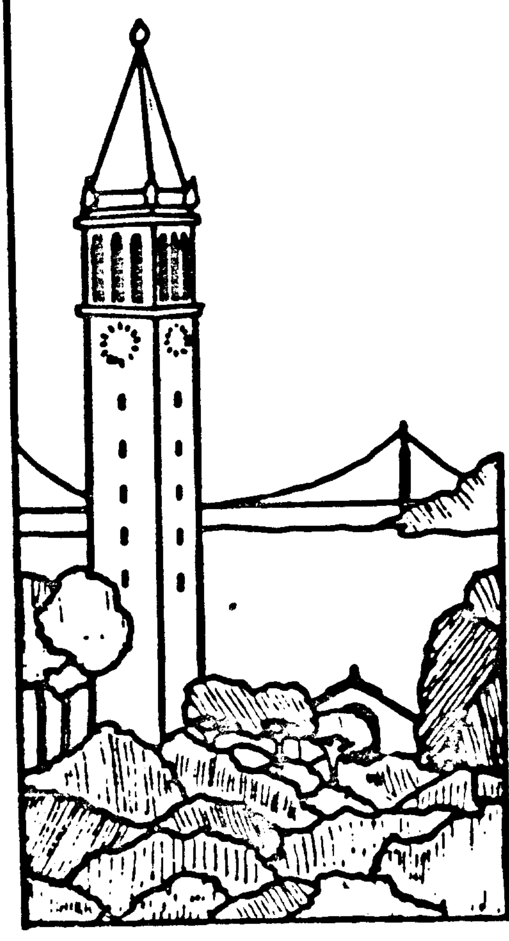

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REPRESENTATION THEOREMS FOR SYMMETRIC AND INTUITIONISTIC ALGEBRAS BY CLASSICAL SETS

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ABSTRACT

All Symmetric and Intuitionistic Algebras are represented by Symmetric or Intuitionistic Algebras of classical sets obtaining, as a corollary, the Bialynicki-Birula and Rasiowa's theorem for De Morgan Algebras.

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KEY WORDS: dual isomorphism, intuitionistic negation, involution, Symmetric Algebra, Intuitionistic Algebra and De Morgan Algebra.

Introduction

Representation theorems for distributive lattices by classical sets and by open sets of a topological space were given by Stone (see, for example [1] or [2]). On its own Bialinycky-Birula and Rasiowa studied the De Morgan Algebras (under the name of Quasi-Boolean Algebras) and gave a representation theorem for these Algebras (see [2]). In this paper three types of negations on distributive lattices are defined taking into account the properties that the connective "not" has in different Logics. This negations have been studied in several papers (see [3], [4], [5]) and define three different types of Algebras. The paper has two sections. The first one deals of negations on a distributive lattices and on a Boolean lattice of classical subsets of a set X. In the second one representation theorems for Symmetric and Intuitionistic Algebras by Algebras of classical sets are given.

We denote by A= (A, $\vee$, $\wedge$) a distributive lattice with universal bounds 0 and 1 and by (P(X), $\cap$,$\cup$,C) the Boolean Algebra of subsets of a set X.

1. Negations in lattices.

Definition 1. A decreasing mapping n: A $\rightarrow$ A is said to be :

- a dual automorphism if n satisfy the De Morgan laws,
- an intuitionistic negation if $n^2 \geq$ Id and n(1)= 0,
- an involution if $n^2 =$ Id.

Definition 2. A mapping c: A $\rightarrow$ A is said to be a closure operartor if:

1) x $\leq$ y imply c(x) $\leq$ c(y),
2) c(x) $\geq$ x for any x ϵ A,
3) c(c(x)) = c(x) for any x ϵ A.

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Definition 3. A set $S \subset A$ will be said to satisfy the minimum condition if for any $x \in A$ there exists $\wedge \{b \in S \mid b \geq x\}$ that belongs to S.

lemma 1. For any closure operator c of A, c(A) satisfy the minimum condition and $c(x) = \wedge \{b \in c(A) \mid b \geq x\}$ (1). Reciprocally, for any $S \subset A$ satisfying the minimum condition, there exists a unique closure operator c such that c(A) = S.

Proof.

a) For any $x \in c(A)$, c(x)=x because if x=c(x'), c(x)=c(c(x'))= c(x')=x.

b) For any $y \in A$, c(y) is given by (1) because $c(y) \in c(A)$ and for any $b \in c(A)$ such that $b \geq y$, $c(y) \leq c(b)=b$.

Reciprocally, if S satisfy the minimum condition, the mapping $c:A \rightarrow A$ defined by (1) is a closure operator as a simple computation show. The unicity of c is a simple consecuence of the first part of this lemma.

Proposition 1. If n is an intuitionistic negation on A, then :

(i) for any $S \subset A$ such that $\vee_{x \in A} x \in A$, $n(\vee_{x \in A} x) = \wedge_{x \in A} n(x)$,

(ii) for any $x, y \in A$, $n(x \wedge y) \geq n(x) \vee n(y)$,

(iii) n^2 is a closure operator on A such that $n^2(A) = n(A)$,

(iv) n(A) is a meet-subsemilattice of A such that satisfy the minimum condition and contains the minimum 0,

(v) the restriction of n to n(A) is an involution $\bar{n}$,

(vi) If S is a subset of A satisfying the minimum condition and containing 0 and if $\bar{n}$ is an involution on S, there exists a unique intuitionistic negation such that $n|_S = \bar{n}$ and $n(A) = \bar{n}(A)$. This negation n is defined by,

$$n(x) = \bar{n}(\wedge \{b \in n(A) \mid b \geq x\}) \qquad (*)$$

Proof.

(i) For every $x \in S$, $\wedge_{x \in S} x \leq x$, then $n(\wedge_{x \in S} x) \geq n(x)$ for every $x \in S$. So,

$$n(\wedge_{x \in S} x) \geq \vee_{x \in S} n(x) \qquad (1.1)$$

For every $x \in S$, $\vee_{x \in S} x \geq x$, then $n(\vee_{x \in S}) \leq n(x)$ for every $x \in S$. So

$$n(\vee_{x \in S} x) \leq \wedge_{x \in S} n(x) \qquad (1.2)$$

For every $x \in S$, $n^2(x) \geq x$, then $\vee_{x \in S} n^2(x) \geq \vee_{x \in S} x$ and by (1.1) we have,

$$\vee_{x \in S} x \leq \vee_{x \in S} n(n(x)) \leq n(\wedge_{x \in S}(n(x))) \qquad (1.3)$$

So applying n to (1.3),

$$n(\vee_{x \in S} x) \geq n^2(\wedge_{x \in S}(n(x))) \geq \wedge_{x \in S} n(x) \qquad (1.4)$$

The inequalities (1.2) and (1.4) prove (i).

(ii) The prove is the same then that given for to prove (1.1).

(iii) By definition of n, n^2 satisfy :

1) If $x \leq y$, $n(x) \geq n(y)$, and so $n^2(x) \leq n^2(y)$,

2) $n^2(x) \geq x$ by definition of n,

3) $n^2(n^2(x)) = n^2(x)$ because,

- n^2 (n(x)) $\geq$ n(x) which imply $n^4(x) \leq n^2(x)$

- $n^2(n^2(x)) \geq n^2(x)$, that is, $n^4(x) \geq n^2(x)$

Then $n^2(n^2(x)) = n^2(x)$

So, n^2 is a closure operator on A.

On the other hand $n(A) = n^2(A)$ because,

- if x= n(x') , $n^2(x) = n(n^2(x')) \leq n(x') = x$. Then n^2 (x)= x and so, x ϵn^2 (A).

- n^2 (A) $\subset$ n(A).

(vi) Given S $\subset$ A such that satisfy the minimum condition and contains 0 and given an involution $\bar{n}$ on S, the mapping n defined by (*) is a negation. On the other hand if n_1 and n_2 were a negations on A such that n_1 (A)= n_2 (A)= S and $n_1 |_S = n_2 |_S = \bar{n}$, $n_1 = n_2$ because if there exists x such that $n_1(x) \neq n_2(x)$, $n_1^2(x) = \bar{n}(n_1(x)) \neq \bar{n}(n_2(x)) = n_2^2(x)$ and then $n_1^2 \neq n_2^2$. So, n_1^2 and n_2^2 would be two different closures operator on A with the same image which is imposible (See lemma 1).

Proposition 2. A mapping n:A $\rightarrow$ A is an involution if, and only if, it is both a dual automorphism and an intuitionistic negation.

Proof.

If n is a dual automorphism and a intuitionistic negation, n(A) = A and by (iv) of proposition 1, n is an involution on n(A) = A.

If n is an involution, obviously n is an intuitionistic negation and n is bijective. So n(a $\vee$ b)= n(a) $\wedge$ n(b) by (i) of proposition 1 and n(a $\wedge$ b)= n(a) $\vee$ n(b) because if n is decreasing, bijective and involutive, there exists n^{-1} and it is also decreasing, bijective and involutive. Then $n^{-1}(n(a) \vee n(b)) = n^{-1}(n(a)) \wedge n^{-1}(n(b)) = a \wedge b$. So, n satisfy the De Morgan laws , that is, n is a dual automorphism.

The representation theorems require to characterize the possible negations that we can define on a Boolean lattice P(X) of classical subsets of a set X. The following propositions characterize these negations.

Definition 3. An automorphism H_s of P(X) is said to be generated by a permutation s of X if it is defined by [H_s (A)](x) = A(s^{-1} (x)) for any A ϵ P(X).

Proposition 3. Any dual automorphism n of P(X) is composition of the complementation C of the boolean algebra P(X) and an automorphism generated by a permutation s of X, that is, n = $H_s \circ$ C = C $\circ$ H_s .

Proof.

Given n, the mapping s:X $\rightarrow$ X defined by {s(x)} = C^{-1} (n({x})) is a permutation of X. So, taking account that n is univocally determined by the image of the atoms or singletons of P(X), a simple computation show that n = C $\circ$ H_s=H_s $\circ$ C.

Corollary 1. A dual automorphism n = $H_s \circ$ C is an involution if , and only if , s^2 = Id.

Proposition 4. An intuitionistic negation n on P(X) is univocally defined by a complete meet-subsemilattice S that contains X and $\emptyset$, and an involution on S.

The proof is omitted because is an easy consecuence of (i) and (iv) of proposition 1 taking account that P(X) is a complete lattice.

In other words, proposition 4 say that, given a meet-subsemilattice S that contains X and $\emptyset$, there is as much intuitionistic negations n on P(X) such that n(P(X)) == S as involutions $\bar{n}$ we can define on S.

2. Representation theorems.

Let A = (A, $\vee$,$\wedge$) be a distributive lattice with universal bounds 0 and 1.

Definition 4. An Algebra (A, $\vee$,$\wedge$,n) is said to be :

- A Symmetric Algebra if n is a dual automorphism ,
- An Intuitionistic Algebra if n is an intuitionistic negation,
- A De Morgan Algebra if n is an involution.

Theorem 1. Any Symmetric Algebra is representable as a subalgebra of a Symmetric Algebra of classical sets.

Proof.

Let X be the set of prime dual ideals of the lattice A. For any F ϵ X, we define s(F) = A - n(F) which also is an element of X. So, if we denote by n' the dual automorphism $H_s \circ C$, (P(X), $\cap$,$\cup$,n') is a Symmetric Algebra.

Let f:(A,$\wedge$,$\vee$,n) $\rightarrow$ (P(X),$\cap$,$\cup$,n') the mapping defined by f(a)={F ϵ X | aϵ F}.

This mapping is a monomorphism because :

1) f(a $\vee$ b)= f(a) $\cup$ f(b),

- if a $\vee$ b ϵ F, a ϵ F or b ϵ F because F is prime. So f(a $\vee$ b) $\subset$ f(a) $\cup$ f(b)
- if a ϵ F or b ϵ F, a $\vee$ b ϵ F. So, f(a) $\cup$ f(b) $\subset$ f(a $\vee$ b).

2) f(a $\wedge$ b) = f(a) $\cap$ f(b) because a $\wedge$ b ϵ F is equivalent to a ϵ F and b ϵ F.

3) f(n(a))= n'(f(a)).

n'(f(a))= (C $\circ$ H_s)({ F ϵ X | a ϵ F })= C ({X- n(F) | a ϵ F}

If F' = X-n(F) , F= n^{-1} (X-F') = X- n^{-1} (F') and n(F) = X - F' . Then,

n'(f(a))= C({F' ϵ X | a ϵ X- n^{-1} (F')}) = C ({F' ϵ X | n(a) ϵ X- F' }) , so

n'(f(A)) = {F' ϵ X | n(a) ϵ F'} = f(n(a)).

So, the algebra A is isomorphic to the subalgebra f(A) of P(X).

Corllary 2. (Bialinycki-Birula and Rasiova [2]) A De Morgan Algebra is representable as a subalgebra of a De Morgan Algebra of classical sets.

Theorem 2. An intuitionistic Algebra is representable as an intuitionistic Algebra of classical sets.

Proof.

As in theorem 1, let X be the set of prime dual ideals of the lattice A and f the mapping f:A $\rightarrow$ P(X) defined by f(A)= {F ϵ X| a ϵ F}. Theorem 1 prove that f is a monomorphism of the lattices.

If we define $\bar{n}$ on f(A) by $\bar{n}$ (P)= (f ∘ n ∘ f^{-1})(P) for every P ϵ f(A), $\bar{n}$ is an intuitionistic negation on f(A) and f ∘ n= $\bar{n}$ ∘ f.

So, (f(A)), $\bigcap, \bigcup, \bar{n}$) is an intuitionistic Algebra of classical sets isomorphic to (A, $\wedge, \vee$, n).

In general (f(A), $\bigcap, \bigcup, \bar{n}$) is not a subalgebra of some Intuitionistic Algebra (P(X), $\bigcap, \bigcup$, n'), that is, the existence of a intuitionistic negation n' on P(X) such that $\bar{n} = n' \mid_{f(A)}$ can not be secured.

Proposition 5. A suficient condition for (f(A), $\bigcap, \bigcup, \bar{n}$) to be a subalgebra of some Intuitionistic Algebra on P(X) is that f(n(A)) = $\bar{n}$ (f(A)) is a complete meet-subsemilattice containing X and $\emptyset$.

Examples.

1) Let M be the set M = $\{ p_1 \cdot p_2 \cdot \cdots p_k \mid p_i$ *is* ' and $p_i \neq p_j$ for *any* $i \neq j \}$

- Let $\wedge$, $\vee$ be the operations

- Let P the set of prime numbers and P' = P $\bigcup$ {0,1}.

P' is a complete inf-subsemilattice of M and n':P' → P' defined by n'(0)= 1, n'(1)= 0 and n'(p)= p for any p ϵ P is an involution on P'. So, by (vi) of proposition 1, there exists an intuitionistic negation n on M such that $n \mid_P$ = n' and n(M) =P'. This negation n is defined by:

$$n(x) = \begin{cases} 1 \text{ if } x = 0 \\ x \text{ if } x \epsilon P \\ 0 \quad otherwise \end{cases}$$

In this case f(n(M)) is obviously a complete inf-subsemilattice and proposition 5 can be applyed. So, the algebra (M, $\wedge, \vee$,n) can be represented by a subalgebra of an intuitionistic algebra on (P(X), $\bigcap, \bigcup$).

2) Let Y be an infinit set and let F(Y) = $[0,1]^Y$ be the set of Fuzzy Sets on Y taking values on [0,1]. If we define $\bigcap$ and $\bigcup$ by $(A \bigcap B)(x) = A(x) \wedge B(x)$ and $(A \bigcup B)(x) = A(x) \vee B(x)$,then (F(Y), $\bigcap, \bigcup$) is a complete and distributive lattice with maximum X and minimum $\emptyset$.

- We denote by σ_x , $\bar{\sigma}_x$, δ_α, σ_x^α and $\bar{\sigma}_x^\alpha$ the Fuzzy Sets defined by :

$\sigma_x(y)$=0 if y $\neq$ x and σ_x (x)= 1

$\bar{\sigma}_x$ = C(σ_x)

δ_α (x)= α for any x ϵ Y

$\sigma_x^\alpha = \sigma_x \wedge \delta_\alpha$ and $\bar{\sigma}_x^\alpha = \bar{\sigma}_x \vee \delta_\alpha$

Let n be the intuitionistic negation defined on F(Y) by the conditions $n \mid_{P(Y)}$=C and n(F(Y)) = P(Y). By (vi) of proposition 1 the negation n is defined by :

$$[n(A)](x) = \begin{cases} 0 \text{ if } A(x) \neq 0 \\ 1 \text{ if } A(x) = 0 \end{cases}$$

Let Z the set of prime dual ideals of F(Y).

lemma 2. The set Z contains the following prime ideals:

- $Z_1 = \{ [\sigma_x^\alpha) \mid x \epsilon X , \alpha \epsilon [0,1] \}$
- $Z_2 = \{ (\sigma_x^\alpha) \mid x \epsilon X , \alpha \epsilon [0,1] \}$
- Z' $= \{ F \epsilon Z \mid \bar{\sigma}_x \epsilon$ F for any $x \epsilon X \}$

Proof.

The proof is a consecuence of the following results :

(i) The join-irreducible elements of F(Y) are the elements of the type σ_x^α for any $x \epsilon X$ an any $\alpha \epsilon [0,1]$. So, Z_1 and Z_2 are prime dual ideals and there are the unique principal duals ideals.

(ii) Any prime dual ideal no principal has contain all the elements $\bar{\sigma}_x$ for any $x \epsilon X$ because for every $x \epsilon X$, $\sigma_x \notin F$ and $\sigma_x \vee \bar{\sigma}_x = X \epsilon F$. So $\bar{\sigma}_x \epsilon F$.

(iii) There exist a prime ideal F containig all elements $\bar{\sigma}_x$ for every $x \epsilon X$ because,

- I = { A ϵ F(Y)| { $x \epsilon X$ | A(x) = 1} is finite } is an ideal.
- D = { A ϵ F(Y)| { $x \epsilon X$ | A(x) $\neq$ 1} is finite} is a dual ideal disjoint with I

Then by Stone theorem (see [3]) there exists a prime dual ideal F containing D and disjoint with I.

Let f:F(Y) $\rightarrow$ P(Z) be the mapping defined by f(A) = $\{ F \epsilon Z \mid A \epsilon F \}$

Lemma 3. The sets f(F(Y)) and f(P(Y)) are no complete sublattices of P(Z).

Proof.

For any $x \epsilon X$, $\sigma_x = \bigcap_{y \neq x} \bar{\sigma}_x$ and ,

$$f(\sigma_x) = \{ [\sigma_x) \mid \alpha \epsilon [0,1] \} \bigcup \{ (\sigma_x) \mid \alpha \epsilon [0,1] \}$$

$$f(\bar{\sigma}_y) = \{ [\sigma_x) \mid \alpha \epsilon [0,1] , y \neq x \} \bigcup \{ (\sigma_x) \mid \alpha \epsilon [0,1], x \neq y \} \bigcup Z'$$

Then , $f(\sigma_x) \neq \bigcap_{x \neq y} f(\bar{\sigma}_y)$

So, f(F(Y)) and f(P(Y)) are sublattices of P(Z) because f is a monomorphism but they are not a complete sublattices of P(Z).

On the other hand n(F(Y)) = P(Y) and f(n(P(Y))) = f(P(Y)). Then in this case proposition 5 can not be applyed.

However the existence of no functions n' on P(Z) such that n' $|_{f(F(Y))} = f \circ n \circ f^{-1}$ can not be secured because the condition of proposition 5 is only suficient.

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