Stopping Times and Tightness II

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Abstract.

To establish weak convergence of a sequence of martingales to a continuous martingale limit, it is sufficient (under the natural uniform integrability condition) to establish convergence of finite-dimensional distributions. Thus in many settings, weak convergence to a continuous limit process can be deduced almost immediately from convergence of finite-dimensional distributions. These results may be technically useful in simplifying proofs of weak convergence, particularly in infinite-dimensional settings. The results rely on a technical tightness condition involving stopping times and predictability of imminent jumps.

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1. Introduction.

One may draw a loose distinction between two methods of proving weak convergence results for stochastic processes. The classical method, e.g. Billingsley (1968), starts by proving convergence of finite-dimensional distributions (f.d.d.'s) and then verifies a tightness condition. The modern approach, e.g. Ethier and Kurtz (1986), starts with a characterization of the limit process, then shows the characterization is "asymptotically true" for the approximating processes, and then argues this must imply weak convergence. One result which is sometimes useful in the modern approach is the following. For $n = 1,2,\ldots;\infty$ let $(X_n(t)\,;\,0 \le t < \infty)$ be real-valued processes. Regard X_n as a random element of the usual function space $D = D([0,\infty), R)$, equipped with its usual (Skorokhod J_1) topology. Let T_n denote a natural stopping time for $X_n(t)$. Then the condition

(1.1) for all $\delta_n \downarrow 0$ and all uniformly bounded (T_n),

$$X_n(T_n + \delta_n) - X_n(T_n) \underset{P}{\to} 0 \quad \text{as} \quad n \to \infty,$$

together with minor side conditions, implies tightness of the sequence (X_n). This result was given in the author's Ph.D. thesis and published in Aldous (1978). Similar results are in Rebolledo (1979), and subsequent extensions and applications are given in Jacod et al (1983), Ethier and Kurtz (1986), Joffe and Metivier (1986), Nikunen (1984), Dawson et al (1987), Walsh (1986).

The purpose of this paper is to exhibit another result from the author's thesis which does not seem to have been rediscovered subsequently. The result relates to the classical method. If we have established convergence of f.d.d.'s, and if the limit process is continuous, then condition (1.1) can be replaced by a much weaker condition of the same type. The precise formulation of the result is deferred to Proposition 2.2, but its most striking consequence is the following. Write $\overset{fdd}{\to}$ for convergence of finite-dimensional distributions, and $\overset{D}{\to}$ for weak convergence of processes.

(1.2) Proposition. *For* $n = 1,2,\ldots$ inf; *let* $(M_n(t) : 0 \le t < \infty)$ *be a martingale. Suppose*

(a) $M_n \overset{fdd}{\to} M_\infty$;

(b) $M_\infty(t)$ *is continuous in* t;

(c) *for each* t, $\{M_n(t) : 1 \le n < \infty\}$ *is uniformly integrable.*

Then $M_n \overset{D}{\to} M_\infty$.

Modern semimartingale theory shows that all integrable discrete-time processes, and most continuous-time processes encountered in practice, can be decomposed as the sum of a martingale and a bounded variation process. In the real-valued case, this gives a decomposition

$$X(t) = M(t) + A(t) + B(t)$$

where M is a martingale, A is an increasing process and B is a decreasing process. Thus in a rather general setting, one may be able to deduce weak convergence to a continuous limit from convergence of f.d.s.'s, using the following corollary.

(1.3) Corollary. *For* $n = 1,2,...;\infty$ *let* $M_n(t)$ *be a martingale,* $A_n(t)$ *an increasing process, and* $B_n(t)$ *a decreasing process. Suppose*

(a) $(M_n, A_n, B_n) \overset{fdd}{\to} (M_\infty, A_\infty, B_\infty)$;

(b) $M_\infty(t)$, $A_\infty(t)$ *and* $B_\infty(t)$ *are continuous;*

(c) *for each* t, $\{M_n(t): 1 \le n < \infty\}$ *is uniformly integrable.*

Then $(M_n, A_n, B_n) \overset{D}{\to} (M_\infty, A_\infty, B_\infty)$ *and in particular* $X_n = M_n + A_n + B_n \overset{D}{\to} X_\infty = M_\infty + A_\infty + B_\infty$.

Some technical remarks are given at the end of the section. There are several potential types of application of these results. One setting where the modern approach has been well-developed is the case where the limit process is a 1 or d-dimensional diffusion. There are powerful general results giving sufficient conditions for convergence, without needing explicitly to prove convergence of f.d.d.'s. But in a concrete example one may have special structure which does enable convergence of f.d.d.'s to be proved, and then weak convergence to be deduced from our results: and this may be easier than seeking to verify hypotheses of general theorems. For instance, a recent preprint of Cox (1987) studies voter models on the d-dimensional torus, and proves (inter alia) that the process giving the density of "1"s at time t can be rescaled to converge to a 1-dimensional diffusion. The proof uses the method of moments to prove convergence of f.d.d.'s, and then verifies a tightness condition. But the processes involved are martingales, and Proposition 1.2 shows that the tightness verification is unnecessary.

For another type of application, recall that to prove $X_n \overset{D}{\to} X_\infty$ it suffices to construct $X_n^k(t)$ such that

$$X_n^k \overset{D}{\to} X_\infty^k \quad \text{as } n \to \infty; \ k \text{ fixed}$$

$$\lim_{k\to\infty} \limsup_{n\to\infty} E \sup_{0\le t\le L} |X_n^k(t) - X_n(t)| = 0; \ L \text{ fixed.} \tag{1.4}$$

Under the kind of "semimartingale" conditions of Corollary 1.3, one can show that in (1.4) the "sup over t" can be replaced by fixed t, giving an easier condition to verify. This is used in Aldous and Shields (1989), in the context of an infinite-dimensional Gaussian diffusion limit process.

A third possible application concerns processes X_t whose values are Schwartz distributions. In this setting, weak convergence reduces to weak convergence of the real-valued processes $< \phi, X_t >$ where ϕ is a smooth function - see Mitoma (1983). Dawson et al (1987) treat a concrete example by first proving convergence of f.d.d.'s and then checking the tightness condition (1.1). In their example the limit process is discontinuous, so our new results do not apply, but there may be similar examples where they are applicable.

Technical remarks. (a) The uniform integrability hypothesis is natural, since under this hypothesis the weak limit of a sequence of martingales must be a martingale; otherwise, the limit can be arbitrary. Of course one can try to apply the results to non-integrable processes (local martingales, semimartingales) by truncating at suitable stopping times. It seems easier to do this on an *ad hoc* basis rather than giving an abstract formalization.

(b) Proposition 1.2 fails if the limit martingale is not continuous. Take (T, ξ_1, ξ_2) independent with T exponential, $P(\xi = 1) = P(\xi = -1) = \frac{1}{2}$ and consider

$$X_n(t) \;=\; \xi_1 1_{(t \ge T)} + \xi_2 1_{(t \ge t+T/n)}.$$

(c) In a rather neglected paper, Loynes (1976) gives Proposition 1.2 under a complicated extra hypothesis on the limit martingale M_∞, and verifies that Brownian motion satisfies this extra hypothesis. Thus applications of our results could presumably be proved using Loynes result and verifying his condition; but the point of our results is to make proofs simple.

(d) Another rather neglected paper, Kharlamov (1976), gives results similar to the criterion (1.1) in Aldous (1978), but less directly applicable.

(e) Corollary 1.3 works because, for increasing or decreasing functions, pointwise convergence to a continuous limit implies (local) uniform convergence. This is not true for bounded variation functions. Thus in the "martingale + bounded variation" decomposition, which is natural in $d > 1$ dimensions, we must explicitly assume that the bounded variation parts converge weakly, as follows.

(1.5) Corollary. *For* $n = 1,2,...;\infty$ *let* $M_n(t)$ *be a* R^d*-valued martingale and let* $A_n(t)$ *be an arbitrary* R^d*-valued process. Suppose*

(a) $(M_n, A_n) \xrightarrow{\text{fdd}} (M_\infty, A_\infty)$;

(b) $M_\infty(t)$ *and* $A_\infty(t)$ *are continuous*;

(c) *for each* t, $\{|M_n(t)| : 1 \le n < \infty\}$ *is uniformly integrable;*

(d) $A_n \xrightarrow{D} A_\infty$.

Then $(M_n, A_n) \xrightarrow{D} (M_\infty, A_\infty)$ *and in particular* $M_n + A_n \xrightarrow{D} M_\infty + A_\infty$.

(f) The stopping times used to prove martingale results (e.g. maximal and upcrossing inequalities) often correspond to natural gambling strategies. Our proof uses a stopping rule of the form ''stop when the process increases significantly above its moving average''. Such rules are familiar to stock market technicians, but have apparently not been used in theoretical probability.

2. The stopping time criterion.

In this section we state and discuss the precise stopping time criterion, Proposition 2.2, and show how the results stated in section 1 are deduced. The proof of Proposition 2.2 is deferred to section 4.

Let $0 < L < \infty$, $0 < \varepsilon < \frac{1}{2}$ and $0 < \delta \le 1$ throughout: thus "for all ε" means "for all $\varepsilon \in (0, \frac{1}{2})$". Let X be a process in D. Define $\Gamma_X(L, \varepsilon, \delta)$ to be the supremum of $\Gamma \ge 0$ such that:

for each stopping time T for X satisfying $P(T \le L) \ge \varepsilon$ we have

(2.1)(a) $$P(X(T + \delta') - X(T) \le \varepsilon \mid T \le L) \ge \Gamma, \quad \text{all } 0 < \delta' \le \delta$$

and

(b) $$P(X(T + \delta') - X(T) \ge -\varepsilon \mid T \le L) \ge \Gamma, \quad \text{all } 0 < \delta' \le \delta.$$

(2.2) Proposition. *For* $n = 1,2,\ldots;\infty$ *let* X_n *be processes in* D. *Suppose*

(a) $X_n \xrightarrow{\text{fdd}} X_\infty$;

(b) X_∞ *is continuous*;

(c) *for each* ε, L *there exists* $\delta > 0$ *such that* $\liminf_{n\to\infty} \Gamma_{X_n}(L, \varepsilon, \delta) > 0$.

Then $X_n \xrightarrow{D} X$.

Remarks. Here is an intuitive interpretation of condition (c). By a "jump" in X we mean a change in level which occurs very quickly (but not necessarily a discontinuity). Conditions (1.1) and (c) both have the rough interpretation "jumps are unpredictable". Less roughly, (1.1) says "if you try to predict a jump, then you are right with probability close to 0". Whereas (c) says "if you try to predict a jump and the sign (+ or -) of the jump, then the probability you are right is not close to 1". Thus (c) is a much weaker condition than (1.1). Incidently, this is the distinction underlying the famous "surprise exam" paradox. If I tell my class there will be a surprise exam one day next week, and then pick the day at random, then the exam is unpredictable (i.e. a surprise) in one sense but not in the other sense.

It is intuitively clear that martingales must satisfy this "no predictable signed jumps" condition, because the increment after a stopping time has mean zero. Lemma 2.3 formalizes this idea.

Aldous (1977) shows that (c) is close to a necessary condition in the setting of Proposition 2.2 (under uniform integrability hypotheses).

(2.3) Lemma. *Let* X *be a martingale. Define* $\phi_X(L, \varepsilon)$ *to be the infimum of* $\phi \geq 1$ *such that, for all stopping times* $0 \leq S_1 \leq S_2 \leq L$,

$$E|X(S_2) - X(S_1)| 1_{(|X(S_2)-X(S_1)| \geq \phi)} \leq \varepsilon.$$

Then

$$\Gamma_X(L, \varepsilon, 1) \geq \frac{1}{4}\varepsilon^2 / \phi_X(L+1, \frac{1}{2}\varepsilon^2).$$

Proof. Fix a stopping time T with $P(T \leq L) \geq \varepsilon$ and fix $0 < \delta' < 1$. Let $U = X(T + \delta') - X(T)$ on $\{T \leq L\}$

$= 0$ on $\{T > L\}$.

Note that we can regard U as being of the form $X(S_2) - X(S_1)$ for $0 \leq S_1 \leq S_2 \leq L+1$ by setting $S_1 = S_2 = L+1$ on $\{T > L\}$. Now

$$\begin{aligned}
0 &= EU \quad \text{(optional sampling theorem)} \\
&= EU1_{(U \geq b)} + EU1_{(T \leq L)}1_{(b > U > -\varepsilon)} + EU1_{(T \leq L)}1_{(U \leq -\varepsilon)}, \quad \text{for any } b > 0 \\
&\leq \frac{1}{2}\varepsilon^2 + bP(T \leq L, U > -\varepsilon) - \varepsilon P(T \leq L, U \leq -\varepsilon) \quad \text{for } b > \phi_X(L+1, \frac{1}{2}\varepsilon^2) \\
&= \frac{1}{2}\varepsilon^2 + (b+\varepsilon)P(T \leq L, U > -\varepsilon) - \varepsilon P(T \leq L) \\
&\leq \frac{1}{2}\varepsilon^2 + 2bP(T \leq L, U > -\varepsilon) - \varepsilon^2.
\end{aligned}$$

Rearranging,

$$\begin{aligned}
P(U \geq -\varepsilon | T \leq L) &\geq P(U \geq -\varepsilon, T \leq L) \geq \frac{1}{4}\varepsilon^2 / b \\
&\geq \frac{1}{4}\varepsilon^2 / \phi_X(L+1, \frac{1}{2}\varepsilon^2) \quad \text{letting } b \downarrow \phi.
\end{aligned}$$

This, and the same result applied to $-U$, establishes the lemma.

Proof of Proposition 1.2. Looking at Proposition 2.2, we see that the only issue is to use the uniform integrability hypothesis (1.2c) to verify the condition (2.2c). It is well known that, starting from a uniformly integrable family, the set of all conditional expectations of members of the family is a uniformly integrable set. Thus the family

$$\{X_n(S) : 1 \leq n < \infty, 0 \leq S \leq L \text{ is a stopping time on } X_n\}$$

is uniformly integrable. So in Lemma 2.3 we have $\sup_n \phi_{X_n}(L, \varepsilon) < \infty$ and then the conclusion of the lemma gives $\inf_n \Gamma_{X_n}(L, \delta, 1) > 0$, verifying (2.2c).

Proof of Corollary 1.3. For monotone processes, convergence of f.d.d.'s to a continuous limit implies weak convergence. Thus $A_n \xrightarrow{D} A$ and $B_n \xrightarrow{D} B$, and by Proposition 1.2 we have $M_n \xrightarrow{D} M$. So each of the sequences (A_n), (B_n), (M_n) is tight in the (local) uniform topology, which implies that the joint processes (A_n, B_n, M_n) are tight, and the result follows.

The proof of Corollary 1.5 is similar.

3. Convergence in measure.

Let μ be a probability measure on $[0,\infty)$ which is equivalent (i.e. mutually absolutely continuous) to Lebesgue measure. For $f, g \in D$ write

(3.1) $$d_\mu(f,g) = \inf\{\varepsilon > 0 : \mu\{t : |f(t) - g(t)| > \varepsilon\} \le \varepsilon\}.$$

So d_μ metrizes the topology of convergence in measure, which is of course weaker than the usual topology on D. Write $\overset{D}{\to}$ (μ-measure) for weak convergence of processes, when D is given this topology. The following result has been given independently several times, apparently first by Grinblat (1976), and recent developments are given in Cremers and Kadelka (1986).

(3.2) Proposition. *If* $X_n \overset{fdd}{\to} X_\infty$ *then* $X_n \overset{D}{\to} X_\infty$ *(μ-measure).*

Now d_μ-convergence implies d_L-convergence, where d_L is defined as at (3.1) with μ replaced by Lebesgue measure on $[0,L]$. Thus an appeal to the Skorokhod representation theorem (or to be precise Dudley's extension of it, since (D, d_μ) is not complete), Theorem 3.1.8 of Ethier and Kurtz (1986), yields

(3.3) Corollary. *Suppose* $X_n \overset{fdd}{\to} X_\infty$. *Then there exist* $(\hat{X}_n)$ *such that*

(a) $\hat{X}_n \overset{D}{=} X_n$; *each* $n = 1,2,\ldots;\infty$

(b) $d_L(\hat{X}_n, \hat{X}_\infty) \to 0$ a.s. *as* $n \to \infty$; *each* L.

We can now describe the idea behind the proof of Proposition 2.2. By Corollary 3.3 we may suppose paths are converging in measure: we want to prove they converge uniformly. How is it possible for a path f in D to be close in measure, but not uniformly, to some continuous path g? This can only happen if f has a "spike", a short time period in which f jumps away from and then returns to a vicinity of g. But then at the apex of the spike, we can predict that the sample path f is going to jump in a certain direction: this is what hypothesis (c) of Proposition 2.2 forbids.

The next section gives the details of this argument.

4. Proof of Proposition 2.2.

Fix L and ε. Write λ for Lebesgue measure. Define

$$\|f\|_{+} = \sup_{0 \le s \le L} f(s)$$

$$w_f(\sigma) = \sup_{\substack{0 \le s_1, s_2 \le L+1 \\ |s_2 - s_1| \le \sigma}} |f(s_2) - f(s_1)|$$

and define $d(f, g)$ as at (3.1) with μ replaced by Lebesgue measure on $[0, L+1]$. The first lemma is designed to pick out an upwards spike, by stopping at the first time that f exceeds a moving average.

(4.1) Lemma. *For* $f \in D$ *and* $0 < \sigma < \varepsilon$ *define*

$$T_\sigma(f) = \inf\{t \le L : \lambda\{r : t - \sigma \le r \le t,\ f(t) - f(r) \ge 5\varepsilon\} \ge \sigma/2\};$$

$$= \infty \text{ if no such } t \le L \text{ exists.}$$

Let A_σ *denote the set of* f *for which there exists some* g *satisfying*

$$\|f - g\|_{+} > 7\varepsilon; \tag{4.2}$$

$$w_g(\sigma) \le \varepsilon; \tag{4.3}$$

$$d(f, g) \le \sigma^2; \tag{4.4}$$

$$f(0) = g(0) = 0. \tag{4.5}$$

Let B_σ *be defined similarly but without (4.2). Then*

$$T_\sigma(f) \le L \text{ for each } f \in A_\sigma; \tag{4.6}$$

$$\lambda\{r : T_\sigma(f) \le r \le T_\sigma(f) + \sigma,\ f(r) - f(T_\sigma(f)) \ge -\varepsilon\} \le \sigma^2 \tag{4.7}$$

for each $f \in B_\sigma$ *such that* $T_\sigma(f) \le L$.

We remark that in the definition of T_σ we take $f(r) = f(0)$ for $r < 0$. Note also that the infimum is attained, so that T_σ defines a natural stopping time.

Proof. Fix $f \in A_\sigma$ and let g satisfy (4.2)-(4.5). By (4.2) there exists $u \le L$ such that

$$f(u) - g(u) > 7\varepsilon. \tag{4.8}$$

So by (4.3), $f(u) - g(r) > 6\varepsilon$ on $u - \sigma \le r \le u$, and so

$$\begin{aligned} &\lambda\{r : u - \sigma \le r \le u,\ f(u) - f(r) < 6\varepsilon - \sigma^2\} \\ &\le \lambda\{r : -\sigma \le r \le L,\ g(r) < f(r) - \sigma^2\} \\ &\le \sigma^2 \text{ by (4.4).} \end{aligned} \tag{4.9}$$

But $0 < \sigma < \varepsilon < 1/2$ and so $\sigma^2 < \varepsilon$, and $\sigma^2 < \sigma/2$. So (4.9) shows that

$$\lambda\{r : u - \sigma \le r \le u,\ f(u) - f(r) \ge 5\varepsilon\} \ge \sigma - \sigma^2$$

$$\geq \sigma/2.$$

Hence u satisfies the condition in the definition of T_σ, and so $T_\sigma(f) \leq u$, proving (4.6).

The proof of (4.7) requires a similar argument to be used twice. Fix $f \in B_\sigma$ with $T_\sigma(f) \leq L$, and let g satisfy (4.3)-(4.5). By definition of T_σ,

$$\lambda\{r: T_\sigma(f) - \sigma \leq r \leq T_\sigma(f),\ f(T_\sigma(f)) - f(r) \geq 5\varepsilon\} \geq \sigma/2.$$

So (4.4) and the estimate $\sigma^2 < \sigma/2$ show that

$$\lambda\{r: T_\sigma(f) - \sigma \leq r \leq T_\sigma(f),\ f(T_\sigma(f)) - g(r) \geq 5\varepsilon - \sigma^2\} > 0.$$

But by (4.3), $g(r) - g(T_\sigma(f)) \geq -\varepsilon$ on the above set, so using the estimate $\sigma^2 < \varepsilon$ we see that

$$f(T_\sigma(f)) - g(T_\sigma(f)) \geq 3\varepsilon.$$

Now using (4.3) again,

$$f(T_\sigma(f)) - g(r) \geq 2\varepsilon \text{ on } \{T_\sigma(f) \leq r \leq T_\sigma(f) + \sigma\}.$$

Hence

$$\lambda\{r: T_\sigma(f) \leq r \leq T_\sigma(f) + \sigma,\ f(r) - f(T_\sigma(f)) \geq \sigma^2 - 2\varepsilon\}$$
$$\leq \lambda\{r: 0 \leq r \leq L + 1,\ f(r) - g(r) \geq \sigma^2\}$$
$$\leq \sigma^2 \text{ by (4.4)}$$

and (4.7) follows.

We now translate Lemma 4.1 into a result about processes.

(4.10) Lemma. *Let* $0 < \sigma < \varepsilon < 1/2$. *Suppose* X *and* Y *are processes such that* $X(0) = Y(0) = 0$ *and*

(4.11) $$P(d(Y,X) \geq \sigma^2) \leq \sigma;$$

(4.12) $$P(w_X(\sigma) > \varepsilon) \leq \varepsilon;$$

(4.13) $$P(\|Y - X\|_+ > 7\varepsilon) \geq 3\varepsilon.$$

Then the stopping time $T = T_\sigma(Y)$ *satisfies*

(4.14) $$P(T \leq L) \geq \varepsilon.$$

And there exists $0 < \delta' < \sigma$ *such that*

(4.15) $$P(Y(T + \delta') \geq Y(T) - \varepsilon \mid T \leq L) \leq \varepsilon^{-1}(2\sigma + P(w_X(\sigma) > \varepsilon)).$$

Proof. Define

$$B = \{\omega: d(X,Y) < \sigma^2,\ w_X(\sigma) \leq \varepsilon\};$$

$$A = B \cap \{\omega : \|Y - X\|_+ > 7\varepsilon\}.$$

Then

$$(4.16) \qquad 1 - P(B) \le \sigma + P(w_X(\sigma) > \varepsilon) \text{ by (4.11)}$$
$$\le 2\varepsilon \text{ by (4.12)},$$

and so $P(A) \ge \varepsilon$, using (4.13).

Recall the definitions of A_σ, B_σ in Lemma 4.1.

If $\omega \in A$, then $Y(\omega) \in A_\sigma$ and so $T(\omega) \le L$ by (4.6), which establishes (4.14).

If $\omega \in B$ then $Y(\omega) \in B_\sigma$. Let θ be distributed uniformly on $[0, \sigma]$ independent of Y. Then (4.7) says:

$$P(Y(T + \theta) - Y(T) \ge -\varepsilon \mid Y) \le \sigma \text{ on } B \cap \{T \le L\}.$$

So

$$P(Y(T + \theta) - Y(T) \ge -\varepsilon \mid T \le L) \le (\sigma + 1 - P(B))/P(T \le L)$$
$$\le \varepsilon^{-1}(2\sigma + P(W_X(\sigma) > \varepsilon))$$

by (4.16) and (4.14). And this inequality must be true for some constant $\delta' \in$ range $(\theta) = [0, \sigma]$, which gives (4.15).

Proof of Proposition 2.2. There is no loss of generality in assuming $X_n(0) = 0$, since we could replace X_n by

$$X_n^*(t) = X_n(t-1), \; t \ge 1$$
$$= tX_n(0), \; 0 \le t \le 1.$$

By Corollary 3.3, there is no loss of generality in assuming $d_{L+1}(X_n, X_\infty) \to 0$ a.s. as $n \to \infty$. Thus we can choose $\sigma_n \downarrow 0$ such that

$$P(d_{L+1}(X_n, X_\infty) \ge \sigma_n^2) \le \sigma_n$$

(note L and ε are still fixed). Since X_∞ is continuous, for n sufficiently large

$$P(w_{X_\infty}(\sigma_n) > \varepsilon) \le \varepsilon.$$

Suppose there are infinitely many n for which

$$(4.18) \qquad P(\|X_n - X_\infty\|_+ > 7\varepsilon) \ge 3\varepsilon.$$

Then we can apply Lemma 4.10 to construct, for these n, stopping times T_n for X_n such that $P(T_n \le L) \ge \varepsilon$ and

$$P(X_n(T_n + \delta_n') \ge X_n(T_n) - \varepsilon \mid T_n \le L)$$
$$(4.19) \qquad \le \varepsilon^{-1}(2\sigma_n + P(w_{X_\infty}(\sigma_n) > \varepsilon)),$$

for some $0 < \delta_n' \le \sigma_n$. Now as $n \to \infty$ the right side of (4.19) tends to 0. Recalling (2.1), this implies

$$\liminf_{n\to\infty} \Gamma_{X_n}(L, \varepsilon, \delta) = 0$$

for all $\delta > 0$, which contradicts hypothesis (2.2c). Thus (4.18) holds for only finitely many n. Applying the same result for $(-X_n)$,

$$\limsup_{n\to\infty} P(\sup_{0\le s\le L} |X_n(s) - X_\infty(s)| > 7\varepsilon) \le 6\varepsilon.$$

Since L and ε are arbitrary, we deduce $X_n \xrightarrow{D} X_\infty$.

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