Representation Theorems for Symmetric and Intuitionistic Algebras by Classical Sets

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Report No. UCB/CSD 87/332

March 19, 1986

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ABSTRACT

All Symmetric and Intuitionistic Algebras are represented by Symmetric or Intuitionistic Algebras of classical sets obtaining, as a corollary, the Bialynicki-Birula and Rasiowa's theorem for De Morgan Algebras.

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KEY WORDS: dual isomorphism, intuitionistic negation, involution, Symmetric Algebra, Intuitionistic Algebra and De Morgan Algebra.

Introduction

Representation theorems for distributive lattices by classical sets and by open sets of a topological space were given by Stone (see, for example [1] or [2]). On its own Bialynicky-Birula and Rasiewa studied the De Morgan Algebras (under the name of Quasi-Boolean Algebras) and gave a representation theorem for these Algebras (see [2]). In this paper three types of negations on distributive lattices are defined taking into account the properties that the connective "not" has in different Logics. This negations have been studied in several papers (see [3], [4], [5]) and define three different types of Algebras. The paper has two sections. The first one deals of negations on a distributive lattices and on a Boolean lattice of classical subsets of a set X. In the second one representation theorems for Symmetric and Intuitionistic Algebras by Algebras of classical sets are given.

We denote by \( A = (A, \lor, \land) \) a distributive lattice with universal bounds 0 and 1 and by \( (P(X), \cap, \cup, \mathcal{C}) \) the Boolean Algebra of subsets of a set X.

1. Negations in lattices.

Definition 1. A decreasing mapping \( n: A \rightarrow A \) is said to be:
- a dual automorphism if \( n \) satisfy the De Morgan laws,
- an intuitionistic negation if \( n^2 \geq \text{Id} \) and \( n(1) = 0 \),
- an involution if \( n^2 = \text{Id} \).

Definition 2. A mapping \( c: A \rightarrow A \) is said to be a closure operator if:
1) \( x \leq y \implies c(x) \leq c(y) \),
2) \( c(x) \geq x \) for any \( x \in A \),
3) \( c(c(x)) = c(x) \) for any \( x \in A \).

* Visiting Scholar at the Computer Science Division at the University of California at Berkeley. Research partially supported by the Comissió Interdepartamental de Recerca i Innovación Tecnológica (C.I.R.I.T.) of the Generalitat de Catalunya, and by N S F Grants IST-8320416 and DCR-8513139
Definition 3. A set $S \subseteq A$ will be said to satisfy the minimum condition if for any $x \in A$ there exists $\bigwedge \{ b \in S \mid b \geq x \}$ that belongs to $S$.

Lemma 1. For any closure operator $c$ of $A$, $c(A)$ satisfy the minimum condition and

\[ c(x) = \bigwedge \{ b \in c(A) \mid b \geq x \} \]  

(1). Reciprocally, for any $S \subseteq A$ satisfying the minimum condition, there exists a unique closure operator $c$ such that $c(A) = S$.

Proof.

a) For any $x \in c(A)$, $c(x) = x$ because if $x = c(x')$, $c(x) = c(c(x')) = c(x') = x$.

b) For any $y \in A$, $c(y)$ is given by (1) because $c(y) \in c(A)$ and for any $b \in c(A)$ such that $b \geq y$, $c(y) \leq c(b) = b$.

Reciprocally, if $S$ satisfy the minimum condition, the mapping $c : A \to A$ defined by (1) is a closure operator as a simple computation show. The unicity of $c$ is a simple consequence of the first part of this lemma.

Proposition 1. If $n$ is an intuitionistic negation on $A$, then:

(i) For any $S \subseteq A$ such that $\forall x \in S$, $n(\forall x \in S \leq x) = \bigwedge_{x \in S} n(x)$,

(ii) For any $x, y \in A$, $n(x \land y) \geq n(x) \lor n(y)$,

(iii) $n^2$ is a closure operator on $A$ such that $n^2(A) = n(A)$,

(iv) $n(A)$ is a meet-subsemilattice of $A$ such that satisfy the minimum condition and contains the minimum 0,

(v) the restriction of $n$ to $n(A)$ is an involution $\bar{n}$,

(vi) If $S$ is a subset of $A$ satisfying the minimum condition and containing 0 and if $\bar{n}$ is an involution on $S$, there exists a unique intuitionistic negation such that $n|_S = \bar{n}$ and $n(A) = \bar{n}(A)$. This negation $n$ is defined by,

\[ n(x) = \bar{n}(\bigwedge \{ b \in n(A) \mid b \geq x \}) \] (*)

Proof.

(i) For every $x \in S$, $\bigwedge_{x \in S} x \leq x$, then $n(\bigwedge_{x \in S} x) \geq n(x)$ for every $x \in S$. So,

\[ n(\bigwedge_{x \in S} x) \geq \bigvee_{x \in S} n(x) \] (1.1)

For every $x \in S$, $\bigvee_{x \in S} x \geq x$, then $n(\bigvee_{x \in S} x) \leq n(x)$ for every $x \in S$. So

\[ n(\bigvee_{x \in S} x) \leq \bigwedge_{x \in S} n(x) \] (1.2)

For every $x \in S$, $n^2(x) \geq x$, then $\bigvee_{x \in S} n^2(x) \geq \bigvee_{x \in S} x$ and by (1.1) we have,

\[ \bigvee_{x \in S} x \leq \bigvee_{x \in S} n(n(x)) \leq n(\bigwedge_{x \in S} n(x)) \] (1.3)

So applying $n$ to (1.3),

\[ n(\bigvee_{x \in S} x) \geq n^2(\bigwedge_{x \in S} n(x)) \geq \bigwedge_{x \in S} n(x) \] (1.4)

The inequalities (1.2) and (1.4) prove (i).

(ii) The prove is the same then that given for to prove (1.1).

(iii) By definition of $n$, $n^2$ satisfy :

1) If $x \leq y$, $n(x) \geq n(y)$, and so $n^2(x) \leq n^2(y)$,
2) \( n^2(x) \geq x \) by definition of \( n \),
3) \( n^2(n^2(x)) = n^2(x) \) because,
   - \( n^2(n(x)) \geq n(x) \) which imply \( n^4(x) \leq n^2(x) \)
   - \( n^2(n^2(x)) \geq n^2(x) \), that is, \( n^4(x) \geq n^2(x) \)

Then \( n^2(n^2(x)) = n^2(x) \)

So, \( n^2 \) is a closure operator on \( A \).

On the other hand \( n(A) = n^2(A) \) because,
- if \( x = n(x') \), \( n^2(x) = n(n^2(x')) \leq n(x') = x \). Then \( n^2(x) = x \) and so, \( x \in n^2(A) \).
- \( n^2(A) \subseteq n(A) \).

(vi) Given \( S \subseteq A \) such that satisfy the minimum condition and contains 0 and given an involution \( \pi \) on \( S \), the mapping \( n \) defined by (*) is a negation. On the other hand if \( n_1 \) and \( n_2 \) were a negations on \( A \) such that \( n_1(A) = n_2(A) = S \) and \( n_1 \upharpoonright S = n_2 \upharpoonright S = \pi \), \( n_1 \neq n_2 \) because if there exists \( x \) such that \( n_1(x) \neq n_2(x) \),
   \( n_1^2(x) = \pi(n_1(x)) \neq \pi(n_2(x)) = n_2^2(x) \) and then \( n_1^2 \neq n_2^2 \). So, \( n_1^2 \) and \( n_2^2 \) would be two different closures operator on \( A \) with the same image which is imposible (See lemma 1).

**Proposition 2.** A mapping \( n:A \rightarrow A \) is an involution if, and only if, it is both a dual automorphism and an intuitionistic negation.

**Proof.**

If \( n \) is a dual automorphism and a intuitionistic negation, \( n(A) = A \) and by (iv) of proposition 1, \( n \) is an involution on \( n(A) = A \).

If \( n \) is an involution, obviously \( n \) is an intuitionistic negation and \( n \) is bijective. So \( n(a \lor b) = n(a) \lor n(b) \) by (i) of proposition 1 and \( n(a \land b) = n(a) \land n(b) \) because if \( n \) is decreasing, bijective and involutive, there exists \( n^{-1} \) and it is also decreasing, bijective and involutive. Then \( n^{-1}(n(a) \lor n(b)) = n^{-1}(n(a)) \lor n^{-1}(n(b)) = a \lor b \). So, \( n \) satisfy the De Morgan laws , that is, \( n \) is a dual automorphism.

The representation theorems require to characterize the possible negations that we can define on a Boolean lattice \( P(X) \) of classical subsets of a set \( X \). The following propositions characterize these negations.

**Definition 3.** An automorphism \( H_\ast \) of \( P(X) \) is said to be generated by a permutation \( s \) of \( X \) if it is defined by \( [ H_\ast (A)](x) = A( s^{-1} (x)) \) for any \( A \in P(X) \).

**Proposition 3.** Any dual automorphism \( n \) of \( P(X) \) is composition of the complementation \( C \) of the boolean algebra \( P(X) \) and an automorphism generated by a permutation \( s \) of \( X \), that is, \( n = H_\ast \circ C = C \circ H_\ast \).

**Proof.**

Given \( n \), the mapping \( s: X \rightarrow X \) defined by \( \{s(x)\} = C^{-1}(n(\{x\})) \) is a permutation of \( X \). So, taking account that \( n \) is univocally determined by the image of the atoms or singletons of \( P(X) \), a simple computation show that \( n = C \circ H_\ast = H_\ast \circ C \).

**Corollary 1.** A dual automorphism \( n = H_\ast \circ C \) is an involution if , and only if , \( s^2 = Id \).
Proposition 4. An intuitionistic negation n on P(X) is univocally defined by a complete meet-subsemilattice S that contains X and \( \mathcal{P} \), and an involution on S.

The proof is omitted because is an easy consequence of (i) and (iv) of proposition 1 taking account that P(X) is a complete lattice.

In other words, proposition 4 say that, given a meet-subsemilattice S that contains X and \( \mathcal{P} \), there is as much intuitionistic negations n on P(X) such that \( n(P(X)) = S \) as involutions \( \pi \) we can define on S.

2. Representation theorems.

Let \( A = (A, \lor, \land) \) be a distributive lattice with universal bounds 0 and 1.

Definition 4. An Algebra \( (A, \lor, \land, n) \) is said to be:

- A Symmetric Algebra if \( n \) is a dual automorphism,
- An Intuitionistic Algebra if \( n \) is an intuitionistic negation,
- A De Morgan Algebra if \( n \) is an involution.

Theorem 1. Any Symmetric Algebra is representable as a subalgebra of a Symmetric Algebra of classical sets.

Proof.

Let \( X \) be the set of prime dual ideals of the lattice A. For any \( F \in X \), we define \( s(F) = A - n(F) \) which also is an element of X. So, if we denote by \( n' \) the dual automorphism \( H_\ast \circ C, (P(X), \cap, \cup) \) is a Symmetric Algebra.

Let \( f(A, \lor, \land, n) \rightarrow (P(X), \cap, \cup, n') \) the mapping defined by \( f(a) = \{ F \in X | a \in F \} \). This mapping is a monomorphism because:

1) \( f(a \lor b) = f(a) \cup f(b) \),
- if \( a \lor b \in F \), \( a \in F \) or \( b \in F \) because F is prime. So \( f(a \lor b) \subset f(a) \cup f(b) \)
- if \( a \in F \) or \( b \in F \), \( a \lor b \in F \). So, \( f(a) \cup f(b) \subset f(a \lor b) \).

2) \( f(a \land b) = f(a) \cap f(b) \) because \( a \land b \in F \) is equivalent to \( a \in F \) and \( b \in F \).

3) \( f(n(a)) = n'(f(a)). \)

\( n'(f(a)) = (C \circ H_\ast)(\{ F \in X | a \in F \}) = C(\{ X - n(F) | a \in F \}) \)

If \( F' = X - n(F) \), \( F = n^{-1}(X) \) and \( n(F) = X' \). Then,

\( n'(f(a)) = C(\{ F' \in X | a \in X - n^{-1}(F') \}) = C(\{ F' \in X | n(a) \in F' \}) \), so

\( n'(f(A)) = \{ F' \in X | n(a) \in F' \} = f(n(a)). \)

So, the algebra A is isomorphic to the subalgebra \( f(A) \) of P(X).

Corollary 2. (Bialinycky-Birula and Rasiova [2]) A De Morgan Algebra is representable as a subalgebra of a De Morgan Algebra of classical sets.

Theorem 2. An intuitionistic Algebra is representable as an intuitionistic Algebra of classical sets.

Proof.

As in theorem 1, let \( X \) be the set of prime dual ideals of the lattice A and \( f \) the mapping \( f : A \rightarrow P(X) \) defined by \( f(A) = \{ F \in X | a \in F \} \). Theorem 1 prove that \( f \) is a monomorphism of the lattices.
If we define $\bar{\pi}$ on $f(A)$ by $\bar{\pi}(P) = (f \circ n \circ f^{-1})(P)$ for every $P \in f(A)$, $\bar{\pi}$ is an intuitionistic negation on $f(A)$ and $f \circ n = \bar{\pi} \circ f$.

So, $(f(A), \bigcap, \bigcup, \bar{\pi})$ is an intuitionistic Algebra of classical sets isomorphic to $(A, \land, \lor, n)$.

In general $(f(A), \bigcap, \bigcup, \bar{\pi})$ is not a subalgebra of some Intuitionistic Algebra $(P(X), \bigcap, \bigcup, n')$, that is, the existence of an intuitionistic negation $n'$ on $P(X)$ such that $\bar{\pi} = n' \mid f(A)$ cannot be secured.

**Proposition 5.** A sufficient condition for $(f(A), \bigcap, \bigcup, \bar{\pi})$ to be a subalgebra of some Intuitionistic Algebra on $P(X)$ is that $f(n(A)) = \bar{\pi}(f(A))$ is a complete meet-subsemilattice containing $X$ and $\mathcal{Q}$.

**Examples.**

1) Let $M$ be the set $M = \{ p_1, p_2, \ldots, p_k \mid p_i \text{ is ' and } p_i \neq p_j \text{ for any } i \neq j \}$

- Let $\land$, $\lor$ be the operations
- Let $P$ the set of prime numbers and $P' = P \cup \{0,1\}$.

$P'$ is a complete inf-subsemilattice of $M$ and $n': P' \to P'$ defined by $n'(0) = 1$, $n'(1) = 0$ and $n'(p) = p$ for any $p \in P$ is an involution on $P'$. So, by (vi) of proposition 1, there exists an intuitionistic negation $n$ on $M$ such that $n \mid P = n'$ and $n(M) = P'$. This negation $n$ is defined by:

$$n(z) = \begin{cases} 1 & \text{if } z = 0 \\ x & \text{if } x \in P \\ 0 & \text{otherwise} \end{cases}$$

In this case $f(n(M))$ is obviously a complete inf-subsemilattice and proposition 5 can be applied. So, the algebra $(M, \land, \lor, n)$ can be represented by a subalgebra of an intuitionistic algebra on $(P(X), \bigcap, \bigcup)$.

2) Let $Y$ be an infit set and let $F(Y) = [0,1]^Y$ be the set of Fuzzy Sets on $Y$ taking values on $[0,1]$. If we define $\bigcap$ and $\bigcup$ by $(A \bigcap B)(x) = A(x) \land B(x)$ and $(A \bigcup B)(x) = A(x) \lor B(x)$, then $(F(Y), \bigcap, \bigcup)$ is a complete and distributive lattice with maximum $X$ and minimum $\mathcal{Q}$.

- We denote by $\sigma_x$, $\vartheta_x$, $\delta_x$, $\sigma_x^*$ and $\vartheta_x^*$ the Fuzzy Sets defined by:

  $\sigma_x(y) = 0$ if $y \neq x$ and $\sigma_x(x) = 1$

  $\vartheta_x = C(\sigma_x)$

  $\delta_x(x) = \alpha$ for any $x \in Y$

  $\sigma_x^* = \sigma_x \land \delta_x$ and $\vartheta_x^* = \sigma_x \lor \delta_x$

Let $n$ be the intuitionistic negation defined on $F(Y)$ by the conditions $n \mid F(Y) = C$ and $n(F(Y)) = P(Y)$. By (vi) of proposition 1 the negation $n$ is defined by:

$$[n(A)](x) = \begin{cases} 0 & \text{if } A(x) \neq 0 \\ 1 & \text{if } A(x) = 0 \end{cases}$$

Let $Z$ the set of prime dual ideals of $F(Y)$. 
Lemma 2. The set \( Z \) contains the following prime ideals:

- \( Z_1 = \{ [\alpha x] \mid x \in X, \alpha \in [0,1] \} \)
- \( Z_2 = \{ (\sigma_x) \mid x \in X, \alpha \in [0,1] \} \)
- \( Z' = \{ F \in Z \mid \sigma_x \in F \text{ for any } x \in X \} \)

Proof.

The proof is a consequence of the following results:

(i) The join-irreducible elements of \( F(Y) \) are the elements of the type \( \sigma_x \) for any \( x \in X \), for any \( \alpha \in [0,1] \). So, \( Z_1 \) and \( Z_2 \) are prime dual ideals and there are the unique principal duals ideals.

(ii) Any prime ideal no principal has contain all the elements \( \sigma_x \) for any \( x \in X \) because for every \( x \in X \), \( \sigma_x \notin F \) and \( \sigma_x \lor \sigma_x = X \in F \). So \( \sigma_x \notin F \).

(iii) There exist a prime ideal \( F \) containing all elements \( \sigma_x \) for every \( x \in X \) because,

- \( I = \{ A \in F(Y) \mid \{ x \in X \mid A(x) = 1 \} \text{ is finite} \} \) is an ideal.
- \( D = \{ A \in F(Y) \mid \{ x \in X \mid A(x) \neq 1 \} \text{ is finite} \} \) is a dual ideal disjoint with \( I \).

Then by Stone theorem (see [3]) there exists a prime dual ideal \( F \) containing \( D \) and disjoint with \( I \).

Let \( f : F(Y) \to P(Z) \) be the mapping defined by \( f(A) = \{ F \in Z \mid A \in F \} \).

Lemma 3. The sets \( f(F(Y)) \) and \( f(P(Y)) \) are no complete sublattices of \( P(Z) \).

Proof.

For any \( x \in X \), \( \sigma_x = \bigcap_{y \neq x} \sigma_y \) and,

\[ f(\sigma_x) = \{ [\sigma_x] \mid \alpha \in [0,1] \} \cup \{ (\sigma_x) \mid \alpha \in [0,1] \} \]

\[ f(\sigma_y) = \{ [\sigma_x] \mid \alpha \in [0,1], y \neq x \} \cup \{ (\sigma_x) \mid \alpha \in [0,1], z \neq y \} \cup Z' \]

Then, \( f(\sigma_x) \neq \bigcap_{y \neq x} f(\sigma_y) \)

So, \( f(F(Y)) \) and \( f(P(Y)) \) are sublattices of \( P(Z) \) because \( f \) is a monomorphism but they are not a complete sublattices of \( P(Z) \).

On the other hand \( n(F(Y)) = P(Y) \) and \( f(n(P(Y))) = f(P(Y)) \). Then in this case proposition 5 can not be applied.

However the existence of no functions \( n' \) on \( P(Z) \) such that \( n' \mid f(P(Y)) = f \circ n \circ f^{-1} \) can not be secured because the condition of proposition 5 is only sufficient.

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