A NOTE ON WEAK STAR UNIFORMITIES

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A NOTE ON WEAK STAR UNIFORMITIES

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Abstract. Consider the set \( \pi(Z) \) of countably additive probabilities on \( Z \), the set of positive integers. Endow \( \pi(Z) \) with the weak star topology. The finitely additive probabilities on \( Z \) form a compactification of \( Z \), which is not the Stone Cech compactification. Indeed, there is a bounded continuous function on \( \pi(Z) \) which cannot be uniformly approximated by polynomials. Furthermore, convolution of finitely additive probabilities is non-commutative.

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1. Introduction.

In (Diaconis and Freedman, 1984, section 5), we investigated some conditions for the consistency of Bayes estimates. A key idea turned out to be the "merging" of two sequences of probabilities \( \{\alpha_n\} \) and \( \{\beta_n\} \), in the sense that \( \alpha_n \) and \( \beta_n \) become indistinguishable from the point of view of integrating bounded continuous functions. A formal treatment involves the uniformities compatible with the weak star topology.

To review briefly, let \((\mathbb{Z},\rho)\) be a metric space. Let \(\alpha_n, \alpha\) be probabilities in \(\mathbb{Z}\). Then \(\alpha_n + \alpha\) weak star iff \(\int f d\alpha_n \to \int f d\alpha\) for all bounded continuous functions on \(\mathbb{Z}\). This defines the weak star topology \(T\) on \(\mathbb{X} = \pi(\mathbb{Z})\), the set of probabilities in \(\mathbb{Z}\). For more information, see Billingsley (1968) or Parthasarathy (1967).

Let \((X,\mathcal{T})\) be any topological space. A uniformity \(U\) is a nonempty collection of subsets of \(X \times X\), satisfying the following conditions:

a) Each member of \(U\) includes the diagonal \(\{(x,x): x \in X\}\).

b) If \(U \in U\) then \(U^{-1} \in U\), where \(U^{-1} = \{(x,y): (y,x) \in U\}\).

c) If \(U \in U\) then \(V \cdot V \subseteq U\) for some \(V \in U\), where \(V \cdot W = \{(x,y): (x,z) \in W \text{ and } (z,y) \in V\}\).

d) If \(U\) and \(V\) are members of \(U\), then \(U \cap V \in U\).

e) If \(U \in U\) and \(U \subseteq V \subseteq X \times X\), then \(V \in U\).

If \(A \subseteq X \times Y\), write \(A[x] = \{y: (x,y) \in A\}\) for the \(x\)-section of \(A\). We will say the uniformity \(U\) is consistent with the topology \(T\) iff for any open set \(W\), and \(x \in W\), there is a \(U \in U\) with \(U[x] \subseteq W\).

The idea is that a real-valued function \(f\) on \(X\) is uniformly continuous iff for all \(\varepsilon > 0\) there is a \(U_{\varepsilon} \in U\) such that \(x, y \in U_{\varepsilon}\) implies \(|f(x) - f(y)| < \varepsilon\). If \(U\) is consistent with \(T\), a uniformly continuous function is continuous.
A metric \( p \) on \( X \) defines a natural uniformity \( U_p \) as follows: \( U \in U_p \) iff \( U \supset \{(x,y): p(x,y) < \delta\} \) for some \( \delta \) positive. Likewise, a family of pseudo-metrics \( \{p_\alpha: \alpha \in A\} \) on \( X \) defines a natural uniformity \( U_A \) as follows: \( U \in U_A \) iff \( U \supset \{(x,y): p_\alpha(x,y) < \delta\} \) for all \( \alpha \in F \), for some positive \( \delta \) and finite \( F \subset A \). For more information, see (Kelley, 1955, pp. 175ff).

Our main result turns out to involve the Stone Cech compactification \( \tilde{X} \) of \( X \). This is the largest possible compactification of \( Y \); any bounded continuous function on \( X \) extends to a continuous function on \( \tilde{X} \). For more information, see (Dunford and Schwartz, 1958, p. 279) or (Kelley, 1955, p. 152).

Let \( Z \) be the set of positive integers, \( \pi(Z) \) the set of countably additive probabilities on \( Z \), and \( \overline{\pi}(Z) \) the set of finitely additive probabilities on \( Z \). Endow \( \pi \) and \( \overline{\pi} \) with the weak star topology. Thus, \( \overline{\pi} \) is a compactification of \( \pi \). The main result of this note is the following proposition, which will be proved in Section 3.

**Proposition 1.1.** \( \overline{\pi}(Z) \) is not the Stone Cech compactification of \( \pi(Z) \).

This issue came up in connection with work reported in Diaconis and Freedman (1984), where we considered two uniformities on \( \pi(Z) \):

- \( U_1 \) induced by the pseudometrics \( \rho_f(\mu, \nu) = |\int f d\mu - \int f d\nu| \) for \( f \in C(Z) \)
- \( U_2 \) induced by the pseudometrics \( \rho_\phi(\mu, \nu) = |\phi(\mu) - \phi(\nu)| \) for \( \phi \in C[\pi(Z)] \).

Here, \( C(X) \) is the set of bounded, continuous functions on \( X \); by \( C(Z) \) we just mean the bounded functions on \( Z \). Clearly, \( U_2 \) is finer than \( U_1 \). That the two are really different is not so obvious.

**Proposition 1.2.** \( U_1 \neq U_2 \).

This is fairly immediate from Proposition 1.1. Indeed, consider the
algebra $A_0$ of functions on $\pi(Z)$ generated by the basic linear functions $\mu + \int f d\mu$, as $f$ varies over $C(Z)$. Thus, $A_0 \subseteq C[\pi(Z)]$. We will call $A_0$ the "polynomials." Of course, any polynomial $\phi \in A_0$ extends to $\overline{\phi} \in C[\overline{\pi(Z)}]$, and

$$\sup \{ |\phi(\mu)| : \mu \in \pi(Z) \} = \sup \{ |\overline{\phi}(\mu)| : \mu \in \overline{\pi(Z)} \}$$

Let $\bar{A} \subseteq C[\pi(Z)]$ be the closure of $A_0$ in the sup norm. As (1) implies, any $\phi \in \bar{A}$ also extends to $\overline{\phi} \in C[\overline{\pi(Z)}]$. Let $\bar{A} = \{ \overline{\phi} : \phi \in A \}$.

**Lemma 1.1.** $\bar{A} = C[\overline{\pi(Z)}]$.

**Proof.** Use the Stone-Weierstrass theorem. \hfill \Box

By Proposition 1.1, $A$ is a proper subset of $C[\pi(Z)]$. Less formally, there are bounded continuous functions $\phi$ on $\pi(Z)$ which cannot be uniformly approximated by the polynomials $A_0$.) Corollary 2.2 below completes the derivation of Proposition 1.2; the object in section 2 is to establish this corollary. (That $A$ separates points and closed sets follows from Lemma 1.1.) Along the way, we discovered that convolution in $\overline{\pi(Z)}$ is noncommutative; we report on this in section 4. Our results can be extended to any noncompact metric space $X$: just identify $Z$ with a sequence $x_j : j \in Z$ having no convergent subsequences.
2. On uniformities.

Let $X$ be a Hausdorff space, completely regular in the sense that the bounded continuous functions separate points and closed sets, i.e., given $x \in X$ and a closed subset $C$ with $x \notin C$, there is a continuous function $f$ with $0 \leq f \leq 1$, $f(x) = 1$, and $f = 0$ on $C$. Let $A$ be a closed subalgebra of $C(X)$, which also separates points and closed sets.

**Lemma 2.1.**

a) If $f(x) = f(y)$ for all $f \in A$, then $x = y$.

b) If $\{x_\alpha\}$ is a net, and $f(x_\alpha) \to f(x)$ for all $f \in A$, then $x_\alpha \to x$.

**Lemma 2.2.** $X$ can be homeomorphically embedded as a subset of a compact Hausdorff space $\overline{X}_A$, such that $A$ is the restriction to $X$ of $C(\overline{X})$.

**Proof.** For $f \in A$, let $I_f$ be the closed interval $[\inf f, \sup f]$. Let $\Omega = \prod_f I_f$, a compact Hausdorff space. Let $\eta$ map $X$ into $\Omega$ as follows:

$$[\eta(x)](f) = f(x)$$

Clearly, $\eta$ is continuous. It is $1 - 1$ by Lemma 2.1a, and $\eta^{-1}$ is continuous by Lemma 2.1b.

Let $\overline{X}$ be the closure in $\Omega$ of $\eta(X)$. Then $\overline{X}$ is compact Hausdorff; for $f \in A$ define $\overline{f}$ on $\overline{X}$ by the formula $\overline{f}(\xi) = \xi(f)$ for $\xi \in \overline{X}$. In particular, $\overline{f}$ is continuous and $\overline{f}[\eta(x)] = f(x)$ extends $f$.

Let $\overline{A} = \{\overline{f}: f \in A\}$. Then $\overline{A}$ is a closed subalgebra of $C(\overline{X})$ which separates points, so $\overline{A} = C(\overline{X})$ by the Stone-Weierstrass theorem. 

**Notes.** $\overline{X}_A$ is unique up to a homeomorphism. See (Dunford and
Schwartz, 1958, Part I, Corollary 27 on page 279). The space $\bar{X}_A$ is essentially the Stone Cech compactification of $X$, relative to $A$ not $C(X)$.

Recall that $A$ is a closed subalgebra of $C(X)$, separating points and closed sets. Let $U_A$ be the uniformity generated by the seminorms $\rho_f(x,y) = |f(x) - f(y)|$ as $f$ varies over $A$. Thus, any $f \in A$ is bounded and $U_A$-uniformly continuous. There are no other such functions.

**COROLLARY 2.1.** If $g$ is bounded and $U_A$-uniformly continuous, then $g \in A$.

**PROOF.** We apply Lemma 2.2, and claim that $g$ extends to $\bar{g} \in C(\bar{X}_A)$. Indeed let $\xi \in \bar{X}_A$, and $x_\alpha \in X$ with $x_\alpha \to \xi$. Then $g(x_\alpha)$ is a Cauchy net of real numbers because $g$ is $U_A$-uniformly continuous, and the net is bounded because $g$ is. Let $\bar{g}(\xi) = \lim_\alpha g(x_\alpha)$. By standard arguments, $\bar{g}$ is well defined and continuous. So $\bar{g} \in C(\bar{X}) = \bar{A}$, and $g \in A$, as required. \hfill \Box

**COROLLARY 2.2.** $U_A$ determines $A$. 
3. The proof of Proposition 1.1.

Let \( X \) be a metric space. Let \( K \) be a closed subset of \( X \). Let \( f \) be a bounded, continuous function on \( K \). The next result is a special case of Tietze's extension theorem. See (Dunford and Schwartz, 1958, pp. 15-17).

**Lemma 3.1.** \( f \) extends to a continuous function \( \bar{f} \) on \( X \), with no change of inf or sup.

**Proof of Proposition 1.1.** We define a subset \( Q \) of \( \pi(Z) \) as follows:

\[ \mu \in Q \text{ iff } \mu = \frac{1}{2} (\delta_j + \delta_k) \text{ for some pair of integers } j, k \text{ with } 1 < j < k, \]

\( j \) even, \( k \) odd. As usual, \( \delta_j \) is point mass at \( j \), so \( \delta_j \in \pi(Z) \).

Let \( \xi \) and \( \zeta \) be remote, finitely additive, \( 0 - 1 \) measures on \( Z \), with \( \xi \) assigning mass 1 to the evens and \( \zeta \) to the odds. So \( \frac{1}{2} (\xi + \zeta) \in \pi(Z) \). Let \( \delta_{\alpha} = \xi \) and \( \delta_{\beta} = \zeta \) weak star: \( \alpha \) and \( \beta \) run through directed sets. So \( \nu_{\alpha\beta} = \frac{1}{2} (\delta_{\alpha} + \delta_{\beta}) + \frac{1}{2} (\xi + \zeta) \) weak star.

We will now construct \( \phi \in C[\pi(Z)] \) such that \( \phi(\nu_{\alpha\beta}) \) fails to converge. More specifically,

\[ \lim_{\alpha} \lim_{\beta} \phi(\nu_{\alpha\beta}) = 1 \]

while

\[ \lim_{\beta} \lim_{\alpha} \phi(\nu_{\alpha\beta}) = 0 \]

To begin with, these equations hold with \( \phi \) replaced by the discontinuous function \( 1_Q \). Indeed, in e.g. (3.1), the double limit is just

\[ \int_Z \int_Z l_Q [\frac{1}{2} (\delta_j + \delta_k)] \zeta(dk) \xi(dj) \]
Without changing anything, we may confine $j$ to the evens and $k$ to the odds. For $j$ even, $l_Q[\frac{1}{2}(\delta_j + \delta_k)] = 1$ except for finitely many odd $k$, so the double integral is 1. Finally, to get $\phi$, smooth $l_Q$ using Lemma 3.1. More specifically, take $K = \{\frac{1}{2}(\delta_j + \delta_k): j, k = 1, 2, \ldots\}$. Then $l_Q$ is continuous on $K$ because the latter has no points of accumulation. □

Note. This $\phi$ is a bounded continuous function on $\pi(Z)$ which does not extend to $\overline{\pi(A)}$, i.e., which cannot be uniformly approximated by polynomials.
4. **Convolutions**

While trying to prove Proposition 1.1, we came across the following point. Let \( \xi \) and \( \zeta \) be finitely additive probabilities on \( \mathbb{Z} \). The convolution \( \xi \ast \zeta \) can be defined as usual

\[
(\xi \ast \zeta)(A) = \int_{\mathbb{Z}} \zeta(A-j)\xi(dj)
\]

where \( A - j = \{a-j: a \in A\} \). This set function is finitely additive; if \( \xi \) and \( \zeta \) are 0–1, so is \( \xi \ast \zeta \). However, \( \ast \) is not commutative.

Here is a preliminary.

**Lemma 4.1.** There is an infinite subset \( S \) of the positive even integers and \( T \) of the odds, such that \( (s,t) \rightarrow (s+t) \) is \( 1-1 \) on \( S \times T \).

**Proof.** Inductively, we define increasing functions \( f \) and \( g \) from \( \mathbb{Z} \) to the evens and odds respectively, such that \( f(j) + g(k) \) determines \( (j,k) \); then \( S = f(\mathbb{Z}) \) and \( T = g(\mathbb{Z}) \). Let \( f(1) = 2 \) and \( g(1) = 3 \).

Suppose \( f(j) \) and \( g(k) \) defined for \( j,k \leq n \). Then

\[
\begin{align*}
  f(n+1) &= f(n) + g(n) - 1 \\
  g(n+1) &= f(n+1) + g(n)
\end{align*}
\]

As is easily seen,

\[
\begin{align*}
  \min_{k=1,\ldots,n} [f(n+1) + g(k)] &> \max_{j=1,\ldots,n} [f(j) + g(k)] \\
  \min_{j=1,\ldots,n+1} [f(j) + g(n+1)] &> \max_{j=1,\ldots,n} [f(j) + g(n+1)]
\end{align*}
\]

**Proposition 4.1.** There are finitely additive 0–1 measures \( \xi \).
and $\zeta$ on $Z$ such that $\xi \ast \zeta \neq \zeta \ast \xi$.

**PROOF.** Construct $S$ and $T$ as in Lemma 4.1. Let $\xi(S) = 1$ and $\zeta(T) = 1$. Let $Q = \{s+t: s \in S$ and $t \in T$ and $s < t\}$. Then

\begin{align*}
(4.1) & \quad (\xi \ast \zeta)(Q) = 1 \\
(4.2) & \quad (\zeta \ast \xi)(Q) = 0
\end{align*}

Only (4.2) will be argued. By definition,

\[
(\xi \ast \xi)(Q) = \int Z (Q-k) \zeta(dk) = \int T (Q-t) \zeta(dt)
\]

because $\zeta(T) = 1$. If $t \in T$, however, $Q-t$ includes only $s \in S$ with $s < t$; this is where we need the fact that $s+t$ determines $s$ and $t$. So $\xi(Q-t) = 0$. \(\square\)

**Remarks.**

i) With bar for Stone Cech, $\overline{Z \times Z}$ seems really bigger than $\overline{Z \times Z}$, by present construction. So a bounded function on $Z \times Z$ is not in the uniform closure of the algebra generated by $(x,y) \rightarrow u(x)$ or $v(y)$, $u$ and $v$ bounded.

ii) Likewise, there seems to be a bounded continuous function on $Z^\infty$ not uniformly approximable by finitary functions, i.e. bounded and of the form $u(x_1, \ldots, x_n)$ as $u$ and $n$ vary, but maybe a new idea is needed, along the following lines.

Let $\mu$ be any remote $0-1$ finitely additive measure on $Z$, and let $\delta_{n_\alpha} \rightarrow \mu$ with $n_\alpha \in Z$. Let $\mu^k$, a finitely additive $0-1$ measure on $Z^k$, be the law of the finite sequence

$$n+1, n+2, \ldots, n+k$$

with $n$ chosen at random from $\mu$. Likewise for $\mu^\infty$ on $Z^\infty$. Define
$a_n, b_n \in \mathbb{Z}^\infty$ as follows:

$$a_n = (n+1, n+2, \ldots, 2n, 2n+1, 2n+2, \ldots)$$

$$b_n = (n+1, n+2, \ldots, 2n, 0, 0, \ldots) .$$

Then $A = \{a_n\}$ and $B = \{b_n\}$ are disjoint closed sets in $\mathbb{Z}^\infty$. Let $f \in C(\mathbb{Z}^\infty)$ with $0 \leq f \leq 1$, $f = 1$ on $A$, $f = 0$ on $B$. Then $f$ is not a uniform limit of finitary functions. Indeed, for any $k$, for all large $\alpha$, the infinite sequences $a(\alpha) = a_n^\alpha$ and $b(\alpha) = b_n^\alpha$ agree to $k$ places. Projected on $\mathbb{Z}^k$, then,

$$\lim_{\alpha} \delta a(\alpha) = \lim_{\alpha} \delta b(\alpha) .$$

On $\mathbb{Z}^\infty$, however, $f(a(\alpha)) \equiv 1$ and $f(b(\alpha)) \equiv 0$.

REFERENCES


