A NOTE ON MARKOV SEMIGROUPS WHICH ARE COMPACT FOR SOME BUT NOT ALL \( t > 0 \)

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1. Introduction

In this note \( \{P_t; t \geq 0\} \) will be a strongly continuous semigroup of transition operators on \( t \) and \( R_\lambda \) will be the corresponding resolvent operator for \( \lambda > 0 \).

The following three statements are correct.

(i) If, for some \( t > 0 \), \( P_t \) is quasi-compact, then it is quasi-compact for all \( t > 0 \).

(ii) \[ (iii) \] If, for some \( \lambda > 0 \), \( \lambda R_\lambda \) is compact [quasi-compact], then it is compact [quasi-compact] for all \( \lambda > 0 \).

Statements (i) and (ii) are very easy to prove, and (iii) is established in each of the accompanying papers by D. Williams and J. G. Basterfield. This note shows that the fourth similar assertion is false by giving an example of a Markov semigroup for which \( P_t \) is compact if \( t > 1 \), but not compact if \( 0 < t < 1 \).

2. The example

The states are labelled 0 and \((m, n)\) for \( n = 1, 2, \ldots \) and \( m = 1, 2, \ldots, n \). Here 0 is an absorbing state and the \((m, n)\)-th state feeds into the \((m - 1, n)\)-th state at rate \( n(m > 1) \), \((1, n)\) feeds into 0 also at rate \( n \).

\[ \text{FIGURE 1} \]

Thus \( p_{(m, n), 0}(t) \) is the distribution function of the sum of \( n \) independent negative exponential random variables, each with mean \( 1/n \); hence, if \( t < 1 \), then \( p_{(m, n), 0}(t) \to 0 \) as \( n \to \infty \) and if \( t > 1 \), then \( p_{(m, n), 0}(t) \to 1 \). It is also clear that \( p_{(m, n), 0}(t) \geq p_{(n, n), 0}(t) \) for \( 1 \leq m \leq n \) and for all \( t \), and that for \( j \geq N \),
\[
(1) \quad \sum_{n=N}^{\infty} \sum_{m=1}^{n} p_{(i,j),(m,n)}(t) = \sum_{m=1}^{i} p_{(i,j),(m,j)}(t) = 1 - p_{(i,i),0}(t).
\]

When \( t < 1 \), if \( 0 < \epsilon < 1 \), then for every given \( N \) we can choose \( j \geq N \) such that \( 1 - p_{(j,j),0}(t) > \epsilon \), that is

\[
(2) \quad \sum_{n=N}^{\infty} \sum_{m=1}^{n} p_{(j,j),(m,n)}(t) > \epsilon,
\]

and so \( P_t \) is not compact. On the other hand, if \( t > 1 \), then given \( \epsilon > 0 \) there exists \( N \) such that \( 1 - p_{(j,j),0}(t) < \epsilon \) for all \( j \geq N \). Then

\[
(3) \quad 0 \leq \sum_{n=N}^{\infty} \sum_{m=1}^{n} p_{(i,j),(m,n)}(t) = 1 - p_{(i,i),0}(t)
\]

\[
\leq 1 - p_{(j,j),0}(t) < \epsilon
\]

for all \( j \geq N \) and all \( i \leq j \), and so \( P_t \) is compact.

3. The spectrum of the semigroup

For such a semigroup, \( R_\lambda \) is not compact for any \( \lambda \), by Williams’ theorem, but the assertions (i), (ii), and (iii) of his theorem remain true.

For when \( P_t \) is compact for some \( t > 0 \), it must be compact for all \( s \geq t \) because \( P_s = P_t P_{s-t} \). Thus for any \( s > 0 \), there exists an integer \( n \) such that \( (P_s)^n = P_{ns} \) is compact. This implies that each nonzero point \( \lambda \) of the spectrum of \( P_s \) is a pole of order \( k \), that the null-space of \( (\lambda - P_s)^k \) is of nonzero finite dimension, and that the spectrum has no nonzero accumulation points ([2], VII.4.6). The proof of (i), (ii), and (iii) now follows as in section 4 of Williams’ paper. Again, each \( P_t \) and \( R_\lambda \) will be quasi-compact so that assertion 1 is approachable.

4. Compactness at the critical value

It is of interest to note that, in general, \( P_t \) may be compact or not compact at the critical value of \( t \). To construct an example of the first case we modify the semigroup of section 2 by taking the rate of departure from the \( (m, n) \)-th state to be \( \frac{n}{1 - a_n} \) instead of \( n \), where \( 0 < a_n < 1 \) and \( a_n \to 0 \) as \( n \to \infty \).

In this case \( p_{(n,n),0}(t) \) is the distribution function of a random variable with mean \( 1 - a_n \) and variance \( (1 - a_n)^2/n \). From the Chebychev inequality we obtain

\[
(4) \quad 0 < 1 - p_{(n,n),0}(1) \leq (1 - a_n)^2/n a_n^2
\]

so that if also \( na_n^2 \to \infty \) as \( n \to \infty \) (for example, if \( a_n = n^{-1/4} \)), then

\[
(5) \quad 1 - p_{(n,n),0}(1) \to 0
\]

and \( P_1 \) is compact. As before, \( P_t \) is not compact for \( 0 < t < 1 \).

For an example in which \( P_t \) is not compact, we can use a similar modification with \( n/(1 + a_n) \) for the rate of departure from \( (m, n) \). We impose the same conditions on \( a_n \).
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REFERENCES

