A FUZZY-SET-THEORETIC APPROACH TO THE COMPOSITIONALITY OF MEANING: PROPOSITIONS, DISPOSITIONS AND CANONICAL FORMS

By

L. A. Zadeh

Cognitive Science Program

Institute of Cognitive Studies

University of California at Berkeley

December 1983

To appear in the *Journal of Semantics*, December 1983

The production of the Berkeley Cognitive Science Report Series is supported by a grant in cognitive science to the University of California at Berkeley from the Alfred P. Sloan Foundation.

A Fuzzy-Set-Theoretic Approach to the Compositionality of Meaning: Propositions, Dispositions and Canonical Forms

L. A. Zadeh

Division of Computer Science
University of California, Berkeley, CA 94720

Abstract

In its traditional interpretation, Frege's principle of compositionality is not sufficiently flexible to have a wide applicability to natural languages. In a fuzzy-set-theoretic setting which is outlined in this paper, Frege's principle is modified and broadened by allowing the meaning of a proposition, *p*, to be composed not from the meaning of the constituents of *p*, but, more generally, from the meaning of a collection of fuzzy relations which form a so-called *explanatory database* that is associated with *p*. More specifically, through the application of test-score semantics, the meaning of *p* is represented as a procedure which tests, scores and aggregates the elastic constraints which are implicit in *p*. The employment of fuzzy sets in this semantics allows *p* to contain fuzzy predicates such as *tall, kind, much richer*, etc.; fuzzy quantifiers such as *most, several, few, usually*, etc.; modifiers such as *very, more or less, quite, somewhat*, etc.; and other types of semantic entities which cannot be dealt with within the framework of classical logic.

The approach described in the paper suggests a way of representing the meaning of dispositions, e.g., *Overeating causes obesity, Icy roads are slippery, Young men like young women*, etc. Specifically, by viewing a disposition, *d*, as a proposition with implicit fuzzy quantifiers, the problem of representing the meaning of *d* may be decomposed into (a) restoring the suppressed fuzzy quantifiers and/or fuzzifying the nonfuzzy quantifiers in the body of *d*; and (b) representing the meaning of the resulting dispositional proposition through the use of test-score semantics.

To place in evidence the logical structure of *p* and, at the same time, provide a high-level description of the composition process, *p* may be expressed in the canonical form "*X is F*," where $X = (X_1, \ldots, X_n)$ is an implicit n-ary variable which is constrained by *p*, and *F* is a fuzzy n-ary relation which may be interpreted as an elastic constraint on *X*. This canonical form and the meaning-composition process for propositions and dispositions are illustrated by several examples among which is the proposition $p \triangleq$ *Over the past few years Naomi earned far more than most of her close friends.*

1. Introduction

It is widely agreed at this juncture that Frege's principle of compositionality has a rather limited validity in application to natural languages (Hintikka (1982)). However, as is well known, its applicability may be extended, as it is done in Montague semantics (Partee (1976)), by the employment of higher-order type-theoretical constructs.

A different approach which is described in this paper is based on a broader interpretation of compositionality which allows the meaning of a proposition to be composed not from the meaning of its constituents, but, more generally, from the meaning of a collection of fuzzy relations in what is referred to as an *explanatory database.* With this interpretation of compositionality, Frege's principle regains much of its validity and, in its modified form, provides a basis for representing the meaning of complex propositions and other types of semantic entities. In particular, it may be used to represent the meaning of propositions containing fuzzy predicates exemplified by *tall, kind, much younger, close friend*, etc.; fuzzy quantifiers such as *most, many, few, several, not very many, frequently, rarely, mostly,* etc.; modifiers such as *very, quite, more or less, somewhat,* etc.; and qualifiers such as *quite true, very unlikely, almost impossible,* etc.

An especially important application of the approach described in this paper relates to the representation of the meaning of *dispositions*, that is, propositions with implicit fuzzy quantifiers. For example, the disposition *Overeating causes obesity* may be viewed as a result of suppressing the fuzzy quantifier *most* in the proposition *Most of those who overeat are obese*. Similarly, the disposition *Young men like young women* may be interpreted as an abbreviation of the proposition *Most young men like mostly young women*. On the other hand, the proposition *Anne never tells a lie* may be interpreted as the *dispositional* proposition *Anne tells a lie very rarely*, in which the fuzzy quantifier *very rarely* may be viewed as a fuzzified version of the nonfuzzy quantifier *never*. In general, a disposition may have a number of different interpretations and the restoration or explicitation of fuzzy quantifiers is an interpretation-dependent process.

2. Test-Score Semantics

The modified Frege's principle underlies a fuzzy-set-based meaning-representation system termed *test-score semantics* (Zadeh (1981)). In this system, a semantic entity such as a proposition, predicate, predicate-modifier, quantifier, qualifier, command, etc., is regarded as a system of elastic constraints whose domain is a collection of fuzzy relations in a database -- a database which describes a state of affairs, a possible world, or, more generally, a set of objects or derived objects in a universe of discourse. The meaning of a semantic entity, then, is represented as a test which when applied to the database yields a collection of partial test scores. Upon aggregation, these test scores lead to an overall vector test score, τ, whose components are numbers in the unit interval, with τ serving as a measure of the compatibility of the semantic entity with the database. In this respect, test-score semantics subsumes both truth-conditional and possible-world semantics as limiting cases in which the partial and overall test scores are restricted to {pass, fail} or, equivalently, {true, false} or {1, 0}.

In more specific terms, the process of meaning representation in test-score semantics involves three distinct phases. In Phase 1, an *explanatory database frame* or *EDF*, for short, is constructed. *EDF* consists of a collection of relational frames, i.e., names of relations, names of attributes and attribute domains whose meaning is assumed to be known. In consequence of this assumption, the choice of *EDF* is not unique and is strongly influenced by the knowledge profile of the addressee of the representation process as well as by the objective of explanatory effectiveness. For example, in the case of the proposition $p \triangleq$ *Over the past few years Naomi earned far more than most of her close friends*, the *EDF* might consist of the following relations: *INCOME [Name; Amount; Year],* which lists the income of each individual identified by his/her name as a function of the variable *Year*; *FRIEND [Name;* μ *],* where μ is the degree to which *Name* is a friend of Naomi; *FEW [Number;* μ *],* where μ is the degree to which *Number* is compatible with the fuzzy number *few*; *MOST [Proportion;* μ *],* in which μ is the degree to which *Proportion* is compatible with the fuzzy quantifier *most*; and *FAR.MORE [Income1; Income2;* μ *],* where μ is the degree to which *Income1* fits the fuzzy predicate *far more* in relation to *Income2*. Each of these relations is interpreted as an elastic constraint on the variables which are associated with it.

In Phase 2, a test procedure is constructed which acts on the relations in the explanatory database and yields the test scores which represent the degrees to which the elastic constraints induced by the constituents of the semantic entity are satisfied. For example, in the case of *p*, the test procedure would yield the test scores for the constraints induced by the relations *FRIEND*, *FEW*, *MOST* and *FAR.MORE*.

In Phase 3, the partial test scores are aggregated into an overall test score, τ, which, in general, is a vector which serves as a measure of the compatibility of the semantic entity with an instantiation of *EDF*. As was stated earlier, the components of this vector are numbers in the unit interval or, more generally, possibility/probability distributions over this interval. In particular, in the case of a proposition, *p*, for which the overall test score is a scalar, τ may be interpreted as the degree of truth of *p* with respect to the explanatory database *ED* (i.e., an instantiation of *EDF*). It is in this sense that test-score semantics may be viewed as a generalization of truth-conditional and model-theoretic semantics.

In summary, the process described above may be regarded as a test which assesses the compatibility of a given proposition, *p*, with an explanatory database, *ED*. What is important to note is that the meaning of *p* is the test itself rather than the overall test score, τ, which it yields.

In effect, the test in question may be viewed as the process by which the meaning of a proposition is composed from the meaning of the constituent relations in the associated explanatory database. As was stated earlier, the essential difference between this approach to compositionality and that of Frege is that, in general, the meaning of a proposition, *p*, is composed not from the meaning of the constituents of *p* but from those of a database, *EDF*, which is constructed for the explicit purpose of explaining or representing the meaning of *p* in terms of a collection of fuzzy relations whose meaning is assumed to be known to the addressee of the representation process.

In some instances, the names of constituent relations in the explanatory database may bear a close relation to the constituents of the proposition. In general, however, the connection may be implicit rather than explicit.

In testing the constituent relations in *EDF*, it is helpful to have a collection of standardized rules for computing the aggregated test score of a combination of elastic constraints $C_1, \ldots, C_k$ from the knowledge of the test scores of each constraint considered in isolation. For the most part, such rules are *default rules* in the sense that they are intended to be used in the absence of alternative rules supplied by the user.

In test-score semantics, the elementary rules of this type are the following:[1]

Rules pertaining to unary modification

If the test score for an elastic constraint C in a specified context is τ, then in the same context the test score for

(a) *not C* is $1 - \tau$ (negation).

(b) *very C* is τ^2 (intensification or concentration).

(c) *more or less C* is $\tau^{\frac{1}{2}}$ (diffusion or dilation).

Rules pertaining to composition

If the test scores for elastic constraints C_1 and C_2 in a specified context are τ_1 and τ_2, respectively, then in the same context the test score for

(a) C_1 and C_2 is τ_1 *andsign* τ_2 , where *andsign* $\stackrel{\Delta}{=}$ min (conjunction).

(b) C_1 *or* C_2 is τ_1 *orsign* τ_2 , where *orsign* $\stackrel{\Delta}{=}$ max (disjunction).

(c) *If* C_1 *then* C_2 is 1 *andsign* $(1 - \tau_1 + \tau_2)$ (implication).

Rule pertaining to quantification

Let Q be a fuzzy quantifier (i.e., a fuzzy number) which is characterized by its membership function μ_Q.

Let A and B by fuzzy subsets of a universe of discourse $U = \{u_1, \ldots, u_n\}$, with respective membership functions μ_A and μ_B.

Define the *sigma-count* (i.e., the cardinality) of A as the real number

$$\Sigma Count(A) \triangleq \Sigma_i \mu_A(u_i) \quad ,$$

where $\mu_A(u_i)$, $i = 1, \ldots, n$, is the grade of membership of u_i in A. [2]

Define the *relative sigma-count* of B in A as the ratio

$$\Sigma Count(B/A) = \frac{\Sigma Count(A \cap B)}{\Sigma Count(A)}$$

$$= \frac{\Sigma_i \mu_A(u_i) andsign \mu_B(u_i)}{\Sigma_i \mu_A(u_i)} \quad .$$

Then, the overall test score for the generic proposition

$$p \triangleq Q\ A\,'s\ are\ B\,'s \quad ,$$

where *A's* and *B's* are generic names of the elements of A and B, is given by

$$\tau = \mu_Q(\Sigma Count(B/A)) \quad .$$

In effect, this expression indicates that the compatibility of p with the denotations of A and B is equal to the degree to which the proportion of *B's* in A -- or, more generally, the degree of containment of A in B -- fits the denotation of Q.

As an illustration of the use of some of these rules in test-score semantics, consider the proposition cited earlier, namely, $p \triangleq$ *Over the past few years Naomi earned far more than most of her close friends*. In this case, we shall assume, as was done earlier, that the constituent relations in the explanatory database are:

EDF $\triangleq$ *INCOME [Name; Amount; Year] +*
FRIEND [Name; μ] +
FEW [Number; μ] +
FAR.MORE [Income1; Income2; μ] +
MOST [Proportion; μ] .

Note that some of these relations are explicit in p; some are not; and that most of the constituent words in p do not appear in *EDF*.

In what follows, we shall describe the process by which the meaning of p may be composed from the meaning of the constituent relations in *EDF*. Basically, this process is a test procedure which tests, scores and aggregates the elastic constraints which are induced by p.

1. Find Naomi's income, IN_i, in $Year_i$, i = 1, 2, 3,..., counting backward from present. In symbols,

$$IN_i \triangleq {}_{Amount}\, INCOME[Name = Naomi\, ; Year = Year_i],$$

which signifies that *Name* is bound to Naomi, *Year* to $Year_i$, and the resulting relation is projected on the domain of the attribute *Amount*, yielding the value of *Amount* corresponding to the values assigned to the attributes *Name* and *Year*.

2. Test the constraint induced by *FEW*:

$$\mu_i \triangleq {}_{\mu}FEW\, [Year = Year_i] \ ,$$

which signifies that the variable *Year* is bound to $Year_i$ and the corresponding value of μ is read by projecting on the domain of μ.

3. Compute Naomi's total income during the past few years:

$$TIN = \Sigma_i \mu_i IN_i,$$

in which the μ_i play the role of weighting coefficients. Thus, we are tacitly assuming that the total income earned by Naomi during a fuzzily specified interval of time is obtained by weighting Naomi's income in year $Year_i$ by the degree to which $Year_i$ satisfies the constraint induced by *FEW* and summing the weighted incomes.

4. Compute the total income of each $Name_j$ (other than Naomi) during the past few years:

$$TIName_j = \Sigma_i \mu_i IName_{ji},$$

where $IName_{ji}$ is the income of $Name_j$ in $Year_i$.

5. Find the fuzzy set of individuals in relation to whom Naomi earned far more. The grade of membership of $Name_j$ in this set is given by

$$\mu_{FM}(Name_j) = {}_{\mu}FAR.MORE[Income\,1 = TIN\, ; Income\,2 = TIName_j].$$

6. Find the fuzzy set of close friends of Naomi by intensifying (Zadeh (1978)) the relation *FRIEND*:

$$CF \triangleq CLOSE.FRIEND \triangleq {}^{2}FRIEND,$$

which implies that

$$\mu_{CF}(Name_j) = ({}_{\mu}FRIEND[Name = Name_j])^2,$$

where the expression

$${}_{\mu}FRIEND[Name = Name_j]$$

represents $\mu_F(Name_j)$, that is, the grade of membership of $Name_j$ in the set of Naomi's friends.

7. Count the number of close friends of Naomi. On denoting the count in question by $\Sigma Count(CF)$, we have:

$$\Sigma Count(CF) = \Sigma_j \mu^2{}_{FRIEND}(Name_j).$$

8. Find the intersection of *FM* with *CF*. The grade of membership of $Name_j$ in the intersection is given by

$$\mu_{FM \cap CF}(Name_j) = \mu_{FM}(Name_j) \wedge \mu_{CF}(Name_j),$$

where the min operator $\wedge$ signifies that the intersection is defined as the conjunction of its operands.

9. Compute the sigma-count of $FM \cap CF$:

$$\Sigma Count(FM \cap CF) = \Sigma_j \mu_{FM}(Name_j) \wedge \mu_{CF}(Name_j).$$

10. Compute the relative sigma-count of *FM* in *CF*, i.e., the proportion of individuals in $FM \cap CF$ who are in *CF*:

$$\rho \triangleq \frac{\Sigma Count(FM \cap CF)}{\Sigma Count(CF)} \quad .$$

11. Test the constraint induced by *MOST*:

$$\tau = {}_{\mu}MOST[Proportion = \rho],$$

which expresses the overall test score and thus represents the compatibility of *p* with the explanatory database.

In general, the relations in *EDF* are context-dependent. As an illustration, consider the proposition

$$p \triangleq Both\ are\ tall \quad ,$$

in which the standards of tallness are assumed to be class-dependent, e.g., depend on whether an individual is male or female. To reflect this, we may express the *EDF* for *p* in the following form:

$$\begin{aligned} EDF \triangleq \quad & POPULATION\ [\ Name;\ Height;\ Sex;\ ]\ + \\ & Indexical\ \rightarrow Name_{\alpha}\ + \\ & Indexical\ \rightarrow Name_{\beta}\ + \\ & TALL\ [\ Height;\ Sex;\ \mu\ ], \end{aligned}$$

in which the notation $Indexical \rightarrow Name_{\alpha}$ *indicates that* $Name_{\alpha}$ *is an indexical* object, i.e., is pointed to by the context. More specifically, we assume (a) that $Name_{\alpha}$ and $Name_{\beta}$ are the names of two individuals in *POPULATION* who are pointed to by the context in which *p* is asserted; and (b) that the relation *TALL* is sex-dependent, with μ representing the degree to which an individual whose height is *Height* and whose sex is *Sex* is *tall*.

For the *EDF* in question, the steps in the test procedure which leads to the overall test score and thereby represents the meaning of *p* may be described as follows:

1. Find the height and sex of $Name_\alpha$ and $Name_\beta$:

$$Height\,(Name_\alpha) \stackrel{\Delta}{=}_{Height} POPULATION\ [Name = Name_\alpha]$$

$$Sex\,(Name_\alpha) \stackrel{\Delta}{=}_{Sex} POPULATION\ [Name = Name_\alpha]$$

$$Height\,(Name_\beta) \stackrel{\Delta}{=}_{Height} POPULATION\ [Name = Name_\beta]$$

$$Sex\,(Name_\beta) \stackrel{\Delta}{=}_{Sex} POPULATION\ [Name = Name_\beta]\ .$$

2. Find the degrees to which $Name_\alpha$ and $Name_\beta$ are tall:

$$\tau_\alpha \stackrel{\Delta}{=}_\mu\ TALL\,[Height = Height(Name_\alpha)\,;\, Sex = Sex(Name_\alpha)]\ ,$$

$$\tau_\beta \stackrel{\Delta}{=}_\mu\ TALL\,[Height = Height(Name_\beta)\,;\, Sex = Sex(Name_\beta)]\ .$$

3. Aggregate the test scores found in 2.:

$$\tau = \tau_\alpha\ andsign\ \tau_\beta\ ,$$

in which we use the min operator (*andsign*) to combine the test scores τ_α and τ_β into the overall test score of τ.

As an illustration of the compositionality of meaning in the case of dispositions, we shall consider, first, the following simple disposition:

$$d \stackrel{\Delta}{=} Claudine\ is\ a\ better\ tennis\ player\ than\ Michael.$$

For concreteness, *d* will be assumed to have the interpretation expressed by the proposition

$$p \stackrel{\Delta}{=} When\ Claudine\ \text{and}\ Michael\ play\ tennis\ ,\ Claudine\ usually\ wins.$$

The *EDF* for *p* is assumed to consist of the relations

$EDF \stackrel{\Delta}{=}$ *PLAY.TENNIS [Outcome] +*
USUALLY [Proportion; μ*].*

The relation *PLAY.TENNIS* represents a tally of the outcomes of n plays between Claudine and Michael, with the variable *Outcome* ranging over the set {*Win*, *Lose*}, and with *Win* implying that Claudine won the game. The relation *USUALLY* is a fuzzy temporal quantifier, with μ representing the degree to which a numerical value of *Proportion* fits the intended meaning of *USUALLY*.

The steps in the test procedure are as follows.

1. Find the proportion of plays won by Claudine:

$$\rho = \frac{1}{n}\ Count(PLAY.TENNIS\ [Outcome = Win\,])\ .$$

2. Test the constraint induced by *USUALLY*:

$$\tau =_{\mu} USUALLY\ [Proportion = \rho]\ .$$

This expression for τ represents the overall test score for *d*.

We can make use of the above result to represent the meaning of a more complex disposition, namely,

$$d \triangleq \textit{Men are better tennis players than women}\ .$$

which will be assumed to be interpreted as the proposition

$$p \triangleq \textit{Most men are better tennis players than most women}\ ,$$

with the associated *EDF* consisting of the relations

$$EDF \triangleq \quad POPULATION\ [M.Name;\ F.Name;\ \mu] + MOST\ [Proportion;\ \mu]\ .$$

For simplicity, we assume that there are n men and n women in *POPULATION*, with μ representing the degree -- computed as in the above example -- to which *M.Name* is a better tennis player than *F.Name*. (More specifically, μ_{ij} is the degree to which $M.Name_i$ is a better tennis player than $F.Name_j$ $, i,\ j = 1, \ldots, n$.)

The steps in the test procedure are as follows:

1. For each $M.Name_i$, find the proportion (i.e., the relative sigma-count) of women tennis players in relation to whom $M.Name_i$ is a better tennis player:

$$\rho_i \triangleq \frac{1}{n} \Sigma_j \mu_{ij}\ .$$

2. For each $M.Name_i$, find the degree to which $M.Name_i$ is a better tennis player than most women:

$$\tau_i \triangleq_{\mu} MOST[Proportion = \rho_i]\ .$$

3. Compute the proportion of men who are better tennis players than most women:

$$\rho = \frac{1}{n} \Sigma_i \tau_i\ .$$

4. Compute the test score for the constraint induced by *MOST*:

$$\tau =_{\mu} MOST[Proportion = \rho]\ .$$

This τ represents the overall test score for *d*.

As an additional illustration, consider the disposition

$$d \triangleq \textit{Young men like young women}$$

which, as stated earlier, may be interpreted as the proposition

$$p \triangleq \textit{Most young men like mostly young women.}$$

The candidate *EDF* for *p* is assumed to consist of the following relations:

$$EDF \triangleq \quad POPULATION\ [\ Name;\ Sex;\ Age\ ] +$$
$$LIKE\ [\ Name1;\ Name2;\ \mu\ ] +$$
$$MOST\ [\ Proportion;\ \mu\ ],$$

in which μ in *LIKE* is the degree to which *Name1* likes *Name2*.

To represent the meaning of *p*, it is expedient to replace *p* with the semantically equivalent proposition

$$q \triangleq \textit{Most young men are } P\ \ ,$$

where *P* is the fuzzy *dispositional* predicate

$$P \triangleq \textit{likes mostly young women}\ \ .$$

In this way, the representation of the meaning of *p* is decomposed into two simpler problems, namely, the representation of the meaning of *P*, and the representation of the meaning of *q* knowing the meaning of *P*.

The meaning of *P* is represented by the following test procedure.

1. Divide *POPULATION* into the population of males, *M.POPULATION*, and population of females, *F.POPULATION*:

   $$M.POPULATION \triangleq {}_{Name\ ,\ Age}\ POPULATION\ [Sex = Male]$$

   $$F.POPULATION \triangleq {}_{Name\ ,\ Age}\ POPULATION\ [Sex = Female]\ ,$$

   where ${}_{Name\ ,\ Age}$ *POPULATION* denotes the projection of *POPULATION* on the attributes *Name* and *Age*. [3]

2. For each $Name_j$, $j = 1, \ldots, l$, in *F.POPULATION*, find the age of $Name_j$:

   $$A_j \triangleq {}_{Age}\ F.POPULATION[Name = Name_j]\ \ .$$

3. For each $Name_j$, find the degree to which $Name_j$ is young:

   $$\alpha_i \triangleq {}_{\mu}\ YOUNG[Age = A_j]\ \ ,$$

where α_i may be interpreted as the grade of membership of $Name_j$ in the fuzzy set, *YW*, of young women.

4. For each $Name_i$, $i = 1, \ldots, k$, in *M.POPULATION*, find the age of $Name_i$:

$$B_i \underline{\underline{\Delta}}_{Age}\ M.POPULATION[Name = Name_i]\ .$$

5. For each $Name_i$, find the degree to which $Name_i$ is young:

$$\delta_i \underline{\underline{\Delta}}_{\mu}\ YOUNG[Age = B_i]\ ,$$

where δ_i may be interpreted as the grade of membership of $Name_i$ in the fuzzy set, *YM*, of young men.

6. For each $Name_j$, find the degree to which $Name_i$ likes $Name_j$:

$$\beta_{ij} \underline{\underline{\Delta}}_{\mu}\ LIKE[Name\,1 = Name_i\ ; Name\,2 = Name_j]\ ,$$

with the understanding that β_{ij} may be interpreted as the grade of membership of $Name_j$ in the fuzzy set, WL_i, of women whom $Name_i$ likes.

7. For each $Name_j$ find the degree to which $Name_i$ likes $Name_j$ and $Name_j$ is young:

$$\gamma_{ij} \underline{\underline{\Delta}}\ \alpha_j\ andsign\ \beta_{ij}\ .$$

Note: As in previous examples, we employ the aggregation operator min (*andsign*) to represent the effect of conjunction. In effect, γ_{ij} is the grade of membership of $Name_j$ in the intersection of the fuzzy sets WL_i and *YW*.

8. Compute the relative sigma-count of young women among the women whom $Name_i$ likes:

$$\rho_i \underline{\underline{\Delta}}\ \Sigma Count(YW/WL_i)$$
$$= \frac{\Sigma Count(YW \cap WL_i)}{\Sigma Count(WL_i)}$$
$$= \frac{\Sigma_j\ \gamma_{ij}}{\Sigma_j\ \beta_{ij}}$$
$$= \frac{\Sigma_j\ \alpha_j\ andsign\ \beta_{ij}}{\Sigma_j\ \beta_{ij}}\ .$$

9. Test the constraint induced by *MOST*:

$$\tau_i \underline{\underline{\Delta}}_{\mu} MOST[Proportion = \rho_i]\ .$$

This test score, then, represents the degree to which $Name_i$ has the property expressed by the predicate

$$P \underline{\underline{\Delta}}\ likes\ mostly\ young\ women\ .$$

Continuing the test procedure, we have:

10. Compute the relative sigma-count of men who have property P among young men:

$$\rho \triangleq \Sigma Count(P/YM)$$

$$= \frac{\Sigma Count(P \cap YM)}{\Sigma Count(YM)}$$

$$= \frac{\Sigma_i \; \tau_i \; andsign \; \delta_i}{\Sigma_i \; \delta_i} \quad .$$

11. Test the constraint induced by *MOST*:

$$\tau =_{\mu} MOST[Proportion = \rho] \; .$$

This test score represents the overall test score for the disposition *Young men like young women.*

3. Canonical Form

The test procedures described in the preceding section provide, in effect, a characterization of the process by which the meaning of a proposition, p, may be composed from the meaning of the constituent relations in the *EDF* which is associated with p. However, the details of the test procedure tend to obscure the higher-level features of the process of composition and thus make it difficult to discern its underlying modularity and hierarchical structure.

The concept of a *canonical form* of p, which plays an important role in PRUF (Zadeh (1978)), provides a way of displaying the logical structure of p and thereby helps to place in a clearer perspective the role of the consecutive steps in the test procedure in the representation of meaning of p. Specifically, as was stated earlier, a proposition, p, may be viewed as a system of elastic constraints whose domain is the collection of fuzzy relations in the explanatory database. In more concrete terms, this implies that p may be represented in the *canonical form*

$$p \rightarrow X \; is \; F$$

where $X = (X_1, \ldots, X_n)$ is an n-ary *base variable* whose components $X_1, \ldots, X_n$ are the variables which are constrained by p; and F -- which is a fuzzy subset of the universe of discourse $U = U_1 \times \cdots \times U_n$, where U_i, $i = 1, \ldots, n$, denotes the domain of X_i -- plays the role of an elastic constraint on X. In general, both the base variables and F are implicit rather than explicit in p.

As a simple illustration, consider the proposition

$$p \triangleq Virginia \; is \; slim.$$

In this case, the base variables are $X_1 \triangleq$ *Height(Virginia)*, $X_2 \triangleq$ *Weight(Virginia)*; the constraint set is *SLIM*; and hence the canonical form of p may be expressed as

$$(Height\,(Virginia),\ Weight\,(Virginia))\ is\ SLIM\ \ ,$$

where *SLIM* is a fuzzy subset of the rectangle $U_1 \times U_2$, with $U_1 \triangleq [0, 200cm]$ and $U_2 = [0, 100kg]$.

If the assertion *"X is F"* is interpreted as an elastic constraint on the possible values of *X*, then the canonical form of *p* may be expressed as the *possibility assignment equation* (Zadeh (1978))

$$\Pi_{(X_1, \ldots, X_n)} = F\ \ ,$$

in which $\Pi_{(X_1, \ldots, X_n)}$ denotes the *joint possibility distribution* of $X_1, \ldots, X_n$. In more concrete terms, this equation implies that the possibility that the variables $X_1, \ldots, X_n$ may take the values $u_1, \ldots, u_n$, respectively, is equal to the grade of membership of the n-tuple $(u_1, \ldots, u_n)$ in *F*, that is,

$$Poss\{X_1 = u_1, \ldots, X_n = u_n\} = \mu_F(u_1, \ldots, u_n)\ \ ,$$

where μ_F denotes the membership function of *F*.

As an illustration, consider the disposition

$$d \triangleq Fat\ men\ are\ kind\ \ ,$$

which may be interpreted as an abbreviation of the proposition

$$p \triangleq Most\ fat\ men\ are\ kind\ \ .$$

Let *FAT* and *KIND* denote the fuzzy sets of *fat men* and *kind men*, respectively, in *U*. Now, the fuzzy quantifier *most* in *p* may be interpreted as a fuzzy characterization of the relative sigma-count of *kind men* in *fat men*. >From this, it follows that the canonical form of *p* may be expressed as

$$\Sigma Count(KIND/FAT)\ is\ MOST$$

or, equivalently, as the possibility assignment equation

$$\Pi_X = MOST\ \ ,$$

where

$$X = \Sigma Count(KIND/FAT)\ \ ,$$

and *MOST* is a fuzzy subset of the unit interval [0,1].

Along the same lines, consider the proposition

$$p \triangleq Most\ big\ men\ are\ not\ very\ agile\ \ .$$

As in the previous example, *BIG* will be assumed to be a fuzzy subset of the rectangle [0, 200 cm] $\times$ [0, 100 kg]. As for the fuzzy predicate *not very agile*, its denotation may be expressed as

$$not\ very\ agile \rightarrow ({}^2AGILE)'\ \ ,$$

where ${}^2AGILE$ represents the denotation of *very agile* and $'$ denotes the complement. More concretely, the membership function of ${}^2AGILE$ is given by

$$\mu_{{}^2AGILE} = (\mu_{AGILE})^2$$

and thus

$$\mu_{({}^2AGILE)'} = 1 - (\mu_{AGILE})^2$$

By relating the denotation of *not very agile* to that of *agile*, the canonical form of *p* may be expressed compactly as

$$p \rightarrow \Sigma Count(({}^2AGILE)'/BIG) \text{ is } MOST \ .$$

As expected, this canonical form places in evidence the manner in which the meaning of *p* may be composed from the meaning of the fuzzy relations *AGILE*, *BIG* and *MOST*.

As a further example, consider the proposition

$$p \triangleq \text{Peggy lives in a small city near San Francisco} \ ,$$

with which we associate the *EDF*

$EDF \triangleq$ *RESIDENCE [Name; City] +*
SMALL.CITY [City; μ *] +*
NEAR [City1; City2; μ *] .*

In *RESIDENCE*, *City* is the city in which *Name* lives; in *SMALL.CITY,* μ is the degree to which *City* is small; and in *NEAR*, μ is the degree to which *City1* is near *City2*.

The fuzzy set of *cities which are near San Francisco* may be expressed as

$$CNSF \triangleq_{City1} NEAR\ [City2 = San\ Francisco] \ ,$$

and hence the fuzzy set of *small cities which are near San Francisco* is given by the intersection

$$SCNSF \triangleq SMALL.CITY \cap CNSF \ ,$$

which is, in effect, the fuzzy constraint set *F* in the canonical form *"X is F."* In terms of this set, then, the canonical form of *p* may be expressed as

$$p \rightarrow Location\ (Residence\ (Peggy)) \text{ is}$$
$$SMALL.CITY \cap_{City1} NEAR[City2 = San\ Francisco].$$

To illustrate a different aspect of canonical forms, consider the proposition

$$p \triangleq \text{Mia had high fever} \ .$$

In this case, we have to assume that the base variable

$$X(t) \stackrel{\Delta}{=} Temperature\ (Mia,\ t)$$

$$\stackrel{\Delta}{=} Temperature\ of\ Mia\ at\ time\ t$$

is time-dependent. Furthermore, the verb *had* induces a fuzzy or, equivalently, elastic constraint on time which may be expressed as

$$had\ fP(b\,3\ t\ is\ PAST\ \ ,$$

with the understanding that PAST is a fuzzy subset of the interval (- ∞ , present time) which is *indexical* in the sense that it is characterized more specifically by the context in which *p* is asserted. Using this interpretation of *PAST*, the canonical form of *p* may be written as

$$p \rightarrow Temperature\ (Mia,\ t\ is\ PAST)\ is\ HIGH\ \ .$$

To conclude our examples, we shall construct canonical forms for two of the propositions considered in Section 2. We begin with the proposition

$$p \stackrel{\Delta}{=} Most\ young\ men\ like\ mostly\ young\ women\ \ .$$

As before, we represent *p* as the proposition

$$p \stackrel{\Delta}{=} Most\ young\ men\ are\ P\ \ ,$$

where *P* is the dispositional predicate *likes mostly young women.* In this way, the canonical form of *p* may be expressed as

$$\Sigma Count(P/YM)\ is\ MOST\ \ ,$$

where *P* is the fuzzy set which represents the denotation of *likes mostly young women* in *M.POPULATION*, and *YM* is the fuzzy subset of *young men* in *M.POPULATION*.

To complete the construction of the canonical form, we must show how to construct *P*. To this end, we shall express in the canonical form the proposition

$$p_i \stackrel{\Delta}{=} Name_i\ is\ P\ \ ,$$

where $Name_i$ is the name of *i*th man in *M.POPULATION*.

As before, let WL_i and *YW* denote, respectively, the fuzzy set of women whom $Name_i$ likes and the fuzzy set of young women in *F.POPULATION*. Then, the canonical form of p_i may be represented as

$$Name_i\ is\ P \rightarrow \Sigma Count(YW/WL_i)\ is\ MOST\ \ .$$

In the above analysis, we have employed a two-stage process to represent the meaning of *p* through the construction of two canonical forms. Alternatively, we can subsume the second form in the first, as follows.

First, we note that, for each $Name_i$, the relative sigma-count $\Sigma Count(YW/WL_i)$ is a number in the interval [0,1]. Let R denote a fuzzy subset of *M.POPULATION* such that

$$\mu_R(Name_i) = \Sigma Count(YW/WL_i) \ .$$

Then, the fuzzy set of *men who like mostly young women* may be represented as

$$P \triangleq MOST(R) \ ,$$

with the understanding that *MOST(R)* should be evaluated through the use of the extension principle (Zadeh (1978)). This implies that the grade of membership of $Name_i$ in P is related to the grade of membership of $Name_i$ in R through the composition

$$\mu_P(Name_i) = \mu_{MOST}(\mu_R(Name_i)) \ , i = 1, \ldots, k \ .$$

Using this representation of P, the canonical form of p may be expressed more compactly as

$$p \rightarrow \ \Sigma Count(MOST(R)/YM) \text{ is } MOST \ .$$

Using the same approach, the canonical form of the proposition

$$p \triangleq \textit{Over the past few years Naomi earned far more than most of her close friends}$$

may be constructed as follows.

First, we construct the canonical form

$$p \rightarrow \Sigma Count(FM/^2F) \text{ is } MOST \ ,$$

where

$$CF \triangleq {}^2F \triangleq \textit{fuzzy set of close friends of Naomi}$$

and

$$FM \triangleq \textit{fuzzy set of individuals in relation whom Naomi earned}$$
$$\textit{far more during the past few years.}$$

Second, we construct the canonical form for the proposition which defines *FM*. Thus,

$$Name_j \text{ is } FM \rightarrow (TIN, \ TIName_j) \text{ is } FAR.MORE \ ,$$

in which the base variables are defined by

$$TIN \triangleq \textit{total income of Naomi during the past few years.}$$
$$= \Sigma_i \mu_{FEW}(i) IN_i$$

and

$$TIName_j \triangleq \textit{total income of } Name_j \textit{ during the past few years.}$$

$$\triangleq \Sigma_i \mu_{FEW}(i) IName_{ji} \quad ,$$

where IN_i is Naomi's income in year $Year_i$, $i = 1,2,3, \ldots,$ and $IName_{ji}$ is $Name_j$'s income in $Year_i$.

It is possible, as in the previous example, to absorb the second canonical form in the first form. The complexity of the resulting form, however, would make it more difficult to perceive the modularity of the meaning-representation process.

Concluding Remark

The fuzzy-set-theoretic approach outlined in the preceding sections is intended to provide a framework for representing the meaning of propositions and dispositions which do not lend themselves to semantic analysis by conventional techniques. The principal components of this framework are (a) the explanatory database which consists of a collection of fuzzy relations; (b) the procedure which tests, scores and aggregates the elastic constraints, and thereby characterizes the process by which the meaning of a proposition is composed from the meaning of the constituent relations in the explanatory database; and (c) the canonical form which represents a proposition as a collection of elastic constraints on a set of base variables which are implicit in the proposition.

Notes

To Walter and Sally Sedelow.

Research supported in part by the NSF Grants ECS-8209679 and IST-8018196.

1. A more detailed discussion of the rules in question may be found in Zadeh (1978).
2. The concept of cardinality is treated in greater detail in Zadeh (1982 b).
3. To obtain the projection in question, all columns other than *Name* and *Age* in the relation *POPULATION [Sex = Female]* should be deleted.

REFERENCES AND RELATED PUBLICATIONS

1. Adams, E.W., The logic of "almost all," *Journal of Philosophical Logic 3* (1974) 3-17.
2. Bartsch, R. and Vennemann, T., *Semantic Structures*, Frankfurt: Attenaum Verlag, 1972.
3. Barwise, J. and Cooper, R., Generalized quantifiers and natural language, *Linguistics and Philosophy* 4 (1981) 159-219.
4. Blanchard, N., Theories cardinales et ordinales des ensembles flou: les multiensembles. Thesis, University of Claude Bernard, Lyon, 1981.
5. Carlstrom, I.F., Truth and entailment for a vague quantifier, *Synthese 30* (1975) 461-495.
6. Carnap, R., *Meaning and Necessity* . University of Chicago Press, 1952.
7. Chomsky, N., *Rules and Representations*. Columbia U. Press, 1980.
8. Cooper, W.S., *Logical Linguistics*. Dordrecht: Reidel, 1978.
9. Cresswell, M.J., *Logic and Languages.* London: Methuen, 1973.
10. Cushing, S., *Quantifier Meanings - A Study in the Dimensions of Semantic Competence.* Amsterdam: North-Holland, 1982.
11. DeLuca, A. and Termini, S., A definition of non-probabilistic entropy in the setting of fuzzy sets theory, *Information and Control* 20 (1972) 301-312.
12. Dubois, D. and Prade, H., *Fuzzy Sets and Systems: Theory and Applications*. New York: Academic Press, 1980.
13. Gallin, D., *Intensional and Higher-Order Modal Logic* North-Holland: Amsterdam, 1975.
14. Goguen, J.A., The logic of inexact concepts, *Synthese 19* (1969) 325-373 .
15. Groenendijk, J., Janssen, T. and Stokjoff, M. (eds.) *Formal Methods in the Study of Language.* Amsterdam: Mathematical Centre, 1981.
16. Hersh, H.M. and Caramazza, A., A fuzzy set approach to modifiers and vagueness in natural language, *J.Experimental Psychology 105* (1976) 254-276.
17. Higginbotham, J. and May, R., Questions, quantifiers and crossing, *The Linguistic Review* 1 (1981) 41-80.
18. Hintikka, J.K., *Logic, Language-Games, and Information: Kantian Themes in the Philosophy of Logic.* Oxford: Oxford University Press, 1973.
19. Hintikka, J., Game-theoretical semantics: insights and prospects *Notre Dame Journal of Formal Logic* 23, (1982) 219-241.
20. Janssen, T.; On Problems Concerning the Quantification Rules in Montague Grammar, in:C. Rohrer (ed.) *Time, Tense and Quantifiers* Tubingen: Niemeyer 1980.

21. Jansen, T., Relative Clause Constructions in Montague Grammar and Compositional Semantics, in J. Groenendijk, T. Janssen and M. Stokhof (eds.) *Formal Methods in the Study of Language*, Mathematical Centre, Amsterdam, 1981.

22. Keenan, E.L., Quantifier structures in English, *Foundations of Language 7* (1971) 255-336.

23. Klement, E.P., An axiomatic theory of operations on fuzzy sets, Institut fur Mathematik, Johannes Kepler Universitat Linz, Institutsbericht 159, 1981.

24. Lakoff, G., Hedges: A study in meaning criteria and the logic of fuzzy concepts, *J. Phil. Logic* 2 (1973) 458-508. Also in: *Contemporary Research in Philosophical Logic and Linguistic Semantics*, Jockney, D., Harper, W. and Freed, B., (eds.). Dordrecht: Reidel, 221-271, 1973.

25. Lambert, K. and van Fraassen, B.C., Meaning relations, possible objects and possible worlds, *Philosophical Problems in Logic* (1970) 1-19.

26. Mamdani, E.H. and Gaines, B.R., *Fuzzy Reasoning and its Applications.* London: Academic Press, 1981.

27 McCarthy, J., Circumscription: A non-monotonic inference rule, *Artificial Intelligence 13* (1980) 27-40.

28. mcCawley, J.D., *Everything that Linguists have Always Wanted to Know about Logic.* Chicago: University of Chicago Press, 1981.

29. McDermott, D.V. and Doyle, J., Non-monotonic logic I. *Artificial Intelligence 13* (1980) 41-72.

30. McDermott, D.V., Nonmonotonic logic II: Nonmonotonic modal theories, *Jour. Assoc. Comp.. Mach. 29* (1982) 33-57.

31. Miller, G.A. and Johnson-Laird, P.N., *Language and Perception*. Cambridge: Harvard University Press, 1976.

32. Mizumoto, M. and Tanaka, K., Some properties of fuzzy numbers, in: *Advances in Fuzzy Set Theory and Applications,* Gupta, M.M., Ragade, R.K. and Yager, R.R., (eds.). Amsterdam: North Holland, 153-164, 1979.

33. Moisil, G.C., *Lectures on the Logic of Fuzzy Reasoning*, Scientific Editions, Bucarest, 1975.

34. Montague, R., *Formal Philosophy*, in: *Selected Papers,* Thomason, R., (ed.). New Haven: Yale University Press, 1974.

35. Morgenstern, C.F., The measure quantifier, *Journal of Symbolic Logic 44* (1979) 103-108.

36. Mostowski, A., On a generalization of quantifiers, *Fundamenta Mathematicae 44* (1957) 17-36.

37. Osherson, D.N. and Smith, E.E., Gradedness and conceptual combination, *Cognition 12R (1982) 299-318.*

38. Partee, B., *Montague Grammar*. New York: Academic Press, 1976.

39. Peterson, P., On the logic of *few, many* and *most, Notre Dame J. of Formal Logic 20* (1979) 155-179.

40. Peterson, P.L., Philosophy of Language, *Social Research* 47 (1980), 749-774.

41. Rescher, N., *Plausible Reasoning*. Amsterdam: Van Gorcum, 1976.

42. Schubert, L.K., Goebel, R.G. and Cercone, N., The structure and organization of a semantic net for comprehension and inference, in: *Associative Networks,* Findler, N.V., (ed.). New York: Academic Press, 122-178, 1979.

43. Scheffler, I., *A Philosophical Inquiry into Ambiguity, Vagueness and Metaphor in Language*. Routledge & Kegan Paul, London, 1981.

44. Searle, J. (ed.) *The Philosophy of Language.* Oxford: Oxford University Press, 1971.

45. Suppes, P., Elimination of quantifiers in the semantics of natural languages by use of extended relation algebras, *Revue Internationale de Philosophie* (1976) 117-118, 243-259.

46. Terano, T. and Sugeno, M., Conditional fuzzy measures and their applications, in: *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, Zadeh, L.A., Fu,K.S., Tanaka, K. and Shimura,M., (eds.) New York: Academic Press, 151-170, 1975.

47. Yager, R.R., Quantified propositions in a linguistic logic, in: *Proceedings of the 2nd International Seminar on Fuzzy Set Theory,* Klement, E.P., (ed.). Johannes Kepler University, Linz, Austria, 1980.

48. Yager, R.R., Some procedures for selecting fuzzy-set-theoretic operators, *Inter. Jour. of General Systems 8* (1982) 115-124.

49. Zadeh, L.A., The concept of a linguistic variable and its application to approximate reasoning, *Information Sciences 8 and 9* (1975), 199-249, 301-357, 43-80.

50. Zadeh, L.A., PRUF -- a meaning representation language for natural languages, *Int. J. Man-Machine Studies 10* (1978) 395-460.

51. Zadeh, L.A., Test-score semantics for natural languages and meaning-representation via PRUF, *Tech.Note 247, AI Center, SRI International,* Menlo Park, CA 1981. Also in *Empirical Semantics*, Rieger, B.B., (ed.). Bochum: Brockmeyer, 281-349, 1981.

52. Zadeh, L.A., Test-score semantics for natural languages, in: *Proceedings of the Ninth International Conference on Computational Linguistics* Prague, 425-430, 1982.

53. Zadeh, L.A. A Computational Approach to Fuzzy Quantifiers in Natural Languages, Memorandum No. UCB/ERL M82/36. University of California, Berkeley, 1982. To appear in *Computers and Mathematics*.

54. Zimmer, A., Some experiments concerning the fuzzy meaning of logical quantifiers, in: *General Surveys of Systems Methodology*, Troncoli, L, (ed.). Louisville: Society for General Systems Research, 435-441, 1982.

55. Zimmermann, H.-J. and Zysno, P., Latent connectives in human decision making, *Fuzzy Sets and Systems 4* (1980) 37-52.