1. Introduction

The main purpose of this note is to discuss the ergodic properties of a certain class of strictly ergodic dynamical systems which appear as subsystems of the shift dynamical system defined on the power space $X = A^Z$, where $Z$ is the set of all integers. We discuss only the cases when the base space $A$ is a finite set. We are particularly interested in two examples of strictly ergodic dynamical systems which are constructed by using certain number-theoretic functions. Among other things it will be shown that there exist a continuum number of strictly ergodic dynamical systems, no two of which are spectrally isomorphic.

2. Strictly ergodic dynamical systems

Let $X = \{x\}$ be a nonempty compact metrizable space, and let $\varphi$ be a homeomorphism of $X$ onto itself. The pair $(X, \varphi)$ is called a dynamical system. A subset $X_0$ of $X$ is said to be $\varphi$-invariant if $\varphi(X_0) = X_0$. If $X_0$ is a nonempty closed $\varphi$-invariant subset of $X$, then $(X_0, \varphi)$ may be considered as a dynamical system, and is called a dynamical subsystem of $(X, \varphi)$. A dynamical system $(X, \varphi)$ is said to be minimal if there is no dynamical subsystem of $(X, \varphi)$ except $(X, \varphi)$ itself, that is if there is no nonempty closed $\varphi$-invariant subset of $X$ except $X$ itself.

Let

$$Z = \{n|n = 0, \pm 1, \pm 2, \cdots\}$$

be the set of all integers. For any point $x_0 \in X$, the set

$$\text{Orb} (x_0) = \{\varphi^n(x)|n \in Z\}$$

is called the orbit of $x_0$, and its closure $\overline{\text{Orb}} (x_0)$ is called the orbit closure of $x_0$. Obviously, $\overline{\text{Orb}} (x_0)$ is a closed $\varphi$-invariant subset of $X$, and hence $(\overline{\text{Orb}} (x_0), \varphi)$ is a dynamical subsystem of $(X, \varphi)$. It is clear that a dynamical system $(X, \varphi)$ is minimal if and only if $\text{Orb} (x_0)$ is dense in $X$ for any $x_0 \in X$.

Let $\mathcal{B} = \{B\}$ be the $\sigma$-field of all Borel subsets $B$ of $X$. It was proved by N. Kryloff and N. Bogoliouboff [6] that, for any dynamical system $(X, \varphi)$, there exists a normalized, countably additive, nonnegative measure $\mu$ defined on $\mathcal{B}$ which is invariant under $\varphi$; that is, $\mu(\varphi(B)) = \mu(B)$ for any $B \in \mathcal{B}$. Such a measure $\mu$ is not necessarily unique. A dynamical system $(X, \varphi)$ is said to be uniquely ergodic.
if such a measure $\mu$ is unique. A dynamical system is said to be strictly ergodic if it is minimal and uniquely ergodic at the same time.

Let $(X, \varphi)$ be a dynamical system, and let $x_0$ be a point of $X$. It was proved by W. H. Gottschalk [1] that $(\text{Orb}(x_0), \varphi)$ is minimal if and only if, for any neighborhood $W$ of $x_0$, there exists a positive integer $n$ such that, for any integer $m \in \mathbb{Z}$, at least one of the points $\varphi^k(x_0)$, $k = m + 1, \cdots, m + n$, belongs to $W$. On the other hand, it was proved by J. C. Oxtoby [10] that $(\text{Orb}(x_0), \varphi)$ is uniquely ergodic if and only if the limit

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} f(\varphi^k(x_0)) = \tilde{f}(x_0)
$$

exists uniformly in $m \in \mathbb{Z}$ for any real-valued continuous function $f$ defined on $X$. By combining these two results, we see that $(\text{Orb}(x_0), \varphi)$ is strictly ergodic if and only if (i) the limit (3) exists uniformly in $m \in \mathbb{Z}$ for any real-valued continuous function $f$ defined on $X$, and if (ii) $\tilde{f}(x_0) > 0$ for any nonnegative continuous function defined on $X$ such that $f(x_0) > 0$.

3. Shift dynamical systems

Let $A = \{a\}$ be a finite set containing more than one element. Let

$$
X = A^\mathbb{Z} = \prod_{n \in \mathbb{Z}} A_n; \quad A_n = A \text{ for all } n \in \mathbb{Z},
$$

be the set of all $A$-valued functions $x$ defined on $\mathbb{Z}$, or equivalently, the set of all two-sided infinite sequences

$$
x = \{a_n | n \in \mathbb{Z}\}; \quad a_n \in A \text{ for all } n \in \mathbb{Z}.
$$

The mapping

$$
\pi_n : x \mapsto a_n = \pi_n(x)
$$

is called the $n$-th projection of the power space $X = A^\mathbb{Z}$ onto the base space $A$, and $a_n$ is called the $n$-th coordinate of $x$.

The space $X$ is a totally disconnected compact metrizable space with respect to the usual direct product topology in which a defining neighborhood of a point $x_0$ of $X$ is given by

$$
W_{n_1, \cdots, n_{\ell}}(x_0) = \{x | \pi_{n_i}(x) = \pi_{n_i}(x_0), i = 1, \cdots, \ell\},
$$

where $\{n_1, \cdots, n_{\ell}\}$ is a finite subset of $\mathbb{Z}$.

A subset $P$ of $X$ is called a primitive set if it is of the form

$$
P = P^{(a)}(\beta) = \{x | \pi_{n+i}(x) = b_i, i = 1, \cdots, \ell\},
$$

where $\beta = (b_1, \cdots, b_{\ell}); b_i \in A_i, i = 1, \cdots, \ell; \text{ and } n \in \mathbb{Z}$. In this expression $\beta$ is called a block of length $\ell$. We do not assume that $b_1, \cdots, b_{\ell}$ are all different. As a special case, we also consider a block $\beta$ of length 0. In this case, we put $P^{(a)}(\beta) = X$ for any $n \in \mathbb{Z}$.

We observe that a primitive set is a special case of a neighborhood $W_{n_1, \cdots, n_{\ell}}(x_0)$
of the form (7) in which \( \{n_1, \ldots, n_\ell\} \) is a consecutive set of integers, namely \( n_i = n + i, i = 1, \ldots, \ell \).

A subset \( E \) of \( X \) is called an \textit{elementary} set if it is a union of a finite number of primitive sets. A neighborhood of the form (7) is clearly an elementary set. It is easy to see that a subset of \( X \) is open and closed at the same time if and only if it is an elementary set.

The family of all primitive subsets of \( X \) is denoted by \( \mathcal{P} \), and the family of all elementary subsets of \( X \) is denoted by \( \mathcal{E} \). Clearly, \( \mathcal{E} \) is a field of subsets of \( X \). The \( \sigma \)-field of subsets of \( X \) generated by \( \mathcal{E} \) is denoted by \( \mathcal{B} \). This \( \mathcal{B} \) is nothing but the \( \sigma \)-field of all Borel subsets of \( X \).

Let \( \varphi \) be a mapping of \( X \) onto itself defined by

\[
\pi_n(\varphi(x)) = \pi_{n+1}(x)
\]

for all \( n \in \mathbb{Z} \).

It is clear that \( \varphi \) is a homeomorphism of \( X \) onto itself. The map \( \varphi \) is called the \textit{shift transformation}, and the dynamical system \( (X, \varphi) \) is called the \textit{shift dynamical system} defined on the power space \( X = A^\mathbb{Z} \). It is clear that \( \varphi \) maps each of \( \mathcal{P}, \mathcal{E} \), and \( \mathcal{B} \) onto itself.

Let \( \mu \) be a normalized, \( \varphi \)-invariant, countably additive nonnegative measure defined on \( \mathcal{B} \). For any block \( \beta = (b_1, \ldots, b_\ell) \), \( \mu(P^{(n)}(\beta)) \) is independent of \( n \in \mathbb{Z} \), and hence we may denote it by \( D(\beta) \). It is then clear that the following conditions are satisfied:

\[
0 \leq D(\beta) \leq 1 \quad \text{for any block } \beta, \quad D(\beta) = 1 \quad \text{if } \beta \text{ is a block of length 0,}
\]

\[
D(\beta) = \sum_{b \in A} D((\beta, b)) = \sum_{b \in A} D((b, \beta)),
\]

where \( (\beta, b) = (b_1, \ldots, b_\ell, b) \) and \( (b, \beta) = (b, b_1, \ldots, b_\ell) \) if \( \beta = (b_1, \ldots, b_\ell) \).

Conversely, assume that \( D(\beta) \) is defined for any block \( \beta \) and that the conditions (10) and (11) are satisfied. Then it is easy to see that there exists a normalized, \( \varphi \)-invariant, countably additive, nonnegative measure \( \mu \) defined on \( \mathcal{B} \) such that \( \mu(P^{(n)}(\beta)) = D(\beta) \) for any block \( \beta \) and \( n \in \mathbb{Z} \).

Now let \( x_0 \) be a point of \( X \). From the result stated at the end of section 2 follows that \( \overline{\text{Orb} (x_0, \varphi) \text{ is strictly ergodic if and only if (i) the limit}} \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=m}^{m+n} \chi_P(\varphi^k(x_0)) = \mu(P)
\]

exists uniformly in \( m \in \mathbb{Z} \) for any primitive set \( P \), where \( \chi_P \) is the characteristic function of \( P \), and if (ii) \( \mu(P) > 0 \) for any primitive set \( P \) with \( x_0 \in P \).

For any block \( \beta = (b_1, \ldots, b_\ell) \), let us put

\[
N(\beta, x_0) = \{n \pi_{n+i}(x_0) = b_i, i = 1, \ldots, \ell\}.
\]

Then the result above can be restated as follows: \( \overline{\text{Orb} (x_0, \varphi) \text{ is strictly ergodic if and only if (i) the limit}} \)

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ \text{the number of integers } k \in N(\beta, x_0) \text{ such that } m + 1 \leq k \leq m + n \right\} = D(\beta)
\]
exists uniformly in \( m \in Z \) for any block \( \beta = (b_1, \cdots, b_t) \) and if (ii) \( D(\beta) > 0 \) for any block \( \beta \) for which \( N(\beta, x_0) \) is not empty.

It is easy to see that the limit \( D(\beta) \) satisfies the conditions (10) and (11), and hence there exists a normalized, \( \varphi \)-invariant, countably additive, nonnegative measure \( \mu \) defined on \( \mathcal{B} \) such that \( \mu(P^{(n)}(\beta)) = D(\beta) \) for any block \( \beta \) and for any \( n \in Z \). This is the same \( \mu \) which appears in the formula (12). It is easy to see that \( \overline{\text{Orb}}(x_0) \) is the carrier of this measure \( \mu \), and that \( \mu \) is nothing but the unique normalized, \( \varphi \)-invariant, countably additive, nonnegative measure for the strictly ergodic dynamical system \((\overline{\text{Orb}}(x_0), \varphi)\).

4. Example 1

We consider the shift dynamical system \((X, \varphi)\) defined on the power space \( X = A^Z \), where the base space \( A \) is a finite set consisting of two elements: \( A = \{-1, +1\} \).

We define a number-theoretic function \( \rho(n) \) by

\[
\rho(n) = (-1)^{n_1+n_2+\cdots+n_k}, \quad n = 0, 1, 2, \ldots
\]

where

\[
n = n_1 + n_2 \cdot 2 + \cdots + n_k \cdot 2^{k-1}
\]

is an expansion of a nonnegative integer \( n \) with base 2. This means that \( n_i = 0 \) or 1 for \( i = 1, \ldots, k \). It is easy to see that \( \{\rho(n)|n = 0, 1, 2, \ldots\} \) is completely determined by the relations

\[
\rho(0) = 1; \quad \rho(2^{n-1} + k) = -\rho(k), \quad k = 0, 1, \ldots, 2^n - 1; n = 1, 2, \ldots.
\]

This sequence \( \{\rho(n)|n = 0, 1, 2, \ldots\} \) has been discussed by many mathematicians [2], [3], [4], [5], [7], [8], [9], [11], [12] in connection with various problems in different parts of mathematics.

We now define a class of more general sequences as follows: let \( \alpha(0 < \alpha \leq 1) \) be a real number, and let

\[
\alpha = \frac{e_1(\alpha)}{2} + \frac{e_2(\alpha)}{2^2} + \cdots + \frac{e_n(\alpha)}{2^n} + \cdots
\]

be a dyadic expansion of \( \alpha \), where \( e_n(\alpha) = 0 \) or 1, \( n = 1, 2, \ldots \). This expansion is unique if we require that there are infinitely many \( n \) for which \( e_n(\alpha) = 1 \). Let us put

\[
\rho_\alpha(n) = (-1)^{n_1 \cdot e_1(\alpha)+n_2 \cdot e_2(\alpha)+\cdots+n_k \cdot e_k(\alpha)}, \quad n = 0, 1, 2, \ldots,
\]

where \((n_1, \ldots, n_k)\) is determined by (16) and \((e_1(\alpha), \ldots, e_k(\alpha))\) is determined by (18). It is easy to see that \( \{\rho_\alpha(n)|n = 0, 1, 2, \ldots\} \) is completely determined by the relations
By comparing (17) with (20), we see that \( \rho(n) \) defined by (15) corresponds to the case of \( \rho(n) \) defined by (19) when \( \epsilon_n(\alpha) = 1 \) for \( n = 1, 2, \ldots \), that is when \( \alpha = 1 \).

We now define \( \rho_a(n) \) for \( n = -1, -2, \ldots \) by

\[
\rho_a(n) = \rho(-n - 1), \quad n = -1, -2, \ldots.
\]

Thus the point \( x_a = \{\rho_a(n)|n \in \mathbb{Z}\} \in X \) is defined for each real number \( \alpha(0 < \alpha \leq 1) \). It is possible to show that, for any block \( \beta = (b_1, \ldots, b_\ell) \), the uniform density \( D_a(\beta) \) exists for the point \( x_a = \{\rho_a(n)|n \in \mathbb{Z}\} \) and that \( D_a(\beta) > 0 \) for any block for which \( N(\beta, x_a) \) is not empty. We see easily that \( D_a(\beta) = \frac{1}{\ell} \) if \( \beta \) is a block of length 1, but it is in general not true that \( D_a(\beta) = \frac{1}{\ell} \) if \( \beta \) is a block of length \( \ell \).

**Theorem 1.** For each real number \( \alpha(0 < \alpha \leq 1) \), \( (\text{Orb} (x_a), \varphi) \) is a strictly ergodic dynamical system.

Let \( \mathcal{B}_a \) be the \( \sigma \)-field of all Borel subsets of \( \text{Orb} (x_a) \) and let \( \mu_a \) be the unique, normalized, \( \varphi \)-invariant, countably additive, nonnegative measure defined on \( \mathcal{B}_a \).

It is clear that \( \varphi \) is an ergodic measure preserving transformation on the measure space \( (\text{Orb} (x_a), \mathcal{B}_a, \mu_a) \).

Let \( \tau \) be a mapping of \( X \) onto itself defined by

\[
\tau(x) = -\pi_n(x) \quad \text{for all} \quad n \in \mathbb{Z}.
\]

It is easy to see that \( \tau \) is a homeomorphism of \( X \) onto itself with period 2 (that is, \( \tau^n(x) = x \) for any \( x \in X \)), and that \( \tau \) commutes with \( \varphi \) (that is, \( \tau \varphi(x) = \varphi \tau(x) \) for any \( x \in X \)). It is also easy to show that \( \tau \) is a homeomorphism of \( \overline{\text{Orb}} (x_a) \) onto itself and that \( \tau \) is a measure preserving transformation on the measure space \( (\overline{\text{Orb}} (x_a), \mathcal{B}_a, \mu_a) \).

Let \( \mathcal{K}_a = L^2(\overline{\text{Orb}} (x_a), \mathcal{B}_a, \mu_a) \) be the complex \( L^2 \)-space over the measure space \( (\overline{\text{Orb}} (x_a), \mathcal{B}_a, \mu_a) \). Let \( V_a^x \), \( V_a^\varphi \) be the unitary operators defined on \( \mathcal{K}_a \) by \( V_a^x f(x) = f(\varphi(x)) \), \( V_a^{\varphi} f(x) = f(\tau(x)) \), respectively. Further, let \( \mathfrak{M}_a^1 \), \( \mathfrak{M}_a^{-1} \) be the closed linear subspaces of \( \mathcal{K}_a \) consisting of all \( f \in \mathcal{K}_a \) such that \( V_a^\varphi f = f, V_a^{\tau} f = -f \), respectively. It is easy to see that \( \mathfrak{M}_a^1 \) and \( \mathfrak{M}_a^{-1} \) are orthogonal to each other and together span the space \( \mathcal{K}_a \). It is also easy to see that both \( \mathfrak{M}_a^1 \) and \( \mathfrak{M}_a^{-1} \) are invariant under \( V_a^x \).

**Theorem 2.** For each real number \( \alpha(0 < \alpha \leq 1) \), \( V_a^x \) has a pure point spectrum on \( \mathfrak{M}_a^1 \), and a continuous singular spectrum on \( \mathfrak{M}_a^{-1} \). Further, for any two real numbers \( \alpha \) and \( \alpha' \) \( (0 < \alpha < \alpha' \leq 1) \), \( V_a^x \) on \( \mathfrak{M}_a^1 \) is spectrally isomorphic with \( V_{a'}^x \) on \( \mathfrak{M}_{a'}^1 \), while \( V_a^x \) on \( \mathfrak{M}_a^{-1} \) and \( V_{a'}^x \) on \( \mathfrak{M}_{a'}^{-1} \) are not spectrally isomorphic if \( \alpha' - \alpha \) is a dyadically irrational number.

Thus we obtained a concrete example of a continuum number of strictly ergodic dynamical systems, no two of which are spectrally isomorphic.
5. Example 2

We now consider the shift dynamical system \((X, \varphi)\) defined on the power space \(X = A^\mathbb{Z}\), where the base space \(A\) is a finite set consisting of four elements: \(A = \{2, 4, 6, 8\}\).

We define a number-theoretic function \(\lambda(n)\) by

\[
\lambda(n) = \text{the last nonzero digit in the decimal expansion of } n!,
\]

\(n = 2, 3, \ldots\)

For example, \(\lambda(2) = 2, \lambda(3) = 6, \lambda(4) = 4, \lambda(5) = 2, \lambda(6) = 2, \ldots\).

If we denote by \(\gamma(n)\) the number of consecutive zeros at the right end of the decimal expansion of \(n!\), then we may write

\[
\frac{n!}{10^{\gamma(n)}} \equiv \lambda(n) \pmod{10}.
\]

We observe that \(\lambda(n)\) is even, and hence \(\lambda(n) \in A\) for \(n = 2, 3, \ldots\). This follows from the fact that if \(n! = 2^a3^b5^c7^d \cdots p^e\) is a representation of \(n\) as the product of powers of a finite number of different prime numbers, then \(a \geq c\), and hence \(\gamma(n) = c\), and consequently, \(\gamma(n) \equiv 2^a3^b5^c7^d \cdots p^e \pmod{10}\).

We now want to find a general rule to compute the values of \(\lambda(n)\). For this purpose, we introduce a cyclic permutation \(T\) of the base space \(A = \{2, 4, 6, 8\}\) of order 4 defined by

\[
T = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 8 & 2 & 6 \end{pmatrix}.
\]

We also consider the set \(B = \{1, 2, 3, 4\}\).

First, let \(n \equiv b \pmod{5}, \ n \geq 3\). In this case, \(\lambda(n)\) is obtained from \(\lambda(n - 1)\) by the relation

\[
\lambda(n) = b \cdot \lambda(n - 1) \pmod{10},
\]

or equivalently, by

\[
\lambda(n) = T^{\eta(b)}\lambda(n - 1),
\]

where \(T\) is a permutation defined by (25), and \(\eta\) is a function defined on \(B\) by

\[
\eta(1) = 0, \ \ \eta(2) = 1, \ \ \eta(3) = 3, \ \ \eta(4) = 2.
\]

If we put \(\lambda(0) = \lambda(1) = 6\), then the relations (26) and (27) hold for \(n = 1\) and \(n = 2\). For this reason, we use these values of \(\lambda(0)\) and \(\lambda(1)\) even though they do not satisfy (23). Thus the relations (26) and (27) are valid for \(n = 1, 2, \ldots\) if \(n \not\equiv 0 \pmod{5}\). If \(n = 0 \pmod{5}\), then the situation is a little more complicated.

Next, let \(n \equiv 0 \pmod{5}\) and \(n/5 \equiv b \pmod{5}\). In this case, we have \(\lambda(n) = \lambda(n - 1) + 1\), and hence \(2\lambda(n) = b \cdot \lambda(n - 1) \pmod{10}\). From this follows that \(T \cdot \lambda(n) = T^{\eta(b)}\lambda(n - 1)\), or equivalently,

\[
\lambda(n) = T^{\eta(b)+3}\lambda(n - 1).
\]

For example, we can compute the value of \(\lambda(15)\) if we know that \(\lambda(14) = 2\).
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Since $15 \equiv 0 \pmod{5}$ and $15/5 \equiv 3 \pmod{5}$, we have $\lambda(15) = T^{s+3}\lambda(14) = T^{s+2} = T^2 = 8$.

Finally, we discuss the general case:

$$n \equiv 0 \pmod{5^k}, \quad n/5^k \equiv b \in B \pmod{5}; \quad k = 0, 1, 2, \ldots.$$  

In this case, we have $\gamma(n) = \gamma(n - 1) + k$, and hence $2^\delta \lambda(n) \equiv b \cdot \lambda(n - 1) \pmod{10}$. From this follows that $T^s \lambda(n) = T^{s(b)} \lambda(n - 1)$, or equivalently,

$$\lambda(n) = T^{t(n)} \lambda(n - 1),$$

where $\xi(n)$ takes one of the four values 0, 1, 2, 3, such that

$$\xi(n) = t(n) + 3k \pmod{4}$$

and $\eta(b)$ is defined by (28).

Equation (31) is a general formula by which we can compute the value of $\lambda(n)$ from that of $\lambda(n - 1)$ for $n = 1, 2, \ldots$. We now want a formula by which we can compute the value of $\lambda(n)$ from that of $\lambda(0)$.

From (31) follows that

$$\lambda(n) = T^{t(n)} \lambda(0),$$

where $\xi(n)$ takes one of the four values 0, 1, 2, 3, such that

$$\xi(n) = \xi(1) + \xi(2) + \cdots + \xi(n) \pmod{4}.$$  

Let now

$$n = c_1 + c_2 5 + \cdots + c_k 5^{k-1}$$

be the expansion of a nonnegative integer $n$ in base 5, where $c_i = 0, 1, 2, 3, 4$ for $i = 1, 2, \ldots, k$. Then

$$\xi(n) = \sum_{m=1}^{n} \xi(m) = \sum_{m=1}^{c_1 5^{k-1}} \xi(m) + \sum_{m=1}^{c_{k-1} 5^{k-1}} \xi(m + c_k 5^{k-1}) + \cdots + \sum_{m=1}^{c_1} \xi(m + c_2 5 + \cdots + c_k 5^{k-1}) \pmod{4}.$$  

If we observe that

$$\xi(m + c_i 5^{i-1} + c_{i+1} 5^i + \cdots + c_k 5^{k-1}) = \xi(m)$$

for $m = 1, 2, \ldots, 5^{i-1} - 1$, then (36) becomes

$$\xi(n) = \sum_{m=1}^{c_1 5^{k-1}} \xi(m) + \sum_{m=1}^{c_{k-1} 5^{k-1}} \xi(m) + \cdots + \sum_{m=1}^{c_1} \xi(m)$$

$$= \xi(c_k 5^{k-1}) + \xi(c_{k-1} 5^{k-2}) + \cdots + \xi(c_1)$$

$$= \xi_k(c_k) + \xi_k(c_{k-1}) + \cdots + \xi_1(c_1) \pmod{4}$$

where

$$\xi_k(c) = \xi(c 5^{k-1}), \quad c = 0, 1, 2, 3, 4; \quad k = 1, 2, \ldots.$$
We calculate the values of $\xi_k(c)$ for $c = 0, 1, 2, 3, 4$ and $k = 1, 2, 3, 4$, and obtain the following table:

<table>
<thead>
<tr>
<th>$c$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1(c)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\xi_2(c)$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\xi_3(c)$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\xi_4(c)$</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

We also observe (by computation) that

\begin{equation}
\xi(5^i) = \xi(5^i) = 0.
\end{equation}

On the other hand, from (32) it follows that

\begin{equation}
\xi(c 5^{i+k}) = \xi(c 5^k), \quad c = 1, 2, 3, 4; k = 0, 1, 2, \ldots.
\end{equation}

From (41) and (42) it is easy to show, by mathematical induction, that

\begin{equation}
\xi_{k+4}(c) = \xi_k(c), \quad c = 0, 1, 2, 3, 4; k = 0, 1, 2, \ldots.
\end{equation}

Thus, (40) and (43) together give all the values of $\xi_k(c)$ for $c = 0, 1, 2, 3, 4$ and $k = 0, 1, 2, \ldots$. Combined with (33) and (38), we have now a fairly simple method to calculate the values of $\lambda(n)$ for $n = 0, 1, 2, \ldots$.

We can restate the above result in the following form. If

\begin{equation}
n = c_1 625 + c_2 625^2 + \cdots + c_k 625^{k-1}
\end{equation}

is the expansion of a nonnegative integer $n$ with base 625 = 54, where $c_i = 0, 1, 2, \ldots, 624$ for $i = 1, 2, \ldots, k$, then there exists a function $\xi^*(c^*)$ defined for $c^* = 0, 1, 2, \ldots, 624$ and taking the values 0, 1, 2, 3, such that

\begin{equation}
\xi(n) = \xi^*(c_1) + \xi^*(c_2) + \cdots + \xi^*(c_k) \pmod{4}.
\end{equation}

In fact, if

\begin{equation}
c^* = c_1 + c_2 5 + c_3 5^2 + c_4 5^3
\end{equation}

is an expansion of $c^*$ with base 5, where $c_i = 0, 1, 2, 3, 4$ for $i = 1, 2, 3, 4$, then

\begin{equation}
\xi^*(c^*) = \xi_1(c_1) + \xi_2(c_2) + \xi_3(c_3) + \xi_4(c_4) \pmod{4}.
\end{equation}

We observe that there is some similarity in the properties of two sequences $\{\rho(n)\mid n = 0, 1, 2, \ldots\}$ and $\{\lambda(n)\mid n = 0, 1, 2, \ldots\}$. For example, formulas (15) and (45) are similar to each other. Instead of the expansion of $n$ with base 2 in (16), we use the expansion with base 625 in (44), and instead of the involution of the set $A = \{-1, +1\}$ which interchanges $-1$ and $+1$, we use the cyclic permutation $T$ of order 4 of our set $A = \{2, 4, 6, 8\}$.

We now define $\lambda(n)$ for $n = -1, -2, \ldots$ by

\begin{equation}
\lambda(n) = \lambda(-n - 1), \quad n = -1, -2, \ldots
\end{equation}

and consider the point $x_0$ in $X \times \mathbb{A}^2$ defined by $x_0 = \{\lambda(n)\mid n \in \mathbb{Z}\}$. It is possible to show that, for any block $\beta = (b_0, \ldots, b_d)$, the uniform density $D(\beta)$ exists for the point $x_0 = \{\lambda(n)\mid n \in \mathbb{Z}\}$ and that $D(\beta) > 0$ for any block $\beta$ for which
$N(\beta, x_0)$ is not empty. We see easily that $D(\beta) = \frac{1}{4}$ for any block of length 1, and that $D(\beta) = \frac{1}{8}$ for any block $\beta$ of length 2, but it is in general not true that $D(\beta) = \frac{1}{4^t}$ for any block of length $t$.

**Theorem 3.** $(\text{Orb} (x_0), \varphi)$ is a strictly ergodic dynamical system.

Let $\mathcal{B}_0$ be the $\sigma$-field of all Borel subsets of $\text{Orb} (x_0)$ and let $\mu_0$ be the unique, normalized, $\varphi$-invariant, countably additive, nonnegative measure defined on $\mathcal{B}_0$.

It is clear that $\varphi$ is an ergodic measure preserving transformation on the measure space $(\text{Orb} (x_0), \mathcal{B}_0, \mu_0)$.

Let $\tau$ be a mapping of $X$ onto itself defined by

$$\pi_n(\tau(x)) = T\pi_n(x) \quad \text{for all} \quad n \in \mathbb{Z}.$$  

It is easy to see that $\tau$ is a homeomorphism of $X$ onto itself with period 4 (that is, $\tau^4(x) = x$ for any $x \in X$), and that $\tau$ commutes with $\varphi$ (that is, $\tau \varphi(x) = \varphi \tau(x)$ for any $x \in X$). It is also easy to see that $\tau$ is a homeomorphism of $\text{Orb} (x_0)$ onto itself, and that $\tau$ is a measure-preserving transformation on the measure space $(\text{Orb} (x_0), \mathcal{B}_0, \mu_0)$.

Let $\mathcal{K}_0 = L^2(\text{Orb} (x_0), \mathcal{B}_0, \mu_0)$ be the complex $L^2$-space over the measure space $(\text{Orb} (x_0), \mathcal{B}_0, \mu_0)$. Let $V_\varphi, V_\tau$ be the unitary operators defined on $\mathcal{K}_0$ by $V_\varphi f(x) = f(\varphi(x))$, $V_\tau f(x) = f(\tau(x))$, respectively. Further, let $\mathfrak{M}_3$ be the closed linear subspace of $\mathcal{K}_0$ consisting of all $f \in \mathcal{K}_0$ such that $V_\varphi f = e^{2\pi \sqrt{k}/d} f, k = 0, 1, 2, 3$. It is easy to see that $\mathfrak{M}_3, k = 0, 1, 2, 3$, are mutually orthogonal and together span the space $\mathcal{K}_0$. It is also easy to see that each $\mathfrak{M}_3$ is invariant under $V_\varphi$.

**Theorem 4.** The operator $V_\varphi$ has a pure point spectrum on $\mathfrak{M}_3$, and continuous singular spectra on $\mathfrak{M}_3, k = 1, 2, 3$.

**References**