A relation between Lévy's stochastic area formula, Legendre polynomials, and some continued fractions of Gauss.

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1. Introduction.

(1.1) The recurrence relation:

\[
\frac{2(\nu + 1)}{x} I_{\nu+1}(x) = I_{\nu}(x) - I_{\nu+2}(x) \quad (\nu > -1; \ x \geq 0)
\]

between modified Bessel functions implies

(1.a) \[ x \frac{I_{\nu+1}}{I_\nu}(x) = \frac{x^2}{2(\nu + 1) + x \frac{I_{\nu+2}}{I_{\nu+1}}(x)} \]

and leads to the continued fraction expansion:

(1.b) \[ x \frac{I_{\nu+1}}{I_\nu}(x) = \frac{x^2}{2(\nu + 1) + \frac{x^2}{2(\nu + 2) + \frac{x^2}{2(\nu + 3) + \cdots}}} \]

a particular case of Gauss's continued fractions for ratios of hypergeometric functions (see Jones and Thron [2], p.211, for example). Formulae (1.a) and (1.b) in the case \( \nu = 1/2 \) are of special interest since:

\[
x \coth x - 1 = x \frac{I_{3/2}}{I_{1/2}}(x)
\]

and therefore:

(1.c) \[ x \coth x - 1 = \frac{x^2}{3} + \frac{x^2}{5} + \frac{x^2}{7} + \cdots \]

(1.2) Let \( k_0(x) = x \coth x - 1 \), and \( h_0(x) = \frac{x}{\text{sh} x} \) (\( x \in \mathbb{R} \)). The functions \( h_0 \) and \( k_0 \) appear in Lévy's formula:

(1.d) \[ E[\exp(\text{i}xS) \mid B(1) = m] = h_0(x)\exp\left(\frac{-|m|^2}{2}k_0(x)\right) \]

expressing the conditional characteristic function of the stochastic area

\[ S \equiv \int_0^1 (B^{(1)}(s)dB^{(2)}(s) - B^{(2)}(s)dB^{(1)}(s)) \]

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of 2-dimensional Brownian motion $B = (B^{(1)}, B^{(2)})$ started at 0, given its position at time 1.

This formula (1.d) plays an important role in various questions, including Bismut's approach [1] to the Atiyah-Singer theorem, and also the asymptotics of the winding numbers for 2-dimensional Brownian motion (Pitman-Yor [6]). Several proofs of formula (1.d) are known, among which:

- Lévy's original proof using the development of Brownian motion along the trigonometric orthogonal basis of $L^2([0,2\pi], ds)$ ([4]);
- an application of Girsanov's theorem, which reduces the problem to determining the semi-group of an Ornstein-Uhlenbeck process;
- an application of Ray-Knight theorem for linear Brownian local times.

These two last proofs are presented in D. Williams [7] (see also Yor [9]), and hinge upon the identity:

$$ E[\exp(ixS) | B(1) = m] = E[\exp - \frac{x^2}{2} \int_0^1 ds |B(s)|^2 | |B(1)| = m]. $$

(1.3) In this paper, we show the following extension of Lévy's formula (1.d).

**Theorem**

Consider the orthogonal decomposition of Brownian motion

$$ B(t) = \sum_{p=0}^{\infty} \beta_p \int_0^t ds P_p(2s-1) \quad (t \leq 1) $$

where: $\beta_p = \int dB(s)P_p(2s-1)$

and $(P_p ; p = 0,1,...)$ is the sequence of Legendre polynomials.

Then:

(i) **With the notation**: $\xi \times \eta = \text{Im}(\overline{\xi} \eta)$, for $\xi, \eta \in \mathbb{C}$, the stochastic area $S$ can be represented as:

$$ S = \sum_{p=0}^{\infty} \beta_p \times \beta_{p+1} $$

where the convergence holds both in $L^2$ and a.s;

(ii) **For any $p \in \mathbb{N}$, we have**:

$$ E[\exp(ixS) | \beta_k = m_k ; 0 \leq k \leq p] = \exp(ix \sum_{k=0}^{p-1} m_k \times m_{k+1}) h_p(x) \exp - \frac{[m_p]^2}{2} k_p(x) $$

where

$$ h_p(x) = \frac{x^{\nu}}{2^{\nu} \Gamma(\nu+1) I_\nu(x)} ; \quad k_p(x) = x \frac{I_{\nu+1}(x)}{I_\nu(x)} ; \quad \nu = p + \frac{1}{2}. $$

(1.f)
In order to show more naturally how the Legendre polynomials are linked with Lévy's stochastic area, we have organized the proof as follows:

- in chapter 2, we prove that, if we represent \((B(t), t \leq 1)\) as:
\[
B(t) = \rho(t) + tB(1), \quad t \leq 1
\]
with \((\rho(t), t \leq 1)\) a Brownian bridge independent of \(B(1)\), and more generally, if this orthogonalization procedure is adequately iterated, then Lévy's formula (1.d) yields a sequence of analogous identities, whose right-hand sides are:
\[
h_p(x)\exp \left( -\frac{|m|^2}{2}k_p(x) \right)
\]
where \(h_p\) and \(k_p\) are defined in (1.f);
- in chapter 3, we identify the orthogonal expansion
\[
B(t) = \sum_{p=0}^{\infty} u_{p+1}(t)\beta_p \quad (t \leq 1)
\]
which is obtained in our orthogonalization procedure as the decomposition (1.e).

2. Lévy's formula and some continued fractions of Gauss.

(2.0) NOTATION.

- If \(Z(t) = X(t) + iY(t), t \leq 1\), is a complex valued continuous semi-martingale, we write:
\[
S_Z = \int_{0}^{1} X(s)dY(s) - Y(s)dX(s)
\]
- If \(m = m^{(1)} + im^{(2)}\), and \(n = n^{(1)} + in^{(2)}\) are two complex numbers, we write \(m \times n\) for \(\text{Im}(mn) = m^{(1)}n^{(2)} - n^{(1)}m^{(2)}\), and \(m \cdot n\) for \(\text{Re}(mn) = m^{(1)}n^{(1)} + m^{(2)}n^{(2)}\).
- For \(\nu > -1\), we note: \(\tilde{I}_\nu(x) = \frac{2^\nu T(\nu + 1)}{x^\nu}I_\nu(x)\)

(2.1) We first reinterpret formula (1.d) in terms of the Brownian bridge \(\rho\) defined in (1.g). Developing \(S\), we obtain:
\[
S = S_\rho + B(1) \times \beta_1, \quad \text{where: } \beta_1 = -2\int_{0}^{1} ds\rho(s),
\]
and formula (1.d) becomes:
\[
E[\exp(ixS_\rho + ixm \times \beta_1)] = h_0(x)\exp \left( -\frac{|m|^2}{2}k_0(x) \right)
\]
so that:
\[
E[\exp(ixS_\rho + in \cdot \beta_1)] = h_0(x)\exp \left( -\frac{|n|^2}{2} \frac{k_0(x)}{x^2} \right).
\]
This formula confirms that $\beta_1$ is a centered 2-dimensional Gaussian variable, with the additional information that:

$$\frac{1}{2} \mathbb{E}(|\beta|) = \lim_{x \to 0} \frac{k_0(x)}{x^2} = \frac{1}{c_0}.$$ 

Moreover, we deduce from (2.a) that:

$$\mathbb{E}[\exp(ixS_p) \mid \beta_1 = m] = h_1(x) \exp \left( -\frac{|m|^2}{2} k_1(x) \right)$$

with:

$$h_1(x) = \frac{h_0(x)x^2}{k_0(x)c_0}; \quad k_1(x) = \frac{x^2}{k_0(x)} - c_0.$$ 

From the recurrence relation (1.a), we get:

$$h_1(x) = \frac{1}{L_{3/2}(x)}; \quad k_1(x) = xL_{5/2}(x); \quad c_0 = 3.$$ 

(2.2) We now iterate the above procedure in defining a sequence of processes $(B_p(t), t \leq 1)$, and of Gaussian variables $(\beta_p)$ via the recurrence relation:

$$\begin{cases}
B_p(t) = B_{p+1}(t) + u_{p+1}(t)\beta_p \\
\beta_p = -2\int_0^1 du_p(s)B_p(s)
\end{cases}$$

with original conditions: $B_0(t) = B(t)$, and $\beta_0 = B(1)$, and the additional requirement that $B_{p+1}(t)$ is orthogonal to $\beta_p$. In order that this recurrence relation be meaningful, we must verify recursively that the functions $(u_p)$ are of bounded variation. Suppose this is so for $u_1, \ldots, u_p$. Then, from the first half of (2.b), using the orthogonality of $\beta_p, \beta_{p-1}, \ldots, \beta_0$, we obtain:

$$u_{p+1}(t)\mathbb{E}[^2 = \mathbb{E}[B(t)\beta_p] = \int_0^t ds\phi_p(s)$$

where $\phi_p(cL^2([0,1],ds))$ is the function appearing in the Wiener representation of $\beta_p \equiv \int dB(s)\phi_p(s)$. Therefore, $u_{p+1}$ is absolutely continuous, and the recurrence is meaningful. Now, from (2.b), we obtain:

$$S_p = S_{p+1} + \beta_p \times \beta_{p+1},$$

where, for simplicity, we have written $S_k$ for $S_{B_k}$ $(k = p, p + 1)$. Consequently, the functions $h_p$ and $k_p$ being defined via the formula:

$$\mathbb{E}[^ixS_p] \mid \beta_p = m] = h_p(x) \exp \left( -\frac{|m|^2}{2} k_p(x) \right)$$
we obtain, much as in (2.1) above, the recurrence formulae:

\[(2.\text{c}) \quad \begin{align*}
(\text{i}) \quad & h_{p+1} = \frac{h_p(x)x^2}{k_p(x)c_p}; \\
(\text{ii}) \quad & k_{p+1}(x) = \frac{x^2}{k_p(x)} - c_p
\end{align*}\]

where \(c_p = \lim_{x \to 0} \frac{x^2}{k_p(x)}.\) Moreover, we also have:

\[(2.\text{d}) \quad \frac{1}{2} E(|\beta_{p+1}|^2) = 1/c_p.\]

We now deduce from the recurrence formula (1.a) that:

\[h_p(x) = \frac{1}{I_{\nu}(x)}; \quad k_p(x) = x\frac{I_{\nu+1}(x)}{I_\nu}; \quad c_p = 2(\nu + 1), \text{ with } \nu = p + 1/2.\]

\[(2.3) \quad \text{For } p > 0, \text{ we introduce the process } V_p \text{ defined by:}\]

\[V_p(t) = \frac{1}{t^p} \int_0^t dB(s) s^p \quad (t > 0), \text{ and } V_p(0) = 0.\]

This is a continuous semimartingale with decomposition:

\[V_p(t) = B(t) - \int_0^t ds V_p(s).\]

Our interest in the process \(V_p\) comes from the fact that, if \((t^a; a \geq 0)\) denotes the family of local times over the whole of \(R_+\) for the Bessel process, call it \(R_p\), with dimension \(c_p = 2p + 3\), then:

\[(2.\text{e}) \quad (t^a; a \geq 0) \overset{\text{[d]}}{=} ([V_p(a)]^2; a \geq 0).\]

This is easily deduced from the particular case \(p = 0\), which is due to D. Williams [8], and is in agreement with Le Gall [3], using deterministic time change, and time-inversion.

We have the following

**Theorem 1:** Let \(p \in \mathbb{N}, \text{ and } \nu = p + \frac{1}{2}.\) Then:

\[E[\exp(\lambda S_{V_p}) \mid \beta_p = m] = E[\exp(\lambda S_{V_p}) \mid V_p(1) = m]\]

\[= \frac{1}{I_\nu(x)} \exp \left(-\frac{|m|^2}{2} x \frac{I_{\nu+1}(x)}{I_\nu(x)} \right).
\]

**Proof:** We have already shown the equality between the first and the last expressions. To prove that the second and the last expressions are equal, we remark that:

\[(2.\text{f}) \quad E[\exp(\lambda S_{V_p}) \mid V_p(1) = m] = E[\exp - \frac{x^2}{2} \int_0^1 |V_p(s)|^2 ds \mid |V_p(1)|^2 = |m|^2] \]
by a classical skew-product argument.

Using the identity in law (2.e), the right-hand side of (2.f) equals:

$$E[\exp -\frac{x^2}{2}\int_0^1 ds1(R_p(s) \leq 1) | t^1 = |m|^2]$$

and, from Pitman-Yor [5], for example, this quantity is equal to the closed form expression presented in Theorem 1.

Theorem 1 may be extended, with no more difficulty, as follows: for any $p > 0$, and $q \in N$, we denote $S^{(p)}$ for $S_{V_p}$, and $S_q^{(p)}$ for $S_{(V_p)_q}$, where $((V_p)_q; q \in N)$ is the sequence of processes appearing in the orthogonalization procedure detailed in (2.2), but now applied to the process $V_p$, instead of $B \equiv V_0$.

The identities stated in theorem 1 now become:

$$E[\exp (ixS_q^{(p)}) | \beta_q^{(p)} = m] = E[\exp (ixS_{V_{pq}}) | V_{p+q}(1) = m]$$

$$= \frac{1}{|\nu(x)|} \exp \left(-\frac{|m|^2}{2} \frac{1}{\nu+1}(x)\right), \text{ where } \nu = p+q+\frac{1}{2}.$$ 

3. Lévy’s formula and Legendre polynomials.

We shall now determine explicitly the functions $(u_p)$ which appear in the recurrence relation (2.b).

Obviously we may, and we shall, assume here that $(B(t), t \leq 1)$ is real-valued.

We need to introduce the Legendre polynomials $(P_n)$ which may be defined by the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n],$$

and constitute an orthogonal basis of $L^2([-1, +1], dx)$.

We now have the following

**Theorem 2:** Let $p \in N$; then:

(i) $E(\beta_p^2) = \frac{1}{2p+1}$;

(ii) $u_{p+1}(t) = (2p+1)\int_0^t dsP_p(2s-1)$.

**Proof:** a) In our proof of Theorem 1, we have already shown that

$$\lambda_p \equiv E(\beta_p^2) = \frac{1}{2p+1}.$$ 

(The difference of $(1/2)$ with formula (2.d) comes from changing dimension 2 to 1). We shall give a direct proof of this below.
b) We now prove that \((u'_{k+1}, k \geq 0)\) is a sequence of orthogonal functions in \(L^2([0,1], ds)\).

The Gaussian variable \(\beta_k\) admits a Wiener representation:

\[
\beta_k = \int_0^1 dB(s)\phi_k(s), \text{ with } \phi_k \in L^2([0,1], ds).
\]

For any \(k\), we deduce from the orthogonal development:

\[
B(t) = B_{k+1}(t) + \sum_{p=0}^k u_{p+1}(t)\beta_p,
\]

that:

\[
\int_0^t u_{k+1}(t)\lambda_k = E[B(t)\beta_k] = \int_0^t ds\phi_k(s)
\]

a formula we already obtained in showing that (2.b) is meaningful. Therefore, \((u'_{k+1} = \frac{1}{\lambda_k}\phi_k; k \geq 0)\) is an orthogonal sequence in \(L^2([0,1], ds)\).

c) We now show the following relations:

\((3.a)\) \hspace{1cm} (i) \(\int_0^1 du_p(s)u_{p+1}(s) = -1/2\); \hspace{1cm} (ii) \(\int_0^1 du_p(s)u_{k+1}(s) = 0 \hspace{1cm} (k > p)\)

which, by integration by parts, may also be written as:

\((3.a')\) \hspace{1cm} (i') \(\int_0^1 du_{p+1}(s)u_p(s) = 1/2\); \hspace{1cm} (ii') \(\int_0^1 du_{k+1}(s)u_p(s) = 0 \hspace{1cm} (k > p)\).

These relations are obtained by writing:

\[
B_p(t) = B_{q+1}(t) + \sum_{k=p}^q u_{k+1}(t)\beta_k \hspace{1cm} (q > p);
\]

Thus:

\[
\beta_p = -2\int_0^1 du_p(s)\{B_{q+1}(s) + \sum_{k=p}^q u_{k+1}(s)\beta_k\}
\]

which implies (3.a), since \(\beta_p, \beta_{p+1}, \cdots, \beta_q, B_{q+1}\) are orthogonal.

d) Next, we remark that the covariance of the process \(B_p\) may be deduced from the orthogonal development: \(B(t) = B_p(t) + \sum_{k=0}^{p-1} u_{k+1}(t)\beta_k\).

We obtain:

\[
E[B_p(t)B_p(s)] = (t-s) - \sum_{k=1}^{p} u_k(t)u_k(s)\lambda_{k-1}.
\]
e) Using our previous remarks, we shall now obtain a simple recurrence formula between $u_{p-1}$, $u_p$ and $u_{p+1}$. We deduce from the equality:

$$B_p(t) = B_{p+1}(t) + u_{p+1}(t)\beta_p$$

that:

$$u_{p+1}(t)\lambda_p = E[B_p(t)\beta_p] = -2\int_0^1 du_p(s)E[B_p(t)B_p(s)]$$

which, using d), and then c), gives:

$$(3.6) \quad u_{p+1}(t)\lambda_p = -2\int_0^t du_p(s)s + 2tu_p(t) + u_{p-1}(t)\lambda_{p-2} \quad (p > 1).$$

For $p = 1$, we have:

$$u_2(t)\lambda_1 = -2\int_0^t ds\{s^2 - st\} = -t(1 - t).$$

In particular, a recurrence argument shows that for every $p \in \mathbb{N}$, $u_p$ is a polynomial of degree $(p + 1)$.

Consequently, using b), we have: $u_{p+1}(t) = \alpha_p P_p^*(t)$, where $\alpha_p$ is a constant to be determined, and

$$P_p^*(t) = (2p+1)^{1/2}P_p(2t - 1) \quad (p \in \mathbb{N})$$

is the orthonormal family in $L^2([0,1], dt)$ which is deduced from the Legendre polynomials ($P_p$).

f) It remains to determine the two sequences $(\alpha_p)$ and $(\lambda_p)$. Writing (3.6) again in terms of $(\alpha_p)$, $(\lambda_p)$ and $(P_p)$, gives the following relation:

$$\lambda_{n+1}\alpha_{n+1}(2n+3)^{1/2}P_{n+1}(x) = \alpha_n(2n+1)^{1/2}P_n(x) + \lambda_{n-1}\alpha_{n-1}(2n-1)^{1/2}P_{n-1}(x)$$

which, when compared with the classical relation:

$$P_{n+1}' = (2n+1)P_n + P_{n-1}'$$

implies:

$$\lambda_p = \frac{1}{2p+1}, \text{ and } \alpha_p = (2p+1)^{1/2} \quad \bullet$$


(4.1) The proof of the theorem stated in the Introduction is obtained by putting together Theorem 1 and Theorem 2. Indeed, since $(P_p : p \in \mathbb{N})$ is an orthogonal basis of $L^2([-1,1], ds)$, we now know that $(\beta_p : p \in \mathbb{N})$ is an orthogonal basis of the Gaussian space generated by $(B(t), t \leq 1)$. Hence, the formula

$$B(t) = B_{k+1}(t) + \sum_{p=0}^k u_{p+1}(t)\beta_p$$
implies (1.e), as \( k \to \infty \).
Likewise, the formula:

\[
S = S_{k+1} + \sum_{p=0}^{k} \beta_p \times \beta_{p+1}
\]

implies

\[
S = \sum_{p=0}^{\infty} \beta_p \times \beta_{p+1}
\]

and the convergence holds both in \( L^2 \) and a.s, since:

\[
( \sum_{p=0}^{k} \beta_p \times \beta_{p+1} ; k \in \mathbb{N} ) \text{ is a}(F_k) \text{ martingale},
\]

where \( F_k \) is the \( \sigma \)-field generated by \( (\beta_0, \beta_1, \cdots, \beta_{k+1}) \). This proves part (i) of the theorem. Part (ii) is then an immediate consequence of theorem 1.

(4.2) To prove formula (1.d), P. Lévy [4] develops Brownian motion along the trigonometric basis of \( L^2([0,1],ds) \), and obtains \( h_0(x) \equiv \frac{x}{\sin x} \), and \( k_0(x) \equiv x \coth x - 1 \) in their classical infinite product representations. On the other hand, we have shown in this paper that, when developing Brownian motion along the Legendre basis, one obtains \( k_0(x) \) in its continued fraction representation (1.c).

(4.3) A number of variants of theorems 1 and 2 can be obtained if we replace the Brownian functional \( S \) by

\[
S(\phi) = \int_0^1 dS_\phi(s), \text{ or by } A(\phi) = \int_0^1 ds \phi(s) |E(s)|^2,
\]

with \( \phi : [0,1] \to \mathbb{R}_+ \) a nice function, and in particular \( \phi(s) = s^k(k \geq 0) \).

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