ON VALUES ASSOCIATED WITH A STOCHASTIC SEQUENCE

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1. Introduction

Let \{zn\} be a sequence of random variables with a known joint distribution. We are allowed to observe the \(z_n\) sequentially, stopping anywhere we please; the decision to stop with \(z_n\) must be a function of \(z_1, \ldots, z_n\) only (and not of \(z_{n+1}, \ldots\)). If we decide to stop with \(z_n\), we are to receive a reward \(x_n = f_n(z_1, \ldots, z_n)\) where \(f_n\) is a known function for each \(n\). Let \(t\) denote any rule which tells us when to stop and for which \(E(x_t)\) exists, and let \(v\) denote the supremum of \(E(x_t)\) over all such \(t\). How can we find the value of \(v\), and what stopping rule will achieve \(v\) or come close to it?

2. Definition of the \(\gamma_n\) sequence

We proceed to give a more precise definition of \(v\) and associated concepts. We assume given always

(a) a probability space \((\Omega, \mathcal{F}, P)\) with points \(\omega\);
(b) a nondecreasing sequence \(\{\mathcal{F}_n\}\) of sub-Borel fields of \(\mathcal{F} \);
(c) a sequence \(\{x_n\}\) of random variables \(x_n = x_n(\omega)\) such that for each \(n \geq 1\), \(x_n\) is measurable \((\mathcal{F}_n)\) and \(E(x_n) < \infty\).

(In terms of the intuitive background of the first paragraph, \(\mathcal{F}_n\) is the Borel field \(\mathcal{B}(z_1, \ldots, z_n)\) generated by \(z_1, \ldots, z_n\). Having served the purpose of defining the \(\mathcal{F}_n\) and \(x_n\), the \(z_n\) disappear in the general theory which follows.) Any random variable (r.v.) \(t\) with values 1, 2, \(\ldots\) (not including \(\infty\)) such that the event \([t = n]\) (that is, the set of all \(\omega\) such that \(t(\omega) = n\)) belongs to \(\mathcal{F}_n\) for each \(n \geq 1\), is called a stopping variable (s.v.); \(x_t = x_t(\omega)\) is then a r.v. Let \(C\) denote the class of all \(t\) for which \(E(x_t) < \infty\). We define the value of the stochastic sequence \(\{x_n, \mathcal{F}_n\}\) to be

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Similarly, for each \( n \geq 1 \) we denote by \( C_n \) the class of all \( t \) in \( C \) such that \( P[ t \geq n ] = 1 \), and set

\[
(2) \quad v_n = \sup_{t \in C_n} E(x_t).
\]

Then

\[
(3) \quad C = C_1 \supset C_2 \supset \cdots \quad \text{and} \quad v = v_1 \geq v_2 \geq \cdots ;
\]

since \( t = n \in C_n \), we have \( v_n \geq E(x_n) > -\infty \).

For any family \( (y_t, t \in T) \) of r.v.'s we define \( y = \text{ess sup}_{t \in T} y_t \) if (a) \( y \) is a r.v. such that \( P[y \geq y_t] = 1 \) for each \( t \) in \( T \), and (b) if \( z \) is any r.v. such that \( P[z \geq y_t] = 1 \) for each \( t \) in \( T \), then \( P[z \geq y] = 1 \). It is known that there always exists a sequence \( \{t_k\} \) in \( T \) such that

\[
(4) \quad \sup_{k} y_{t_k} = \text{ess sup}_{t \in T} y_t.
\]

We may therefore define for each \( n \geq 1 \) a r.v. \( \gamma_n \) measurable \((\mathbb{F}_n)\) by

\[
(5) \quad \gamma_n = \text{ess sup}_{t \in C_n} E(x_t|\mathbb{F}_n);
\]

then \( \gamma_n \geq x_n \) (equalities and inequalities are understood to hold up to sets of P-measure 0) and \( E(\gamma_\infty^-) \leq E(x_\infty^-) < \infty \).

It might seem more natural to consider, instead of \( C_n \), the larger class \( \check{C}_n \) of all s.v.'s \( t \) such that \( P[t \geq n] = 1 \) and \( E(x_t) \) exists, that is \( E(x^-) \) and \( E(x^+) \) not both infinite. However, this would yield the same \( v_n \) and \( \gamma_n \). For if \( t \in \check{C}_n \), define

\[
(6) \quad t' = \begin{cases} t & \text{if } E(x_t|\mathbb{F}_n) \geq x_n, \\ n & \text{otherwise}. \end{cases}
\]

Then setting \( A = [E(x_t|\mathbb{F}_n) \geq x_n] \), we have

\[
(7) \quad E(x_{t'}) \leq E(x_{t}^-) + \int_A x_t^-.
\]

But \(-\infty < \int_A x_t \leq \int_A x_t, \text{so } \int_A x_t^- < \infty \). Hence, \( E(x_{t'}) < \infty \) and \( t' \in C_n \). Now \( E(x_t|\mathbb{F}_n) = \max (x_n, E(x_t|\mathbb{F}_n)) \geq E(x_t|\mathbb{F}_n) \), and hence \( E(x_{t'}) \geq E(x_t) \). It follows that \( v_n \) and \( \gamma_n \) are unchanged if we replace \( C_n \) by \( \check{C}_n \) in their definitions.

3. Some lemmas

**Lemma 1.** For each \( n \geq 1 \) there exists a sequence \( \{t_k\} \) in \( C_n \) such that

\[
(8) \quad x_n \leq E(x_{t_k}|\mathbb{F}_n) \uparrow \gamma_n \quad \text{as } \quad k \to \infty.
\]

**Proof.** Choose \( \{t_k\} \) in \( C_n \) with \( t_1 = n \) such that \( \gamma_n = \sup_k E(x_{t_k}|\mathbb{F}_n) \). By lemmas 2 and 3 below, we can assume that (8) holds.

**Lemma 2.** For any \( t \in C_n \), define \( t' = \text{first } k \geq n \) such that \( E(x_t|\mathbb{F}_k) \leq x_k \). Then
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(a) $t' \leq t$, $t' \in C_n$,
(b) $E(x_t|\mathcal{F}_n) \geq E(x_t|\mathcal{F}_n)$,
(c) $t' > j \geq n \Rightarrow E(x_t|\mathcal{F}_j) > x_j$.

PROOF. If $t = j \geq n$, then $E(x_t|\mathcal{F}_j) = x_j$, so $t' \leq j$; hence, $t' \leq t$. Now

$$E(x_{t'}) = \sum_{k=n}^{\infty} \int_{[t'=k]} x_k \leq \sum_{k=n}^{\infty} \int_{[t'=k]} E^-(x_t|\mathcal{F}_k) \leq \sum_{k=n}^{\infty} \int_{[t'=k]} E(x_t|\mathcal{F}_k) = E(x_{t'}) < \infty,$$

so that $t' \in C_n$. Hence (a) holds. For any $A \in \mathcal{F}_j$ with $j \geq n$,

$$\int_{A[t' \geq j]} x_{t'} = \sum_{k=j}^{\infty} \int_{A[t'=k]} x_k \geq \sum_{k=j}^{\infty} \int_{A[t'=k]} E(x_t|\mathcal{F}_k) = \int_{A[t' \geq j]} x_t.$$

Putting $j = n$ gives (b). For $t' > j$ we obtain $E(x_{t'}|\mathcal{F}_j) \geq E(x_t|\mathcal{F}_j) > x_j$, which gives (c).

Any $t' \in C_n$ satisfying (c) of lemma 2 will be called $n$-regular.

LEMMA 3. Let $\{t_i\}^n_{i=1} \subset C_n$ be $n$-regular for some fixed $n \geq 1$, and define $\tau_i = \max (t_1, \cdots, t_i)$. Then $\tau_i \in C_n$ is $n$-regular and

$$\max_{1 \leq k \leq i} E(x_t|\mathcal{F}_n) \leq E(x_{\tau_i}|\mathcal{F}_n) \leq E(x_{\tau_{n+i}}|\mathcal{F}_n).$$

PROOF. That $\tau_i \in C_n$ is clear. For $j \geq n$ and $A \in \mathcal{F}_j$,

$$\int_{A[\tau_i \geq j]} x_{\tau_i} = \sum_{k=j}^{\infty} \left( \int_{A[\tau_i = k \geq \tau_{i+1}]} x_{\tau_{i+1}} + \int_{A[\tau_i = k < \tau_{i+1}]} x_k \right) \leq \sum_{k=j}^{\infty} \left( \int_{A[\tau_i = k \geq \tau_{i+1}]} x_{\tau_{i+1}} + \int_{A[\tau_i = k < \tau_{i+1}]} x_{\tau_{i+1}} \right) = \int_{A[\tau_i \geq j]} x_{\tau_{i+1}}.$$

For $j = n$, this gives

$$E(x_{\tau_{i+1}}|\mathcal{F}_n) \geq E(x_{\tau_i}|\mathcal{F}_n) \geq \cdots \geq E(x_{\tau_1}|\mathcal{F}_n) = E(x_t|\mathcal{F}_n),$$

and hence, by symmetry,

$$E(x_{\tau_i}|\mathcal{F}_n) \geq \max_{1 \leq k \leq i} E(x_t|\mathcal{F}_n).$$

To prove that $\tau_i$ is $n$-regular, we observe by the above that

$$\tau_i \geq j \Rightarrow E(x_{\tau_i}|\mathcal{F}_j) \leq E(x_{\tau_{i+1}}|\mathcal{F}_j).$$

Since $t_i$ is $n$-regular,

$$t_i < j \Rightarrow x_j < E(x_{\tau_i}|\mathcal{F}_j) = E(x_{\tau_{i+1}}|\mathcal{F}_j) \leq \cdots \leq E(x_{\tau_i}|\mathcal{F}_j),$$

and by symmetry,

$$\tau_i > j \Rightarrow x_j < E(x_{\tau_i}|\mathcal{F}_j).$$
4. The fundamental theorem

**Theorem 1.** The following relations hold:

(a) \( \gamma_n = \max (x_n, E(\gamma_{n+1}|\mathcal{F}_n)) \), \( n \geq 1 \)

(b) \( E(\gamma_n) = v_n \).

**Proof.** (a). Given any \( t \in C_n \), let \( t' = \max (t, n+1) \in C_{n+1} \) and set \( A = [t = n] \), and \( I_A \) is indicator function of \( A \). Then

\[
E(x_t|\mathcal{F}_n) = I_A \cdot x_n + I_{\bar{A}} \cdot E(x_{t'}|\mathcal{F}_n) \\
= I_A \cdot x_n + I_{\bar{A}} \cdot E(E(x_{t'}|\mathcal{F}_{n+1})|\mathcal{F}_n) \\
\leq I_A \cdot x_n + I_{\bar{A}} \cdot E(\gamma_{n+1}|\mathcal{F}_n) \leq \max (x_n, E(\gamma_{n+1}|\mathcal{F}_n)).
\]

To prove the reverse inequality, choose, by lemma 1, \( \{t_k\}_1^\infty \in C_{n+1} \) such that

\[
x_{n+1} \leq E(x_{t_k}|\mathcal{F}_{n+1}) \uparrow \gamma_{n+1} \quad \text{as} \quad k \to \infty;
\]

then by the monotone convergence theorem for conditional expectations,

\[
E(\gamma_{n+1}|\mathcal{F}_n) = E(\lim_{k \to \infty} E(x_{t_k}|\mathcal{F}_{n+1})|\mathcal{F}_n) = \lim_{k \to \infty} E(x_{t_k}|\mathcal{F}_n) \leq \gamma_n.
\]

And since \( t = n \) is in \( C_n \), \( x_n = E(x_n|\mathcal{F}_n) \leq \gamma_n \). This completes the proof of (a).

(b). Since for each \( t \) in \( C_n \), \( E(x_t|\mathcal{F}_n) \leq \gamma_n \), \( E(x_t) \leq E(\gamma_n) \), so \( v_n \leq E(\gamma_n) \).

Now choose \( \{t_k\}_1^\infty \) in \( C_n \), according to lemma 1; then

\[
E(\gamma_n) = \lim_{k \to \infty} E(x_{t_k}) \leq v_n.
\]

**Lemma 4.** If \( t \in C \), then

\[
t \geq n \Rightarrow E(x_t|\mathcal{F}_n) \leq \gamma_n \quad \text{and} \quad E(x_t^-|\mathcal{F}_n) \geq \gamma_t^-.
\]

**Proof.** Set \( t' = \max (t, n) \in C_n \). By definition of \( \gamma_n \),

\[
t \geq n \Rightarrow E(x_t|\mathcal{F}_n) = E(x_{t'}|\mathcal{F}_n) \leq \gamma_n,
\]

and hence

\[
t \geq n \Rightarrow E(x_t^-|\mathcal{F}_n) \geq E^-(x_t|\mathcal{F}_n) \geq \gamma_t^-.
\]

5. The r.v. \( \sigma \)

We define the r.v.

\[
\sigma = \text{first } n \geq 1 \text{ such that } x_n = \gamma_n \quad (= \infty \text{ if no such } n \text{ exists}).
\]

In general, \( P[\sigma < \infty] < 1 \), so that \( \sigma \) is not always a s.v.

**Lemma 5.** If \( t \in C \), then \( t' = \min (t, \sigma) \in C \) and \( E(x_{t'}) \geq E(x_t) \).  

**Proof.** From lemma 4 we have

\[
E(x_t^-) = \int_{[t'=t]} x_t^- + \sum_{n=1}^\infty \int_{[t>n=\sigma]} x_t^- \geq \int_{[t'=t]} x_t^- + \sum_{n=1}^\infty \int_{[t>n=\sigma]} \gamma_t^- \\
= \int_{[t'=t]} x_t^- + \sum_{n=1}^\infty \int_{[t>n=\sigma]} x_t^- = E(x_{t'}),
\]

so that $t' \in C$. The same argument without the $-$ and with reversed inequality proves the inequality $E(x_t) \leq E(x_{t'})$.

A s.v. $t \in C$ is optimal if $v = E(x_t)$. A s.v. $t$ in $C$ is regular if it is 1-regular; that is, if for each $n \geq 1$, $t > n \Rightarrow E(x_t[1:n]) > x_n$.

**THEOREM 2.** (a) If $\sigma \in C$ and is regular, then it is optimal. (b) If $v < \infty$ and an optimal s.v. exists, then $\sigma \in C$ and is optimal and regular; moreover, $\sigma$ is the minimal optimal s.v. and

$$\sigma \geq n \Rightarrow E(x_t[1:n]) = E(\gamma_t[1:n]) = \gamma_n \quad (n \geq 1).$$

**Proof.** (a) If $\sigma \in C$ and is regular, then $\sigma > n \Rightarrow E(x_\sigma[1:n]) > x_n$ for each $n \geq 1$. And for any $t \in C$, $\sigma = n$, $t \geq n \Rightarrow E(x_t[1:n]) \leq \gamma_n = x_n$ by lemma 4. Hence by lemma 1 of [1], $\sigma$ is optimal.

(b) Since $v < \infty$, $v_n = E(\gamma_n) < \infty$ for each $n \geq 1$. Let $s$ in $C$ be any optimal s.v., set $A = [s = n < \sigma]$, and suppose $P(A) > 0$. Then

$$\int_A \gamma_n > \int_A x_n + \epsilon \quad \text{for some } \epsilon > 0.$$ 

Choose $\{t_k\}^n$ in $C_n$ by lemma 1; then $\int_A x_{t_k} \uparrow \int_A \gamma_n$, so that we can find $k$ so large that $\int_A x_{t_k} > \int_A \gamma_n - \epsilon$. Set

$$s' = \begin{cases} s & \text{off } A, \\ t_k & \text{on } A; \end{cases}$$

then it is easy to see that $s'$ is a s.v. in $C$. But

$$E(x_{s'}) = \int_{u=A} x_s + \int_A x_{t_k} > \int_{u=A} x_s + \int_A x_n = E(x_s),$$

a contradiction. Hence $P(A) = 0$, and thus $P[\sigma \leq s] = 1$, so $\sigma$ is a s.v. By lemma 5, $\sigma = \min (s, \sigma)$ is in $C$ and $\sigma$ is optimal and minimal.

For any $n \geq 1$, let $A = [E(x_\sigma[1:n]) < \gamma_n, \sigma > n] \in F_n$. If $P(A) > 0$, then $\int_A \gamma_n > \int_A x_s$, since $E(\gamma_n) \leq E(\gamma_1) = v < \infty$. By lemma 1, there exists $t$ in $C_n$ such that $\int_A x_t > \int_A x_s$. Define

$$\tau = \begin{cases} t & \text{on } A, \\ \sigma & \text{off } A; \end{cases}$$

then it is easy to see that $\tau$ is a s.v. in $C$ and $E(x_\tau) > E(x_s) = v$, a contradiction. Hence $P(A) = 0$, and by lemma 4,

$$\sigma > n \Rightarrow E(\gamma_\sigma[1:n]) = E(x_\sigma[1:n]) = \gamma_n > x_n,$$

so $\sigma$ is regular and the last part of (b) holds.

6. Bounded stopping variables

The r.v.’s $\gamma_n$ and the constants $v_n$ are in general impossible to compute directly. To this end we define for any $N \geq 1$ and $1 \leq n \leq N$ the expressions

$$C^N_n = \text{all } t \in C_n \text{ such that } P[t \leq N] = 1; v^N_n = \sup_{t \in C^N_n} E(x_t);$$

$$\gamma^N_n = \text{ess sup}_{t \in C^N_n} E(x_t[1:n]).$$
(35) \(-\infty < E(x_n) = v_n \leq v_n^{n+1} \leq \cdots \leq v_n\) and \(x_n = \gamma_n^N \leq \gamma_n^{n+1} \leq \cdots \leq \gamma_n\), so that we can define

(36) \(v_n' = \lim_{N\to\infty} v_n^N, \quad \gamma_n' = \lim_{N\to\infty} \gamma_n^N\),

and we have

(37) \(-\infty < E(x_n) \leq v_n' \leq v_n, \quad x_n \leq \gamma_n' \leq \gamma_n\).

By the argument of theorem 1 applied to the finite sequence \(\{x_n\}_1^N\), we have

(38) \(\gamma_n^N = x_N, \quad \gamma_n' = \max(x_n, E(\gamma_n^N|\mathcal{F}_n)), \quad (n = 1, \ldots, N - 1),\)

and \(E(\gamma_n^N) = v_n^N\), so that \(\gamma_n^N\) and \(v_n^N\) are computable by recursion. By the monotone convergence theorem for expectations and conditional expectations, \(E(\gamma_n') = v_n'\) and

(39) \(\gamma_n' = \max(x_n, E(\gamma_n'+1|\mathcal{F}_n)), \quad (n \geq 1).\)

Hence \(\{\gamma_n'\}_1^\infty\) satisfies the same recursion relation as does \(\{\gamma_n\}_1^\infty\). (In [2], \(\gamma_n^N = \beta_n^N, \gamma_n' = \beta_n).\)

**Theorem 3.** If the condition \(A^-: E(\sup_n x_n^-) < \infty\) holds, then

(40) \(\gamma_n' = \gamma_n \quad \text{and} \quad v_n' = v_n, \quad (n \geq 1).\)

**Proof.** For any \(t \in C_n\) and \(A \in \mathcal{F}_n\),

(41) \(\int_{A[t \leq N]} x_t \leq \int_A x_{\min(t, N)} + \int_{A[t > N]} x_N.\)

Since \(E(x_{\min(t, N)}|\mathcal{F}_n) \leq \gamma_n^N \leq \gamma_n'\),

(42) \(\int_{A[t \leq N]} x_t \leq \int_A \gamma_n' + \int_{A[t > N]} (\sup_m x_m^-).\)

Letting \(N \to \infty\),

(43) \(\int_A x_t \leq \int_A \gamma_n', \quad E(x_t|\mathcal{F}_n) \leq \gamma_n', \quad \gamma_n \leq \gamma_n'\),

so \(\gamma_n = \gamma_n'\) and \(v_n = v_n'\).

**Corollary.** If \(A^-\) holds and \(\{x_n\}_1^\infty\) is Markovian, and \(\mathcal{F}_n = \mathcal{B}(x_1, \ldots, x_n)\), then \(\gamma_n = E(\gamma_n|x_n).\)

**Proof.** The Markovian property of \(\{x_n\}_1^\infty\) implies (by downward induction on \(n\)) \(\gamma_n^N = E(\gamma_n^N|x_n)\) which entails \(\gamma_n' = E(\gamma_n'|x_n) \), and then \(\gamma_n = E(\gamma_n|x_n).\) (The assumption \(A^-\) will be dropped in the corollary to theorem 9.)

7. Supermartingales

A sequence \(\{y_n\}_1^\infty\) of r.v.'s is a **supermartingale** (or lower semimartingale) if for each \(n \geq 1, y_n\) is measurable \((\mathcal{F}_n), E(y_n)\) exists, \(-\infty \leq E(y_n) \leq \infty,\) and \(E(y_{n+1}|\mathcal{F}_n) \leq y_n.\) We shall denote by \(D\) the class of all supermartingales \(\{y_n\}_1^\infty\) such that \(y_n \geq x_n\) for each \(n \geq 1.\) The sequences \(\{\gamma_n\}_1^\infty\) and \(\{\gamma_n'\}_1^\infty\) are in \(D.\)
THEOREM 4. The sequence \( \{y_i'\} \) is the minimal element of \( D \).

PROOF. For any \( \{y_n\} \) in \( D \),
\[
y_n \geq x_n = \gamma_n^n,
\]
(44) 
\[
y_{n-1} \geq E(y_n|\mathcal{F}_{n-1}) = E(\gamma^n_{n-1}|\mathcal{F}_{n-1}),
\]
so that
\[
y_{n-1} \geq \max (x_{n-1}, E(\gamma^n_{n-1}|\mathcal{F}_{n-1})) = \gamma^n_{n-1}, \ldots, y_i \geq \gamma^n_i, \ldots
\]
(45) 
\[
y_i \geq \lim_{n \to \infty} \gamma^n_i = \gamma'_i, \quad (i \geq 1).
\]

We shall define various types of "regularity" for elements of \( D \), according to the class of s.v.'s \( t \) for which \( E(y_t) \) is assumed to exist and the relation
\[
t \geq n \Rightarrow E(y_t|\mathcal{F}_n) \leq y_n, \quad (n \geq 1)
\]
(46) to hold. An element \( \{y_n\} \) of \( D \) is said to be
(a) regular if for every s.v. \( t \), \( E(y_t) \) exists and (46) holds;
(b) semiregular if for every s.v. \( t \) such that \( E(y_t) \) exists, (46) holds;
(c) \( C \)-regular if for every s.v. \( t \in C \) (for which \( E(y_t) \) necessarily exists),
(46) holds.

Clearly, for elements of \( D \), regular \( \Rightarrow \) semiregular \( \Rightarrow \) \( C \)-regular.

We shall use the notation \( A^+ : E(\sup_n x^n_t) < \infty, A^* : E(x_t) \) exists for every s.v. \( t \). Clearly, \( A^+ \Rightarrow A^* \Leftarrow A^-.

LEMMA 6. If \( A^* \) holds, then for any \( \epsilon > 0 \) and \( n \geq 1 \), there exists \( s \in C_n \) such
\[
E(x_s|\mathcal{F}_n) > \gamma_n - \epsilon \quad \text{on} \quad [\gamma_n < \infty].
\]
(47)

PROOF. Choose \( \{t_n\} \) in \( C_n \) by lemma 1. On \( [\gamma_n < \infty] \) define \( \alpha = \text{first } k \geq 1 \)
such that \( E(x_{\alpha}|\mathcal{F}_n) > \gamma_n - \epsilon \), and set
\[
s = \begin{cases} \{t_n \text{ on } [\gamma_n < \infty] \\ \{n \text{ elsewhere.} \end{cases}
\]
(48) 

Then \( E(x_s) \) exists, and on \( [\gamma_n < \infty], E(x_s|\mathcal{F}_n) > \gamma_n - \epsilon \). Hence,
\[
E(x_s) \geq \int_{[\gamma_n < \infty]} (\gamma_n - \epsilon) + \int_{[\gamma_n = \infty]} x_s > -\infty,
\]
(49) 
so that \( s \in C_n \).

LEMMA 7. (a) Condition \( A^- \) implies \( E(\gamma_t^-) = E((\gamma_t')^-) < \infty \) for every s.v. \( t \), and
(b) condition \( A^+ \) implies \( E((\gamma_t')^+) \leq E(\gamma_t^+) < \infty \) for every s.v. \( t \).

PROOF. (a) Since by theorem 3 \( x_s \leq \gamma_t' = \gamma_n, \gamma_t^- = (\gamma_t')^- \leq \sup x_n^- \).

(b) Since
\[
\gamma_t^+ = \text{ess sup}_{t \in C_n} E^+(x_t|\mathcal{F}_n) \leq E(\sup_j x^+_j|\mathcal{F}_n),
\]
(50) 
then
\[
E((\gamma_t')^+) \leq E(\gamma_t^+) = \sum_{n=1}^\infty \int_{[t=n]} \gamma_t^+ \leq \sum_{n=1}^\infty \int_{[t=n]} E(\sup_j x^+_j|\mathcal{F}_n) = E(\sup_j x^+_j).
\]
(51)
Theorem 5. (a) If \( \{y_n\}_{n=1}^\infty \in D \) and is \( C \)-regular, then \( y_n \geq \gamma_n \) for each \( n \geq 1 \); (b) \( A^* \Rightarrow \{\gamma_n\}_{n=1}^\infty \) is semiregular; (c) \( A^- \) or \( A^+ \Rightarrow \{\gamma_n\}_{n=1}^\infty \) is regular; (d) \( \{\gamma_n\}_{n=1}^\infty \) is \( C \)-regular.

Proof. (a) If \( \{y_n\}_{n=1}^\infty \in D \) and is \( C \)-regular, then

\[
\gamma_n = \text{ess sup}_{t \in C_n} E(x_t | \mathcal{F}_n) \leq \text{ess sup}_{t \in C_n} E(y_t | \mathcal{F}_n) \leq y_n.
\]

(b) Let \( \tau \) be any s.v. such that \( P[\tau \geq n] = 1 \) and \( E(\gamma_{\tau}) \) exists. For arbitrary \( \epsilon > 0, k \geq n \), and \( m \geq 1 \), setting \( A_m = [\gamma_n < m] \), we have

\[
m \geq \int_{A_m} \gamma_n \geq \int_{A_k} \gamma_{n+1} \geq \cdots \geq \int_{A_k} \gamma_k \geq \cdots,
\]

so that \( \gamma_k < \infty \) on \( A_m \). Hence, \( \gamma_k < \infty \) on \( A = [\gamma_n < \infty] \). By lemma 6, we can choose \( t_k \in C_k \) such that

\[
E(x_{t_k} | \mathcal{F}_k) > \gamma_k - \epsilon
\]
on \( A \).

Define

\[
t = \begin{cases} 
t_k & \text{on } A[\tau = k], \\
\tau & \text{off } A.
\end{cases}
\]

Then \( E(x_t) \) exists, and on \( A \),

\[
E(x_t | \mathcal{F}_n) = E \left( \sum_{k=n}^\infty I_{[t=k]} E(x_{t_k} | \mathcal{F}_k) | \mathcal{F}_n \right) \geq E \left( \sum_{k=n}^\infty I_{[t=k]} (\gamma_k - \epsilon) | \mathcal{F}_n \right)
\]

\[
= E(\gamma_{\tau} | \mathcal{F}_n) - \epsilon;
\]

and therefore on \( A \), by the remark preceding lemma 1,

\[
\gamma_n = \text{ess sup}_{t \in C_n} E(x_t | \mathcal{F}_n) \geq E(\gamma_{\tau} | \mathcal{F}_n) - \epsilon
\]

(recall that \( \mathcal{C}_n \) = all s.v.'s \( t \geq n \) such that \( E(x_t) \) exists). Hence,

\[
\gamma_n \geq E(\gamma_{\tau} | \mathcal{F}_n)
\]
on \( \Omega \).

Now let \( t \) be any s.v. such that \( E(\gamma_{\tau}) \) exists. Set \( \tau = \max(t, n) \). Then if \( E(\gamma_{t}^+) = \infty \), \( E(\gamma_{\tau}^-) < \infty \), and hence

\[
E(\gamma_{\tau}^-) \leq \int_{[t > n]} \gamma_{\tau}^- + \int_{[t \leq n]} \gamma_{\tau}^- < \infty,
\]

while if \( E(\gamma_{t}^+) < \infty \), then

\[
E(\gamma_{t}^+) \leq \int_{[t > n]} \gamma_{t}^+ + \int_{[t \leq n]} \gamma_{t}^+ < \infty,
\]

since

\[
\int_{[t \leq n]} \gamma_t = \sum_{k=1}^n \int_{[t = k]} \gamma_t \leq \sum_{k=1}^n \int_{[t = k]} \gamma_n = \int_{[t \leq n]} \gamma_n.
\]

Hence \( E(\gamma_{\tau}) \) exists. By the previous result, \( \gamma_n \geq E(\gamma_{\tau} | \mathcal{F}_n) \), and hence,

\[
t \geq n \Rightarrow \gamma_n \geq E(\gamma_{\tau} | \mathcal{F}_n) = E(\gamma_{\tau} | \mathcal{F}_n).
\]

(c) This statement follows from (b) and lemma 7.
(d) For $0 \leq b < \infty$, let $x_n(b) = \min(x_n, b)$, and let $\gamma_n^b (\leq \gamma_n)$ denote $\gamma_n$ for the sequence $\{x_n(b)\}$. As $b \to \infty$, $-x_n^{-} \leq \gamma_n^b \uparrow \gamma_n$, say, where $\gamma_n \leq \gamma_n$, and for any $t$ in $C_n$, $x_t(b) \geq -x_t^{-}$, so that $E(x_t(b)|\mathcal{F}_n) \uparrow E(x_t|\mathcal{F}_n)$. Since $\gamma_n \geq \gamma_n^b \geq E(x_t(b)|\mathcal{F}_n)$, $\gamma_n \geq E(x_t|\mathcal{F}_n)$, and hence $\gamma_n \geq \gamma_n$, $\gamma_n = \gamma_n$. Now if $t \in C$, then by (e), $t \geq n \Rightarrow E(\gamma_t^b|\mathcal{F}_n) \leq \gamma_n^b \leq \gamma_n$. As $b \to \infty$, since $\gamma_t^b \geq -x_t^{-}$ and $E(x_t^{-}) < \infty$, $t \geq n \Rightarrow E(\gamma_t^b|\mathcal{F}_n) \leq \gamma_n$, so $\{\gamma_n^b\}$ is $C$-regular.

**COROLLARY 1.** (a) The sequence $\{\gamma_n\}^\infty_1$ is the minimal $C$-regular element of $D$.

(b) Condition $A^*$ implies that $\{\gamma_n\}^\infty_1$ is the minimal semiregular element of $D$.

(c) Either $A^-$ or $A^+$ implies that $\{\gamma_n\}^\infty_1$ is the minimal regular element of $D$.

We remark that under $A^-$, $E(\sup_n \gamma_n) < \infty$. Hence, by a well-known theorem, $\{\gamma_n\}^\infty_1$ is regular, and similarly for $\{\gamma_n\}^\infty_1$. By theorems 4 and 5(a), $\{\gamma_n\}^\infty_1 = \{\gamma_n\}^\infty_1$, which gives an alternative proof of theorem 3.

**COROLLARY 2.** If $\gamma_n^b = \text{ess sup}_B E(\min (x_t, b)|\mathcal{F}_n)$, then

$$\gamma_n = \lim_{b \to \infty} \gamma_n^b.$$  

(63) $$(n \geq 1).$$

8. Almost optimal stopping variables

**LEMMA 8.** If $v < \infty$, then for any $\epsilon > 0$, $P[x_n \geq \gamma_n - \epsilon, \text{i.o.}] = 1$.

**Proof.** Since $\infty > v = E(\gamma_1) \geq E(\gamma_2) \geq \cdots$, we have $P[\gamma_n < \infty] = 1$ for each $n \geq 1$. Choose any $\epsilon > 0$ and $r > 0$, and define for $n \geq 1$,

$$B_n = \left[ E(x_t|\mathcal{F}_n) > \gamma_n - \frac{\epsilon}{r} \right],$$

where $\{t_n\}$ is chosen by lemma 1 for each $n \geq 1$ so that $t_n \in C_n$ and $P(B_n) > 1 - 1/r$ (convergence a.e. $\Rightarrow$ convergence in probability). Define

$$B = [x_n < \gamma_n - \epsilon \text{ for all } n \geq m]$$

where $m$ is any fixed positive integer. Then

$$x_n \leq \gamma_n - \epsilon I_B$$

for $n \geq m$,

so on $B_n$ for any $n \geq m$,

$$\gamma_n - \frac{\epsilon}{r} < E(x_t|\mathcal{F}_n) \leq E(\gamma_t|\mathcal{F}_n) - \epsilon P(B|\mathcal{F}_n)$$

$$\leq \gamma_n - \epsilon P(B|\mathcal{F}_n)$$

by theorem 5(d).

Hence on $B_n$, $P(B|\mathcal{F}_n) \leq 1/r$, and therefore $P(BB_n) \leq 1/r$. It follows that $P(B) \leq P(BB_n) + P(1 - B_n) \leq (1/r) + (1/r) = (2/r)$. Since $r$ can be arbitrarily large, $P(B) = 0$, and therefore,

$$P[x_n \geq \gamma_n - \epsilon \text{ for some } n \geq m] = 1$$

and

$$P[x_n \geq \gamma_n - \epsilon, \text{i.o.}] = \lim_{m \to \infty} 1 = 1.$$

**THEOREM 6.** For any $\epsilon > 0$, define

$$s = \text{first } n \geq 1 \text{ such that } x_n \geq \gamma_n - \epsilon (s = \infty \text{ if no such } n \text{ exists}).$$

(70)
Assume the following: (a) $P[s < \infty] = 1$,  
(b) $E(x_n)$ exists,  
(c) $\lim\inf_{n \to \infty} \int_{[s > n]} E^+(\gamma_{n+1} | \mathcal{F}_n) = 0$.  

Then $E(x_n) \geq v - \epsilon$.

**Proof.** We can assume $E(x_n) < \infty$. Since $\gamma_s \leq x_s + \epsilon$, $E(\gamma_s) < \infty$. Now

\[(71) \quad v = E(\gamma_1) = \int_{[s=1]} \gamma_s + \int_{[s > 1]} E(\gamma_2 | \mathcal{F}_1) = \int_{[s=1]} \gamma_s + \int_{[s=2]} \gamma_s + \int_{[s > 2]} E(\gamma_3 | \mathcal{F}_2) = \cdots = \int_{[1 \leq s \leq n]} \gamma_s + \int_{[s > n]} E(\gamma_{n+1} | \mathcal{F}_n).
\]

Letting $n \to \infty$, $v \leq E(\gamma_s) \leq E(x_s) + \epsilon$.

**Corollary.** For any $\epsilon \geq 0$, define $s$ by (70). Then  
(i) for $\epsilon > 0$, $A^+ \Rightarrow P[s < \infty] = 1$ and $E(x_s) \geq v - \epsilon$;  
(ii) for $\epsilon = 0$, $\{A^+, P[s < \infty] = 1\} \Rightarrow E(x_s) = v$.

**Proof.** Condition $A^+$ implies $v < \infty$, and by lemma 8, this implies that $P[s < \infty] = 1$. Condition $A^+$ also implies (b) and (c).

**Theorem 7.** Let $\{\alpha_n\}$ be any sequence of r.v.'s such that $\alpha_n$ is $(\mathcal{F}_n)$ measurable and $E(\alpha_n)$ exists for each $n \geq 1$, and such that

(a) $\alpha_n = \max (x_n, E(\alpha_{n+1} | \mathcal{F}_n))$,  
(b) $P[x_n \geq \alpha_n - \epsilon \text{ i.o.}] = 1$ for every $\epsilon > 0$,  
(c) $\{E^+(\alpha_{n+1} | \mathcal{F}_n)\}$ is uniformly integrable,  
(d) either $E(\sup_n \alpha_n) < \infty$, or $A^+$ holds.

Then for each $n \geq 1$, $\alpha_n \leq \gamma_n$.

**Proof.** For $m \geq 1$, $A \in \mathcal{F}_m$, and $\epsilon > 0$, define $t =$ first $n \geq m$ such that $x_n \geq \alpha_n - \epsilon$. Then $P[m \leq t < \infty] = 1$. If the first part of (d) holds, then $E(\alpha_t^-) < \infty$, and since $x_t \geq \alpha_t - \epsilon$, it follows that $E(x_t^-) < \infty$, and hence, by theorem 5(d),

\[(72) \quad \int_A \alpha_t \leq \int_A x_t + \epsilon \leq \int_A \gamma_t + \epsilon \leq \int_A \gamma_m + \epsilon.
\]

If $A^+$ holds, then $E(\alpha_t^+) \leq E(x_t^+) + \epsilon < \infty$, and the same result follows from theorem 5(c). Now

\[(73) \quad \int_A \alpha_m = \int_{[t=m]} \alpha_t + \int_{[t > m]} \alpha_{m+1} + \cdots = \int_{[m \leq t \leq m+h]} \alpha_t + \int_{[t > m+h]} \alpha_{m+h+1} \leq \int_{[m \leq t \leq m+h]} \alpha_t + \int_{[t > m+h]} E^+(\alpha_{m+h+1} | \mathcal{F}_{m+h}).
\]

Letting $k \to \infty$, it follows from (c) that

\[(74) \quad \int_A \alpha_m \leq \int_A \alpha_t \leq \int_A \gamma_m + \epsilon,
\]

so since $\epsilon$ was arbitrarily small, $\int_A \alpha_m \leq \int_A \gamma_m$, and therefore, $\alpha_m \leq \gamma_m$.  

Corollary. Assume that $A^-$ holds. If $\{\alpha_n\}^\infty_1$ is any sequence such that $\alpha_n$ is measurable ($\mathcal{F}_n$), $E(\alpha_n)$ exists for each $n \geq 1$, and (a), (b), and (c) hold, then

$$\alpha_n = \gamma_n.$$  

Proof. By theorems 7, 3, and 4, since $A^-$ implies (d),

$$\gamma_n' \leq \alpha_n \leq \gamma_n = \gamma_n'.$$

9. A theorem of Dynkin

We next prove a slight generalization of a theorem of Dynkin [3]. Let $\{z_n\}^\infty_1$ be a homogeneous discrete time Markov process with arbitrary state space $Z$. For any nonnegative measurable function $g(\cdot)$ on $Z$, define the function $P_g(\cdot)$ by

$$P_g(z) = E(g(z_{n+1})|z_n = z),$$

and set

$$Q_g = \max (g, P_g), \quad Q_{g}^{k+1} = Q(Q_{g}^k), \quad (k \geq 0), \quad Q_{g}^0 = g.$$  

Then $g \leq Q_g \leq Q_{g}^2 \leq \cdots$, so

$$h = \lim_{N \to \infty} Q_{g}^N$$

exists. Let $\mathcal{F}_n = \mathcal{B}(z_1, \cdots, z_n)$ and consider the sequence $\{x_n\}^\infty_1$ with $x_n = g(z_n)$.

Theorem 8. For the process defined above, $\sup E(g(z_i)) = E(h(z_i))$.

Proof. By theorem 3,

$$\gamma_1 = \gamma_1' = \lim_{N \to \infty} \gamma_{N}',$$

where

$$\gamma_{N} = g(z_N),$$

$$\gamma_{N-1} = \max (g(z_{N-1}), E(g(z_N)|z_{N-1})) = Q_g(z_{N-1}),$$

$$\gamma_{N-2} = \max (g(z_{N-2}), E(Q_g(z_{N-1})|z_{N-2})) = \max (g(z_{N-2}), PQ_g(z_{N-2}))$$

$$= \max (g(z_{N-2}), P_g(z_{N-2}), PQ_g(z_{N-2})) = Q_{g}^2(z_{N-2}),$$

$$\vdots$$

$$\gamma_{1}' = Q_{g}^{N-1}g(z_1) \to h(z_1) \quad \text{as} \quad N \to \infty.$$  

Hence $\gamma_1 = h(z_1)$ and $v = E(\gamma_1) = E(h(z_1))$.

10. The triple limit theorem

Lemma 9. Assume $A^+$ holds, and define

$$x_n(a) = \max (x_n, -a), \quad (0 \leq a < \infty),$$

(82)

$$\gamma_n^a = \text{ess sup}_{P[n \geq 1]} E(x_i(a)|\mathcal{F}_n).$$

Then

$$\gamma_n = \lim_{a \to \infty} \gamma_n^a.$$
PROOF. Since $\gamma_n^* = \max (x_n(a), E(\gamma_{n+1}^*/\mathcal{F}_n))$ and $\gamma_n(a) \downarrow \gamma_n^*$, say, as $a \to \infty$, where $\gamma_n^* \geq \gamma_n$, it follows from $A^+$ that $\gamma_n^* = \max (x_n, E(\gamma_{n+1}^*/\mathcal{F}_n))$. For any $\epsilon > 0$ and $m \geq 1$, define $s = \text{first } n \geq m$ such that $x_n \geq \gamma_n^* - \epsilon (= \infty$ if no such $n$ exists). Then $\{\gamma_{\min (s,n)}^*\}_{n=m}^\infty$ is a martingale, since

$\begin{align*}
E(\gamma_{\min (s,n)}^*) &= I_{[s > n]}E(\gamma_{s+1}^*|\mathcal{F}_n) + I_{[s \leq n]}E(\gamma_n^*|\mathcal{F}_n) \\
&= I_{[s > n]}\gamma_n^* + I_{[s = m]}\gamma_m^* + \cdots + I_{[s = n]}\gamma_n^* = \gamma_{\min (s,n)}^*.
\end{align*}$

Since $E((\gamma_{\min (s,n)}^*)^+) \leq E(\sup x_n^+) < \infty$, and since $E((\gamma_n^*)^-) < \infty$, we have by a martingale convergence theorem,

$\begin{align*}
\gamma_{\min (s,n)}^* &\to \text{a finite limit as } n \to \infty, \\
\gamma_n^* &\to \text{a finite limit on } [s = \infty] \text{ as } n \to \infty.
\end{align*}$

But on $[s = \infty]$, $\gamma_n^* > x_n + \epsilon$ for $n \geq m$, so

$\limsup_n x_n \leq \gamma_n^* - \epsilon \quad \text{on } [s = \infty].$

Since $\gamma_n^* \leq E(\sup_{j \geq m} x_j(a)|\mathcal{F}_n)$ for $n \geq m$,

$\limsup_n \gamma_n^* \leq \limsup_n \gamma_n^* \leq \sup_n x_j(a),$

and hence,

$\limsup_n \gamma_n^* \leq \limsup_n x_n(a) = \max (\limsup_n x_n, -a),$

and

$\limsup_n \gamma_n^* \leq \limsup_n x_n,$

but $\gamma_n^* \geq x_n$. Hence,

$\limsup_n \gamma_n^* = \limsup_n x_n,$

contradicting (87) unless $P[s = \infty] = 0$. Hence,

$P[x_n \geq \gamma_n^* - \epsilon, \text{ i.o.}] = 1,$

and by theorem 7, $\gamma_n^* \leq \gamma_n$. Therefore, $\gamma_n^* = \gamma_n$.

THEOREM 9. The random variables $\gamma_n$ are equal to

$\gamma_n = \lim_{b \to \infty} \lim_{a \to -\infty} \lim_{N \to \infty} \gamma_N^*(a, b),$

where

$\gamma_N^*(a, b) = \text{ess sup } E(x_t(a, b)|\mathcal{F}_n)$

and

$x(a, b) = \begin{cases} a & \text{if } x < a, \\ x & \text{if } a \leq x \leq b, \\ b & \text{if } x > b. \end{cases}$

PROOF. This follows from lemma 9, theorem 3, and corollary 2 of theorem 5.

COROLLARY 1. The values $v_n$ are equal to
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(96) \[ \lim_{b \to a} \lim_{N \to \infty} v_n^*(a, b). \]

Corollary 2. If \( \{x_n\} \) is Markovian and \( \mathcal{F}_n = \mathcal{B}(x_1, \ldots, x_n) \), then

(97) \[ \gamma_n = E(\gamma_n | x_n). \]

If the \( x_n \) are independent, then

(98) \[ E(\gamma_{n+1} | \mathcal{F}_n) = E(\gamma_{n+1}) = v_{n+1}, \]

and the \( v_n \) satisfy the recursion relation

(99) \[ v_n = E(\max(x_n, v_{n+1})), \quad (n \geq 1). \]

Proof. By induction \( \gamma_n^*(a, b) = E(\gamma_n^*(a, b) | x_n) \) from \( n = N \) down, as in the proof of the corollary of theorem 3. Letting \( N, a, b \) become infinite yields (97).

Under independence,

(100) \[ E(\gamma_{n+1} | \mathcal{F}_n) = E(E(\gamma_{n+1} | x_{n+1}) | \mathcal{F}_n) = E(\gamma_{n+1}) = v_{n+1}. \]

And from \( \gamma_n = \max(x_n, E(\gamma_{n+1} | \mathcal{F}_n)) = \max(x_n, v_{n+1}) \), we obtain (99) on taking expectations.

11. Remarks on the independent case

Theorem 10. Let the \( \{x_n\} \) be independent with \( \mathcal{F}_n = \mathcal{B}(x_1, \ldots, x_n) \). Set \( s = \text{first } n \geq 1 \) such that \( x_n \geq \gamma_n - \epsilon \) for \( \epsilon > 0 \) (= \( \infty \) if no such \( n \) exists). Then

(101) \[ v < \infty \Rightarrow P[s < \infty] = 1, \]

and if in addition \( E(x_n) \) exists, then

(102) \[ E(x_n) \geq v - \epsilon. \]

Proof. By lemma 8 and theorem 6, since by (87)

(103) \[ \int_{[s > n]} E^+(\gamma_{n+1} | \mathcal{F}_n) = \int_{[s > n]} v_{n+1}^+ P[s > n] \leq v^+ P[s > n] \to 0. \]

We remark that when \( \epsilon = 0 \) the conditions \( v < \infty, P[s < \infty] = 1, E(x_n) \) exists, imply \( E(x_n) = v \).

Theorem 11. Let the \( \{x_n\} \) be independent with \( \mathcal{F}_n = \mathcal{B}(x_1, \ldots, x_n) \), and let \( \{\alpha_n\} \) be any sequence of r.v.'s such that \( \alpha_n \) is measurable (\( \mathcal{F}_n \)) and \( E(\alpha_n) \) exists, \( n \geq 1 \). If

(a) \( \alpha_n = \max(x_n, E(\alpha_{n+1} | \mathcal{F}_n)), (n \geq 1), \)

(b) \( P(x_n \geq \alpha_n - \epsilon \text{ i.o.}) = 1 \) for every \( \epsilon > 0, \)

(c) \( E(\alpha_{n+1} | \mathcal{F}_n) = c_n = \text{constant, with } E(\alpha_1) = c_1 < \infty, \)

(d) \( A^+ \text{ holds, or } \lim inf_{n} E(x_n) > -\infty, \)

then

(104) \[ \alpha_n \leq \gamma_n, \quad (n \geq 1). \]

Proof. Define \( A \) and \( t \) as in theorem 7. Since
\[ c_n = E \{ \max (x_{n+1}, c_{n+1}) \mid \mathcal{F}_n \} \geq c_{n+1}, \]

we have
\[
\int_A \alpha_m = \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t > m+k]} \alpha_{m+k+1} \\
= \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t > m+k]} c_{m+k} \\
\leq \int_{A[m \leq t \leq m+k]} \alpha_t + c_t P[t > m+k].
\]

Hence under \( A^+ \) (or \( A^- \)),
\[
\int_A \alpha_m \leq \liminf_{k \to \infty} \int_{A[m \leq t \leq m+k]} \alpha_t \leq \liminf_{k \to \infty} \int_{A[m \leq t \leq m+k]} x_t + \epsilon \\
\leq \liminf_{k \to \infty} \int_{A[m \leq t \leq m+k]} \gamma_t + \epsilon = \int_A \gamma_t + \epsilon \leq \int_A \gamma_m + \epsilon
\]
by theorem 5(c), so \( \alpha_m \leq \gamma_m \). If the second part of (d) holds, then since \( c_n \downarrow c \), say, where \( c \geq \liminf_n E(x_n) > -\infty \), and \( x_t \geq c_t - \epsilon \geq c - \epsilon \), it follows that \( E(x^-) < \infty \), so theorem 5(d) yields the same conclusion.

REMARKS. 1. Lemmas 2 and 3 are slight extensions of lemmas 1 and 2 of [2].

2. Theorem 1 has been proved independently by G. Haggstrom [4] when \( E|x_n| < \infty \) and \( E(\sup_n x_n^+) < \infty \), as have theorem 4, corollary 1(c) of theorem 5 under \( A^+ \), and the corollary of theorem 6. The latter was also proved by J. L. Snell [5].

3. We are greatly indebted to Mr. D. Siegmund for improvements in the statement and proof of many of our results. In particular, theorem 9 is largely due to him.

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