A NOTE ON RANDOM TRIGONOMETRIC POLYNOMIALS

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1. General remarks

This note is a postscript to our paper [1]. It deals with a problem having close connection with the topics discussed there, and uses similar methods. However, to make the note more readable, we make it self-contained at the expense of a repetition of some of the arguments in [1]. For the sake of proper perspective we begin by restating some of the results of that paper.

Consider a general trigonometric polynomial of order \( n \),

\[
\frac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),
\]

with, say, real coefficients. Let \( \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots \) be the Rademacher functions

\[
\varphi_n(t) = \text{sign} \sin 2\pi nt, \quad 0 \leq t \leq 1,
\]

which represent independent random variables taking values \( \pm 1 \), each with probability \( 1/2 \). We write

\[
P_n(x, t) = \frac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \varphi_k(t),
\]

\[
M_n(t) = \max_x |P_n(x, t)|.
\]

One of the problems discussed in [1] was that of the order of magnitude of \( M_n(t) \) for \( n \to \infty \) and almost all \( t \) (this presupposes, of course, that the \( a_k \) and \( b_k \) are defined for all \( k \)). It turns out (see pp. 270–271 in [1]) that, if the series \( \sum (a_k^2 + b_k^2) \) diverges, and

\[
R_n = \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + b_k^2),
\]

then

\[
\limsup_{n \to \infty} \frac{M_n(t)}{\sqrt{R_n \log n}} \leq 2
\]

for almost all \( t \).

This result was obtained under the sole assumption that \( \sum (a_k^2 + b_k^2) \) diverges. If we want to obtain an estimate for \( M_n(t) \) from below we must introduce further restrictions on \( a_n, b_n \). Write

\[
T_n = \sum_{k=1}^{n} (a_k^2 + b_k^2).
\]
It was shown in [1] that, if
\[ \frac{T_n}{n^{\frac{1}{2}}} = O \left( \frac{1}{n} \right), \tag{1.8} \]
then
\[ \liminf_{n \to \infty} \frac{M_n(t)}{\sqrt{R_n \log n}} \geq \frac{1}{2 \sqrt{6}}, \tag{1.9} \]
for almost all \( t \) [incidentally, we can also then in the right-hand side of (1.6) replace 2 by 1]. Thus, under the hypothesis (1.8), \( M_n(t) \) is for almost all \( t \) strictly of order \((R_n \log n)^{1/2}\).

Clearly (1.8) implies the divergence of \( \sum (\alpha_k^2 + \beta_k^2) \), and is, in turn, a consequence of this divergence if \((\alpha_k^2 + \beta_k^2)^{1/2}\) is bounded above and away from 0. In particular, the \( M_n(t) \) for the series
\[ \frac{1}{2} + \sum_{\nu=1}^{\infty} \varphi_{\nu}(t) \cos \nu t \tag{1.10} \]
for almost all \( t \) are strictly of order \((n \log n)^{1/2}\).

We add, parenthetically, that the problem of whether there exists at least one \( t = t_0 \) (not diadically rational), such that (1.10) satisfies \( M_n(t_0) = O(\sqrt{n}) \), seems to be open.

We now pass to the proper topic of this note. Subdivide the interval \((0, 2\pi)\) into \( 2n + 1 \) equal parts and write
\[ a_\nu = a^{(\omega)} = \frac{2\pi \nu}{2n+1}, \quad \nu = 0, 1, \ldots, 2n. \tag{1.11} \]
Consider the trigonometric interpolating polynomial of order \( n \) which at the point \( a_\nu \) takes the value \( \varphi_{\nu}(t) \), \( \nu = 0, 1, \ldots, 2n \). Such a polynomial exists and is uniquely determined. We denote it by \( I_n(x, t) \) or sometimes, for brevity, by \( I \), and write
\[ M_n(t) = \max_{\nu} \left| I_n(x, t) \right| \tag{1.12} \]
[thus \( M_n(t) \) no longer has the meaning (1.4)]. We are going to prove the following result.

**Theorem.** For almost all \( t \) we have
\[ \limsup_{n \to \infty} \frac{M_n(t)}{(\log n)^{1/2}} \leq 2. \tag{1.13} \]

2. **Proof of the theorem**

Denote by \( D_n(x) \) the Dirichlet kernel
\[ D_n(x) = \frac{\sin \left( \frac{n+\frac{1}{2}}{2} \right) x}{2 \sin \frac{1}{2} x}. \tag{2.1} \]
We have then the classical formula
\[ I_n(x, t) = \frac{1}{2n+1} \sum_{\nu=0}^{2n} \varphi_{\nu}(t) D_n(x - a_\nu). \tag{2.2} \]

For any finite sum \( S = \sum a_\nu \varphi_{\nu}(t) \) of Rademacher functions with real coefficients, and any positive \( \lambda \), we have
\[ \int_0^1 e^{\lambda t} dt = \prod \int_0^1 e^{\lambda x \varphi} dt = \prod \frac{1}{2l} \left( e^{\lambda x} + e^{-\lambda x} \right) = \prod \left( 1 + \frac{\lambda^2 a^2_{2l}}{2l} + \frac{\lambda^4 a^4_{4l}}{4l} + \cdots \right), \tag{2.3} \]
and since $(2p)! \geq 2^p p!$, the last product does not exceed

$$\prod_{p=0}^{\infty} \left( \frac{\lambda^{2p} 2^{2p}}{2^p p!} \right) = \prod_{p=0}^{\infty} e^{\lambda^{2p}/2p}.$$

This leads to the very well known inequality

$$\int_0^1 e^{\lambda x^2} x^r \, dx \leq e^{\lambda x^2/2}.$$

If we apply it to (2.2) we get

$$\int_0^1 e^{\lambda |x|} \, dx < \int_0^1 \left( e^{\lambda x} + e^{-\lambda x} \right) \, dx \leq 2 \exp \left[ \frac{\lambda^2}{2 (2n+1)^2} \sum_{p=0}^{2n} D^p(x-a_p) \right].$$

Now, $D^p_a$ being a trigonometric polynomial of order $2n$, its discrete average over any system of $2n+1$ equidistant points is the same. Hence

$$\frac{2}{2n+1} \sum_{p=0}^{2n} D^p(x-a_p) = \frac{2}{2n+1} \sum_{p=0}^{2n} D^p(a_p) = \frac{2n+1}{2},$$

by (2.1) and (1.11), so that

$$\frac{4}{(2n+1)^2} \sum_{p=0}^{2n} D^p(x-a_p) = 1$$

and (2.6) takes the form

$$\int_0^1 e^{\lambda |x|} \, dx \leq 2 e^{\lambda/2}.$$

Integrating this with respect to $x$ and inverting the order of integration we obtain

$$\int_0^1 d\int_0^{2\pi} e^{\lambda |x|} \, dx \leq 4\pi e^{\lambda/2}.$$

Our next step will be to deduce from this an estimate for the integral

$$\int_0^1 e^{M_n(x)} \, dx.$$

This deduction is based on the very well known theorem of S. Bernstein which asserts that for any trigonometric polynomial $T(x)$ of order $n$ we have

$$\max_{x} |T'(x)| \leq n \max_{x} |T(x)|.$$

[A proof of this theorem may be found, for example, in [2] (see p. 90, Vol. 2).]

Fix $t$, write $M = M_n(t)$ and denote by $x_0 = x_0(t)$ a point $x$ at which $|I|$ attains its maximum $M_n(t)$. Take any number $\theta$ positive and less than 1, and consider the interval $x_0 \leq x \leq x_0 + (1-\theta)n$. Since the slope of the curve $y = I$ does not exceed $M$, the value of $|I|$ in the interval just written cannot change more than $(1-\theta)M$, and so is at least $\theta M$ in that interval. When in the inner integral (2.10) we replace the interval of integration $(0, 2\pi)$ by $[x_0, x_0 + (1-\theta)n]$, it follows that

$$\int_0^1 e^{\lambda M_n(t)} \cdot \frac{1-\theta}{n} \, dt \leq 4\pi e^{\lambda/2},$$

or

$$\int_0^1 e^{\lambda M_n(t)} \, dt \leq \frac{4\pi n}{1-\theta} e^{\lambda/2} = \frac{4\pi}{1-\theta} e^{\lambda/2 + \log n}.$$
So far $\lambda$ has been arbitrary. We now set $\lambda = (2c \log n)^{1/2}$, where $c$ will be determined in a moment. We obtain successively

\[
(2.15) \quad \int_0^1 e^{\lambda M_n(t)} dt < \frac{4\pi}{1 - \theta} e^{(c+1) \log n},
\]

and

\[
(2.16) \quad \int_0^1 e^{\lambda M_n(t) - (c+2+\epsilon) dt} < \frac{4\pi}{1 - \theta} e^{-(1+\epsilon) \log n} = \frac{4\pi}{1 - \theta} n^{-1-\epsilon},
\]

where $\epsilon > 0$. Since the series with terms $n^{-1-\epsilon}$ converges, the sum of the integrals on the left of (2.16) is finite. This implies that the series with terms $\exp[\lambda \theta M_n - (c + 2 + \epsilon)]$ converges, for almost all $t$, and in particular that, for almost all $t$ and $n$ large enough,

\[
(2.17) \quad M_n(t) \leq \frac{(c + 2 + \epsilon) \log n}{\theta (2c \log n)^{1/2}} = \frac{1}{\theta \sqrt{2}} \sqrt{\log n} \cdot \frac{c + 2 + \epsilon}{\sqrt{c}}.
\]

Selecting now for $c$ the value 2 (which minimizes the sum $c^{1/2} + 2c^{-1/2}$) we deduce that

\[
(2.18) \quad \limsup_{n \to \infty} \frac{M_n(t)}{\log n^{1/2}} \leq \frac{1}{\theta} (2 + \frac{3}{2} \epsilon)
\]

for almost all $t$. Since we may take $\epsilon$ arbitrarily small and $\theta$ arbitrarily close to 1, (1.13) follows and the theorem is established.

It is very likely that for almost all $t$, $M_n(t)$ is exactly of the order $(\log n)^{1/2}$ but, so far, this is an open problem.

REFERENCES
