SOLVING PIECEWISE-LINEAR EQUATIONS FOR RESISTIVE NETWORKS

by

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Abstract

Nonlinear resistive networks can be characterized by the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ where $\underset{\sim}{f}(\cdot)$ is a continuous piecewise-linear mapping of $\mathcal{R}^n$ into itself. The n-dimensional Euclidean space is divided into a finite number of regions, and, in each region say region R_m, we can express $\underset{\sim}{f}$ by $\underset{\sim}{J}^{(m)} \underset{\sim}{x} + \underset{\sim}{w}^{(m)}$ where $\underset{\sim}{J}^{(m)}$ is a constant n×n Jacobian matrix and $\underset{\sim}{w}^{(m)}$ is a constant n-vector. In this paper we obtain the following results: If all the Jacobian determinants in the unbounded regions have the same sign, the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ has at least one solution and an algorithm is developed, which obtains one or more solutions in a finite number of steps. The work represents a generalization of early work by Fijisawa, Kuh and Ohtsuki and relaxes the condition imposed on the function. For example, in the bounded regions, the Jacobian matrices can be singular and the sign of Jacobian determinants can be arbitrary.

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I. Introduction

Most electrical engineers are familiar with the technique of piecewise-linear approximation of a nonlinear function. For example, in the analysis of a simple electronic circuit, diode characteristics can be represented by continuous, linear segments. This often gives considerable insight to the problem, and, as a result, yields quick solution. During the past decade, major advances have been made on the analysis of general nonlinear resistive networks based on the piecewise-linear approach [1]-[16]. Because of the generality of the approach and the specific results, we believe that piecewise-linear analysis will become useful not only in nonlinear networks but also in many related fields such as structural analysis, flow networks, mathematical economics, numerical integration and nonlinear system problems in general.

Consider a nonlinear network or system which is characterized by the equation

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y} \tag{1}$$

where $\underset{\sim}{f}$ maps the real n-dimensional Euclidean space $\mathcal{R}^n$ into itself. $\underset{\sim}{x}$ is a point in $\mathcal{R}^n$ and represents a set of chosen variables of a given network or system and $\underset{\sim}{y}$ is an arbitrary point in $\mathcal{R}^n$ and represents the input. By specifying $\underset{\sim}{f}$ as a continuous, piecewise-linear function, we can express $\underset{\sim}{f}$ as follows:

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{J}^{(m)} \underset{\sim}{x} + \underset{\sim}{w}^{(m)} , \quad m = 1,2,\ldots,\ell \tag{2}$$

where $\underset{\sim}{J}^{(m)}$ is a constant n×n matrix (called Jacobian matrix for convenience) and $\underset{\sim}{w}^{(m)}$ is a constant n-vector, both defined in region R_m.

The whole space $\mathcal{R}^n$ is divided into a finite number (ℓ) of polyhedral regions by a finite number of hyperplanes. A typical boundary hyperplane in the $\underset{\sim}{x}$-space can be characterized by the following equation:

$$\underset{\sim}{n}^T \underset{\sim}{x} = \text{constant} \tag{3}$$

where $\underset{\sim}{n}$ is the normal vector of the hyperplane. The continuity of $\underset{\sim}{f}$ imposes an important constraint between Jacobian matrices in neighboring regions, namely:

$$\underset{\sim}{J} - \underset{\sim}{J}' = \underset{\sim}{c}\ \underset{\sim}{n}^T \tag{4}$$

where $\underset{\sim}{J}$ and $\underset{\sim}{J}'$ are Jacobian matrices of neighboring regions R and R', respectively, sharing the common boundary defined by the normal vector $\underset{\sim}{n}$, and $\underset{\sim}{c}$ is an arbitrary constant n-vector. Thus $\underset{\sim}{c}\ \underset{\sim}{n}^T$ is a dyad and it turns out that eq. (4) represents a key property of a continuous, piecewise-linear function.

In previous work, necessary and sufficient conditions for $\underset{\sim}{f}$ to be a homeomorphism have been established. A continuous function $\underset{\sim}{f}$ is said to be homeomorphic from $\mathcal{R}^n$ onto itself if and only if the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ has a unique solution for all $\underset{\sim}{y}$. Fujisawa and Kuh [7] have shown that if $\underset{\sim}{f}$ is homeomorphic then the algorithm due to Katzenelson always converges, thus the solution can be found for any input $\underset{\sim}{y}$. Furthermore, Fujisawa, Kuh and Oktsuki [5] have demonstrated that as long as all Jacobian determinants det $\underset{\sim}{J}^{(i)}$, $i = 1,2,\ldots\ell$ have the same sign (the property is referred to as the "sign condition") there exists at least one solution to the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ and the Katzenelson algorithm also converges. Nevertheless, the "sign condition" on Jacobian determinants is a rather severe restriction for many problems. In this paper, we shall deal with continuous, piecewise-linear functions which do not obey the "sign condition" in general. Moreover, singular Jacobian matrices will be allowed. We shall develop the condition under which solutions of eq. (1) exist and can be obtained.

A related problem is also treated in the present paper. Recall that Katzenelson's algorithm depends on the tracing of a solution curve in the $\underset{\sim}{x}$-space. When the solution curve reaches a multiple boundary, called a corner, where more than two regions meet, Fujisawa and Kuh have introduced a perturbation method to by-pass the corner. The method works if the given function satisfies the "sign condition" on the Jacobian determinants. It turns out that for a general function, a new look at the corner problem is essential. In this paper we have developed a general theory to handle the corner problem.

In order to make the present paper reasonably self-contained, we will first review briefly the Katzenelson algorithm in Sec. 2. The dynamic behavior of the solution curve is next discussed when the "sign condition" constraint is removed. What is crucial in this case is to ensure that the solution curve in the $\underset{\sim}{x}$-space always crosses a boundary which separates two regions having Jacobian determinants with opposite signs. The technique used is a generalization of that first introduced by Kuh and Hajj [2]. Two simple examples are given to illustrate several difficulties of the problem when the "sign condition" is not satisfied.

In Section 3 we develop the general theory first with two assumptions, namely: the solution curve never hits a corner and all Jacobian matrices are nonsingular. We find that under the more relaxed condition that, in all unbounded regions, the Jacobian determinants have the same sign, one or more solutions exist and can be obtained by tracing the solution curve. It should be pointed out that the condition is not as restrictive as it seems because most physical systems behave like a passive element when any of its variables becomes unbounded, which implies that the Jacobian determinants in the unbounded regions are all positive.

In Section 4, we deal with the corner problem by means of a new perturbation method. The basic result is that, theoretically, an initial point can always be picked in the x-space for which the solution curve does not hit any corners. Computationally, it gives a method to perturb the solution curve if the solution curve hits a corner.

In Section 5, we discuss in detail the problem of singular Jacobian matrices and show how to handle the problem. The method depends on the result of Section 4.

With the results of Sections 4 and 5, we can finally state a general theorem of solving piecewise-linear equations. Our algorithm will lead to one or more solutions if the piecewise-linear function has the property that all Jacobian determinants in the unbounded regions have the same sign.

II. Dynamic behavior of solution curves

The basic problem is to obtain one or more solutions of eq. (1) for a given input $\tilde{y}^*$ where $\tilde{f}$ is continuous and piecewise linear. If all Jacobian determinants have the same sign, the Katzenelson algorithm always converges and can be illustrated by means of Fig. 1. We first choose an arbitrary initial point in the $\tilde{x}$-space, say $\tilde{x}_0$ in region R_0. In R_0, the equation which characterizes eq. (1) is

$$f(x) = J^{(0)} x + w^{(0)} = y \tag{5}$$

We may use eq. (5) to compute the image $y_0 = f(x_0)$

$$y_0 = J^{(0)} x_0 + w^{(0)} \tag{6}$$

Denote the line segment joining y_0 and y^* in the y-space by L_y. The problem is then reduced to one of determining a continuous curve in the x-space, starting with x_0, which is the inverse image of L_y. The curve is called the solution curve in the x-space. The beginning point is x_0 and the end point of the solution curve is the solution x^*. Thus we only need to trace the solution curve to obtain the solution x^*.

Let us take a look at the properties of the solution curve. The portion of the solution curve which lies in R_0 is determined by

$$x_0(\lambda) = x_0 + \lambda\, d_0 \tag{7}$$

where

$$d_0 = {J^{(0)}}^{-1} (y^* - y_0) \tag{8}$$

and $\lambda \geq 0$ is a parameter. If $x_0(1)$ happens to be in R_0, then $x_0(1) = x^*$ is the desired solution. The line segment joining x_0 and $x_0(1)$ is the solution curve and the algorithm terminates. If otherwise, the value of λ has to be determined such that $x_0(\lambda)$ lies on the boundary of R_0. Denote such a value of λ by λ_0 and define $x_1 = x_0(\lambda_0)$ and $y_1 = f(x_1)$. The line segment joining x_0 and x_1 is then the first portion of the desired solution curve. The next step is to extend the solution curve beyond x_1 into region R_1.

Assuming that x_1 lies on a simple boundary hyperplane, between the two regions R_0 and R_1, Fujisawa and Kuh proved that if det $\underset{\sim}{J}^{(0)}$ and det $\underset{\sim}{J}^{(1)}$ have the same sign, the solution curve in the $\underset{\sim}{x}$-space will indeed enter R_1 as $\underset{\sim}{y}_0$ moves forward $\underset{\sim}{y}^*$ through $\underset{\sim}{y}_1$. We have next in region R_1, the solution curve

$$\underset{\sim}{x}_1(\lambda) = \underset{\sim}{x}_1 + \lambda \underset{\sim}{d}_1 \qquad , \qquad \lambda \geq 0 \tag{9}$$

where

$$\underset{\sim}{d}_1 = \underset{\sim}{J}^{(1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_1) \tag{10}$$

Since the total number of regions is finite, the method will converge eventually to a solution $\underset{\sim}{x}^*$ which is the inverse image of $\underset{\sim}{y}^*$. The crucial point here is that the solution curve will never reenter a region which has already been traced. This is due to the fact that all Jacobian matrices are nonsingular, and the proof is simple. Suppose that the solution curve enters region R_j at $\underset{\sim}{x}_j$ and leaves it at $\underset{\sim}{x}_{j+1}$ and that the curve later reenters the same region at $\underset{\sim}{x}_k$ as shown in Fig. 2. Clearly the three points $\underset{\sim}{x}_j$, $\underset{\sim}{x}_{j+1}$ and $\underset{\sim}{x}_k$ do not lie on a line. Therefore the two vectors $\underset{\sim}{x}_{j+1} - \underset{\sim}{x}_j$ and $\underset{\sim}{x}_k - \underset{\sim}{x}_j$ are linearly independent, whereas their images under linear mapping $\underset{\sim}{J}^{(j)} \underset{\sim}{x} + \underset{\sim}{w}^{(j)}$ are both constant multipliers of $\underset{\sim}{y}^* - \underset{\sim}{y}_0$ and hence are linearly dependent. This contradicts the assumption that $\underset{\sim}{J}^{(j)}$ is nonsingular.*

It should be noted that $\underset{\sim}{x}_1 = \underset{\sim}{x}_0(\lambda_0)$ may lie on more than one boundary as shown in Fig. 3, or stated in another way, $\underset{\sim}{x}_1$ is at a corner. We shall postpone the discussion of the corner problem until Section 4. Meantime,

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*It whould be noted that we only need the nonsingularity of Jacobians to prove this result. This does not rule out the situation that a solution curve in a region can be retraced, which implies that a region is reentered at $\underset{\sim}{x}_j$. It will be seen later that when the Jacobian determinants have different signs, a modified algorithm can lead to a cyclic solution curve.

we assume that the solution curve will not hit any corners.

If we remove the condition that all Jacobian determinants have the same sign, the problem becomes much more complicated. Let us consider the solution curve in the $\underset{\sim}{x}$-space which transverses in region R starting from $\underset{\sim}{x}_j$ and along the direction $\underset{\sim}{d}_j$. The solution curve reaches the point $\underset{\sim}{x}_{j+1}$ at the boundary which separates region R and region R'. In [7] Fujisawa and Kuh derived the following equation

$$\det \underset{\sim}{J}\, \underset{\sim}{n}^T \underset{\sim}{J}^{-1} = \det \underset{\sim}{J}'\, \underset{\sim}{n}^T \underset{\sim}{J}'^{-1} \tag{11}$$

This equation prescribes completely the local behavior of the soltuion curve at $\underset{\sim}{x}_{j+1}$, and thus the strategy of our algorithm. The proof of boundary crossing in the x-sapce for the case that the Jacobian determinants have the same sign is essentially based on eq. (11). Furthermore, eq. (11) also suggests what we should do for the case in which the Jacobian determinants of neighboring regions have opposite signs. With reference to Fig. 4, assume that the Jacobian determinants of $\underset{\sim}{J}$ and $\underset{\sim}{J}'$ have opposite signs. The solution curve in the x-space has been traced from $\underset{\sim}{x}_j$ in region R along the direction $\underset{\sim}{d}_j$ and reaches $\underset{\sim}{x}_{j+1}$ on the simple boundary between R and R'. In the y-space, the corresponding points $\underset{\sim}{y}_j$ and $\underset{\sim}{y}_{j+1}$ are marked. In order to force the solution curve in the $\underset{\sim}{x}$-space into region R', we must reverse the direction of traversing at $\underset{\sim}{y}_{j+1}$ away from $\underset{\sim}{y}^*$ as shown in Fig. 4. The proof of this together with the case which has determinants with same sign is given in Lemma 1 below, stated in a fairly general form.

<u>Lemma 1</u> The solution curve will enter the region R' at $\underset{\sim}{x}_{j+1}$ and traverse region R' along the direction $\underset{\sim}{d}_{j+1}$ given by either (i) or (ii):

(i) $\underset{\sim}{d}_{j+1} = \underset{\sim}{J}'^{-1} (\underset{\sim}{y}^* - \underset{\sim}{y}_{j+1})$ if

(a) $\underset{\sim}{d}_j = \underset{\sim}{J}^{-1}(\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ and $(\det \underset{\sim}{J}) (\det \underset{\sim}{J}') > 0$, or

(b) $\underset{\sim}{d}_j = -\underset{\sim}{J}^{-1}(\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ and $(\det \underset{\sim}{J}) (\det \underset{\sim}{J}') < 0$;

(ii) $\underset{\sim}{d}_{j+1} = - \underset{\sim}{J}'^{-1}(\underset{\sim}{y}^* - \underset{\sim}{y}_{j+1})$ if

(a) $\underset{\sim}{d}_j = \underset{\sim}{J}^{-1}(\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ and $(\det \underset{\sim}{J}) (\det \underset{\sim}{J}') < 0$, or

(b) $\underset{\sim}{d}_j = - \underset{\sim}{J}^{-1}(\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ and $(\det \underset{\sim}{J}) (\det \underset{\sim}{J}') > 0$.

<u>Proof</u>: Since the solution curve reaches $\underset{\sim}{x}_{j+1}$ on the boundary from $\underset{\sim}{x}_j$, we have $\underset{\sim}{n}^T \underset{\sim}{d}_j > 0$ where $\underset{\sim}{n}$ is the normal vector at $\underset{\sim}{x}_{j+1}$ of the hyperplane. Furthermore, the solution curve will enter and traverse R' if and only if $\underset{\sim}{n}^T \underset{\sim}{d}_{j+1} > 0$. Suppose $\underset{\sim}{y}_j$ and $\underset{\sim}{y}_{j+1}$ are represented by the following equations

$$\underset{\sim}{y}_j = \underset{\sim}{y}_0 + \mu_j (\underset{\sim}{y}^* - \underset{\sim}{y}_0), \quad 1 > \mu_j > 0$$

$$\underset{\sim}{y}_{j+1} = \underset{\sim}{y}_0 + \mu_{j+1}(\underset{\sim}{y}^* - \underset{\sim}{y}_0), \quad 1 > \mu_{j+1} > 0$$

then

$$\underset{\sim}{y}^* - \underset{\sim}{y}_j = (1-\mu_j) (\underset{\sim}{y}^* - \underset{\sim}{y}_0)$$

$$\underset{\sim}{y}^* - \underset{\sim}{y}_{j+1} = (1-\mu_{j+1}) (\underset{\sim}{y}^* - \underset{\sim}{y}_0)$$

The proof then follows immediately from eq. (11). It is important to note, from lemma (1), that if $(\det \underset{\sim}{J}) (\det \underset{\sim}{J}') < 0$, we must reverse the direction of traversing at $\underset{\sim}{y}_{j+1}$ in the y-space along L_y in order to make $\underset{\sim}{n}^T \underset{\sim}{d}_j$ and $\underset{\sim}{n}^T \underset{\sim}{d}_{j+1}$ both positive.

From the computation point of view, the simplified statement given in the following lemma is more useful.

Lemma 2. If $\underset{\sim}{x}_0$ is an interior point of the region R_0 then $\underset{\sim}{d}_j$ is given by

(i) $\underset{\sim}{J}^{(j)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ if $(\det \underset{\sim}{J}^{(j)})(\det \underset{\sim}{J}^{(0)}) > 0$, or

(ii) $-\underset{\sim}{J}^{(j)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ if $(\det \underset{\sim}{J}^{(j)}) (\det \underset{\sim}{J}^{(0)}) < 0$.

Proof: Since $\underset{\sim}{x}_0$ is an interior point of R_0,

$$\underset{\sim}{d}_0 = \underset{\sim}{J}^{(0)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_0)$$

It follows, from Lemma 1, that

$$\underset{\sim}{d}_1 = \underset{\sim}{J}^{(1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_1) \text{ if } (\det \underset{\sim}{J}^{(1)}) (\det \underset{\sim}{J}^{(0)}) > 0, \text{ or}$$

$$\underset{\sim}{d}_1 = -\underset{\sim}{J}^{(1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_1) \text{ if } (\det \underset{\sim}{J}^{(1)})(\det \underset{\sim}{J}^{(0)}) < 0.$$

We want to prove this lemma by induction. First assume that the lemma is true for the j-1th region, that is,

$$\underset{\sim}{d}_{j-1} = \underset{\sim}{J}^{(j-1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_{j-1}) \text{ and } (\det \underset{\sim}{J}^{(j-1)}) (\det \underset{\sim}{J}^{(0)}) > 0 \text{ or}$$

$$\underset{\sim}{d}_{j-1} = -\underset{\sim}{J}^{(j-1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_{j-1}) \text{ and } (\det \underset{\sim}{J}^{(j-1)}) (\det \underset{\sim}{J}^{(0)}) < 0.$$

From Lemma 1, we have $\underset{\sim}{d}_j = \underset{\sim}{J}^{(j)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ if

(a) $\underset{\sim}{d}_{j-1} = \underset{\sim}{J}^{(j-1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_{j-1})$ and $(\det \underset{\sim}{J}^{(j)}) (\det \underset{\sim}{J}^{(j-1)}) > 0$ or

(b) $\underset{\sim}{d}_{j-1} = -\underset{\sim}{J}^{(j-1)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_{j-1})$ and $(\det \underset{\sim}{J}^{(j)}) (\det \underset{\sim}{J}^{(j-1)}) < 0$.

Thus if $(\det \underset{\sim}{J}^{(j)}) (\det \underset{\sim}{J}^{(0)}) > 0$, clearly

$$(\det \underset{\sim}{J}^{(j)}) \ (\det \underset{\sim}{J}^{(j-1)}) \ (\det \underset{\sim}{J}^{(j-1)}) \ (\det \underset{\sim}{J}^{(0)}) > 0$$

we conclude $\underset{\sim}{d}_j = \underset{\sim}{J}^{(j)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_j)$ if $(\det \underset{\sim}{J}^{(j)})\ (\det \underset{\sim}{J}^{(0)}) > 0$. Thus (i) is proven. Using the same argument, we can prove (ii). The advantage of Lemma 2 is that we only need to look into a new region and compare its Jacobian determinant with that of the initial region.

The above algorithms take care of the solution curve at a local point on the boundary when Jacobian determinants have different signs. On the other hand the global behavior of the solution curve needs to be investigated. It turns out that if we do not have additional properties imposed on the continuous piecewise linear function the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ may not have a solution. Furthermore, the algorithm of tracing a solution curve may not work even if soluctions do exist. Before we derive the main result in the following section, which imposes further conditions on $\underset{\sim}{f}$ and guarantees the convergence of the algorithm, we will present two examples to illustrate two possible difficulties which have not been encountered up to now.

Example 1. Consider a continuous, piecewise-linear function $\underset{\sim}{f}$ specified by the following equations with the regions shown in Fig. 5.

In R_1 $\quad \underset{\sim}{y} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \underset{\sim}{x}$, $\det \underset{\sim}{J}^{(1)} = 1.$

In R_2, $\quad \underset{\sim}{y} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underset{\sim}{x}$, $\det \underset{\sim}{J}^{(2)} = 1$

In R_3, $\quad \underset{\sim}{y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underset{\sim}{x}$, $\det \underset{\sim}{J}^{(3)} = -1$

In R_4, $\quad \underset{\sim}{y} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} \underset{\sim}{x} \quad , \quad \det \underset{\sim}{J}^{(4)} = 1.$

Since the problem is simple, it is possible to determine the complete mapping of the 2-dimensional space. The mapping $\underset{\sim}{f}(\mathcal{R}^2)$ is shown in Fig. 5 in the $\underset{\sim}{y}$-space in order to have a better understanding of the problem at hand. Note the overlapping in the second quadrant of the $\underset{\sim}{y}$-space. Thus for eacy $\underset{\sim}{y}$ in the second quadrant, there exist three solutions which lie in R_2, R_3 and R_4 of the $\underset{\sim}{x}$-space.

Let us assume that the given input vector is $\underset{\sim}{y}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and we wish to find the solution of the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. We choose arbitrarily the initial point $\underset{\sim}{x}_0 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$ in region R_3. $\underset{\sim}{y}_0$ is calculated to be $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$. A line segment is drawn from $\underset{\sim}{y}_0$ to $\underset{\sim}{y}^*$ as shown in Fig. 5. Following the algorithm just described, we obtain the first segment of the solution curve in the $\underset{\sim}{x}$-space. It is given by

$$\underset{\sim}{x}_0(\lambda) = \underset{\sim}{x}_0 + \lambda\, \underset{\sim}{d}_0 \quad , \quad 0 \leq \lambda \leq \lambda_0$$

where

$$\underset{\sim}{d}_0 = \underset{\sim}{J}^{(3)^{-1}} (\underset{\sim}{y}^* - \underset{\sim}{y}_0) = \begin{pmatrix} 3/2 \\ 2 \end{pmatrix}$$

and λ_0 represents the maximum value of λ for which $\underset{\sim}{x}_0(\lambda)$ is in R_3. λ_0 is found to be 1/3 and $\underset{\sim}{x}_1 = \underset{\sim}{x}_0(\frac{1}{3}) = \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix}$, $\underset{\sim}{y}_1 = \underset{\sim}{f}(\underset{\sim}{x}_1) = \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix}$. The next region is identified as R_4. Since $(\det \underset{\sim}{J}^{(3)})\ (\det \underset{\sim}{J}^{(4)}) < 0$, in order to enter R_4, we traverse in the $\underset{\sim}{y}$-space from $\underset{\sim}{y}_1$ away from $\underset{\sim}{y}^*$. The direction $\underset{\sim}{d}_1$ is given by Lemma 2, $-\underset{\sim}{J}^{(4)^{-1}}(\underset{\sim}{y}^* - \underset{\sim}{y}_1) = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$. The second segment of the solution curve is $\underset{\sim}{x}_1(\lambda) = \underset{\sim}{x}_1 + \lambda \underset{\sim}{d}_1$, $\lambda \geq 0$. It enters R_4 as shown in the figure, but will not reach any boundary hyperplane. Thus the algorithm fails to find a solution. Of course, as we can see, in the y-space, $\underset{\sim}{y}^*$ is never reached. Instead, the curve will pass by $\underset{\sim}{y}_0$ and become unbounded.

Example 2. Consider the continuous, piecewise-linear function given by the following, together with the regions indicated in Fig. 6a.

In R_1 $\quad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \underset{\sim}{x}$, $\quad \det \underset{\sim}{J}^{(1)} = -1$

In R_2 $\quad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \underset{\sim}{x}$, $\quad \det \underset{\sim}{J}^{(2)} = 1$

In R_3 $\quad \underset{\sim}{y} = \begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix} \underset{\sim}{x}$, $\quad \det \underset{\sim}{J}^{(3)} = 1$

In R_4 $\quad \underset{\sim}{y} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 5 \\ 15 \end{pmatrix}$, $\quad \det \underset{\sim}{J}^{(4)} = 1$

In R_5 $\quad \underset{\sim}{y} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 10 \\ 20 \end{pmatrix}$, $\quad \det \underset{\sim}{J}^{(5)} = 3$

In R_6 $\quad \underset{\sim}{y} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 10 \\ 20 \end{pmatrix}$, $\quad \det \underset{\sim}{J}^{(6)} = 1$

In R_7 $\quad \underset{\sim}{y} = \begin{pmatrix} 2 & 1 \\ 2.5 & 1.5 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 5 \\ 12.5 \end{pmatrix}$, $\quad \det \underset{\sim}{J}^{(7)} = .5$

$$\text{In } R_8 \qquad \underset{\sim}{y} = \begin{pmatrix} 2 & 1 \\ 2.5 & 3.5 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 5 \\ 12.5 \end{pmatrix}, \quad \det \underset{\sim}{J}^{(8)} = 4.5$$

$$\text{In } R_9 \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underset{\sim}{x}, \quad \det \underset{\sim}{J}^{(9)} = 1$$

$$\text{In } R_{10} \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underset{\sim}{x}, \quad \det \underset{\sim}{J}^{(10)} = -1$$

The images of these regions in the $\underset{\sim}{y}$-space are shown in Fig. 6b. Again it is noted that there is an overlapping in the $\underset{\sim}{y}$-space as shown by the shaded area. For every $\underset{\sim}{y}$ in this area, there exist multiple solutions.

Let us consider the given input vector $\underset{\sim}{y}^* = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$. Suppose that $\underset{\sim}{x}_0$ is chosen to be $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ in R_7. $\underset{\sim}{y}_0$ is found to be $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$. The solution curve corresponding to $L_y = \{\underset{\sim}{y} | \underset{\sim}{y} = \underset{\sim}{y}_0 + \lambda(\underset{\sim}{y}^* - \underset{\sim}{y}_0)\}$, can be traced by the algorithm given in this section. It is found that the solution $\underset{\sim}{x}^* = \begin{pmatrix} 13 \\ 2 \end{pmatrix}$ in four steps as shown in the figure. Next, suppose that $\underset{\sim}{x}_0$ is chosen to be $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ in R_1, then $\underset{\sim}{y}_0 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ which happens to be in the shaded area. If we go through with our algorithm, we find that the solution curve is a cyclic curve shown in Fig. 6a. The corresponding traversing in the y-space is marked. Thus $\underset{\sim}{y}^*$ can never be reached. This example points out that a solution curve in the $\underset{\sim}{x}$-space can reenter a region previously traversed and form a cyclic curve when Jacobian determinants have different signs. It should be noted that this does not contradict the early statement and proof since the region is reentered at the same point $\underset{\sim}{x}_j$ and the solution curve is retraced. Thus the choice of initial point is of primary importance in order to avoid the situation as illustrated.

III. The main result

The main purpose of this paper is to obtain conditions not as restrictive as the "sign condition," yet under which a convergent algorithm can be developed to obtain one or more solutions of piecewise-linear equations. In this connection, the two examples give us considerable insight as to the nature of the conditions we are looking for. First let us present the following lemma concerning existence of solutions.

Lemma 3. Let $\underset{\sim}{f}(\cdot)$ maps from $\mathcal{R}^n$ into $\mathcal{R}^n$ be continuous and piecewise-linear. Let the Jacobian determinants of all unbounded regions have the same sign, then there exists a constant $M > 0$ such that

(i) there exists at least one solution to the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ if $\|\underset{\sim}{y}\| \geq M$, and

(ii) there exists no solution in any bounded regions.

Proof The proof follows from that given in Appendix 1 of [5]. Since $\underset{\sim}{f}$ is continuous, the image of the union of all bounded regions is contained in a ball $B(0,M_1)$ in the $\underset{\sim}{y}$-space. Suppose (i) is not true, then there exists an unbounded boundary hyperplane between $f(\mathcal{R}^n)$ and $\mathcal{R}^n/\underset{\sim}{f}(\mathcal{R}^n)$. However, $\underset{\sim}{f}$ is a local homeomorphism at any point on this simple boundary hyperplane since the determinants of Jacobians of all unbounded regions have the same sign. This is the desired contradiction. Therefore, $M > M_1 > 0$ exists such that both (i) and (ii) are true.

The condition (ii) of this lemma is of primary importance in developing a convergent algorithm. Let us assume that the initial point $\underset{\sim}{x}_0$ is chosen such that $\|\underset{\sim}{y}_0\| \geq M$ as given in Lemma 3, where $\underset{\sim}{x}_0$ is an interior point of a region. Since there are no other solutions of $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}_0$ in any bounded region, in traversing from $\underset{\sim}{y}_0$ to $\underset{\sim}{y}^*$, we are certain that the curve will not come back to $\underset{\sim}{y}_0$ and go beyond $\underset{\sim}{y}_0$. Thus if we assume that the solution curve does not hit any corners, the solution curve cannot become cyclic. Since regions previously entered cannot be reentered and since the total number of regions is finite, we conclude that the following algorithm will converge in a maximum of ℓ steps where ℓ is the total number of regions.

Algorithm I

Step 1: Choose $\underset{\sim}{x}_0$, an interior point of R_0, such that $\|\underset{\sim}{y}_0\| \geq M$, where M is defined in Lemma 3. Set $j = 0$.

Step 2: Compute $\underset{\sim}{d}_j$ according to Lemma 2.

Step 3: Compute $\underset{\sim}{x}_{j+1} = \underset{\sim}{x}_j + \lambda_j \underset{\sim}{d}_j$, $\underset{\sim}{y}_{j+1} = \underset{\sim}{f}(\underset{\sim}{x}_{j+1})$ where $\lambda_j > 0$ is the maximum value such that $\{\underset{\sim}{x}(\lambda) = \underset{\sim}{x}_j + \lambda \underset{\sim}{d}_j,\ 0 \leq \lambda \leq \lambda_j\}$ is in R_j.

Step 4: If $\underset{\sim}{J}^{(j)} \underset{\sim}{d}_j = \underset{\sim}{y}^* - \underset{\sim}{y}_j$ and $\lambda_j \geq 1$, then $\underset{\sim}{x}^* = \underset{\sim}{x}_j + \underset{\sim}{d}_j$ is a solution. Stop.

Step 5: Otherwise, identify region R_{j+1}. Set $j = j + 1$ and go to Step 2.

To conclude, we state the following theorem:

Theorem 1: Algorithm I will find a solution in finite number of steps if

(i) All Jacobian matrices are nonsingular,

(ii) the determinants of Jacobian matrices in all unbounded regions have the same sign, and

(iii) the solution curve does not hit any corners.

It is possible to modify the algorithm slightly so that once a solution is found, the algorithm will continue from that point to find other solutions without choosing a new initial point. The details will not be given here.

In the next two chapters we will deal with, first, the corner problem, and then the problem involving singular Jacobian matrices. We will remove the conditions in (i) and (iii) of Theorem 1.

IV. The corner problem.

This section is devoted to the study of the corner problem. First, we will give an example to illustrate one possible difficulty which arises when the solution curve hits corners. Next we will study whether it is possible to choose the initial point so that the solution curve will never hit any corners. Intuitively, we can see that this is always possible since there are only finite number of corners. Furthermore, since corners are at places where hyperplanes meet; therefore, if we deal with a two dimensional space, for example, boundary hyperplanes are straight lines and corners are points. Clearly, it is possible to choose an initial point in the $\underset{\sim}{x}$-space such that the solution curve avoids all corners. This is proven in this section for the general case.

Let us first review the problem at hand. When a solution curve reaches a corner, the previous algorithm cannot determine the next region to be entered. Suppose that, by means of a perturbation technique, we can determine the next region to be entered, we can then use the corner as the next starting point to continue the tracing of the solution curve in the proper region.

Example 3. Let the continuous, piecewise-linear function be defined by the following equations with the regions shown in Fig. 7.

$$\text{In } R_1 \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underset{\sim}{x} \qquad\qquad \det \underset{\sim}{J}_{*}^{(1)} = 1$$

$$\text{In } R_2 \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \underset{\sim}{x} \qquad\qquad \det \underset{\sim}{J}^{(2)} = 1$$

$$\text{In } R_3 \qquad \underset{\sim}{y} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \qquad \det \underset{\sim}{J}^{(3)} = 2$$

$$\text{In } R_4 \qquad \underset{\sim}{y} = \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \qquad \det \underset{\sim}{J}^{(6)} = -2$$

$$\text{In } R_5 \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \underset{\sim}{x} \qquad\qquad \det \underset{\sim}{J}^{(5)} = -1$$

$$\text{In } R_6 \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underset{\sim}{x} \qquad\qquad \det \underset{\sim}{J}^{(6)} = -1$$

Let the input vector be $\underset{\sim}{y}^{*} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, and the initial point be $\underset{\sim}{x}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in R_3. Thus $\underset{\sim}{y}_0 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ $\underset{\sim}{x}_1$ and $\underset{\sim}{y}_1$ can be calculated to be $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, respectively. We use a parallel perturbation and find R_2 as one possible next region. Therefore we start with $\underset{\sim}{x}_1$ and use the equation for R_2 to determine the next portion of the solution curve. x_2 and y_2 are found to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Thereafter, the solution curve traverses R_1 and R_6 and R_5 successively and returns to the point $\underset{\sim}{x}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. And, once again R_2 is found to be a possible region to be entered. Thus a cyclic solution curve emerges. Actually, for this problem no solution exists for the input $\underset{\sim}{y}^* = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$. This example points out nevertheless an important property. When a solution curve hits a corner, it can become cyclic although the initial point $\underset{\sim}{x}_0$ is not on the cyclic portion of the solution curve. This could not have happened if the solution curve never hits a corner.

In the following, we will demonstrate that it is indeed possible to choose an $\underset{\sim}{x}_0$ for which the solution curve never hits any corners. As indicated in Section 1, a boundary hyperplane can be represented as $H_x = \{\underset{\sim}{x} | \underset{\sim}{n}^T \underset{\sim}{x} = \gamma\}$, where $\underset{\sim}{n}$ is the normal vector to the hyperplane and γ is a constant. A corner is a subset of the intersection of two or more hyperplanes in the $\underset{\sim}{x}$-space. Suppose that the Jacobian matrices are non-singular, the image of any boundary hyperplane of region R is a hyper-plane H_y in the $\underset{\sim}{y}$-space. The image of a corner in the $\underset{\sim}{x}$-space is a subset of the intersection of two or more hyperplanes in the $\underset{\sim}{y}$-space. For convenience, the image of a boundary hyperplane in the $\underset{\sim}{x}$-space is called a boundary hyperplane in the $\underset{\sim}{y}$-space. Similarly, the image of a corner in the $\underset{\sim}{x}$-space is called a corner in the $\underset{\sim}{y}$-space. Thus in the $\underset{\sim}{y}$-space, $H_y = \{\underset{\sim}{y} | \underset{\sim}{p}^T \underset{\sim}{y} = q\}$ and a corner can be represented as a subset of $C_y = \{\underset{\sim}{y} | \underset{\sim}{P}^T \underset{\sim}{y} = \underset{\sim}{Q}\}$, where $\underset{\sim}{P}$ is an $n \times 2$ matrix and $\underset{\sim}{Q}$ a 2-dimensional vector.

Also, for convenience, we transfer the coordinate of $\underset{\sim}{y}^*$ to be at the origin, hence, we are dealing with the equation $\underset{\sim}{f}(\underset{\sim}{x}) = 0$. The purpose of the development to follow is to locate a straight line $L_y = \{\underset{\sim}{y} | \underset{\sim}{y} = \mu \underset{\sim}{a}\}$ where $\underset{\sim}{a}$ is a unit vector and μ is a positive parameter, such that L_y does not intersect any corners in the $\underset{\sim}{y}$-space. Thus the solution curve in the $\underset{\sim}{x}$-space corresponding to L_y will not meet any corners. Since a corner is represented by a subset of C_y, we need to find a vector $\underset{\sim}{h}$ such that $\underset{\sim}{h}^T \underset{\sim}{y} = 0$ for all $\underset{\sim}{y} \in C_y$. Thus if $\underset{\sim}{h}^T \underset{\sim}{a} \neq 0$, L_y defined by $\{\underset{\sim}{y} | \underset{\sim}{y} = \mu \underset{\sim}{a}\}$ will not intersect Cy. It is obvious that $\underset{\sim}{h}$ can always be determined because Cy is contained in Sp(Cy), the span of Cy which is an (n-1) dimensional subspace assuming that Cy does not contain the origin. Since the number of corners is finite, we expect that $\underset{\sim}{a}$ can be found by induction. This result is stated below as a theorem.

<u>Theorem 2</u>: Let $S = \{Cy_i\}_{i=1}^{N}$ be the set of corners in the $\underset{\sim}{y}$-space. Suppose that the origin is not contained in the union of Cy_i's, then there exists a unit vector $\underset{\sim}{a}$ for which $Ly = \{\underset{\sim}{y} | \underset{\sim}{y} = \mu \underset{\sim}{a}\}$ does not intersect Cy_i for all i.

<u>Proof</u>: First we need to characterize a unit vector $\underset{\sim}{h}_i$ corresponding to Cy_i such that $\underset{\sim}{h}_i^T \underset{\sim}{y} = 0$ for all $\underset{\sim}{y} \in Cy_i$. This can be done easily. Since $Cy_i = \{y | \underset{\sim}{P}_i^T \underset{\sim}{y} = \underset{\sim}{Q}_i\}$, $Sp(Cy_i)$ is an (n-1) dimensional subspace. Let $\underset{\sim}{y}_1^{(i)}, \underset{\sim}{y}_2^{(i)}, \ldots, \underset{\sim}{y}_{n-1}^{(i)}$ constitute a basis for $Sp(Cy_i)$, which can be determined. Then $\underset{\sim}{h}_i$ is defined by the following

$$\begin{bmatrix} \underset{\sim}{y}_1^{(i)T} \\ \underset{\sim}{y}_2^{(i)T} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{y}_{n-1}^{(i)T} \end{bmatrix} \underset{\sim}{h}_i = 0 \quad \text{and } \underset{\sim}{h}_i^T \underset{\sim}{h}_i = 1 \tag{12}$$

Next we need to construct the unit vector $\underset{\sim}{a}$ from $\underset{\sim}{h}_i$, i=1,2,...,N. This, we will do by induction. Let $\underset{\sim}{a}_1 = \underset{\sim}{h}_1$. It is obvious that $\underset{\sim}{h}_1^T \underset{\sim}{a}_1 \neq 0$. Suppose that $\underset{\sim}{h}_j^T \underset{\sim}{a}_i \neq 0$ for $1 \leq j \leq i$, we want to show that $\underset{\sim}{a}_{i+1}$ can be computed for which $\underset{\sim}{h}_j^T \underset{\sim}{a}_{i+1} \neq 0$ for $1 \leq j \leq (i+1)$. There are two cases to be considered:

(i) $\underset{\sim}{h}_{i+1}^{T} \underset{\sim}{a}_i \neq 0$, we simply define $\underset{\sim}{a}_{i+1} = \underset{\sim}{a}_i$.

(ii) $\underset{\sim}{h}_{i+1}^{T} \underset{\sim}{a}_i = 0$, we perturb $\underset{\sim}{a}_i$ to obtain $\underset{\sim}{a}_{i+1}$ according to

$$\underset{\sim}{a}_{i+1} = \underset{\sim}{a}_i + \underset{\sim}{v} \tag{13}$$

where

$$\underset{\sim}{h}_{i+1}^{T} \underset{\sim}{v} \neq 0$$

Furthermore, the magnitude of $\underset{\sim}{v}$ should be small enough so that it does not concel the effect of previous perturbations, if any. Let

$$m = \min_{1 \leq j \leq i} |\underset{\sim}{h}_j^{T} \underset{\sim}{a}_i| \tag{14}$$

$$M = \max_{1 \leq j \leq i+1} |\underset{\sim}{h}_j^{T} \underset{\sim}{h}_{i+1}| \tag{15}$$

We define

$$\underset{\sim}{a}_{i+1} = \underset{\sim}{a}_i + \frac{m}{2MK} \underset{\sim}{h}_{i+1} \quad , \quad K \geq 1 \tag{16}$$

It is easy to see that $\underset{\sim}{h}_j^{T} \underset{\sim}{a}_{i+1} \neq 0$ for $1 \leq j \leq i+1$. Premultiplying the above equation by $\underset{\sim}{h}_j^{T}$, we obtain

$$\underset{\sim}{h}_j^{T} \underset{\sim}{a}_{i+1} = \underset{\sim}{h}_j^{T} \underset{\sim}{a}_i + \frac{m}{2MK} \underset{\sim}{h}_j^{T} \underset{\sim}{h}_{i+1} \tag{17}$$

For $j = i+1$, $\underset{\sim}{h}_{i+1}^{T} \underset{\sim}{a}_{i+1}$ is clearly positive. For $1 \leq j \leq i$,

$$|\underset{\sim}{h}_j^{T} \underset{\sim}{a}_{i+1}| \geq |\underset{\sim}{h}_j^{T} \underset{\sim}{a}_i| - \frac{m}{2MK} |\underset{\sim}{h}_j^{T} \underset{\sim}{h}_{i+1}| \geq m - \frac{m}{2MK} M \geq \frac{m}{2} > 0 \tag{18}$$

Since the total number of corners is N which is finite, we have demonstrated that a_N can always be obtained such that

$$\underset{\sim}{h}_j^T \underset{\sim}{a}_N \neq 0 \quad \text{for} \quad 1 \leq j \leq N \tag{19}$$

Thus the unit vector $\underset{\sim}{a} = \frac{a_N}{\|\underset{\sim}{a}_N\|}$ has the property that $Ly = \{\underset{\sim}{y} | \underset{\sim}{y} = \mu \underset{\sim}{a}\}$ does not intersect any corners. This completes the proof of theorem 2.

From the computational point of view, it is certainly impractical to compute $\underset{\sim}{a}$ according to the suggested procedure. Since it is not a frequent event that the solution curve will hit a corner, we should only carry out a perturbation if and when the soltuion curve hits a corner. Thus if a corner represented by $Cy_i = \{\underset{\sim}{y} | \underset{\sim}{P}_i^T \underset{\sim}{y} = \underset{\sim}{Q}_i\}$ is hit, we need to perturb the vector $\underset{\sim}{a}$ by a small vector $\underset{\sim}{v} = \frac{m}{2MK} \underset{\sim}{h}_i$ where $\underset{\sim}{h}_i$ is perpendicular to the subspace spanned by Cy_i.

With this, we have overcome the corner problem. The condition (iii) given in Theorem 1 that the solution curve does not hit any corners has therefore been justified.

V. Singular Jacobian Matrices

As seen from the development of various results so far, the assumption of the nonsingularity of all Jacobian matrices plays a major role. For example, without this assumption , the direction $\underset{\sim}{d}_j$ in region R_j cannot be defined by our algorithm; furthermore, the solution curve can reenter a region which has been traversed previously. This fact can be illustrated as follows: Intuitively, when we encounter a region R_j with singular Jacobian $\underset{\sim}{J}^{(j)}$, we may wish to choose $\underset{\sim}{d}_j$ according to $\underset{\sim}{J}^{(j)} \underset{\sim}{d}_j = 0$. This implies that in the y-space $\underset{\sim}{y}_{j+1} = \underset{\sim}{y}_j$, that is, the solution curve traverses in the x-space across region R_j via $\underset{\sim}{d}_j$ when in the $\underset{\sim}{y}$-space the image stands still. Obviously, we run into difficulty in that there may exist more than one vector which satisfies the equation $J^{(j)} d_j = 0$. Then, as shown in Fig. 8, if d_1 and d_2 are two such vectors which both satisfy the equation, the solution curve can indeed reenter region R_j and become cyclic. Before pursuing to the the development, we again illustrate with an example the complication involved.

<u>Example 4</u>. Consider the continuous, piecewise-linear function as follows

In R_1 $\quad y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\qquad$ $\det J^{(1)} = 0$

In R_2 $\quad y = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} x$ $\qquad$ $\det J^{(2)} = 0$

In R_3 $\quad y = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} x$ $\qquad$ $\det J^{(3)} = 0$

In R_4 $\quad y = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} x$ $\qquad$ $\det J^{(4)} = 0$

In R_5 $\quad y = \begin{pmatrix} 1 & 1 \\ -\frac{2}{3} & -\frac{1}{4} \end{pmatrix} x + \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ $\qquad$ $\det J^{(5)} = \frac{5}{12}$

In R_6 $\quad y = \begin{pmatrix} -1 & 1 \\ 0 & -\frac{1}{4} \end{pmatrix} x + \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ $\qquad$ $\det J^{(6)} = \frac{1}{4}$

In R_7 $\quad y = \begin{pmatrix} -1 & 2 \\ 0 & -\frac{9}{4} \end{pmatrix} x + \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ $\qquad$ $\det J^{(7)} = \frac{9}{4}$

$$\text{In } R_8 \qquad \underset{\sim}{y} = \begin{pmatrix} -\frac{1}{5} & 1 \\ -1 & -1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad \det \underset{\sim}{J}^{(8)} = \frac{6}{5}$$

$$\text{In } R_9 \qquad \underset{\sim}{y} = \begin{pmatrix} \frac{9}{5} & 1 \\ -\frac{11}{5} & -1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad \det \underset{\sim}{J}^{(9)} = \frac{2}{5}$$

$$\text{In } R_{10} \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \qquad \det \underset{\sim}{J}^{(10)} = 1$$

$$\text{In } R_{11} \qquad \underset{\sim}{y} = \begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{4} \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \qquad \det \underset{\sim}{J}^{(11)} = \frac{5}{4}$$

The regions in the $\underset{\sim}{x}$-space are shown in Fig. 9a, the images in the $\underset{\sim}{y}$-space are shown in Fig. 9b. It is seen that the four regions R_1, R_2, R_3, and R_4 into which singular Jacobian matrices are mapped become a straight line in the $\underset{\sim}{y}$-space. The solution curves corresponding to three different cases are shown. Note specially that the solution curve in $\underset{\sim}{x}$-space corresponding to any point on the line segment $\{\underset{\sim}{y} | \underset{\sim}{y} = \mu \begin{pmatrix} 2 \\ 2 \end{pmatrix},\ 0 \le \mu \le 1\}$ is a closed curve going through regions R_1, R_2, R_3 and R_4.

There are two methods to overcome this problem. The first one is to perturb the Jacobian matrices which are singular such that the perturbed function is sufficiently close to the original function, yet contains no singular matrices. This method will not be given in this paper. We shall present in this section the second method in which we prove that it is always possible to choose an initial point in the $\underset{\sim}{x}$-space such that the solution curve will be noncyclic and furthermore will not reenter a region previously traversed. It turns out that the key to this approach is to distinguish between singular Jacobians which are of rank (n-1) and those with rank less than (n-1).

Let a singular region be a region whose Jacobian is singular. If the rank of $\underset{\sim}{J}$ for region R is (n-1), there exists a vector $\underset{\sim}{p}$ such that $\underset{\sim}{p}^T \underset{\sim}{J} = 0$. Consequently, the image of R is a subset of the hyperplane $\{\underset{\sim}{y} | \underset{\sim}{p}^T \underset{\sim}{y} = \underset{\sim}{p}^T \underset{\sim}{w}\}$ because $\underset{\sim}{y} = \underset{\sim}{J}\, \underset{\sim}{x} + \underset{\sim}{w}$ and $\underset{\sim}{p}^T \underset{\sim}{y} = \underset{\sim}{p}^T \underset{\sim}{J}\, \underset{\sim}{x} + \underset{\sim}{p}^T \underset{\sim}{w} = \underset{\sim}{p}^T \underset{\sim}{w}$. On the other hand, if the rank of $\underset{\sim}{J}$ is (n-2) or less, there exist at least two vectors $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_2$, $\underset{\sim}{p}_1^T \underset{\sim}{p}_2 \neq 0$, such that $\underset{\sim}{p}_1^T \underset{\sim}{J}$ and $\underset{\sim}{p}_2^T \underset{\sim}{J}$ are both zero. Let $\underset{\sim}{P} = (\underset{\sim}{p}_1, \underset{\sim}{p}_2)$ and $\underset{\sim}{Q} = (\underset{\sim}{p}_1, \underset{\sim}{p}_2)^T \underset{\sim}{w}$. The image of R is then a subset of $\{\underset{\sim}{y} | \underset{\sim}{P}^T \underset{\sim}{y} = \underset{\sim}{Q}\}$ which behaves exactly like a corner. This suggests that we can use the result of the previous section to deal with those singular regions which are of rank (n-2) or less. All we need to do is to ensure that Ly in the $\underset{\sim}{y}$-space does not intersect with any singular regions whose Jacobians are of rank (n-2) or less. Theorem 2 shows that this can always be done. Therefore, we shall persue immediately to the case in which all singular Jacobian matrices are of rank (n-1). First, we need to understand some basic properties of singular Jacobian matrices in connection with continuous, piecewise-linear functions.

<u>Lemma 4</u>. Let the rank of $\underset{\sim}{J}$ for region R be (n-1). Let Ly = $\{\underset{\sim}{y} | \underset{\sim}{y} = \mu \underset{\sim}{a},\ \mu \geq 0\}$ intersect the image of region R under the mapping $\underset{\sim}{f}$, $\underset{\sim}{f}(R)$. Then either (i) Ly intersects $\underset{\sim}{f}(R)$ at one and only one point, or (ii) Ly is a subset of an (n-1) dimensional subspace which contains $\underset{\sim}{f}(R)$.

<u>Proof:</u> Suppose that Ly intersects $\underset{\sim}{f}(R)$ at two points, namely: $\underset{\sim}{y}_1 = \mu_1 \underset{\sim}{a}$, $\underset{\sim}{y}_2 = \mu_2 \underset{\sim}{a}$, $\mu_1 \neq \mu_2$, then there eixsts a vector $\underset{\sim}{p}$ such that $\underset{\sim}{p}^T \underset{\sim}{y}_1 = \underset{\sim}{p}^T \underset{\sim}{y}_2$. This implies $\underset{\sim}{p}^T \underset{\sim}{a} = 0$ since $\mu_1 \neq \mu_2$. Therefore, Ly is a subset of the subspace $\{\underset{\sim}{y} | \underset{\sim}{p}^T \underset{\sim}{y} = 0\}$ which contains $\underset{\sim}{f}(R)$. This completes the proof.

In the two dimensional case, this lemma is illustrated by Fig. 10. Note that $\underset{\sim}{f}(R)$ is a line segment, thus it either intersects an Ly which connects to the origin or is a subset of an Ly.

Let us consider the case in which Ly is a subset of an (n-1) dimensional subspace which contains $\underset{\sim}{f}(R)$. Obviously, we can treat this case like the corner problem, since it is always possible to find an Ly which does not intersect the image of such a region. Therefore, we are left with the remaining case that Ly hits a singular region of rank (n-1) at precisely one point. The following lemma tells us that, for this case, there is a unique direction $\underset{\sim}{d}$ in R for which $\underset{\sim}{J}\,\underset{\sim}{d} = 0$, and furthermore the solution curve will not reenter this region afterwards.

<u>Lemma 5</u>. Let the Jacobian $\underset{\sim}{J}$ of region R be of rank (n-1). Let $Ly = \{\underset{\sim}{y} | \underset{\sim}{y} = \mu \underset{\sim}{a},\ \mu \geq 0\}$ intersect $\underset{\sim}{f}(R)$ at one and only one point. Let the solution curve in the $\underset{\sim}{x}$-space enter R at $\underset{\sim}{x}_j$ and leave R at $\underset{\sim}{x}_{j+1}$, then $\underset{\sim}{x}_{j+1} = \underset{\sim}{x}_j + \lambda \underset{\sim}{d}_j$ where $\underset{\sim}{J}\,\underset{\sim}{d}_j = 0$. Furthermore, the solution curve in the $\underset{\sim}{x}$-space cannot reenter region R through a point other than $\underset{\sim}{x}_j$ and $\underset{\sim}{x}_{j+1}$.

<u>Proof</u>: Suppose that the solution curve reenters region R at $\underset{\sim}{x}_k$, then $\underset{\sim}{y} = \underset{\sim}{J}\,\underset{\sim}{x}_j + \underset{\sim}{w} = \underset{\sim}{J}\,\underset{\sim}{x}_{j+1} + \underset{\sim}{w} = \underset{\sim}{J}\,\underset{\sim}{x}_k + \underset{\sim}{w}$. Since $\underset{\sim}{x}_j$, $\underset{\sim}{x}_{j+1}$ and $\underset{\sim}{x}_k$ are not on a straight line, $(\underset{\sim}{x}_j - \underset{\sim}{x}_k)$ and $(\underset{\sim}{x}_{j+1} - \underset{\sim}{x}_k)$ are linearly independent. But $\underset{\sim}{J}(\underset{\sim}{x}_j - \underset{\sim}{x}_k) = 0$ and $\underset{\sim}{J}(\underset{\sim}{x}_{j+1} - \underset{\sim}{x}_k) = 0$ imply that the rank of $\underset{\sim}{J}$ is less than (n-1). Thus $\underset{\sim}{x}_k$ cannot exist and $\underset{\sim}{d}_j$ is the only vector which satisfies the equation $\underset{\sim}{J}\,\underset{\sim}{d}_j = \underset{\sim}{J}(\underset{\sim}{x}_{j+1} - \underset{\sim}{x}_j) = 0$. This completes the proof.

The final item is to investigate the property of boundary crossing in the x-space when a singular region is reached. In this connection, it is important to present the following lemma:

Lemma 6. Let J and J' be n×n matrices, and

$$J' = J + c\, n^T \tag{20}$$

If det $J \neq 0$ and det $J' = 0$, then the rank of J' is (n-1).

Proof: Suppose that the rank of J' is equal to or less than (n-2), then there exist two vectors, d_1 and d_2, such that $J'd_1 = J'd_2 = 0$ and $d_1^T d_2 = 0$. From eq (20), and the above we obtain

$$J'\, d_1 = 0 = J\, d_1 + c\, n^T d_1$$

$$J'\, d_2 = 0 = J\, d_2 + c\, n^T d_2$$

Since J is nonsingular, $J\, d_1$ and $J\, d_2$ are non-zero, thus $n^T d_1$ and $n^T d_2$ are non-zero. Let

$$\hat{d}_1 = d_1 | n^T d_1 \text{ and } \hat{d}_2 = d_2 | n^T d_2$$

Again, from eq. (20) and the above, we have

$$J'_1\, \hat{d}_1 = J\, \hat{d}_1 + c\, n^T \hat{d}_1 = J\, \hat{d}_1 + c$$

$$J'_2\, \hat{d}_2 = J\, \hat{d}_2 + c\, n^T \hat{d}_2 = J\, \hat{d}_2 + c$$

Therefore $J\, \hat{d}_1 = J\, \hat{d}_2$, which implies that $\hat{d}_1 = \hat{d}_2$. This contradicts $d_1^T\, d_2 = 0$.

From this lemma, we know that a singular region with rank (n-1) is entered from a regular region and departs to a regular region. Let us consider the solution curve in the x-space traversing through regions R, R' and R", respectively as shown in Fig. 11. We assume that regions R and R" are regular and region R' is singular with a Jacobian $\underset{\sim}{J}'$ of rank (n-1). The portion of the solution curve in region R as indicated by the vector $\underset{\sim}{d}$ is determined by Lemma 2. When the boundary point $\underset{\sim}{x}_{j+1}$ is reached, $\underset{\sim}{d}'$ is determined according to $\underset{\sim}{J}'\underset{\sim}{d}' = 0$. The next boundary point is reached at $\underset{\sim}{x}_{j+2}$, where the solution curve enters region R". The image for the portion of solution curve in region R' is a single point $\underset{\sim}{y}_{j+1} = \underset{\sim}{f}(\underset{\sim}{x}_{j+1}) = \underset{\sim}{y}_{j+2} = \underset{\sim}{f}(\underset{\sim}{x}_{j+2})$. The solution curve then start from $\underset{\sim}{x}_{j+2}$ and $\underset{\sim}{d}''$ is again determined by Lemma 2. This concludes the discussion on singular matrices.

VI. Conclusion

In conclusion, we present a summary of our results in terms of the following theorem and algorithm.

Theorem 3. Let $\underset{\sim}{f}(\cdot)$ be a continuous, piecewise-linear function which maps $\mathcal{R}^n$ into $\mathcal{R}^n$.

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{J}^{(m)} \underset{\sim}{x} + \underset{\sim}{w}^{(m)} \quad , \; m = 1,2,\ldots,\ell.$$

Let $\underset{\sim}{J}^{(m)}$ in all unbounded regions be nonsingular, and furthermore, their determinants all have the same sign. Then algorithm II below leads to a solution of $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ for any given $\underset{\sim}{y}^*$ in a finite number of steps.

Algorithm II.

Step 1: Use Algorithm I to trace the solution curve.

Step 2: If the solution curve hits a corner or a singular region whose image is a subset of an (n-1) dimensional subspace in the $\underset{\sim}{y}$-space, use the perturbation method as given in Section 4 to find a new initial point.

Go to step 1.

Step 3: If the solution curve hits a singular region other than those given above, the direction $\tilde{d}$ is defined by $\tilde{J}\ \tilde{d} = 0$. Go to Step 1.

With some minor modification, it is possible to continue the tracing of the solution curve once a solution is obtained. This enables us to obtain multiple solutions. However, this in no way guarantees that all solutions can be found. It is still an open question to obtain the conditions under which all solutions can be determined.

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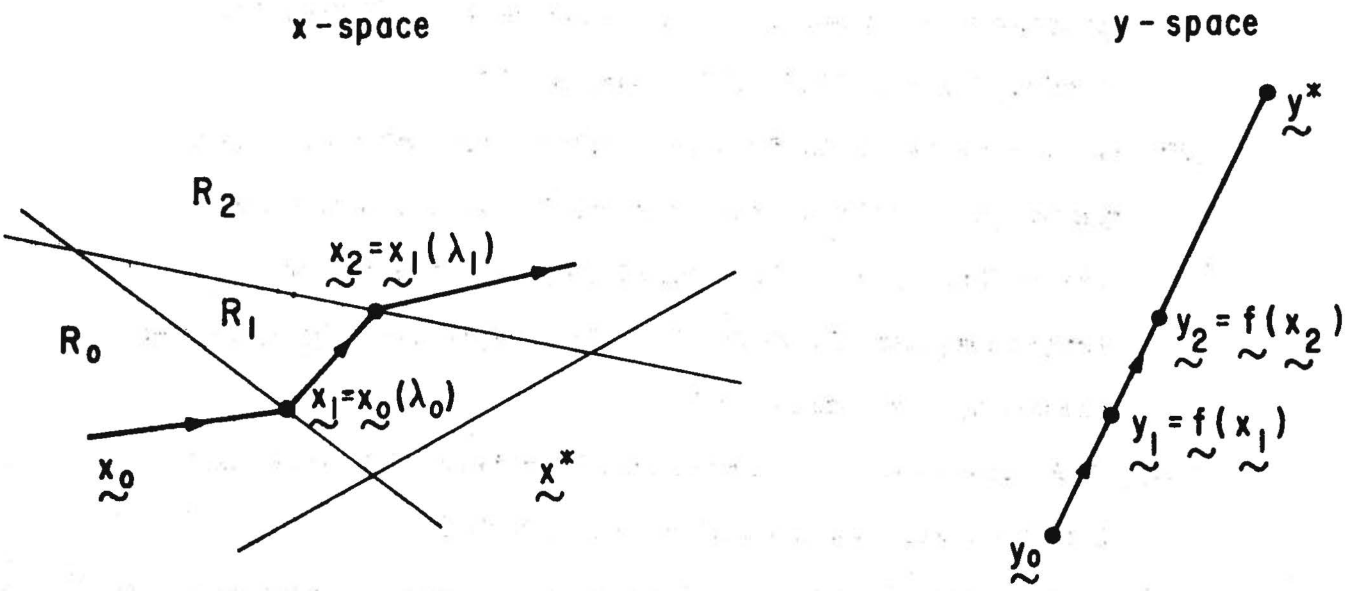

Fig. 1. The solution curve in the y-space is on the straight line connecting $\tilde{y}$ and $\tilde{y}^*$. The solution curve in the x-space is a continuous piecewise-linear curve.

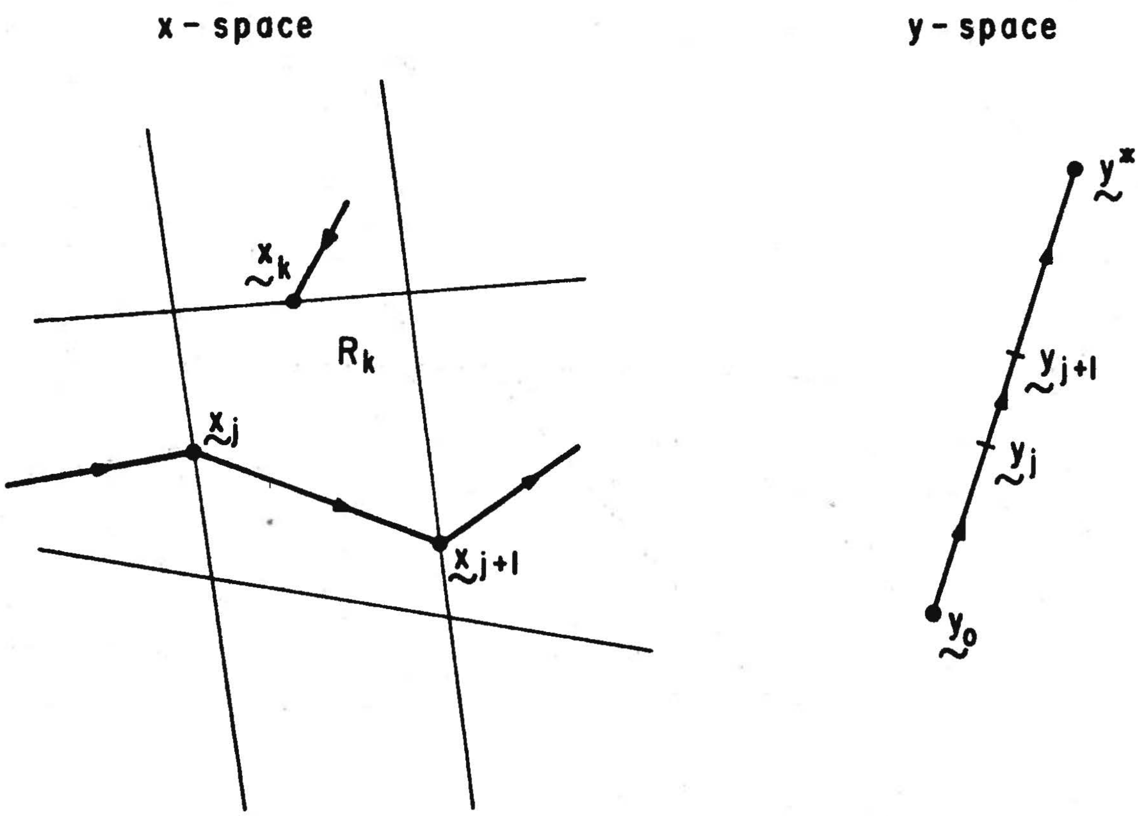

Fig. 2. The solution curve in the x-space will not reenter a region whose Jacobian is nonsingular.

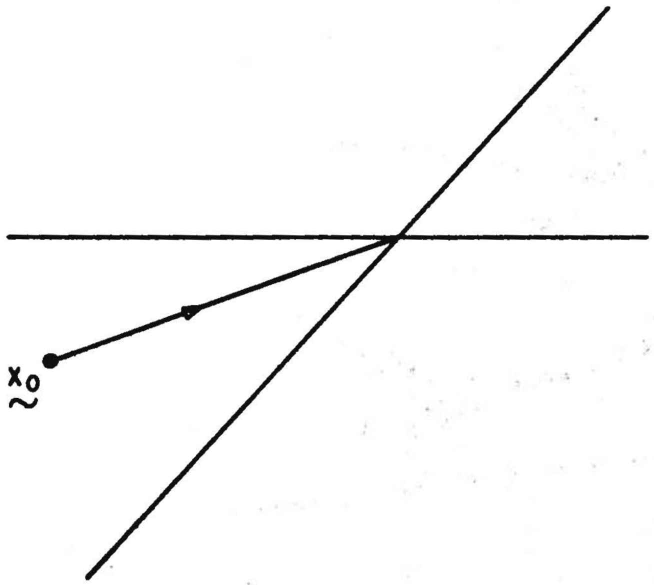

Fig. 3. The solution curve in the x-space reaches a corner.

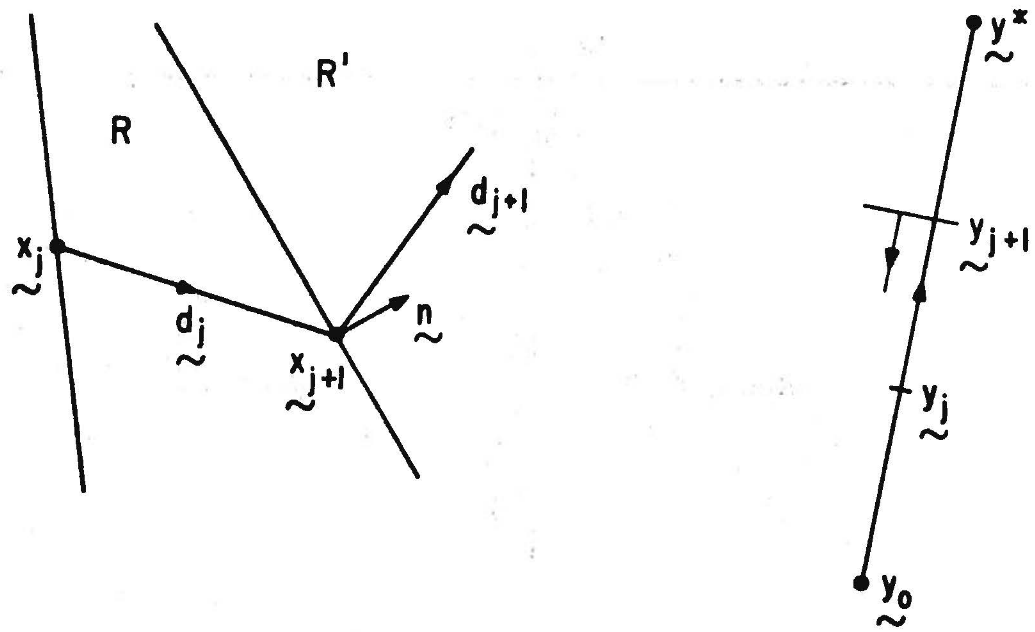

Fig. 4. The solution curve in the y-space reverses direction, i.e., moves away from y^*.

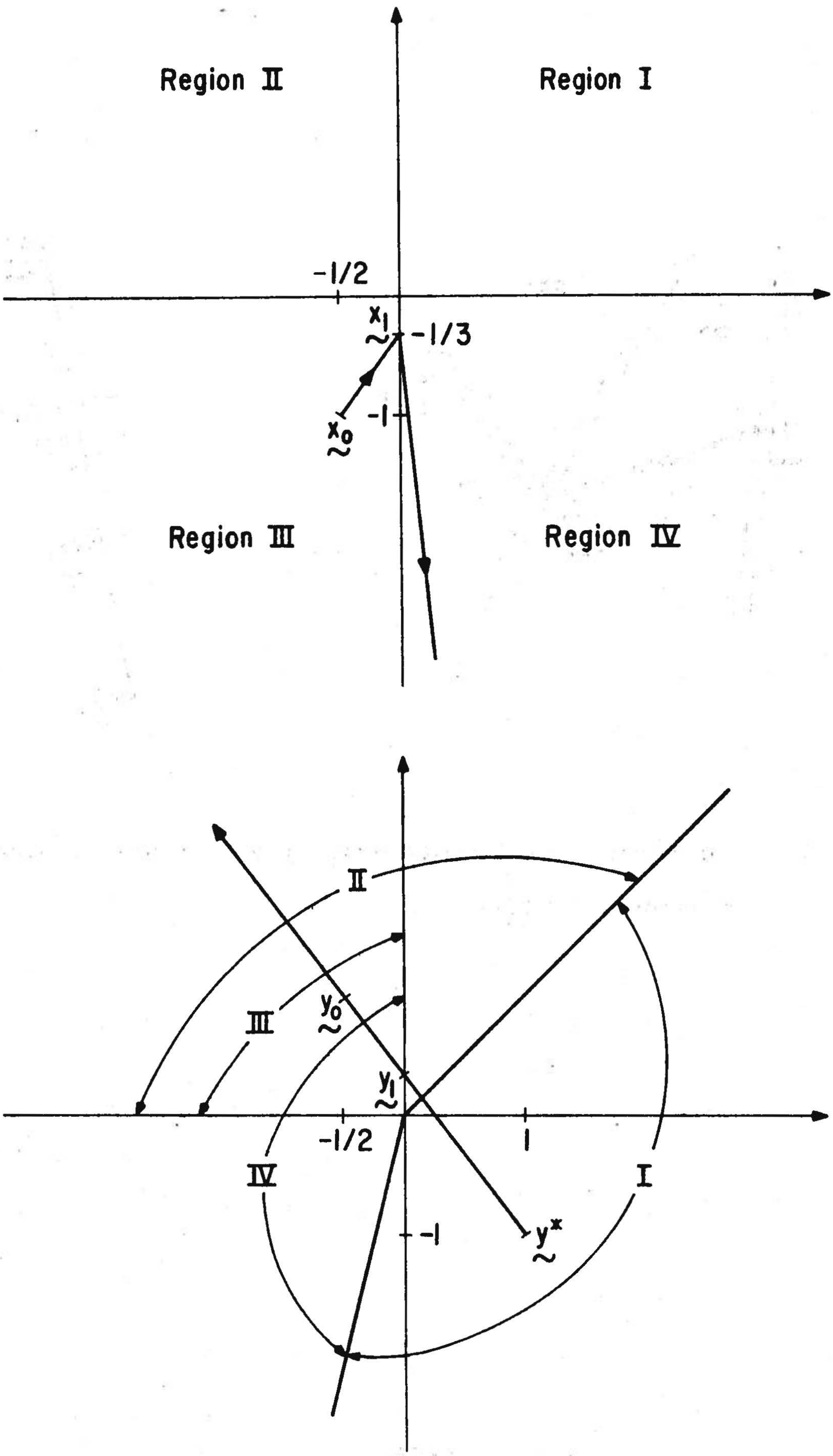

Fig. 5. The solution curves go to infinity without finding a solution.

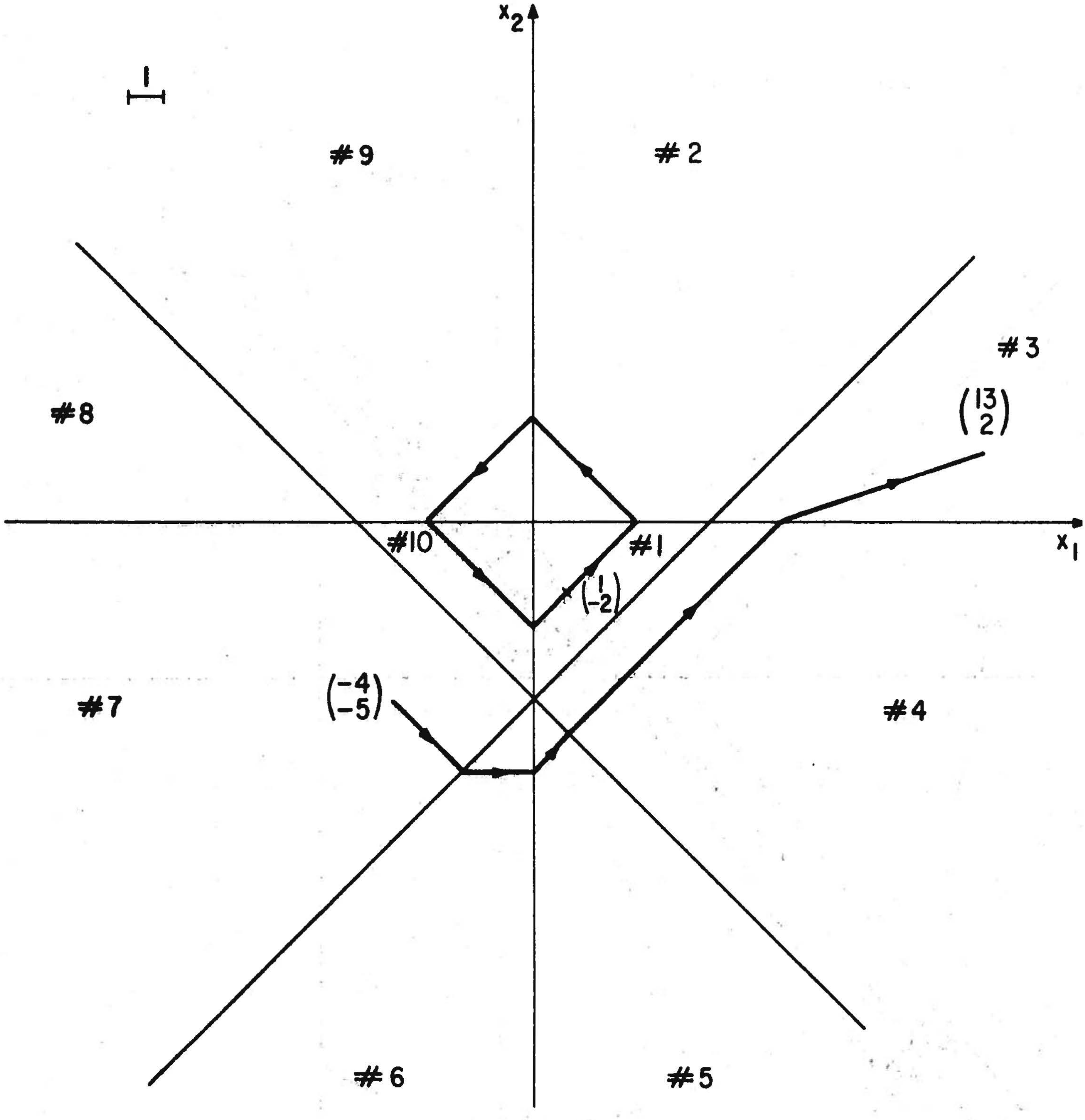

Fig. 6. (a) Solution curves in the x-space starting from different initial points. One of them is cyclic.

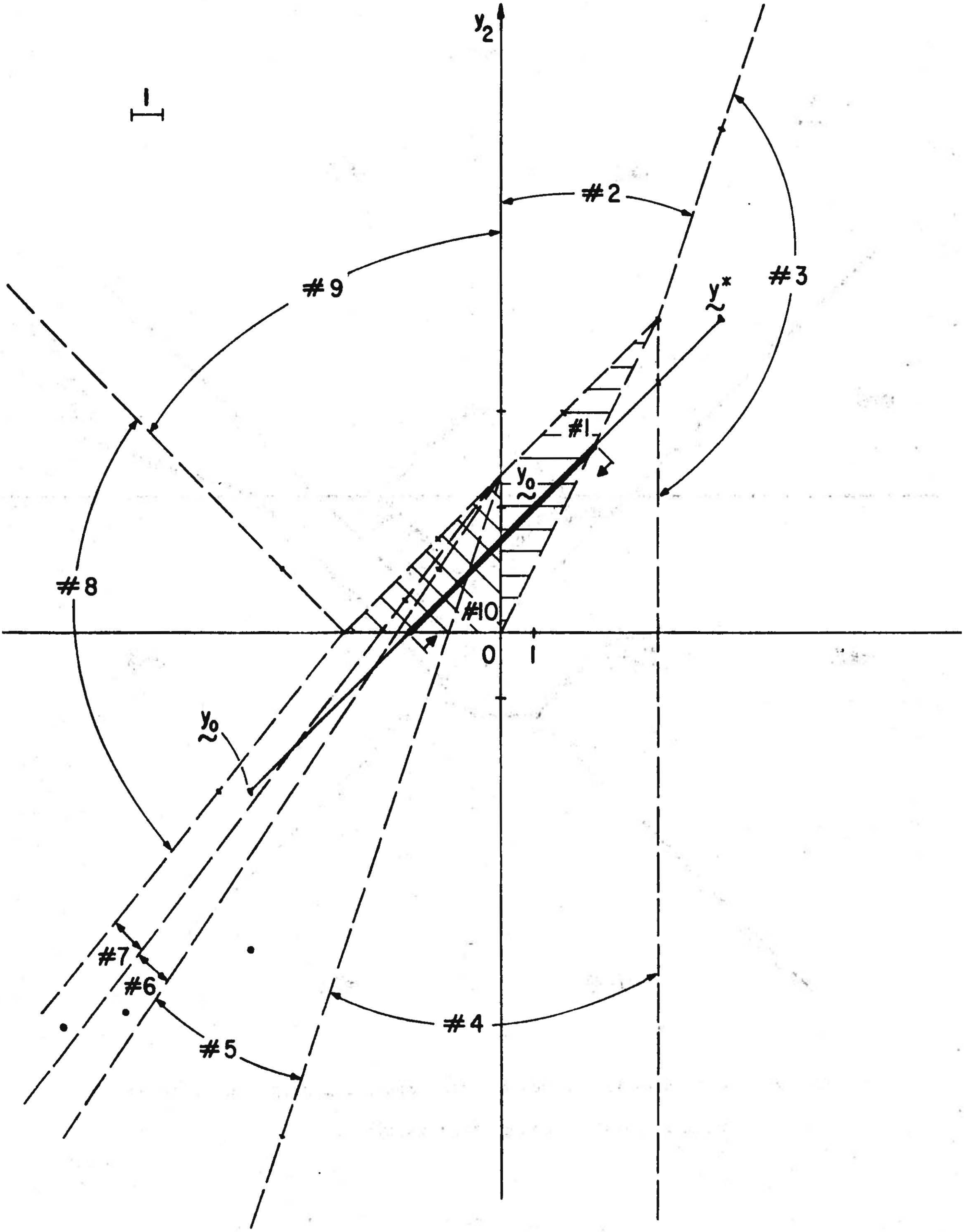

Fig. 6. (b) Solution curves in the y-space.

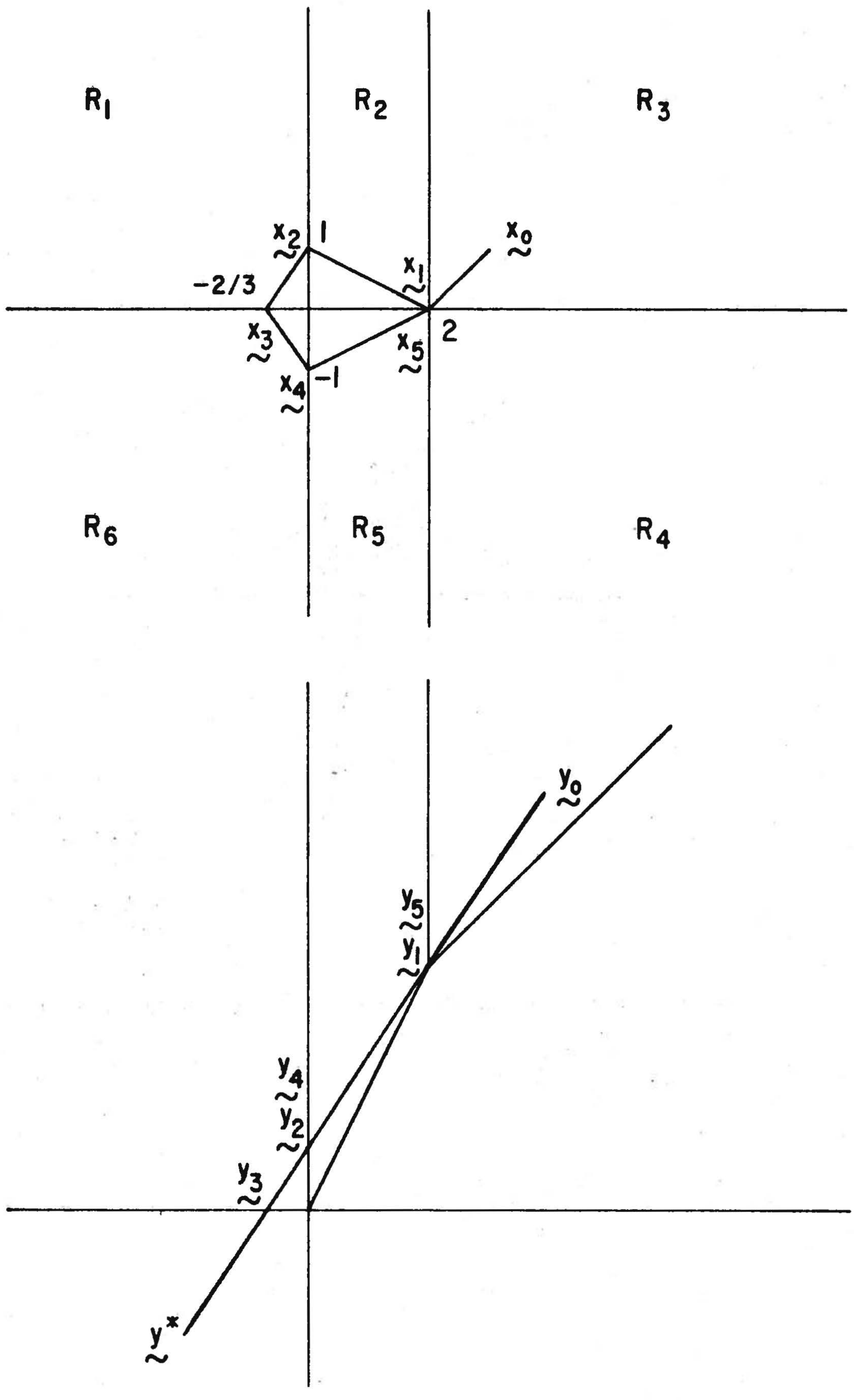

Fig. 7. Cyclic solution curve through a corner.

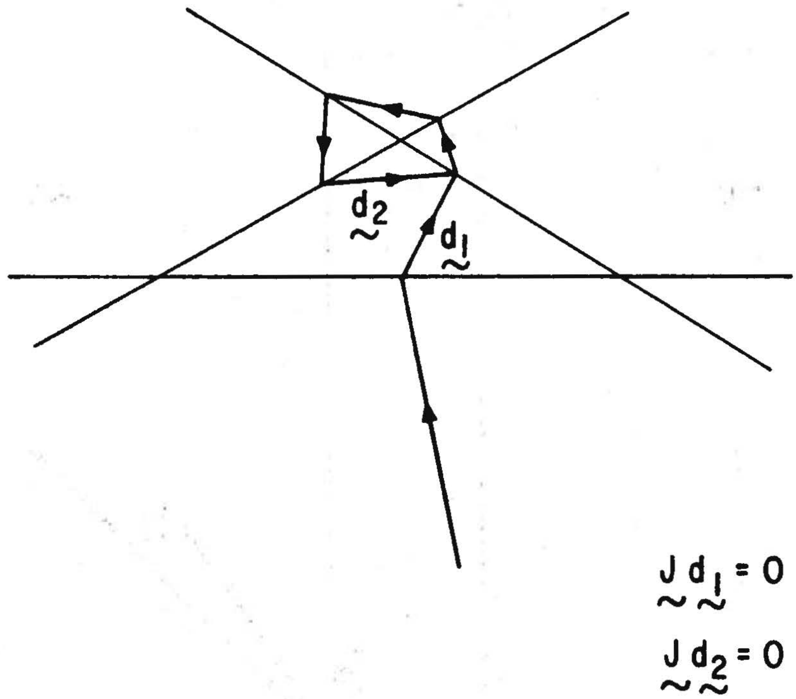

Fig. 8. Cyclic solution curve through a region whose Jacobian is singular.

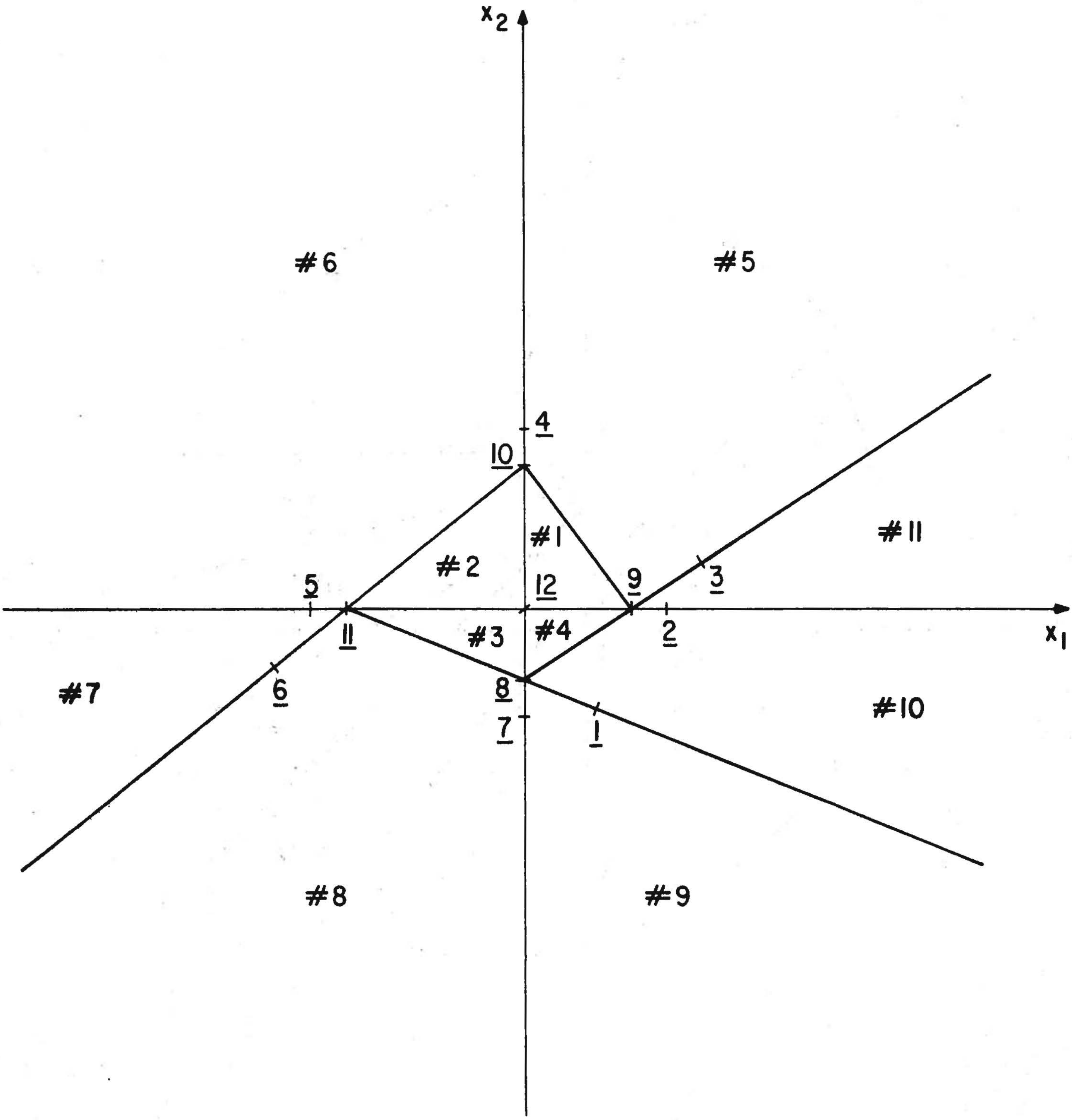

Fig. 9. (a) The x-space is divided into eleven regions. The Jacobians of all bounded regions are singular.

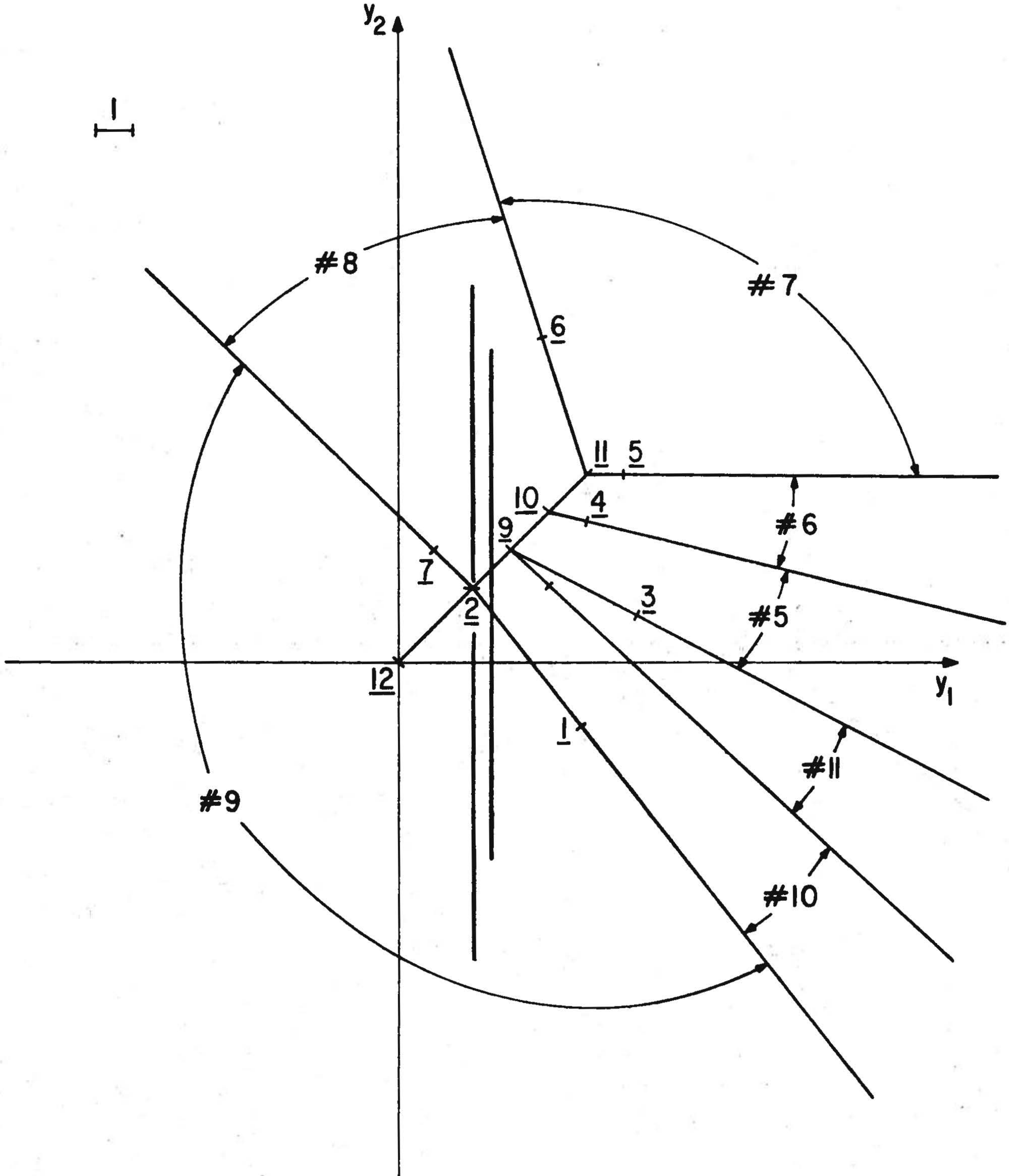

Fig. 9. (b) Images of 11 regions in the y-space.

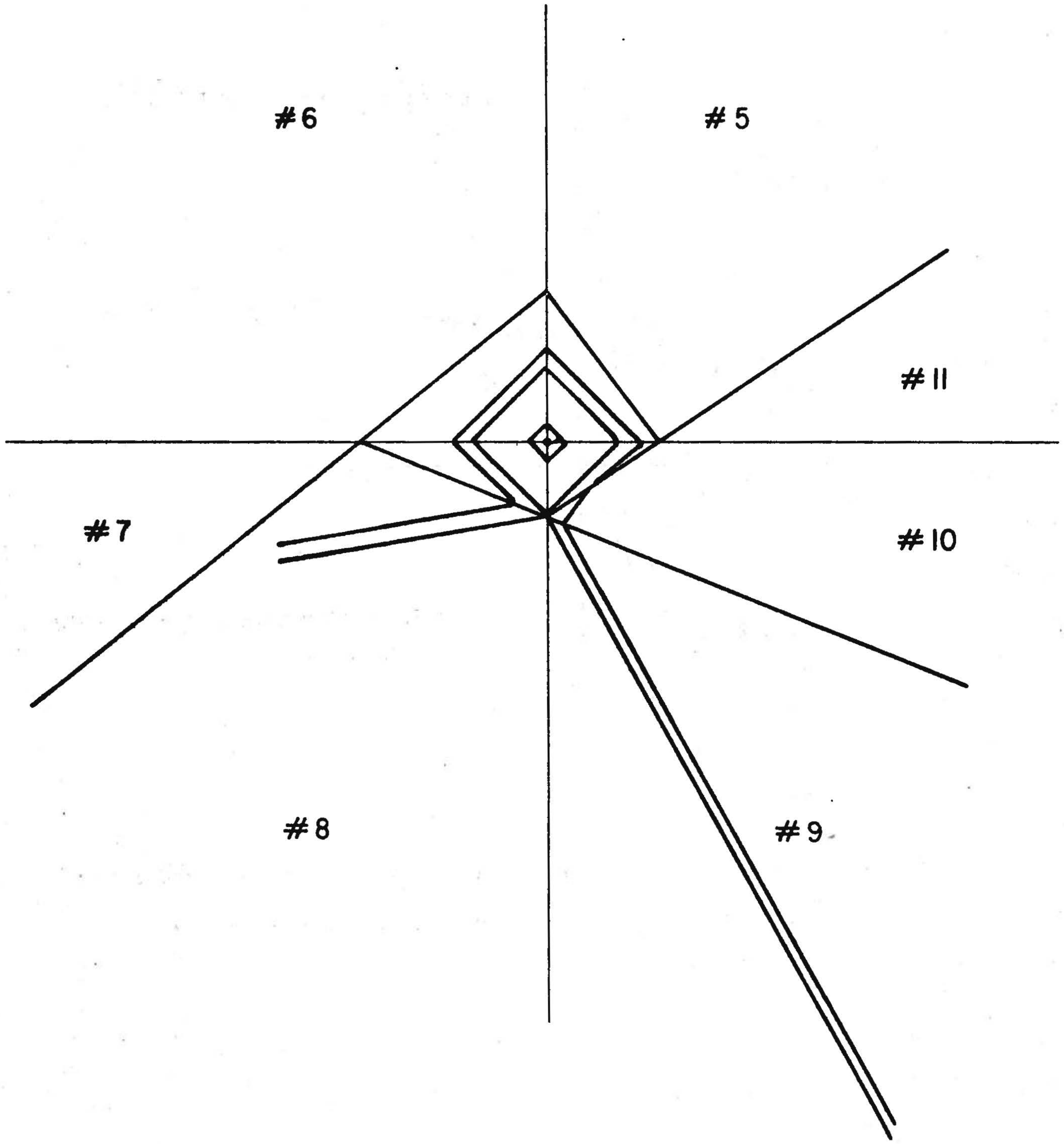

Fig. 9. (c) Solution curves through singular regions.

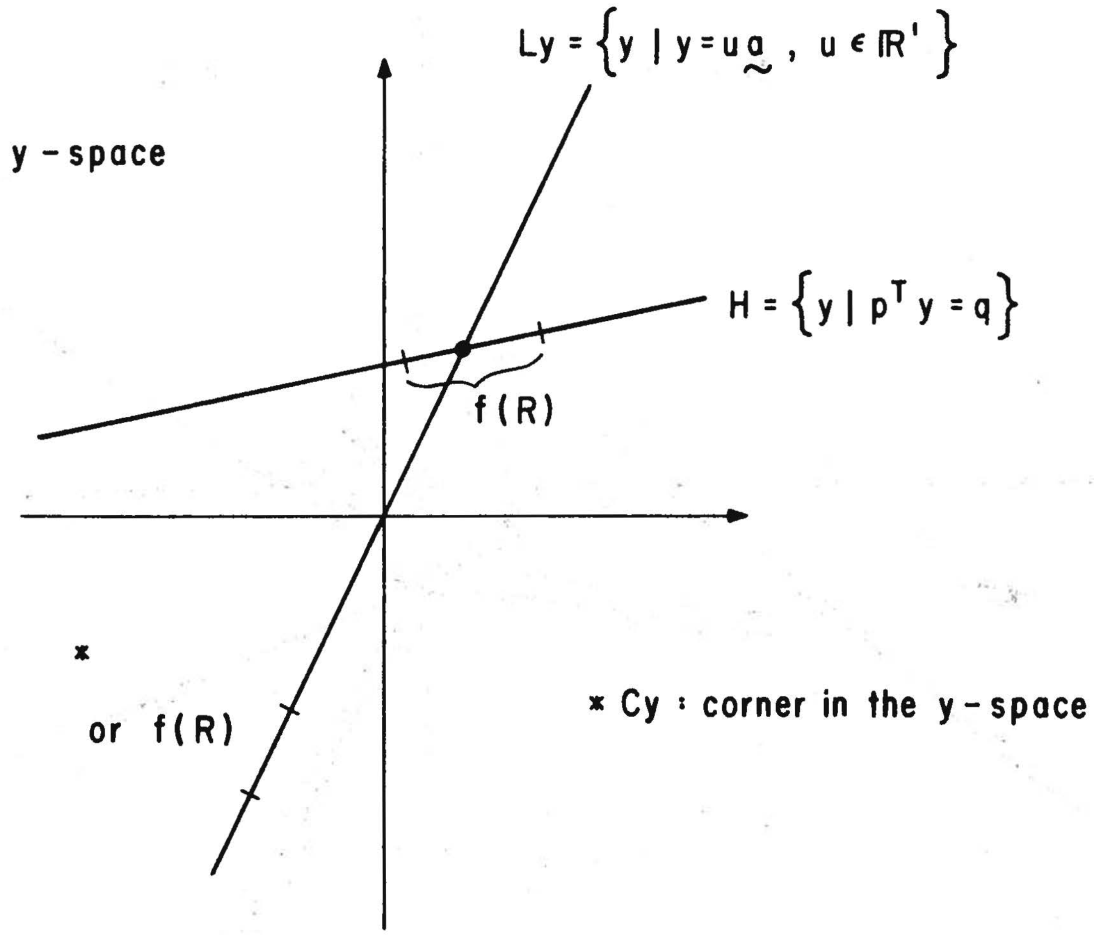

Fig. 10. Intersections of a straight line passing through the origin in the y-space and the images of regions whose Jacobian are singular.

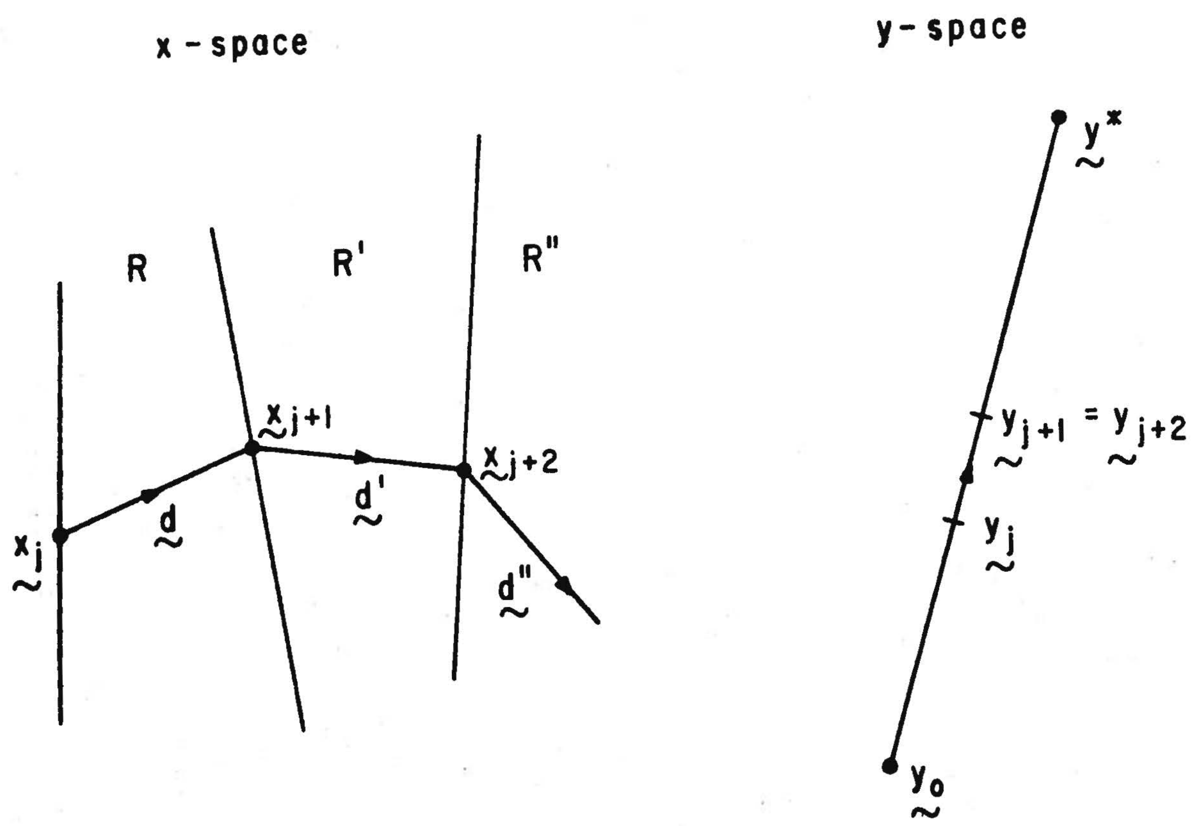

Fig. 11. Solution curves going through a region whose Jacobian has rank n-1.