Solving the Weighted Parity Problem for Gammoids by Reduction to Graphic Matching

Po Tong
E. L. Lawler
*V. V. Vazirani***

Computer Science Division
and Electronic Research Laboratory
University of California
Berkeley, CA 94720

ABSTRACT

It is shown that the weighted parity problem for gammoids, with matching and transversal matroids as special cases, can be reduced to the weighted graphic matching problem. Since the cycle matroid of a series parallel graph is a gammoid, this means that it is possible to solve the weighted parity problem for the cycle matroid of a series parallel graph by graphic matching.

Keywords and phrases: matching matroid, transversal matroid, gammoid, matroid parity problem, graphic matching, polynomial algorithm.

This research was supported by NSF Grants MCS78-20054 and MCS79-03767 administered through the Electronics Research Laboratory, the University of California at Berkeley.
**Also supported by IBM Graduate Fellowship.

1. INTRODUCTION

We assume the reader is familiar with basic matroid definitions and terminology, including the notions of transversal matroids, matching matroids, and graphic matroids. For reference, see Aigner [1979], Lawler [1976], Welsh [1976].

Let $M = (S,\mathcal{I})$ be a matroid whose ground set S is partitioned into two-element subsets called pairs. A *parity set* $P \subseteq S$ is a union of pairs. The ***matroid parity problem*** (also known as the matroid matching problem (Lovász [1978]) or the matchoid problem (Jenkyns [1974])) asks for an independent parity set of maximum size. In the weighted version of the problem, the pairs have specified weights and a maximum-weight independent parity set is sought. The matroid parity problem is a common generalization of the graphic matching problem and the matroid intersection problem (cf. Lawler [1976]).

Lovász has provided a polynomial-time algorithm for the unweighted matroid parity problem in the case that the matroid is linearly representable and, moreover, an explicit linear representation of the matroid is given. However, Lovász's algorithm is rather complicated and admits of no known generalization to the weighted case. The purpose of this paper is to show that some special cases of the weighted matroid parity problem can be solved by reducing them to graphic matching problems. In particular, we shall exhibit such reductions for matching and transversal matroids and for gammoids. The cycle matroid of a series parallel graph is a gammoid. This means that the weighted version of the "spanning tree parity problem" (cf. Garey and Johnson [1979]) in the special case of series parallel graphs can be solved as a weighted matching problem. It appears that the cases dealt with in this paper are the only ones (aside from the weighted matching problem itself) for which polynomial-bounded algorithms have been found for the weighted matroid parity problem.

2. MATCHING MATROIDS

Let $G = (V,E)$ be a graph and $S \subseteq V$ be a given subset of its vertices. Then $M = (S,\mathcal{I})$ is a matroid, where $I \subseteq S$ is in if and only if there exists a matching in G covering all the vertices in I. Any matroid induced in this way is a ***matching matroid*** (Edmonds and Fulkerson [1965]).

Suppose the vertices in S are partitioned into pairs and that a positive weight $w(s,s')$ is specified for each pair $\{s,s'\} \subseteq S$. The objective of the parity problem for the matching matroid is to find a matching in G that covers a subset of pairs of maximum weight. The parity problem can be reduced to a weighted matching problem as follows.

First assume, without loss of generality, that G contains no edges incident to two vertices in $V - S$. (If there are such edges, delete them from the graph, without affecting the matching matroid that is induced.) Next add to G the set of edges

$$E' = \{(s,s') \mid \{s,s'\} \textit{ a pair}; \; s,s' \textit{ nonadjacent}\}$$

thereby obtaining the graph $G' = (V, E \cup E')$. Assign weights to the edges of G' as follows:

(*i*) Give weight $w(s,s')$ to an edge $(s,s') \in E'$.

(*ii*) Give weight $w(s_1,s'_1) + w(s_2,s'_2)$ to an edge $(s_1,s_2) \in E$, where $s_1,s_2 \in S$ and $\{s_1,s'_1\},\{s_2,s'_2\}$ are the unique pairs containing s_1,s_2. (Thus if $\{s_1,s_2\}$ is a pair, edge (s_1,s_2) receives weight $2w(s_1,s_2)$.)

(*iii*) Give weight $w(s,s')$ to an edge $(s,t) \in E$, where $s \in S, t \in V-S$ and $\{s,s'\}$ is the unique pair containing s.

As a simple example, consider the graph G indicated in the upper part of Figure 1. $S = \{s_1 s'_1, \ldots, s_4, s'_4\}$, and there are four pairs $\{s_i, s'_i\}$ with $w(s_i, s'_i) = i$, for $i = 1, \ldots, 4$. The graph G' and its edge weights are as shown in the lower part of the figure. Edges in E' are indicated by dashed lines.

For each pair $\{s,s'\} \subseteq S$ there is an edge (s,s') in G' with positive weight. It follows that any maximum-weight matching in G' covers at least one vertex of each pair. Moreover, it is not difficult to verify that the total weight of a maximum-weight matching is equal to the sum of the weights of all pairs (a constant) plus the weight of all pairs of vertices covered by edges of the matching in E. Thus a maximum-weight matching N in G' yields a matching $N \cap E$ in G that covers a maximum-weight subset of pairs, and we have the following.

Theorem 1 Let N be any maximum-weight matching in $G' = (V, E \cup E')$. Then the set of pairs of S covered by $N \cap E$ is a maximum-weight independent parity set for the matching matroid induced by $G = (V,E)$, with $S \subseteq V$.

In the example shown in Figure 1, a maximum-weight matching indicated with bold lines) contains edges $(s_1, s'_1), (s_2, s'_2), (s'_3, s_4)$ and (s'_4, t_2), which have total weight 15. The pairs covered by edges in E are $\{s_1, s'_1\}$ and $\{s_4, s'_4\}$ with total weight 5. Note that 15 (total weight of matching) = 10 (total weight of all pairs) + 5 (weight of covered pairs).

The weighted matching problem for $G' = (V, E \cup E')$ can be solved in $0(|V|^3)$ time (Lawler [1976]), and this establishes an immediate upper bound on the time required to solve the parity problem for the matching matroid induced on $G = (V,E)$. However, we can reduce this upper bound to $0(|V| + |E| + |S|^3)$, by making use of the following theorem, the bipartite version of which is well known (cf. Aigner [1979], p. 378.)

Theorem 2 Let N be a matching in $G = (V,E)$ that covers a maximum number of the vertices in $S \subseteq V$. Then the graph obtained by deleting vertices in $V-S$ not covered by N induces the same matching matroid as G.

Proof: Exactly the same as that for Proposition 7.66 in Aigner [1979].

Assuming that the graph G is given in edge-list form, with random access to the edge list of any given vertex, a matching N covering a maximum number of vertices in S can be constructed in $0(|S|^3)$ time: At most $|S|$ augmentations are necessary. For each augmentation, apply a labeling procedure (Lawler [1976]) to find an augmenting path from an exposed vertex in S. At most $2|S|$ vertices need to be labeled, and at most $|S|^2$ edges must be inspected for each augmentation. Thus the desired matching can be found in $0(|S|^3)$ time. Deletion of exposed vertices in G requires at most $0(|V| + |E|)$ time.

After deletion of vertices, we have $|V| \leq 2|S|$. Thus solution of the weighted matching problem on G' can now be accomplished in $0(|S|^3)$ time.

3. TRANSVERSAL MATROIDS

If the graph $G = (V,E)$ defining a matching matroid on $S \subseteq V$ is bipartite, i.e. each edge extends between a vertex in S and a vertex in $V-S$, then the matching matroid is said to be ***transversal*** (Edmonds and Fulkerson [1965]), given a graph $G = (V,E)$ and $S \subseteq V$, one can construct a bipartite graph that induces the same matching matroid as G. Thus every matching matroid has a transversal representation.

The reduction of the parity problem to weighted matching is somewhat simpler when the graph G is bipartite. The graph G' (which is tripartite) is such that the only edges (s,s') extending between vertices s,s' in S are edges in E' with weight $w(s,s')$. This means that if all pairs $\{s,s'\} \subseteq S$ have equal weight, then all edges of G' have equal weight as well. In other words, the unweighted parity problem for transversal matroids reduces to the unweighted matching problem. Assuming $|V| \leq 2|S|$, this latter problem can be solved in $O(|S|^{\frac{1}{2}}|E|)$ time (Micali and Vazirani [1980]) instead of $O(|S|^3)$ time.

We should also note the special case in which the parity problem is actually an intersection problem, with transversal matroids specified by two bipartite graphs $G_1 = (S, T_1, E_1)$ and $G_2 = (S, T_2, E_2)$. In this case the graph G' is formed by adding edges between corresponding vertices of S in G_1 and G_2. Thus G' is itself bipartite and the matching problem is simpler to solve. This reduction to a bipartite matching problem is essentially the same as Fulkerson's [1971] reduction of the common transversal problem to a network flow problem.

4. GAMMOIDS

Gammoids are ordinarily defined in terms of linkings in directed graphs. For our purposes, a *strict gammoid* is simply the dual of a transversal matroid and a *gammoid* is a restriction of a strict gammoid. Gammoids are closed under duality and taking of minors, but transversal matroids are not. Thus every transversal matroid is a gammoid, but not conversely. (See Aigner [1979] and Welsh [1976] for background on gammoids and for a discussion of the Ingleton-Piff theorem which justifies our definition.)

Let us consider the dual of a transversal matroid, i.e. a strict gammoid. (We could deal with matching matroids defined by nonbipartite graphs, but for simplicity choose not to.) Suppose the transversal matroid is defined by a bipartite graph $\overline{G} = (\overline{S}, \overline{T}, \overline{E})$. Without loss of generality, we may assume that each vertex of $\overline{T}$ is covered by a maximum matching. (Recall the discussion in Section 2.) A subset $I \subseteq \overline{S}$ is independent in the dual of the transversal matroid if and only if there is a maximum matching in $\overline{G}$ that leaves the vertices in I exposed.

In order to obtain a representation for the dual that is more similar to what we are accustomed, we create a second copy of the vertex set $\overline{S}$, call this second copy S, and provide a set of edges E between corresponding vertices $s \in S$ and $\overline{s} \in \overline{S}$. The result is a bipartite graph $G = (S \cup \overline{T}, \overline{S}, \overline{E} \cup E)$, as shown in Figure 2. A set $I \subseteq S$ is independent in the dual of the transversal matroid induced by $\overline{G}$ if and only if there is a matching in G that covers all the vertices in I and in $\overline{T}$. A restriction of this dual matroid is obtained by simply deleting vertices, as appropriate, from S.

Hereafter a gammoid will be represented by a bipartite graph $G = (S', T, E)$ and a subset $S \subseteq S'$ such that there is a matching in G covering all vertices in $S' - S$. A subset $I \subseteq S$ is independent in the gammoid induced by G if and only if there exists a matching covering all the vertices in I *and* all the vertices in $S' - S$. If $S = S'$, then the gammoid is transversal. Note that there is a polynomial time algorithm to transform the representation of a gammoid in terms of linkings to our chosen representation (see Aigner [1979]).

Note also that the matroid induced by $G = (S', T, E)$, with $S \subseteq S'$, may be viewed as the contraction to S of the transversal matroid on S' induced by G. As is well known, a matroid is a gammoid if and only if it is the contraction of a transversal matroid.

In order to reduce the parity problem for gammoids to the weighted matching problem, a simple generalization of our previous reduction suffices: In forming the graph G', all edges incident to vertices in S are assigned weights as before. However, each edge incident to a vertex in $S' - S$ is assigned weight W, where W is a large number. (W is chosen to be sufficiently large to insure that a maximum-weight matching in G' covers all vertices in $S'-S$.) By the same reasoning as that in Section 2, we have:

Theorem 3 Let N be any maximum-weight matching in $G' = (S', T, E \cup E')$. Then the set of pairs of S covered by $N \cap E$ is a maximum-weight independent parity set for the gammoid induced by $G = (S', T, E)$, with $S \subseteq S'$.

5. SERIES AND PARALLEL EXTENSIONS

Let $M = (S, \mathcal{I})$ be a matroid and $e \in S$. The *series extension of M at e by $e' \notin S$* is the matroid on $S \cup e'$ which has as its bases all sets of the form $B \cup e'$, where B is a base of M, and $B \cup e$, where B is a base of M, with $e \notin B$. The *parallel extension of M at e by $e' \notin S$* is the matroid on $S \cup e'$ which has as its bases all bases of B and all sets of the form $(B - e) \cup e'$, where B is a base of M, with $e \in B$. A *series parallel extension* of M is any matroid derived from M by a finite sequence of series and parallel extensions.

It is well known that any series or parallel extension of a gammoid is a gammoid and any series extension of a transversal matroid is transversal (but a parallel extension is not necessarily transversal). Here we give explicit constructions demonstrating these results, in terms of our chosen representation of a gammoid.

Let $G = (S', T, E)$, with $S \subseteq S'$, represent a gammoid. Let us construct the series extension by e' at e. To do this, we simply add a vertex e' to S (and S') and a vertex t to T with edges (e', t) and (e, t), as shown in Figure 3. If B is a base of the gammoid, now $B \cup e'$ becomes a base, by adding edge (e', t) to the matching defining B. And if B, with $e \notin B$, is a base of the gammoid, now $B \cup e$ is a base, by adding edge (e, t). Note that if $S = S'$, so that the gammoid is transversal, the series extension is transversal as well.

The construction for the parallel extension by e' at e is slightly more complicated. The existing vertex e is renamed s' and becomes a member of $S' - S$. Add two new vertices e and e' to S and a vertex t to T, with edges $(e', t), (e, t)$, and (s', t), as shown in Figure 4. Note that no matching can now cover both e and e'. If B is a base of the gammoid, with $e \notin B$, the edge (s', t) must be added to the matching defining B in order to obtain a valid matching (covering all vertices in $S' - S$) in the graph for the parallel extension. If B is a base of the gammoid, with $e \in B$, then the matching defining B covers s' (since this was e) and either edge (e, t) or (e', t) can be added to the matching, making B and $(B - e) \cup e'$ bases in the parallel extension, as required. Note that if the gammoid is transversal, the parallel extension has not been given a transversal representation ($S' - S \neq \phi$ for the parallel extension) and may indeed not be transversal.

6. SERIES PARALLEL GRAPHS

The concept of series and parallel extension of a matroid is a generalization of series and parallel extension of a graph. (Series extension is the subdivision of an edge e to produce two edges e and e'. Parallel extension introduces a new edge e' incident to the same vertices as e.) Minty [1966] defines a graph to be *series parallel* if it is obtained by series parallel extension of a bridge or a loop (i. e. a single edge between two distinct vertices or a loop at a single vertex). It is well known that the cycle matroid of a series parallel graph is a gammoid. Moreover, it is essentially true Duffin [1965] that a matroid is a binary gammoid if and only if it is the cycle matroid of a series parallel graph. (See Aigner [1979] or Welsh [1976] for a more precise and complete statement of this result.)

There are linear-time algorithms for testing an arbitrary graph to determine whether or not it is series parallel and if it is, to find a sequence of series and parallel extensions by which it is derived from a bridge or loop. (cf. Valdes, et al [1982].) Given such a sequence of series and parallel extension, a bipartite graph representing the matroid of the series parallel graph is easily obtained by the construction given in the previous section. As an example, bipartite representations of two dual series parallel graphs are obtained by successive series and parallel extensions, as shown in Figure 5. (Actually both matroids are transversal.)

It follows from our discussion that it is possible to reduce the weighted parity problem for graphic matroids to the weighted matching problem in the special case of series parallel graphs, (cf. the "spanning tree parity problem" Garey and Johnson [1979].)

References

M. Aigner, *Combinatorial Theory*, Spring-Verlag, New York, 1979.

R. Duffin, "Topology of Series-Parallel Networks," *J. Math. Anal. Appl.*, 10 (1965) 303-318.

J. Edmonds and D. R. Fulkerson, "Transversals and Matroid Partition," *J. Res. Nat. Bur. Stand.*, 69B (1965) 147-153.

D. R. Fulkerson, "Disjoint Common Partial Transversals of Two Families of Sets," *Studies in Pure Mathematics, Festschrift for R. Rado* (L. Mirsky, ed.), Academic Press (1971), 107-113.

M. R. Garey and D. S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness*, W. H. Freeman and Company, San Francisco, 1979.

T. A. Jenkyns, *Matchoids: A Generalization of Matchings and Matroids*, Ph.D. Thesis, University of Waterloo, 1974.

E. L. Lawler, *Combinatorial Optimization: Networks and Matroids*, Holt, Rinehart and Winston, San Francisco, 1976.

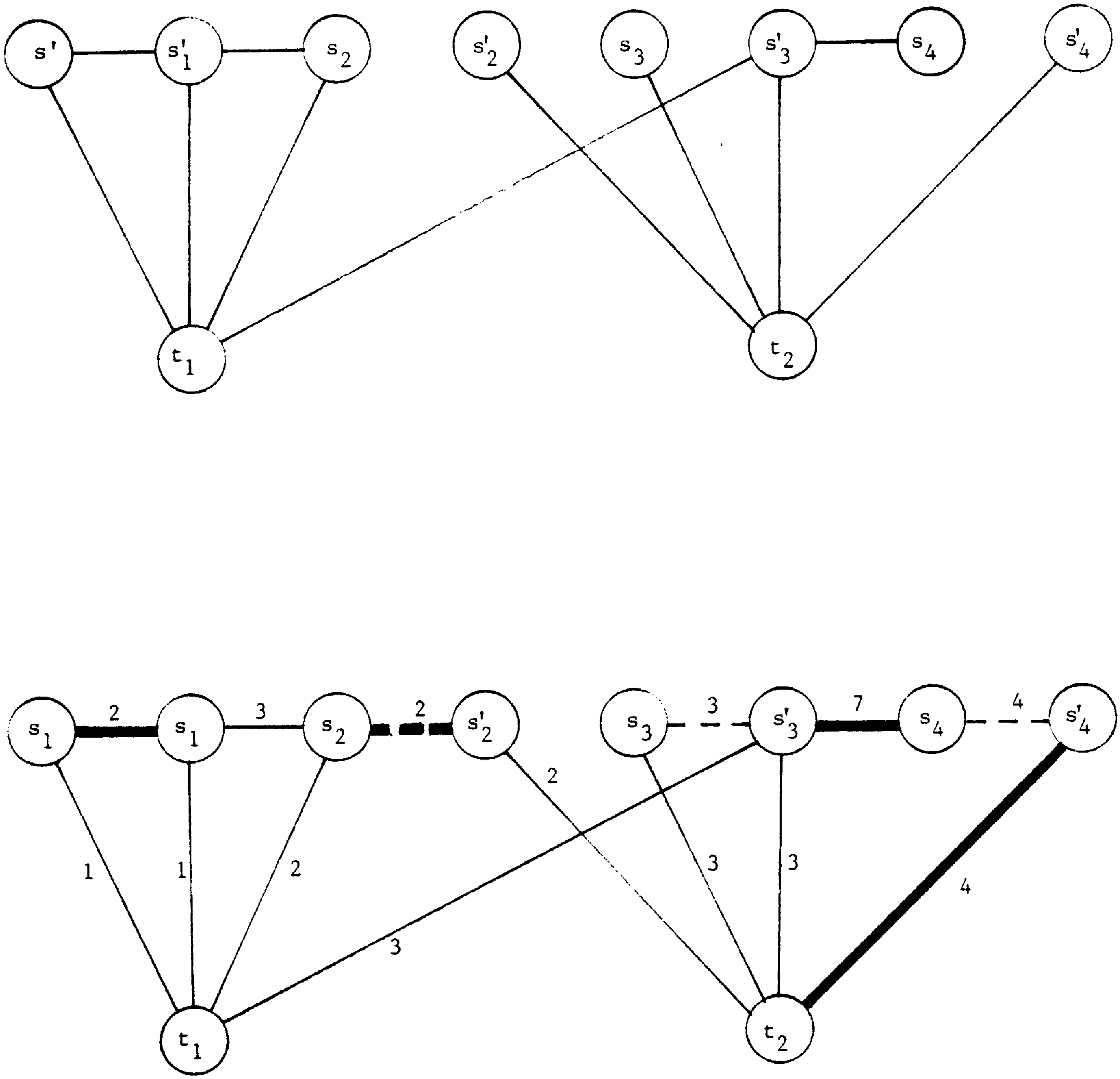
L. Lovász, "The Matroid Matching Problem," *Algebraic Methods in Graph Theory*, for the Colloquia Mathematica Societatis János Bolyai, Szeged (Hungary), 1978.

S. Micali and V. V. Vazirani, "A $O(\sqrt{V}E)$ Algorithm for Finding Maximum Matchings in General Graphs," *21 st Annual FOCS Conference*, (1980) 17-27.

G. J. Minty, "On the Axiomatic Foundations of the Theories of Directed Linear Graphs, Electrical Networks and Network Programming," *J. Math. Mech.*, 15 (1966) 485-520.

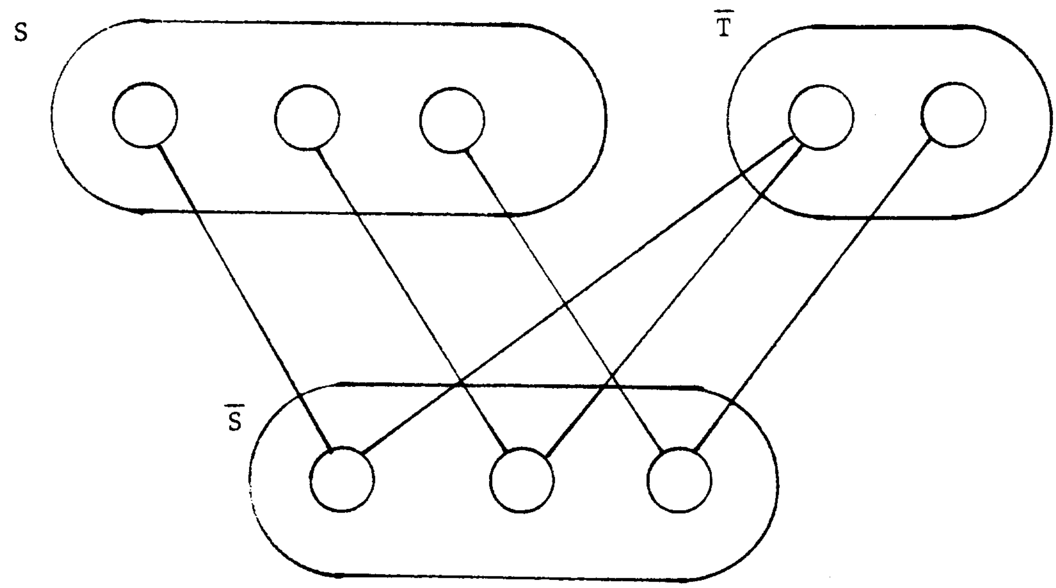
J. Valdes, R. E. Tarjan and E. L. Lawler, "The Recognition of Series Parallel Digraphs," to appear in *SIAM J. Computing*.

D. J. A. Welsh, *Matroid Theory*, Academic Press, London/New York, 1976.



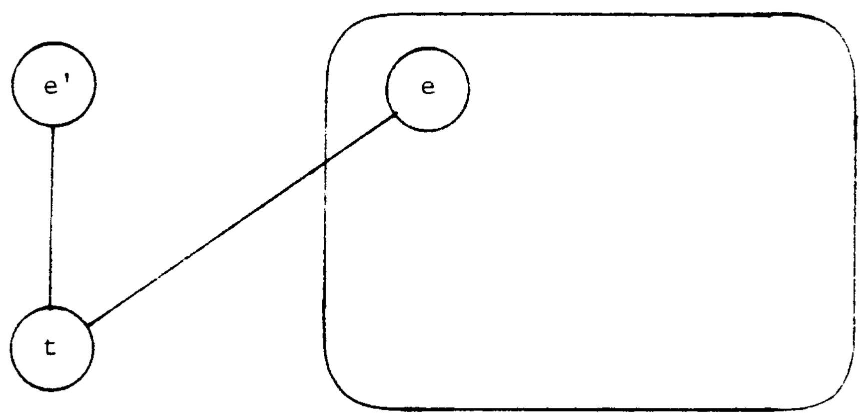
Example of reduction to weighted matching problem

Figure 1



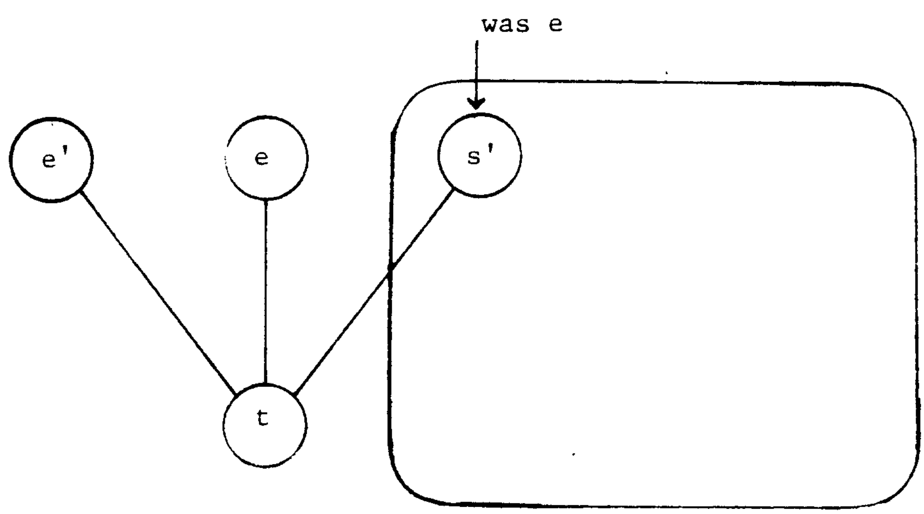
Bipartite graph for strict gammoid

Figure 2



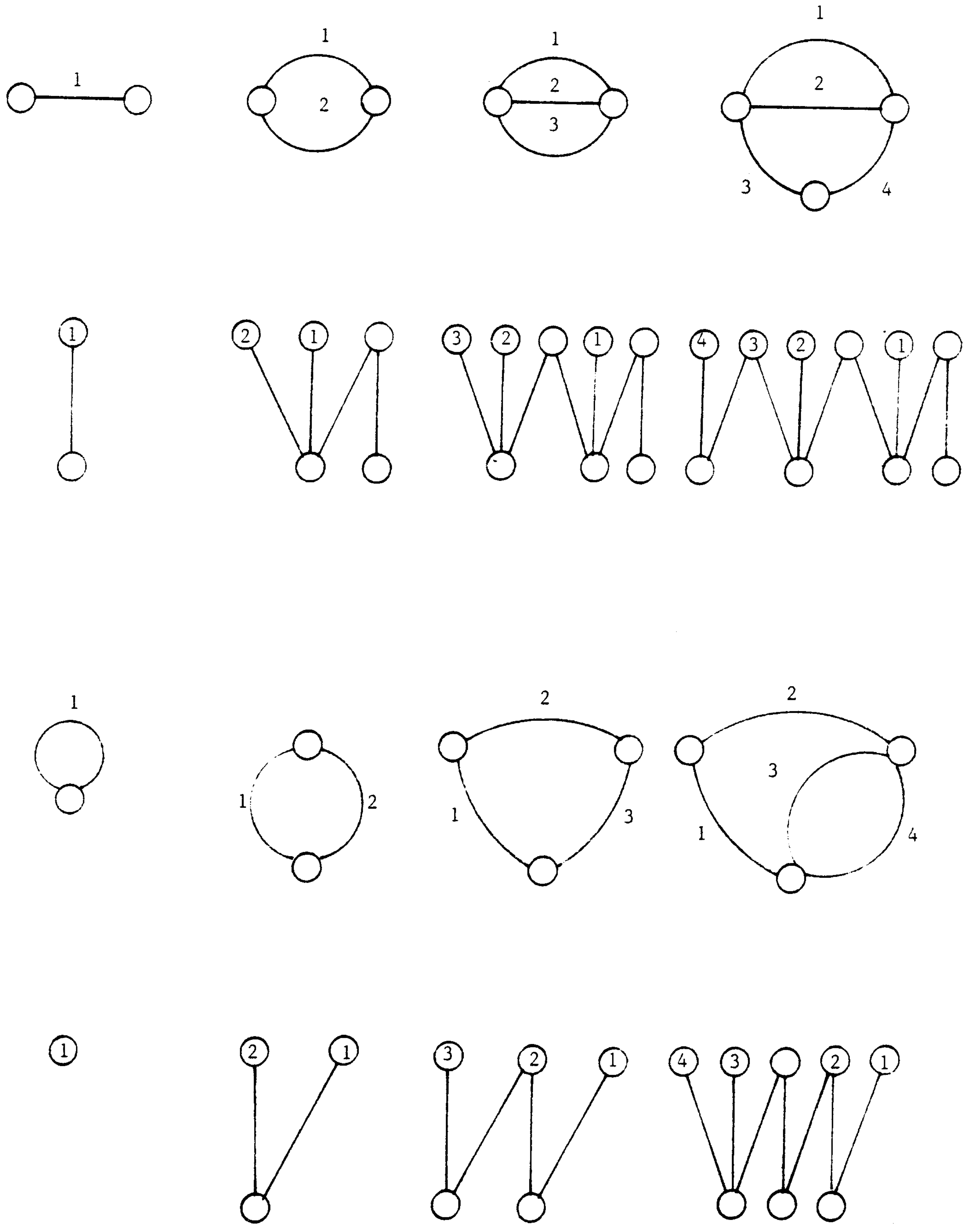
Graph for series extension

Figure 3



Graph for parallel extension

Figure 4



Obtaining representations of matroids of series parallel graphs

Figure 5