DE FINETTI'S THEOREM IN CONTINUOUS TIME

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Abstract. This paper gives a simpler proof of theorems characterizing mixtures of continuous time Markov chains, and mixtures of processes with stationary, independent increments.

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Running head. De Finetti's Theorem.

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1. Introduction. This paper gives a simpler proof of two theorems in Freedman (1963), characterizing mixtures of Markov chains in continuous time with recurrent, stable states the (stationarity condition is eliminated), as well as mixtures of processes with stationary independent increments. To state the first result, let $I$ be a countable set--the state space. Let $Ω$ be the set of all functions from $[0,∞)$ to $I$, with the product $σ$-field $F$. Let $\{X_t\}$ be the coordinate process on $Ω$. The law $P$ of an $I$-valued stochastic process is thus a probability of $F$. Fix $i_0 \in I$, the starting state.

Let $Π$ be the set of standard stochastic semigroups $P$ on $I_p \subset I$, such that $i_0 \in I_p$ and $I_p$ is a single recurrent class of stable states; $Π$ is a standard Borel space. Let $P_{i0}$ be the law of a Markov chain starting from $i_0$ and moving according to $P$, so $P_{i0}$ is a probability on $F$.

**THEOREM.** $P = \int_Π P_{i0} \mu(dP)$ for some probability $μ$ on $Π$ iff:

i) $P\{X_0 = i_0\} = 1$

ii) $\{X_t\}$ has no fixed points of discontinuity

iii) $P\{X_n = i_0 \text{ for infinitely many integers } n\} = 1$

iv) For each $h > 0$, the $P$-law of $\{X_{nh}: n = 0,1,2,...\}$ depends only on the transition counts, in the sense of Freedman (1962) or Diaconis and Freedman (1980).

The mixing measure $μ$ is unique.

For the second result, let $I$ be the real line with the Borel $σ$-field, $Ω$ the set of the functions from $[0,∞)$ to $I$, and $F$ the product $σ$-field in $Ω$. Again, let $\{X_t\}$ be the coordinate process on
$\Omega$, and $P$ a probability on $F$. This time, let $P \in \Pi$ be the law of a process with stationary, independent increments, starting from 0, continuous in probability. Again, $\Pi$ is a standard Borel space.

**THEOREM 2.** $P = \int_{\Pi} P \mu(dP)$ for some probability $\mu$ on $\Pi$ iff:

i) $P\{X_0 = 0\} = 1$

ii) $\{X_t\}$ is continuous in $P$-probability

iii) for each $h > 0$, the $P$-law of $\{X_{nh} - X_{(n-1)h}: n=1,2,...\}$ is exchangeable.

The mixing measure $\mu$ is unique.

In both theorems, necessity is obvious, and the uniqueness of $\mu$ follows from corresponding results in discrete time. Sufficiency is proved by approximation through the binary rationals, and only $h$ of the form $1/2^k$ are used. It is shown that conditional on a certain remote $\sigma$-field, the process is Markov (Theorem 1) or has stationary, independent increments (Theorem 2). The two proofs are very similar. That for Theorem 1 is given in section 2. The modifications for Theorem 2 are sketched in section 3, which also characterizes mixtures of Brownian motions or Poisson processes. A connection is then made with the theory of the Laplace transform, analogous to the connection between de Finetti's theorem for coin-tossing and the Hausdorff moment problem. Theorem 2 could easily be extended to processes taking values in a Euclidean space, or even a locally compact second countable abelian group, but such generalizations will not be discussed here. It is worth noting that neither theorem requires smoothness conditions on the sample paths.
2. The proof of Theorem 1. The following easy fact will be useful.

**Lemma 2.1.** Let $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{F}$ be $\sigma$-fields in $\Omega$. Let $Q(\omega, A)$ be an rcd given $\mathcal{F}$ and given $\mathcal{F}_1$. If $\mathcal{F}$ is any other $\sigma$-field with $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{F}_1$, then $Q(\omega, A)$ is an rcd given $\mathcal{F}$.

**Proof.** $Q(\cdot, A)$ is $\mathcal{F}_0$-measurable, and therefore $\mathcal{F}$-measurable. Likewise, for any $B \in \mathcal{F}_1$,

$$\int_B Q(\omega, A) \, P(d\omega) = P(A \cap B).$$

The display holds a fortiori for any $B \in \mathcal{F}$. \hfill $\square$

Turn now to the theorem. Let $P$ satisfy the conditions. Consider $h = 1/2^k$. Let $\tau_{n,h}$ be the time of the $n^{th}$ visit to $i_0$ by $\{X_{mh} : m = 0, 1, 2, \ldots\}$, with $\tau_0,h = 0$. Let $\{Y_{n,h} : n = 0, 1, \ldots\}$ be the $n^{th}$ $i_0$-block in $\{X_{mh}\}$, viz., the finite string

$$\{X_{\tau_{n,h}, mh} : 0 \leq m < \tau_{(n+1),h} = \tau_{n,h}\}.$$ 

By convention, an $i_0$-block starts at $i_0$, and ends just before the next $i_0$. Let $F_h$ be the tail $\sigma$-field of the process of pairs $\{Y_{n,h}, \tau_{n,h} : n = 0, 1, 2, \ldots\}$. Let $F_0 = \cup_h F_h$: here as elsewhere only $h$ of the form $1/2^k$ are contemplated.

**Lemma 2.2.** $F_h$ increases as $h$ decreases.

The easy proof is omitted.

**Lemma 2.3.** Given $F_h$, the process $\{X_{nh}\}$ is a Markov chain with stationary transition matrix $P_{h,\omega}$ and state space $I_{h,\omega}$; here $i_0 \in I_{h,\omega} \subseteq I$ and $I_{h,\omega}$ is a single recurrent class. Furthermore, $P_{h,\omega}$
is \( F_h \) measurable and unique a.e.

**PROOF.** This follows from Diaconis and Freedman (1980), because \( F_h \) is intermediate between the tail \( \sigma \)-field and the exchangeable \( \sigma \)-field of \( \{Y_{n,h}\} \): see Lemma 2.1. □

**LEMMA 2.4.** Let \( h \leq \frac{1}{2} \). Then \( P_{h,\omega}^2 = P_{2h,\omega} \text{ a.e.} \)

**PROOF.** By Lemma 2.3, given \( F_h \), the process \( \{X_{(2n)h}\} = \{X_{n(2h)}\} \) is conditionally Markov with transitions \( P_{h,\omega}^2 \). It remains only to show that \( P_{h,\omega}^2 \) is \( F_{2h} \)-measurable. But, e.g., for \( i_1 \neq i_0 \), \( P_{h,\omega}^2(i_0,i_1) \) is the limiting relative frequency of the \( i_0 \)-blocks of \( \{X_{n2h}\} \) which begin \( (i_0,i_1) \), and is therefore \( F_{2h} \)-measurable. The balance of the argument is omitted. □

Of course, \( j \in I_{h,\omega} \) iff \( j \) is reachable from \( i_0 \) relative to \( P_{h,\omega}^n \), i.e., \( (P_{h,\omega}^n)(i_0,j) > 0 \) for some \( n \). If \( j \in I_{2h,\omega} \), then

\[
0 < P_{2h,\omega}^n(i_0,j) = P_{h,\omega}^n(i_0,j), \quad j \in I_{h,\omega} \text{ i.e., } I_{h,\omega} \supset I_{2h,\omega}.
\]

In principle, \( I_{h,\omega} \) could have subclasses of period 2 relative to \( P_{h,\omega} \); then \( I_{2h,\omega} \) would be strictly smaller; as will be seen, however, in fact this cannot happen.

**LEMMA 2.5.** As \( h \to 0 \), \( P_{h,\omega}(i_0,i_0) \to 1 \) in probability.

**PROOF.** \( \int P_{h,\omega}(i_0,i_0) P(d\omega) = P\{X_h = i_0\} \to 1 \) as \( h \to 0 \) by conditions (i-ii). □

**LEMMA 2.6.** For \( P \)-almost all \( \omega \):

a) \( P_{h,\omega}(i_0,i_0) \to 1 \) as \( h \to 0 \) rapidly

b) \( P_{h,\omega}(i_0,i_0) > 0 \) for all \( h \)
c) \( I_{h, \omega} = I_{1, \omega} \)

PROOF. Claim a) follows from Lemma 2.5; then b) is immediate, because
\( P_{h, \omega}(i_0, i_0) \geq P_{h', \omega}(i_0, i_0) h/h' \) for \( h' < h \). To prove c), let \( j \in I_{h, \omega} \), so \( P^n_{h, \omega}(i_0, j) > 0 \) for some \( n \). Find \( m \) such that \((n+m)h\) is an integer. Now
\[
P^{(n+m)h}(i_0, j) = P^n_{h, \omega}(i_0, j) \geq P_{h, \omega}(i_0, i_0)^m P^n_{h, \omega}(i_0, j) > 0.
\]

Let \( I_\omega = I_{h, \omega} \). Plainly, \( \{P_{h, \omega}: h = 1/2^k \text{ and } k \text{ is a nonnegative integer}\} \) extends to a unique semigroup \( \{P_r, \omega\} \) of stochastic matrices on \( I_\omega \), where \( r \) runs through the nonnegative binary rationals \( R \), and \( P_0, \omega \) is the identity matrix by convention. Recall that \( F_h \) increases to \( F_0 \) as \( h \) decreases.

**LEMMA 2.7.** Given \( F_0 \), the process \( \{X_r: r \in R\} \) is conditionally Markov, with stationary transitions \( \{P_r, \omega\} \).

PROOF. This is immediate from the forward martingale convergence theorem. □

The next objective is to extend \( \{P_r, \omega\} \) to a standard stochastic semigroup on \([0, \infty)\).

**LEMMA 2.8.** For \( P\)-a.a. \( \omega \), for each \( j \in I_\omega \), \( P_r, \omega(j, j) \rightarrow 1 \) as \( r \rightarrow 0 \). (The convergence need not be uniform in \( j \).)

PROOF. Fix \( j \) and \( n \). Given \( X_n = j \), with probability 1, the process \( \{X_r\} \) must stay in \( j \) on the interval \([n, n+\epsilon]\), where \( \epsilon > 0 \) is random. This remains true given \( F_0 \), and the lemma follows: convergence a.e. implies convergence in probability. □
LEMMA 2.9. For $P$-a.a. $\omega$:

a) $P_{*,\omega}(j,k)$ is uniformly continuous for each $j \in I_\omega$, and extends to a continuous function $P_{t,\omega}(j,k)$ of nonnegative real $t$.

b) $P_{t,\omega}$ is a substochastic matrix.

c) $P_{t+s,\omega}(i,k) \geq \sum_j P_{t,\omega}(i,j)P_{s,\omega}(j,k)$.

PROOF. The argument is standard:

Claim a). From the semigroup property, 

$$P_{r+s,\omega}(j,k) = \sum_i P_{s,\omega}(j,i)P_{r,\omega}(i,k)$$

So 

$$P_{r+s,\omega}(j,k) - P_{r,\omega}(j,k) = [P_{s,\omega}(j,j) - 1]P_{r,\omega}(j,k) + \sum_{i \neq j} P_{s,\omega}(j,i)P_{r,\omega}(i,k)$$

Now 

$$|P_{r+s,\omega}(j,k) - P_{r,\omega}(j,k)| \leq 1 - P_{s,\omega}(j,j).$$

Claims b) and d) follow by Fatou's lemma. \qed
LEMMA 2.10. Fix a sequence of times $0 = t_0 < t_1 < \ldots < t_n$ and states $i_0, i_1, \ldots, i_n$. Let

$$A = \{X_{t_m} = i_m \text{ for } m = 0, \ldots, n\}$$

Let $B \in F_0$. Then

$$P\{A \cap B\} = \int_A \prod_{m=0}^{n-1} p_{t_{m+1} - t_m, \omega}(i_m, i_{m+1}) P(d\omega)$$

PROOF. Equation (2.1) holds for binary rational $t$ by Lemma 2.7. Now approximate real $t$ by binary rationals. The left side of (2.1) converges by condition (ii) of the theorem; the right side, by Lemma 2.9a and dominated convergence.

We do not yet know that $\{P_{t, \omega}\}$ is a stochastic semigroup, so (2.1) does not say that $\{X_t\}$ is Markov given $F_0$.

LEMMA 2.11. For each $t$ and each $j \in I_\omega$,

$$\sum_k p_{t, \omega}(j, k) = 1 \text{ a.e. } P.$$

PROOF. Define $G_j = \{w: j \in I_\omega\}$ and $G_{j, n} = \{w: P^n_{1, \omega}(i_0, j) > 0\}$, so $G_j = \bigcup_{n=1}^{\infty} G_{j, n}$. Fix $j$ and $n$. Now Lemma 2.10 shows:

$$P(X_n = j) = \int_{G_{j, n}} p^n_{1, \omega}(i_0, j) P(d\omega)$$

$$P(X_n = j \text{ and } X_{n+t} = k) = \int_{G_{j, n}} p^n_{1, \omega}(i_0, j)p_{t, \omega}(j, k) P(d\omega)$$

The sum on $k$ of the left side of (2.3) equals the left side of (2.2); in view of Lemma 2.9b),

$$\sum_k p_{t, \omega}(j, k) = 1 \text{ a.e. on } G_{j, n}.$$

The balance of the argument is omitted.
LEMMA 2.12. For $P$-a.a. $\omega$, \( \{ P_{t,\omega} \} \) is a standard stochastic semigroup.

PROOF. Let \( H_\omega = \{ t: \sum_j P_{t,\omega}(i,j) = 1 \text{ for all } i \in I_\omega \} \). By Lemma 2.11 and Fubini's theorem, for a.a. $\omega$, the complement of $H_\omega$ is a Lebesgue-null set. On the other hand, $H_\omega$ is closed under addition, by Lemma 2.9c). Thus, $H_\omega = [0, \infty)$, i.e., $P_{t,\omega}$ is a stochastic matrix. Now sum the inequality in Lemma 2.9c) over $k$, to see that equality holds, i.e., $P_{t,\omega}$ is a semigroup. That $P_{t,\omega}$ is standard follows from Lemma 2.9a).

Clearly, $i_0 \in I_\omega$ and $I_\omega \subseteq I$ is a single recurrent class of stable states for $P_{t,\omega}$. Lemma 2.10 implies the following result, which gives the theorem.

PROPOSITION 2.1. Given $F_0$, the process \( \{ X_t \} \) is conditionally Markov with transition \( \{ P_{t,\omega} \} \).

The argument really shows the existence of $E \in F_0$, such that $P(E) = 1$ for all $P$ satisfying the conditions of the theorem, and the existence of the standard stochastic semigroup \( \{ P_{t,\omega} \} \) for all $\omega \in E$.

Mixtures of processes with instantaneous states can probably be characterized the same way, replacing condition (ii) by continuity in probability.

The definition of $F_h$ may seem a bit complicated, but neither the tail $\sigma$-field nor the exchangeable $\sigma$-fields are nested. We discuss the tail $\sigma$-field in discrete time. Let $X_0, X_1, X_2, \ldots$ be I-valued, starting form $i_0$, with
infinitely many visits to $i_0$, on even times. Let $\Sigma_1$ be the tail
$\sigma$-field of the $i_0$-blocks in $\{X_0, X_1, X_2, \ldots\}$. Let $\Sigma_1$ be the tail
$\sigma$-field of the $i_0$-blocks of $\{X_0, X_2, X_4, \ldots\}$. If $\Sigma_1 \supset \Sigma_2$ then atoms of
$\Sigma_1$ cannot split atoms of $\Sigma_2$. This is false by example.

Let $i_1, i_2, \ldots$ be any infinite sequence in $I - \{i_0\}$. Consider the $X_n$-sequence

$$X_{2n} = i_0 \text{ for all } n$$

$$X_1 = i_0, \ X_3 = i_1, \ X_5 = i_0, \ X_7 = i_2, \ X_9 = i_0, \ X_{11} = i_3, \ldots$$

The $i_0$-blocks of $X_n$ are then

$$i_0, i_0, i_0i_1, i_0, i_0i_2, i_0, i_0, i_0i_3, \ldots$$

An atom of $\Sigma_1$ consists of all $X_n$-sequences whose $i_0$-blocks eventually
agree with this. Consider the clock time for the start of the doublets
$i_0i_j$. For an $\omega$ in our atom, this is eventually even, or eventually odd.
If even, the $i_0$-blocks in $\{X_{2n}\}$ end up identically $i_0i_0i_0, \ldots$ If odd,
then $i_0$-blocks in $\{X_{2n}\}$ end up agreeing from some point on with
$i_0i_1, i_0i_2, i_0i_3, \ldots$ Thus a $\Sigma_1$-atom splits a $\Sigma_2$-atom and $\Sigma_1 \supset \Sigma_2$. 
3. The proof of Theorem 2. This follows the same general pattern for Theorem 1, but is much easier. For $F_h$, use the tail or exchangeable $\sigma$-field of $\{X_{nh} - X_{(n-1)h}: n=1,2,...\}$. Clearly, $F_n$ increases as $h \to 0$; call the limit $F_0$. Given $F_h$, the differences are iid with common distribution $F_{h,\omega}$; and $F_{h,\omega} \ast F_{h,\omega} = F_{2h,\omega}$. Then we have a convolution semigroup $S_\omega = \{F_{r,\omega}: r \in \mathbb{R}\}$, and given $F_0$, the process $\{X_r: r \in \mathbb{R}\}$ has stationary independent increments governed by $S_\omega$. Now there is a simplification.

**Lemma 3.1.** If $\{X_r: r \in \mathbb{R}\}$ has stationary, independent increments, then for each fixed real $t$, as $r \to t$, $X_r$ converges a.e.

**Proof.** This is well known; the restriction of the time domain to a countable set is crucial. Perhaps the simplest direct argument involves considering the martingale

$$\exp(iuX_r)E\{\exp(iuX_r)\}$$

where $\exp(x) = e^x$.

**Lemma 3.2.** Let $Q(\omega,A)$ be a regular conditional $P$-distribution for $\{X_r: r \in \mathbb{R}\}$ given $F_0$. Suppose that for each real $t$, as $r \to t$, $\{X_r\}$ is fundamental in $Q(\omega,\cdot)$ probability. Suppose too that $\{X_t: 0 \leq t \text{ real}\}$ is continuous in $P$-probability.

a) $Q(\omega,\cdot)$ extends to $F$; call the extension $\bar{Q}(\omega,\cdot)$.

b) $\bar{Q}(\omega,A)$ is an rcd for $\{X_t\}$ given $F_0$.

**Proof.** Fix positive real times $t_1 < t_2 < ... < t_k$ and bounded continuous functions $f_1,\ldots,f_k$. Let $r_j \to t_j$ through $R$. Now the $\bar{Q}(\omega,d\omega')$ integral of $\prod_{j=1}^k f_j[X_{t_j} (\omega')]$ can be defined as

$$\lim \int \prod_{j=1}^k f_j[X_{r_j} (\omega')] Q(\omega,d\omega')$$

Integrate over $F \in F_0$ and use dominated convergence:
\[
\int_{F} \prod_{j=1}^{k} f_{j}(X_{t_{j}(\omega)}) \, \tilde{Q}(\omega, d\omega') = \lim_{j \rightarrow \infty} \int_{F} \prod_{j=1}^{k} f_{j}(X_{t_{j}(\omega)}) \, Q(\omega, d\omega')
\]
\[
= \lim_{j \rightarrow \infty} \int_{F} \prod_{j=1}^{k} f_{j}(X_{t_{j}(\omega)}) \, P(d\omega)
\]
\[
= \int_{F} \prod_{j=1}^{k} f_{j}(X_{t_{j}(\omega)}) \, P(d\omega)
\]
by the continuity assumption on P. 

Two special cases of Theorem 2 are worth considering.

1) Suppose \( \{X_{t}\} \) has continuous sample paths. Then, \( Q_{\omega} \) is defined in the space of sample paths on the binary rationals; it assigns measure 1 to the paths which are uniformly continuous on compacts. Thus, \( \tilde{Q}_{\omega} \) concentrates on the continuous sample paths. And, a process with continuous paths and exchangeable increments is a scale-drift mixture of Brownian motions: indeed, a process with continuous sample paths and stationary, independent increments is a Brownian motion.

2) Suppose \( \{X_{t}\} \) has exchangeable increments and sample paths which are counting functions. Then it is a mixture of Poisson processes.

David Aldous and Persi Diaconis remark that the last observation enables us to develop the theory of the Laplace transform, just as de Finetti's theorem for coin tossing solves the Hausdorff moment problem (Feller, 1971, p. 228). For instance, let \( L \) be defined on \( [0, \infty) \). When is there a probability \( \mu \) on \( [0, \infty) \) such that:

\[
L(t) = \int_{0}^{\infty} e^{-\lambda t} \mu(d\lambda)
\]

Necessary conditions are that \( L(0) = 1 \) and \( L \) is \( C_{\infty} \); while \( L' < 0, \ L'' > 0 \), etc.

According to Bernstein's theorem, these conditions are also sufficient: for instance, see (Feller, 1971, p. 439). Here is a rough sketch of the idea for a probabilistic proof. Initially, we tried to construct a process
\{X_t\} with exchangeable increments and counting sample functions, having
\(X(0) = 0\) and \(L(t) = \text{prob}\{X_t = 0\}\). This seemed hard to do. Instead,
we made a "completely exchangeable" process of binary trees \(T_0, T_1, T_2, \ldots\)
of random variables; the variables take two values, 0 and 1. More
specifically, \(T_n = \{X_{ns}\}\) where \(X_{ns} = 0\) or 1 and the node \(ns\) consists
of the nonnegative integer \(n\) followed by a finite string \(s\) (perhaps
empty) of 0's and 1's. These \(T_n\) are required to be exchangeable. Also,
each \(T_n\) splits into \(T_{n0}\) and \(T_{n1}\), the left half and right half.
These are required to be exchangeable too. And so forth. That is our
definition of "completely exchangeable." Technically, \(T_{n0}\) for example
is the tree \(\{T_{n0s}\}\), where 0s denotes 0 followed by the string \(s\).

We require that each variable be the maximum of the variables at the
two successor nodes, so \(X_{ns} = X_{ns0} \lor X_{ns1}\). Finally, we require

\[\text{Prob}\{1\text{st} \ j \text{ variables at level } k \text{ are 0}\} = L(j/2^k)\].

Here, the nodes are ordered lexicographically: the first three nodes
at level 0 are 0, 1, 2; the first five nodes at level 1 are 00, 01, 10,
11, 20; and so on.

Informally, the nodes correspond to intervals, e.g., the node \(n\) corresponds
to the interval \([n, n+1]\), the node \(n_0\) to \([n, n+\frac{1}{2}]\), the node
\(n_{01}\) to \([n+\frac{1}{4}, n+\frac{3}{4}]\), and so forth. The variable \(X_{ns} = 0\) iff there is no dot
in the corresponding interval for the desired counting process.

Formally, we can construct the tree distributions consistently down
to any finite level, and then use the Kolmogorov consistency theorem to
get the infinite trees. For example, for levels 0 and 1, we define
\(\{\xi_n: n = 0, 1, \ldots\}\) to be exchangeable and

\[P(\xi_0 = \cdots = \xi_{N-1} = 0) = L(N/2)\]
See (Feller, 1971, p. 228). We use the pair $\xi_0, \xi_1$ for $X_{00}, X_{01}$; the pair $\xi_2, \xi_3$ for $X_{10}, X_{11}$. And so on. We set $X_0 = X_{00} \lor X_{01}, X_1 = X_{10} \lor X_{11}$. This is consistent because

$$P\{X_0 = \cdots = X_{N-1} = 0\} = P\{\xi_0 = \xi_1 = \cdots = \xi_{2N-2} = \xi_{2N-1} = 0\}$$

$$= L(2N \cdot 2^N) = L(N)$$

We now condition on the tail $\sigma$-field $\Sigma$ of $\{T_n\}$. The fragments are iid trees. Clearly, $T_n$ is a 1-1 function of $(T_{n0}, T_{n1})$. So the tail $\sigma$-field of $T_n$ equals the tail $\sigma$-field of $T_{ns}$, where $ns$ is lexicographically ordered along any fixed level $k$, i.e., strings $s$ of length $k = 0, 1, \ldots$. Given $\Sigma$, we have at level $k$ iid variables $X_{ns}$, which are 0 with probability $p_{k,\omega}$. Clearly, $p_{k,\omega} = p_{k+1,\omega}$, so $p_{k,\omega} = (p_{0,\omega})^{1/2^k}$. If $p_{0,\omega} = 0$ then $p_{k,\omega} = 0$ for all $k$; then let $\omega = \infty$, else let $p_{0,\omega} = \exp(-\lambda_\omega)$ and then $p_{k,\omega} = \exp(\lambda_\omega/2^k)$, where $0 \leq \lambda_\omega < \infty$. Now for $j \geq 1$,

$$L(j/2^k) = P\{1^{st} j \text{ variables at level } k \text{ are 0}\}$$

$$= \int_{\lambda<\infty} e^{-\lambda j/2^k} dP$$

On $\lambda = \infty$, all variables are 1, and $L = 0$. Thus, for $t > 0$,

$$L(t) = \int_{\lambda<\infty} e^{-\lambda t} dP.$$

Let $t \to 0$, so $P(\lambda<\infty) = L(0+) = 1$. This completes the proof of the sufficiency of the condition for $L$ to be a Laplace transform, using de Finetti's theorem for trees--but not Theorem 2. For a derivation through the Martin boundary, see Watanabe (1960).
4. **Other literature.** Theorem 2 goes back to Buhlmann (1960). For another exposition, see Aldous (1984, sec. 10). For a stopping-time approach to Theorems 1 and 2, see Kallenberg (1982).
REFERENCES


32. BREIMAN, L. (May 1984). Nail finders, edifices, and Oz.


43. BICKEL, P. J. and YAHAV, J. A. (1985). On estimating the number of unseen species: how many executions were there?


56. LE CAM, L. and YANG, G. L. (January 1986). Replaced by No. 68.


64. O'SULLIVAN, F. (May 1986). Nonparametric Estimation in the Cox Proportional Hazards Model.


71. BICKEL, P.J. and YAHAV, J.A. (July 1986). Richardson Extrapolation and the Bootstrap.


77. O'SULLIVAN, F. (September 1986). Deconvolution of episodic hormone data.


90. DABROWSKA, D.M. (March 1987). Rank tests for matched pair experiments with censored data.
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