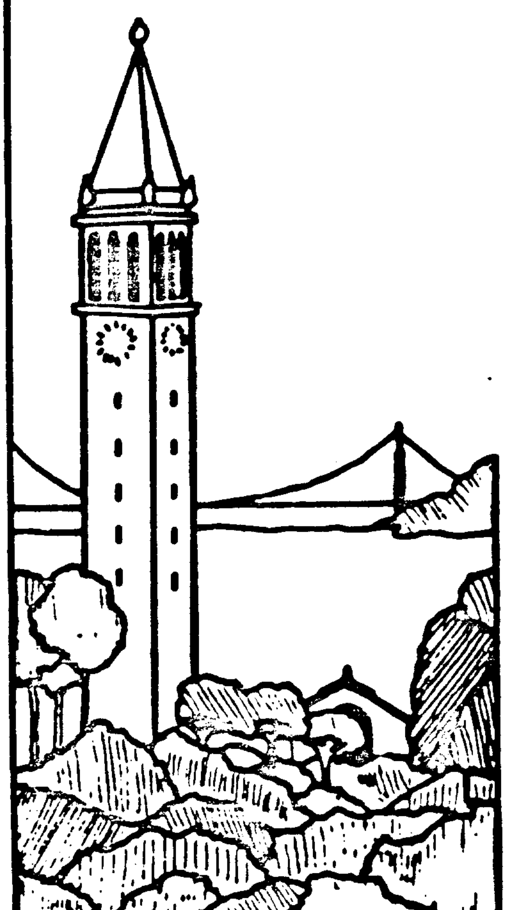
Representation of Fuzzy Symmetric Relations

L. Valverde and S.V. Ovchinnikov



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Computer Science Division (EECS)
University of California
Berkeley, California 94720

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L. Valverde

Dept. de Matemàtiques i Estadística
Universitat Politècnica de Catalunya
Avda. Diagonal, 649. 08028 Barcelona. Spain

S.V. Ovchinnikov

Mathematics Department
San Francisco State Univ.
1600 Halloway Ave., San Francisco, CA 94132

ABSTRACT

The representation theorem for F-transitive fuzzy relations is used to prove that the set of reflexive, symmetric and F-transitive fuzzy relations on a set X, F being an Archimedean t-norm, is dense in the set of Z-transitive relations on X. It is also shown that any similarity relation can be represented as a limit of a sequence of transitive relations with respect to Archimedean t-norms.

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*L. Valverde**

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Universitat Politècnica de Catalunya
Avda. Diagonal, 649. 08028 Barcelona. Spain

S.V. Ovchinnikov

Mathematics Department
San Francisco State Univ.
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1. Introduction and Preliminaries

There are several interesting issues related to the transitive property of fuzzy binary relations which have been adressed by some authors. Thus, for instance, both epistemological and technical requirements have induced to consider different types of transitivity [3,6], depending on the particular properties fulfilled by the operation used in the definition of that property. As it is well known, among the possible ways to define transitivity, the most used is the so-called **F-transitivity:** A fuzzy binary relation on a set X, i.e. a map R from XxX into [0,1], is called F-transitive if

$$F(R(u,v),R(v,w)) \leq R(u,w) \tag{1.1}$$

for all u,v,w in X, where F stands for a t-norm ([4,7]).

As it is well known, with any Zadeh's similarity relation (i.e. when F(a,b)=Min(a,b)) an ultrametric is associated, a fact which enables to deal with a distance on the set where the relation is defined. In order to refine the ultrametric triangular inequality Ruspini introduced likeness relations [3] for which the associated metric is the restriction of the Euclidian metric to the unit interval, and so forth. (see [7]).

On the other hand, one of the major problems related to the actual use of fuzzy transitive relations lies precisely in the dificulty to obtain transitive relations. The transitive closure method carries on a number of major problems, like the need of storage and computer-time required and, after all, no one is satisfied with the results it yields, because there is no way to control the distorsion that its application produces on the data sample. The representation theorem for fuzzy transitive relations ([2,7]) make possible to avoid most of the above mentioned problems, since it provides a way to generate fuzzy transitive relations in a more efficient way than the transitive closure method does. For

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instance, the use of this theorem no longer requires a reflexive and symmetric fuzzy relation as initial data; it requires both less storage and less computer-time and it produces less distorsion on the data sample. Moreover, as it is shown in this paper, this theorem is also an useful tool to deal with some other aspects of transitive fuzzy relations.

At this point, it should be noticed that through the transitive property a correspondence can be established between the set of t-norms and the set $\Lambda(X)$ of fuzzy binary relations on a set X, namely the correspondence which associates with any t-norm F the set $\Lambda_F(X)$ of F-transitive relations on X. As one of the well known features of this correspondence, we mention its order-reversing property, i.e. if $F \leq F'$, then $\Lambda_{F'}(X) \subseteq \Lambda_F(X)$. Then, since $F \leq Min$, for any t-norm F, it turns out that the set of similarity relations on a set X is included in $\Lambda_F(X)$, for any t-norm F.

On the other hand, if Z stands for the minimal t-norm, i.e.

$$Z(a,b) = \begin{cases} Min(a,b), & \text{if } Max(a,b)=1 \\ 0, & otherwise, \end{cases}$$

we have that $\Lambda_F(X) \subset \Lambda_Z(X)$ for any t-norm F. In fact, as it is easy to show, $\Lambda_Z(X)$ is the set symmetric fuzzy relations for which the crisp relation $u \equiv v$ if $R(u,v)=1$ is an equivalence relation, i.e. those fuzzy symmetric relations which are **strict reflexive** ($R(u,v)=1$ iff $u=v$) up to an equivalence relation on X.

From this standpoint, a question arises how sparse are the F-transitive relations in the set $\Lambda_Z(X)$. The above mentioned representation theorem allows us to prove that the set $\Lambda_A(X)$ of F-transitive relations, F being an Archimedean t-norm, is dense in $\Lambda_Z(X)$, i.e. any strict reflexive symmetric fuzzy relation is the point-wise limit of a sequence of relations in $\Lambda_A(X)$. To prove it, first we show that the representation theorems, which were proved only for the continuous t-norm case, also hold for the t-norm Z. The above result follows from the fact that the t-norm Z can be expressed as the limit of monotonic sequences of Archimedean t-norms, and the stability of the representation theorem. Moreover, since the t-norm Min may also be expressed as a limit of a sequence of Archimedean t-norms, it turns out that similarity relations can also be obtained as limits of relations in $\Lambda_A(X)$.

Throughout this paper the standard notations and conventions related to fuzzy binary relations and t-norms are used. However, it is convenient to remind that, given a t-norm F, the **quasi-inverse** of F is the function $F^{\wedge}$ from $[0,1]\times[0,1]$ into $[0,1]$ defined by

$$F^{\wedge}(a \mid b) = Sup\{\alpha\epsilon[0,1];\ F(\alpha,a) \leq b\}. \tag{1.2}$$

Let it be noticed that it follows, from boundary condition $F(a,0) = 0$, that the quasi-inverse of a t-norm F always exists whether it is continuous or not. Thus, for instance, the quasi-inverse of the t-norm Z is given by

$$Z^{\wedge}(a \mid b) = \begin{cases} b, & \text{if } a=1 \\ 1, & otherwise. \end{cases}$$

A comprehensive list of the properties of quasi-inverses can be found in [7]. Also in [7] the representation theorem is found in the following form:

Theorem 1.1. Let R be a map from XxX into [0,1] and let F be a continuous t-norm. Then R is a reflexive symmetric and F-transitive fuzzy relation on X (**F-indistinguishability relation**) if, and only if, there exists a family $\{h_j\}_{j\epsilon J}$ of fuzzy subsets of X, such that

$$R(u,v) = Inf_{j\epsilon J} F\hat{}(h_j(u) \vee h_j(v) \mid h_j(u) \wedge h_j(v)) \tag{1.3}$$

for all u,v in X.

Thus, if $\{h_j\}_{j\epsilon J}$ is a family of fuzzy subsets of a given set X, then

$$R(u,v) = Inf_{j\epsilon J} Min\hat{}(h_j(u) \vee h_j(v) \mid h_j(u) \wedge h_j(v)) = Inf_{j\epsilon J}(h_j(u) \longleftrightarrow h_j(v)),$$

where

$$a \longleftrightarrow b = \begin{cases} Min(a,b), & \text{if } a \neq b \\ 1, & otherwise; \end{cases}$$

is a similarity relation on X. Analogously,

$$R(u,v) = Inf_{j\epsilon J} \mid h_j(u) - h_j(v) \mid ,$$

is a likeness relation, i.e. R is F-transitive, F being the Lukasiewicz t-norm F(a,b) = Max(a+b-1,0).

As it will be shown in the following section, this theorem also holds for Z-transitive relations.

2. On Z-transitive relations.

In this section some features of Z-transitive relations are explored, specifically the validity of the representation theorem for such relations. In the proof of the above mentioned representation theorem the continuity of the t-norm plays an essential role. However due to the special features of the t-norm Z, it turns out that this theorem can be proven without using continuity, that is, we have

Theorem 2.1. Let R be a fuzzy binary relation on a non-empty set X. Then R is reflexive, symmetric and Z-transitive if, and only if, there exists a family $\{h_j\}_{j\epsilon J}$ of fuzzy subsets of X such that

$$R(u,v) = Inf_{j\epsilon J} Z\hat{}(h_j(u) \vee h_j(v) \mid h_j(u) \wedge h_j(v)) \tag{2.1}$$

Proof. The "only if" part works as in the continuous case, i.e. if R is symmetric and Z-transitive, then both

$$R(u,v) \leq Z\hat{}(R(u,w) \mid R(v,w))$$

and

$$R(u,v) \leq Z\hat{}(R(v,w) \mid R(u,w)),$$

hold for all w in X. Now, equality (2.1) follows from the reflexivity of R and by considering J=X and $h_w(u)=R(u,w)$, for all $u,w \epsilon X$.

In what concerns to the converse, the proof works as follows: First, it is clear that the relation R defined by (2.1) is reflexive and symmetric; thus, R will be Z-transitive if the crisp relation u ≡ v iff R(u,v)=1 is an equivalence relation. It is easy to check this fact taking into account that if R(u,v)=1 then, for any $j \in J$, either $h_j(u) \vee h_j(v) = 1$ or $h_j(u) = h_j(v) = 1$. ∎

In other words, given a family $\{h_j\}_{j \in J}$ of fuzzy subsets of a given set X, then

$$R(u,v) = \begin{cases} Inf_{j \in J_{uv}}(h_j(u) \wedge h_j(v)), & \text{if } J_{uv} \neq \phi, \\ 1, & otherwise, \end{cases}$$

is a Z-indistinguishability relation, where $J_{uv} = \{j \in J; h_j(u) \vee h_j(v) = 1\}$.

Let it be noticed that, when the functions h_j are the columns of a strict reflexive fuzzy relation S, then

$$Inf_{w \in X} Z^*(S(u,w) \vee S(v,w) \mid S(u,w) \wedge S(v,w)) = S(u,v) \wedge S(v,u),$$

i.e. the generated relation is exactly the greatest symmetric relation contained in S.

From theorem (2.1) it follows that, from the structural point of view, strict reflexive and symmetric fuzzy relations (i.e. Z-indistinguishability relations) may be treated as F-indistinguishability relations, F being a continuous t-norm. Thus, for instance, we can associate a generalized metric with any of such Z-indistinguishability relations and use it to describe, as it is shown in [7], the fuzzy cluster coverages given by the relation. But, in this case, the obtained generalized metric in [0,1] is a pseudo-metric, such that $m(a,b)=0$ for all $a,b \neq 1$. In other words, the only meaningful information given by the Z-relation is the one given by the equivalence relation associated with it.

From this standpoint, it makes sense to approximate Z-relations by means of F-indistinguishability relations, F being a continuous t-norm. To enforce that appreciation, it should be noticed that the t-norm Z is the point-wise limit of monotonic sequences of Archimedean t-norms, therefore if the representation theorem is stable under that kind of limits such approximation would be possible. This topic is adressed in the following section.

3. Stability of the representation theorems.

The first question which should be pointed out when dealing with point-wise limits of t-norms is related to the stability of associativity under that kind of limits. As it is proven in [5], even for monotonic sequences of t-norms, continuity is requiered to guarantee the associativity of the limit function. The first result we present in this section extends the above mentioned result of Thorp [5], i.e. associativity is stable under point-wise limits of sequences of continuous t-norms:

Theorem 3.1. Let $F(a,b) = \lim_{n \to \infty} F_n(a,b)$ for any a,b in [0,1]. If for any n, F_n is a **continuous t-norm,** then F is a **t-norm.**

Proof. a) From $F(a,b) = \lim_{n \to \infty} F_n(a,b)$, it follows that for any $\delta > 0$, there exists $n_1(\delta)$ such that, for any $n \geq n_1(\delta)$, it is

$$| F_n(a,b) - F(a,b) | < \delta.$$

b) From continuity of F_n, it follows that for any $\epsilon>0$ there exists $\delta>0$ ($\delta(a,y,n)$) such that, for any b in [0,1] with $| b-y | <\delta$ it is

$$| F_n(a,y) - F_n(a,b) | < \frac{\epsilon}{4}.$$

c) From (a) and (b) it follows that there exists n_1(a,b,c,n) such that

$$| F_n(a,F_m(b,c)) - F_n(a,F(b,c)) | < \frac{\epsilon}{4}.$$

d) From $F(a,F(b,c)) = \lim_{n\to\infty} F_n(a,F(b,c))$ it follows that for any $\epsilon>0$ there exists n_2(a,b,c) such that

$$| F_n(a,F(b,c)) - F(a,F(b,c)) | < \frac{\epsilon}{4}.$$

e) So, from (c) and (d) it follows that for any $\epsilon>0$ it is

$$| F_n(a,F_m(b,c)) - F(a,F(b,c)) | < \frac{\epsilon}{2}.$$

for any $n \geq n_2(a,b,c)$ and for any $m \geq n_1(a,b,c,n)$.

Consequently, for any $\epsilon>0$ there exists n_0 such that

$$| F_p(a,F_p(b,c)) - F(a,F(b,c)) | < \frac{\epsilon}{2}.$$

for any $p \geq n_0$ $(=Max(n_1,n_2))$.

f) Similarly, it can be proven that, for any $\epsilon>0$, there exists m_0 such that

$$| F_m(F_m(a,b),c) - F(F(a,b),c) | < \frac{\epsilon}{2},$$

for any $m \geq m_0$.

g) Now, from (e) , (f) and associativity of F_n it follows that

$$| F(a,F(b,c) - F(F(a,b),c) | < \epsilon,$$

for any $\epsilon>0$, i.e.

$$F(a,F(b,c)) = F(F(a,b),c).$$

In addition, as it is shown in [5], when $\{F_n\}_{n\in N}$ is an increasing sequence, the the limit is a left-continuous t-norm. It is easy to check that the limit of a decreasing sequence of continuous t-norms is a right-continuous t-norm. ∎

Next step is to check if the quasi-inverse of a t-norm which is the limit of a sequence of continuous t-norms is the limit of the sequence of quasi-inverses. First of all, let it be noticed that this property holds for increasing sequences of continuous t-norms:

Theorem 3.2. Let $\{F_n\}_{n\in N}$ be a **increasing** sequence of continuous t-norms; and let be $F(a,b) = Sup_{n\in N} F_n(a,b)$. Then $\{F_n^\wedge\}_{n\in N}$ is a **decreasing** sequence such that $F^\wedge(a \mid b) = Inf_{n\in N} F_n^\wedge(a \mid b)$.

Proof. It is easy to check, $\{F_n^{\wedge}\}_{n\in N}$ is a decreasing sequence. Since $F_n \leq F$, for any a,b in [0,1] it is $F^{\wedge}(a \mid b) \leq F_n^{\wedge}(a \mid b)$ for any $n \in \mathbf{N}$, i.e. $F^{\wedge}(a \mid b) \leq Inf_{n\in N} F_n(a \mid b)$. Now, suppose that $\alpha \leq Inf_{n\in N} F_n^{\wedge}(a \mid b)$, then $F_n(a,\alpha) \leq b$ for any $n \in \mathbf{N}$, i.e. $F(a,\alpha) = Sup_{n\in N} F_n(a,\alpha) \leq b$, that is $\alpha \leq F^{\wedge}(a \mid b)$. ∎

This is the case, for instance, of the sequence of Archimedean t-norms given by

$$F_n(a,b) = f_n^{[-1]}(f_n(a)+f_n(b)),$$

f_1 being a continuous and strictly decreasing function from [0,1] into $\mathbf{R}^+$ with $f_1(1) = 0$; and

$$f_n(a) = (f(a))^n.$$

However, the above mentioned property does not hold, in general, for arbitrary sequences of t-norms. In fact, it may fail even for decreasing sequences of continuous t-norms, as it is shown in the following example:

Example. For a given $\alpha \epsilon (0,1)$, let f be a continuous and strictly decreasing function from $[\alpha,1]$ into $\mathbf{R}^+$ with f(1)=0 and, for any $n \in \mathbf{N}$, let be

$$f_n(a) = (f(a))^{\frac{1}{n}}$$

Now, consider the sequence of continuous t-norms defined by

$$F_n(a,b) = \begin{cases} f_n^{[-1]}(f_n(a)+f_n(b)), & \text{if } a,b \epsilon [\alpha,1], \\ Min(a,b), & otherwise, \end{cases}$$

where

$$f_n^{[-1]}(a) = \begin{cases} f_n^{-1}(a), & \text{if } a \epsilon [0,f(\alpha)], \\ \alpha, & otherwise. \end{cases}$$

In that case,

$$F(a,b) = Inf_{n\in N} F_n(a,b) = \begin{cases} \alpha, & \text{if } a,b \epsilon [\alpha,1), \\ Min(a,b), & otherwise \end{cases}.$$

Then, it is easy to check that, for any $n \in \mathbf{N}$ and for any $a_0 > \alpha$, it is $F_n^{\wedge}(a_0 \mid \alpha) = \alpha$ and $F^{\wedge}(a_0 \mid \alpha) = 1$.

However, as it is easy to prove, if $\alpha = 0$, then $\{F_n\}_{n\in N}$ is a decreasing sequence of Archimedean t-norms whose limit is the t-norm Z, for which $\{F_n^{\wedge}\}_{n\in N}$ is an increasing sequence with $Z^{\wedge}$ as its limit.

At this point we can prove the stability of the representation theorems for monotonic sequences of continuous t-norms for which

$$\lim_{n\to\infty} F_n^{\wedge} = F^{\wedge},$$

being $F = \lim_{n\to\infty} F_n$. To this end, let $\{h_j\}_{j\in J}$ be a family of fuzzy subsets of a non-empty set X; given a monotonic sequence of continuous t-norms $\{F_n\}_{n\in N}$, let $\{R_n\}_{n\in N}$ be the sequence of F_n -indistinguishability relations on X generated by the family $\{h_j\}_{j\in J}$, i.e.

$$R_n(u,v) = Inf_{j\epsilon J} F_n\hat{}(\alpha_j \mid \beta_j),$$

where

$$\alpha_j = h_j(u) \vee h_j(v), \text{ and } \beta_j = h_j(u) \wedge h_j(v),$$

then

Theorem 3.3. $\{R_n\}_{n\epsilon N}$ is a monotonic sequence of indistinguishability relations whose limit is the F-indistinguishability relation on X generated by the family $\{h_j\}_{j\epsilon J}$.

Proof. First, assume that $\{F_n\}_{n\epsilon N}$ is a decreasing sequence of continuous t-norms, then $\{R_n\}_{n\epsilon N}$ is an increasing sequence of F-transitive relations, where $F = Inf_{n\epsilon N} F_n$. Now, let be

$$S(u,v) = Sup_{n\epsilon N} R_n(u,v),$$

then S is a F-indistinguishability relation on X, because

$$F(S(u,v),S(v,w)) \leq F(Sup_n R_n(u,v), Sup_m R_m(v,w))$$

$$\leq F_p(Sup_n R_n(u,v), Sup_m R_m(v,w))$$

$$= Sup_{n,m} F_p(R_n(u,v), R_m(v,w)) \leq Sup_{n,m} F_p(R_{Max(n,m)}(u,v), R_{Max(n,m)}(v,w))$$

$$\leq_{(p \geq Max(n,m))} Sup_{n,m} R_{Max(n,m)}(u,w) = S(u,w).$$

Now, let R be the F-indistinguishability relation on X generated by the family $\{h_j\}_{j\epsilon J}$. Since $F_n\hat{} \leq F\hat{}$, it is clear that $R \geq R_n$ for any $n \epsilon N$, i.e $R \geq S$. Suppose that

$$\alpha < R(u,v) = Inf_{j\epsilon J} F\hat{}(\alpha_j \mid \beta_j),$$

then for any $j \epsilon J$,

$$\alpha < F\hat{}(\alpha_j \mid \beta_j) = Sup_{n\epsilon N} F_n\hat{}(\alpha_j \mid \beta_j),$$

therefore there exists n_0 such that

$$\alpha < F_n\hat{}(\alpha_j \mid \beta_j),$$

for any $n \geq n_0$ and for any $j \epsilon J$, i.e.

$$\alpha \leq Inf_{j\epsilon J} F_n\hat{}(\alpha_j \mid \beta_j) = R_n(u,v)$$

for any $n \geq n_0$, thus $\alpha \leq S(u,v) = Sup_{n\epsilon N} R_n(u,v)$, i.e. $S = R$.

Similar arguments can be used in the case of being $\{F_n\}_{n\epsilon N}$ an increasing sequence of continuous t-norms. In that case, $S(u,v) = Inf_{n\epsilon N} R_n(u,v)$, is the F-indistinguishability relation generated by the given family of fuzzy subsets of X, where $F(a,b) = Sup_{n\epsilon N} F_n(a,b)$. ∎

It is worth noting that, from the above results, it follows that if $Z = Inf_{n\epsilon N} F_n$, $Z\hat{} = Sup_{n\epsilon N} F_n\hat{}$ and R is a Z-indistinguishability relation on a set X, then

$$R_n(u,v) = Inf_{w\epsilon X} F_n\hat{}(R(u,w) \vee R(v,w) \mid R(u,w) \wedge R(v,w)), \tag{3.1}$$

is an increasing sequence of indistinguishability relations whose limit is the relation R, that is, the following theorem holds:

Theorem 3.4. For any X, the set $\Lambda_A(X)$ is dense in $\Lambda_Z(X)$.

If X is a finite set or so is the set $\{\alpha\epsilon[0,1];\ \alpha=R(u,v)$ for some $u,v\epsilon X\}$, it is possible, for any $\epsilon>0$ to determine n_0 such that

$$|\ R_n(u,v) - R(u,v)\ | < \epsilon,$$

for all $n \geq n_0$.

Finally, let it be noticed that, if $\{F_n\}_{n\epsilon N}$ is an increasing sequence of Archimedean t-norms such that $Sup_{n\epsilon N} F_n(u,v) = Min(u,v)$ and R is a F_1 -indistinguishability relation, then (3.1) defines a sequence of indistinguishability relations whose limit is a similarity relation.

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