On ℓ-adic Cohomology of Artin stacks: L-functions, Weights, and the Decomposition theorem

by

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Abstract

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We develop the notion of stratifiability in the context of derived categories and the six operations for stacks in [26, 27]. Then we reprove Behrend's Lefschetz trace formula for stacks, and give the meromorphic continuation of the L-series of $\mathbb{F}_q$-stacks. We give an upper bound for the weights of the cohomology groups of stacks, and as an application, prove the decomposition theorem for perverse sheaves on stacks with affine diagonal, both over finite fields and over the complex numbers. Along the way, we generalize the structure theorem of ι-mixed sheaves and the generic base change theorem to stacks. We also give a short exposition on the lisse-analytic topoi of complex analytic stacks, and give a comparison between the lisse-étale topos of a complex algebraic stack and the lisse-analytic topos of its analytification.

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Chapter 1

Introduction

In topology there is the famous Lefschetz-Hopf trace formula, which roughly says that if $f : X \to X$ is an endomorphism of a compact connected oriented space X with isolated fixed points, then the number of fixed points of f, counted with multiplicity, is equal to the alternating sum of the traces of f^* on the singular cohomology groups $H^i(X, \mathbb{Q})$.

There is also a trace formula in algebraic geometry, for schemes over finite fields, due to Grothendieck. It says that if X_0 is a scheme over $\mathbb{F}_q$, separated and of finite type, and F_q is the q-geometric Frobenius map, then

$$\#X_0(\mathbb{F}_q) = \sum_{i=0}^{2\dim X_0} (-1)^i \mathrm{Tr}(F_q, H^i_c(X, \overline{\mathbb{Q}}_\ell)),$$

where $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is the ℓ-adic cohomology with compact support. In fact he proved the trace formula for an arbitrary constructible sheaf. See [17, 38, 11].

Behrend conjectured the trace formula for smooth algebraic stacks over $\mathbb{F}_q$ in his thesis and [2], and proved it in [3]. However, he used ordinary cohomology and arithmetic Frobenius (rather than compact support cohomology and geometric Frobenius) to prove the "dual statement", probably because at that time the theory of dualizing complexes of algebraic stacks, as well as compact support cohomology groups of stacks, were not developed. Later Laszlo and Olsson developed the theory of the six operations for algebraic stacks [26, 27], which makes it possible to reprove the trace formula, and remove the smoothness assumption in Behrend's result. Also we want to work with a fixed isomorphism of fields $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, which is a more general setting.

Once we have the trace formula, we get a factorization of the zeta function into a possibly infinite product of L-factors, and from this one can deduce the meromorphic continuation of the zeta functions, generalizing a result of Behrend ([2], 3.2.4). Furthermore, to locate the zeros and poles of the zeta functions, we give a result on the weights of cohomology groups of stacks.

Drinfeld observed that, for a complex elliptic curve E, Gabber's decomposition theorem fails for the proper morphism $f : \mathrm{Spec}\ \mathbb{C} \to BE$. This is essentially due to the failure of the upper bound on weights in [9] for the cohomology groups of BE. In the weight theorem, we will see that for stacks with affine automorphism groups, the usual upper bound in [9] still applies. As an application, we verify that the proof in [4] of the decomposition theorem can be generalized to such stacks.

We briefly mention the technical issues. For the trace formula, there are three difficulties. As pointed out in [3], a big difference between schemes and stacks is the following. If $f : X_0 \to Y_0$ is a morphism of $\mathbb{F}_q$-schemes of finite type, and $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$, then f_*K_0 and $f_!K_0$ are also bounded complexes. Since often we are mainly interested in the simplest case when K_0 is a sheaf concentrated in degree 0 (for instance, the constant sheaf $\overline{\mathbb{Q}}_\ell$), and D_c^b is stable under f_* and $f_!$, it is enough to consider D_c^b only. But for a morphism $f : \mathscr{X}_0 \to \mathscr{Y}_0$ of $\mathbb{F}_q$-algebraic stacks of finite type, f_* and $f_!$ do not necessarily preserve boundedness. For instance, the cohomology ring $H^*(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ is the polynomial ring $\overline{\mathbb{Q}}_\ell[T]$ with $\deg(T) = 2$. So for stacks we have to consider unbounded complexes, even if we are only interested in the constant sheaf $\overline{\mathbb{Q}}_\ell$. In order to define the trace of the Frobenius on cohomology groups, we need to consider the convergence of the complex series of the traces. This leads to the notion of an ι-convergent complex of sheaves (see (4.1.1)).

Another issue is the following. In the scheme case one considers bounded complexes, and for any bounded complex K_0 on a scheme X_0, there exists a stratification of X_0 that "trivializes the complex K_0" (i.e. the restrictions of all cohomology sheaves $\mathscr{H}^i K_0$ to each stratum are lisse). But in the stack case we have to consider unbounded complexes, and in general there might be no stratification of the stack that trivializes every cohomology sheaf. This leads to the notion of a stratifiable complex of sheaves (see (7.3.1)). We need the stratifiability condition to control the dimensions of cohomology groups (see (3.2.5)). All bounded complexes are stratifiable (3.1.4v).

Also we will have to impose the condition of ι-mixedness, due to unboundedness. For bounded complexes on schemes, the trace formula can be proved without using this assumption, although the conjecture of Deligne ([9], 1.2.9) that all sheaves are ι-mixed is proved by Laurent Lafforgue. See the remark (2.2.5.1).

For the decomposition theorem on complex algebraic stacks, the main difficulties are the comparison between the derived categories on the lisse-étale topos of the algebraic stack and on the lisse-analytic topos of the associated analytic stack, as well as the comparison between the derived categories with prescribed stratification of the complex stack and of the stack over a field of positive characteristic. To prove the second comparison, we also need to generalize the generic base change theorem (cf. [11]) to stacks.

We briefly introduce the main results of this paper.

Fixed point formula.

Theorem 1.0.1. *Let $\mathscr{X}_0$ be an Artin stack of finite type over $\mathbb{F}_q$. Then the series*

$$\sum_{n\in\mathbb{Z}}(-1)^n \mathrm{Tr}(F_q, H_c^n(\mathscr{X}, \overline{\mathbb{Q}}_\ell)),$$

regarded as a complex series via ι, is absolutely convergent, and its limit is "the number of $\mathbb{F}_q$-points of $\mathscr{X}_0$", namely

$$\#\mathscr{X}_0(\mathbb{F}_q) := \sum_{x\in[\mathscr{X}_0(\mathbb{F}_q)]} \frac{1}{\#\mathrm{Aut}_x(\mathbb{F}_q)}.$$

Here F_q denotes the q-geometric Frobenius. To generalize, one wants to impose some condition (P) on complexes $K_0 \in D_c^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ such that

(1) (P) is preserved by $f_!$;

(2) if a complex K_0 satisfies (P), then the "naive local terms" are well-defined, and

(3) trace formula holds in this case.

The condition (P) on K_0 turns out to be a combination of three parts: ι-convergence (which implies (2) for K_0), ι-mixedness and stratifiability (which, together with the first part, implies (2) for $f_!K_0$). See (4.1.2) for the general statement of the theorem. These conditions are due to Behrend [3].

Meromorphic continuation.

The rationality in Weil conjecture was first proved by Dwork, namely the zeta function $Z(X_0, t)$ of every variety X_0 over $\mathbb{F}_q$ is a rational function in t. Later, this was reproved using the fixed point formula [17, 16]. Following ([2], 3.2.3), we define the zeta functions of stacks as follows.

Definition 1.0.2. *For an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ of finite type, define the* zeta function

$$Z(\mathscr{X}_0, t) = \exp\Big(\sum_{v\geq 1}\frac{t^v}{v}\sum_{x\in[\mathscr{X}_0(\mathbb{F}_{q^v})]}\frac{1}{\#\mathrm{Aut}_x(\mathbb{F}_{q^v})}\Big),$$

as a formal power series in the variable t. Here $[\mathscr{X}_0(\mathbb{F}_{q^v})]$ is the set of isomorphism classes of objects in the groupoid $\mathscr{X}_0(\mathbb{F}_{q^v})$ (see 1.0.7 below).

Notice that in general, the zeta function is not rational (cf. §7). Behrend proved that, if $\mathscr{X}_0$ is a smooth algebraic stack, and it is a quotient of an algebraic space by a linear algebraic group, then its zeta function $Z(\mathscr{X}_0, t)$ is a meromorphic function in the complex t-plane; if $\mathscr{X}_0$ is a smooth Deligne-Mumford stack, then $Z(\mathscr{X}_0, t)$ is a rational function ([2], 3.2.4, 3.2.5). These results can be generalized as follows.

Theorem 1.0.3. *For every $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ of finite type, its zeta function $Z(\mathscr{X}_0, t)$ defines a meromorphic function in the whole complex t-plane. If $\mathscr{X}_0$ is Deligne-Mumford, then $Z(\mathscr{X}_0, t)$ is a rational function.*

See (5.2.3.1) and (5.3.1) for the general statement.

A theorem of weights.

One of the main results in [9] is that, if X_0 is an $\mathbb{F}_q$-scheme, separated and of finite type, and $\mathscr{F}_0$ is an ι-mixed sheaf on X_0 of punctual ι-weights $\leq w \in \mathbb{R}$, then for every n, the punctual ι-weights of $H^n_c(X, \mathscr{F})$ are $\leq w + n$. The cohomology groups are zero unless $0 \leq n \leq 2 \dim X_0$. We will see later (5.2.2.1) that the upper bound $w + n$ for the punctual ι-weights does not work in general for algebraic stacks. We will give an upper bound that applies to all algebraic stacks. Deligne's upper bound of weights still applies to stacks for which all the automorphism groups are affine.

Theorem 1.0.4. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack of finite type, and let $\mathscr{F}_0$ be an ι-mixed sheaf on $\mathscr{X}_0$, with punctual ι-weights $\leq w$, for some $w \in \mathbb{R}$. Then the punctual ι-weights of $H^n_c(\mathscr{X}, \mathscr{F})$ are $\leq \dim \mathscr{X}_0 + \frac{n}{2} + w$, and they are congruent mod $\mathbb{Z}$ to weights that appear in $\mathscr{F}_0$. If $n > 2 \dim \mathscr{X}_0$, then $H^n_c(\mathscr{X}, -) = 0$ on sheaves. If for all integers $v \geq 1$ and all points $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, the automorphism groups Aut_x are linear algebraic groups, then the punctual ι-weights of $H^n_c(\mathscr{X}, \mathscr{F})$ are $\leq n + w$.*

Decomposition theorem.

The estimation of weights for stacks with affine automorphism groups in (1.0.4) enables us to follow [4] and prove the stack version of the decomposition theorem over $\mathbb{F}_q$.

Theorem 1.0.5. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack of finite type, with affine automorphism groups.*

(i) For any ι-pure complex $K_0 \in D^b_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, the complex K on $\mathscr{X}$ induced by K_0 by extension of scalars is isomorphic to the direct sum of the shifted perverse cohomology sheaves $({}^p\mathscr{H}^i K)[-i]$.

(ii) Any ι-pure $\overline{\mathbb{Q}}_\ell$-perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ is geometrically semi-simple, i.e. the induced $\mathscr{F}$ is semi-simple in the category of perverse sheaves on $\mathscr{X}$.

After proving the generic base change and some necessary comparisons, one can give the decomposition theorem for complex algebraic stacks. We only state a simple version here; see (8.3.2.4) for the general one.

Theorem 1.0.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of complex algebraic stacks of finite type, and $\mathcal{X}$ is smooth. Then $f_*\underline{\mathbb{C}}_{\mathcal{X}}$ is isomorphic to the direct sum of the ${}^p\mathscr{H}^i(f_*\underline{\mathbb{C}}_{\mathcal{X}})[-i]$'s, and each shifted summand is a semi-simple $\mathbb{C}$-perverse sheaf on $\mathcal{Y}$.*

Organization. In chapter 2, we review some preliminaries on derived categories of ℓ-adic sheaves on algebraic stacks over $\mathbb{F}_q$ and ι-mixed complexes, and show that ι-mixedness is stable under the six operations.

In chapter 3, we develop the notion of stratifiable complexes in the context of Laszlo and Olsson's ℓ-adic derived categories, and prove its stability under the six operations.

In chapter 4, we discuss convergent complexes, and show that they are preserved by $f_!$, and reprove the trace formula for stacks in terms of compacted supported cohomology. These two theorems are stated and proved in [3] in terms of ordinary cohomology and arithmetic Frobenius.

In chapter 5, we discuss convergence of infinite products of formal power series, and give some examples of zeta functions of stacks. In one example we give the functional equation of the zeta functions and independence of ℓ of Frobenius eigenvalues for proper varieties with quotient singularities (5.2.3.2). Then we prove the meromorphic continuation of the L-series of ι-mixed stratifiable convergent complexes on stacks.

In chapter 6, we prove the weight theorem (1.0.4), and give some simple applications on the existence of rational points on stacks. We also discuss "independence of ℓ" for stacks, and prove (6.3.5) that for the quotient stack $[X_0/G_0]$, where X_0 is a proper smooth variety and G_0 is a linear algebraic group acting on X_0, the Frobenius eigenvalues on its cohomology groups are independent of ℓ.

In chapter 7, we review Drinfeld's counter-example, and complete the proof of the structure theorem for ι-mixed sheaves on stacks. Then we follow [4] and generalize the decomposition theorem for perverse sheaves on stacks over finite fields, using weight theory.

In chapter 8, we prove the generic base change for f_* and $R\mathscr{H}om$, and use it to prove a comparison between bounded derived categories with prescribed stratification over the complex numbers and over an algebraic closure of a finite field, as well as a comparison between the lisse-étale topos and the lisse-analytic topos of a $\mathbb{C}$-stack, and finally we finish the proof of the decomposition theorem over $\mathbb{C}$.

Notations and Conventions 1.0.7. We fix a prime power $q = p^a$ and an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_q$ with q elements. Let F or F_q be the q-geometric Frobenius, namely the q-th root automorphism on $\mathbb{F}$. Let ℓ be a prime number, $\ell \neq p$, and fix an isomorphism of fields $\overline{\mathbb{Q}}_\ell \xrightarrow{\iota} \mathbb{C}$. For simplicity, let $|\alpha|$ denote the complex absolute value $|\iota\alpha|$, for $\alpha \in \overline{\mathbb{Q}}_\ell$.

In this paper, by an Artin stack (or an algebraic stack) over a base scheme S, we mean an S-algebraic stack in the sense of M. Artin ([25], 4.1) *of finite type*. When we want the more general setting of Artin stacks *locally of finite type*, we will mention that explicitly.

Objects over $\mathbb{F}_q$ will be denoted with an index $_0$. For instance, $\mathscr{X}_0$ will denote an $\mathbb{F}_q$-Artin stack; if $\mathscr{F}_0$ is a lisse-étale sheaf (or more generally, a Weil sheaf (2.2.1)) on $\mathscr{X}_0$, then $\mathscr{F}$ denotes its inverse image on $\mathscr{X} := \mathscr{X}_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

We will usually denote algebraic stacks over a base scheme S by

- $\mathcal{X}, \mathcal{Y} \cdots$, if the base S is unspecified,
- $\mathscr{X}_0, \mathscr{Y}_0 \cdots$ (resp. $\mathscr{X}, \mathscr{Y} \cdots$) if the base is $\mathbb{F}_q$ (resp. $\mathbb{F}$),

and complex analytic stacks are usually denoted by $\mathfrak{X}, \mathfrak{Y} \cdots$.

For a field k, let $\mathrm{Gal}(k)$ denote its absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. By a variety over k we mean a separated reduced k-scheme of finite type. Let $W(\mathbb{F}_q)$ be the *Weil group* $F_q^{\mathbb{Z}}$ of $\mathbb{F}_q$.

For a profinite group H, by $\overline{\mathbb{Q}}_\ell$-representations of H we always mean finite-dimensional continuous representations ([9], 1.1.6), and denote by $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(H)$ the category of such representations.

For a scheme X, we denote by $|X|$ the set of its closed points. For a category $\mathscr{C}$ we write $[\mathscr{C}]$ for the collection of isomorphism classes of objects in $\mathscr{C}$. For example, if $v \geq 1$ is an integer, then $[\mathscr{X}_0(\mathbb{F}_{q^v})]$ denotes the set of isomorphism classes of $\mathbb{F}_{q^v}$-points of the stack $\mathscr{X}_0$. This is a finite set.

For $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$ we will write $k(x)$ for the field $\mathbb{F}_{q^v}$. For an $\mathbb{F}_q$-scheme X_0 (always of finite type) and $x \in |X_0|$, we denote by $k(x)$ the residue field of x. In both cases, let $d(x)$ be the degree of the field extension $[k(x) : \mathbb{F}_q]$, and $N(x) = q^{d(x)} = \#k(x)$. Also in both cases let $x : \mathrm{Spec}\ \mathbb{F}_{q^v} \to \mathscr{X}_0$ (or X_0) be the natural map ($v = d(x)$), although in the second case the map is defined only up to an automorphism in $\mathrm{Gal}(k(x)/\mathbb{F}_q)$. Given a $K_0 \in D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (cf. §2), the pullback $x^*K_0 \in D_c(\mathrm{Spec}\ k(x), \overline{\mathbb{Q}}_\ell) = D_c(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(k(x))))$ gives a complex $K_{\overline{x}}$ of representations of $\mathrm{Gal}(k(x))$, and we let F_x be the geometric Frobenius generator $F_{q^{d(x)}}$ of this group, called "the local Frobenius".

Let V be a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space and φ an endomorphism of V. For a function $f : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, we denote by $\sum_{V,\varphi} f(\alpha)$ the sum of values of f in α, with α ranging over all the eigenvalues of φ on V with multiplicities. For instance, $\sum_{V,\varphi} \alpha = \mathrm{Tr}(\varphi, V)$.

A 0×0-matrix has trace 0 and determinant 1. For $K \in D_c^b(\overline{\mathbb{Q}}_\ell)$ and an endomorphism φ of K, we define (following [11])

$$\mathrm{Tr}(\varphi, K) := \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{Tr}(H^n(\varphi), H^n(K))$$

and

$$\det(1 - \varphi t, K) := \prod_{n \in \mathbb{Z}} \det(1 - H^n(\varphi)t, H^n(K))^{(-1)^n}.$$

For unbounded complexes K we use similar notations, if the series (resp. the infinite product) converges (resp. converges term by term; cf. (5.1.2)).

For a map $f : X \to Y$ and a sheaf $\mathscr{F}$ on Y, we sometimes write $H^n(X, \mathscr{F})$ for $H^n(X, f^*\mathscr{F})$. We will write $H^n(\mathscr{X})$ for $H^n(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$, and $h^n(\mathscr{X}, \mathscr{F})$ for $\dim H^n(\mathscr{X}, \mathscr{F})$, and ditto for $H^n_c(\mathscr{X})$ and $h^n_c(\mathscr{X}, \mathscr{F})$.

For the theory of dualizing complexes on algebraic stacks, as well as the theory of the six operations on stacks, we follow [26, 27]. For the theory of perverse sheaves on schemes and stacks, we follow [4, 28]. When discussing perverse t-structures, we will always take the middle perversity.

The derived functors $Rf_*, Rf_!, Lf^*$ and $Rf^!$ are usually abbreviated as $f_*, f_!, f^*, f^!$. But we do not use similar abbreviations for $\otimes^L$, $R\mathscr{H}om$ and $RHom$; namely we reserve $\otimes$, $\mathscr{H}om$ and Hom for the ordinary sheaf tensor product, sheaf Hom and Hom group respectively.

Chapter 2

Derived category of ℓ-adic sheaves and mixedness.

We briefly review the definition in [26, 27] for derived category of ℓ-adic sheaves on stacks. Then we generalize the structure theorem of ι-mixed sheaves in [9] to algebraic stacks (2.2.4.1). Finally we show that ι-mixedness is stable under the six operations. By a result of Lafforgue (2.2.5.1), this is automatic, but we still want to give a much more elementary argument.

2.1 Review of ℓ-adic derived categories for stacks.

In this section, we briefly review the definition in [26, 27] for derived category of ℓ-adic sheaves on stacks.

2.1.1. Let Λ be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and residual characteristic ℓ. Let $\Lambda_n=\Lambda/\mathfrak{m}^{n+1}$, and let $\Lambda_\bullet$ be the pro-ring $(\Lambda_n)_n$. We take the base scheme S to be a scheme that satisfies the following condition denoted (LO): it is noetherian affine excellent finite-dimensional, in which ℓ is invertible, and all S-schemes of finite type have finite ℓ-cohomological dimension. We denote by $\mathcal{X},\mathcal{Y}\cdots$ Artin stacks locally of finite type over S.

Consider the ringed topos $\mathscr{A}=\mathscr{A}(\mathcal{X}):=\mathrm{Mod}(\mathcal{X}^{\mathbb{N}}_{\text{lis-ét}},\Lambda_\bullet)$ of projective systems $(M_n)_n$ of $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$ such that M_n is a Λ_n-module for each n, and the transition maps are Λ-linear. An object $M\in\mathscr{A}$ is said to be *AR-null*, if there exists an integer $r>0$ such that for every integer n, the composed map $M_{n+r}\to M_n$ is the zero map. A complex K in $\mathscr{A}$ is called *AR-null*, if all cohomology systems $\mathscr{H}^i(K)$ are AR-null; it is called *almost AR-null*, if for every U in $\text{Lis-ét}(\mathcal{X})$, the restriction of $\mathscr{H}^i(K)$ to $\text{Ét}(U)$ is AR-null. Let $\mathscr{D}(\mathscr{A})$ be the ordinary derived category of $\mathscr{A}$.

Definition 2.1.2. *An object $M = (M_n)_n \in \mathscr{A}$ is* adic *if all the M_n's are constructible, and for every n, the natural map*

$$\Lambda_n \otimes_{\Lambda_{n+1}} M_{n+1} \to M_n$$

is an isomorphism. It is called almost adic *if all the M_n's are constructible, and for every object U in* Lis-ét$(\mathcal{X})$*, the restriction $M|_U$ is AR-adic, i.e. there exists an adic $N_U \in \mathrm{Mod}(U^{\mathbb{N}}_{\text{ét}}, \Lambda_\bullet)$ and a morphism $N_U \to M|_U$ with AR-null kernel and cokernel.*

A complex K in $\mathscr{A}$ is a λ-complex *if $\mathscr{H}^i(K) \in \mathscr{A}$ are almost adic, for all i. Let $\mathscr{D}_c(\mathscr{A})$ be the full triangulated subcategory of $\mathscr{D}(\mathscr{A})$ consisting of λ-complexes, and let $D_c(\mathcal{X}, \Lambda)$ be the quotient of $\mathscr{D}_c(\mathscr{A})$ by the thick full subcategory of almost AR-null complexes. This is called the* derived category of Λ-adic sheaves on $\mathcal{X}$.

Remark 2.1.2.1. (i) $D_c(\mathcal{X}, \Lambda)$ is a triangulated category with a natural t-structure, and its heart is the quotient of the category of almost adic systems in $\mathscr{A}$ by the thick full subcategory of almost AR-null systems. One can use this t-structure to define the subcategories $D^{\dagger}_c(\mathcal{X}, \Lambda)$ for $\dagger = \pm, b$.

If $\mathcal{X}/S$ is of finite type (in particular, quasi-compact), it is clear that $K \in \mathscr{D}(\mathscr{A})$ is AR-null if it is almost AR-null. Also if $M \in \mathscr{A}$ is almost adic, the adic system N_U and the map $N_U \to M|_U$ in the definition above are unique up to unique isomorphism, for each U, so by ([25], 12.2.1) they give an adic system N of Cartesian sheaves on $\mathcal{X}_{\text{lis-ét}}$, and an AR-isomorphism $N \to M$. This shows that an almost adic system is AR-adic, and it is clear ([16], p.234) that the natural functor

$$\Lambda\text{-Sh}(\mathcal{X}) \to \text{heart } D_c(\mathcal{X}, \Lambda)$$

is an equivalence of categories, where Λ-Sh$(\mathcal{X})$ denotes the category of Λ-adic (in particular, constructible) systems.

(ii) $D_c(\mathcal{X}, \Lambda)$ is different from the ordinary derived category of $\mathrm{Mod}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ with constructible cohomology; the latter is denoted in [27] by $\mathscr{D}_c(\mathcal{X}, \Lambda)$. Here $\mathrm{Mod}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ denotes the abelian category of $\Lambda_{\mathcal{X}}$-modules, not adic sheaves Λ-Sh$(\mathcal{X})$.

(iii) When $S = \mathrm{Spec}\, k$ for k a finite field or an algebraically closed field, and $\mathcal{X} = X$ is a separated S-scheme, ([27], 3.1.6) gives a natural equivalence of triangulated categories between $D^b_c(X, \Lambda)$ and Deligne's definition $\mathscr{D}^b_c(X, \Lambda)$ in ([9], 1.1.2).

2.1.3. Let $\pi : \mathcal{X}^{\mathbb{N}}_{\text{lis-ét}} \to \mathcal{X}_{\text{lis-ét}}$ be the morphism of topoi, where π^{-1} takes a sheaf F to the constant projective system $(F)_n$, and π_* takes a projective system to the inverse limit. This morphism induces a morphism of ringed topoi $(\pi^*, \pi_*) : \mathscr{A} \to \mathrm{Mod}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$. The functor $R\pi_* : \mathscr{D}_c(\mathscr{A}) \to \mathscr{D}_c(\mathcal{X}, \Lambda)$ vanishes on almost AR-null objects ([27], 2.2.2), hence factors through $D_c(\mathcal{X}, \Lambda)$. In ([27], 3.0.8), the normalization functor is defined to be

$$K \mapsto \widehat{K} := L\pi^* R\pi_* K : \ D_c(\mathcal{X}, \Lambda) \to \mathscr{D}_c(\mathscr{A}).$$

This functor plays an important role in defining the six operations [27]. For instance:

• For $F \in D_c^-(\mathcal{X}, \Lambda)$ and $G \in D_c^+(\mathcal{X}, \Lambda)$, $R\mathscr{H}om(F, G)$ is defined to be the image of $R\mathscr{H}om_{\Lambda_\bullet}(\widehat{F}, \widehat{G})$ in $D_c(\mathcal{X}, \Lambda)$.

• For $F, G \in D_c^-(\mathcal{X}, \Lambda)$, the derived tensor product $F \otimes^L G$ is defined to be the image of $\widehat{F} \otimes^L_{\Lambda_\bullet} \widehat{G}$.

• For a morphism $f : \mathcal{X} \to \mathcal{Y}$ and $F \in D_c^+(\mathcal{X}, \Lambda)$, the derived direct image f_*F is defined to be the image of $f_*^{\mathbb{N}} \widehat{F}$.

Let E_λ be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathscr{O}_\lambda$. Following [27] we define $D_c(\mathcal{X}, E_\lambda)$ to be the quotient of $D_c(\mathcal{X}, \mathscr{O}_\lambda)$ by the full subcategory consisting of complexes K such that, for every integer i, there exists an integer $n_i \geq 1$ such that $\mathscr{H}^i(K)$ is annihilated by λ^{n_i}. Then we define

$$D_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell) = 2\text{-colim}_{E_\lambda} D_c(\mathcal{X}, E_\lambda),$$

where E_λ ranges over all finite subextensions of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$, and the transition functors are

$$E_{\lambda'} \otimes_{E_\lambda} - : D_c(\mathcal{X}, E_\lambda) \to D_c(\mathcal{X}, E_{\lambda'})$$

for $E_\lambda \subset E_{\lambda'}$.

2.2 ι-mixedness.

In this section, we take $S = \operatorname{Spec} \mathbb{F}_q$. We recall some notions of weights and mixedness from [9], generalized to $\mathbb{F}_q$-algebraic stacks $\mathscr{X}_0$.

2.2.1. Frobenius endomorphism. For an $\mathbb{F}_q$-scheme X_0 (not necessarily of finite type), let $F_{X_0} : X_0 \to X_0$ be the morphism that is identity on the underlying topological space and q-th power on the structure sheaf $\mathscr{O}_{X_0}$; this is an $\mathbb{F}_q$-morphism. Let $\sigma_{X_0} : X \to X$ be the induced $\mathbb{F}$-morphism $F_{X_0} \times \mathrm{id}_{\mathbb{F}}$ on $X = X_0 \otimes \mathbb{F}$.

By functoriality of the maps F_{X_0}, we can extend it to stacks. For an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$, define $F_{\mathscr{X}_0} : \mathscr{X}_0 \to \mathscr{X}_0$ to be such that for every $\mathbb{F}_q$-scheme X_0, the map

$$F_{\mathscr{X}_0}(X_0) : \mathscr{X}_0(X_0) \longrightarrow \mathscr{X}_0(X_0)$$

sends x to $x \circ F_{X_0}$. We also define $\sigma_{\mathscr{X}_0} : \mathscr{X} \to \mathscr{X}$ to be $F_{\mathscr{X}_0} \times \mathrm{id}_{\mathbb{F}} : \mathscr{X} \to \mathscr{X}$.

Weil complexes. A *Weil complex* K_0 *on* $\mathscr{X}_0$ is a pair (K, ϕ), where $K \in D_c(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ and $\phi : \sigma^*_{\mathscr{X}_0} K \to K$ is an isomorphism. We also call K_0 a *Weil sheaf* if K is a sheaf. Let $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be the category of Weil complexes on $\mathscr{X}_0$; it is a triangulated category with the standard t-structure, and its core is the category of Weil sheaves.

In this article, when discussing stacks over $\mathbb{F}_q$, by a "sheaf" or "complex of sheaves", we usually mean a "Weil sheaf" or "Weil complex", whereas a "lisse-étale sheaf or complex" will be an ordinary constructible $\overline{\mathbb{Q}}_\ell$-sheaf or complex on the lisse-étale site of $\mathscr{X}_0$.

For $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, it is a fixed point of $\sigma^v_{\mathscr{X}_0}$, hence there is a "local Frobenius automorphism" $F_x : K_{\overline{x}} \to K_{\overline{x}}$ for every Weil complex K_0.

ι-Weights and ι-mixedness. Recall that ι is a fixed isomorphism $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$. For $\alpha \in \overline{\mathbb{Q}}_\ell^*$, let $w_q(\alpha) := 2\log_q |\iota\alpha|$, called the ι-weight of α relative to q. For a real number β, a sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ is said to be punctually ι-pure of weight β, if for every integer $v \geq 1$ and every $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, and every eigenvalue α of F_x acting on $\mathscr{F}_{\overline{x}}$, we have $w_{N(x)}(\alpha) = \beta$. We say $\mathscr{F}_0$ is ι-mixed if it has a finite filtration with successive quotients punctually ι-pure, and the weights of these quotients are called the punctual ι-weights of $\mathscr{F}_0$. A complex $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be ι-mixed if all the cohomology sheaves $\mathscr{H}^i K_0$ are ι-mixed.

Torsion. For $b \in \overline{\mathbb{Q}}_\ell^*$, let $\overline{\mathbb{Q}}_\ell^{(b)}$ be the Weil sheaf on Spec $\mathbb{F}_q$ of rank one, where F acts by multiplication by b. This is an étale sheaf if and only if b is an ℓ-adic unit ([9], 1.2.7). For an algebraic stack $\mathscr{X}_0/\mathbb{F}_q$, we also denote by $\overline{\mathbb{Q}}_\ell^{(b)}$ the inverse image on $\mathscr{X}_0$ of the above Weil sheaf via the structural map. If $\mathscr{F}_0$ is a sheaf on $\mathscr{X}_0$, we denote by $\mathscr{F}_0^{(b)}$ the tensor product $\mathscr{F}_0 \otimes \overline{\mathbb{Q}}_\ell^{(b)}$, and say that $\mathscr{F}_0^{(b)}$ is deduced from $\mathscr{F}_0$ by torsion, or a generalized Tate twist. Note that the operation $\mathscr{F}_0 \mapsto \mathscr{F}_0^{(b)}$ adds the weights by $w_q(b)$. For an integer d, the usual Tate twist $\overline{\mathbb{Q}}_\ell(d)$ is $\overline{\mathbb{Q}}_\ell^{(q^{-d})}$. We denote by $\langle d \rangle$ the operation $(d)[2d]$ on complexes of sheaves, where $[2d]$ means shifting $2d$ to the left. Note that ι-mixedness is stable under the operation $\langle d \rangle$.

Lemma 2.2.2. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack.*

(i) If $\mathscr{F}_0$ is an ι-mixed sheaf on $\mathscr{X}_0$, then so is every sub-quotient of $\mathscr{F}_0$.

(ii) If $0 \to \mathscr{F}_0' \to \mathscr{F}_0 \to \mathscr{F}_0'' \to 0$ is an exact sequence of sheaves on $\mathscr{X}_0$, and $\mathscr{F}_0'$ and $\mathscr{F}_0''$ are ι-mixed, then so is $\mathscr{F}_0$.

(iii) The full subcategory of ι-mixed complexes in $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is a triangulated subcategory with induced t-structure, denoted by $W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. Similarly, let $D_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be the full subcategory of ι-mixed lisse-étale complex in $D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$.

(iv) If f is a morphism of $\mathbb{F}_q$-algebraic stacks, then f^* on complexes of sheaves preserves ι-mixedness.

(v) If $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ is an open immersion and $i : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$ is its complement, then $K_0 \in D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is ι-mixed if and only if j^*K_0 and i^*K_0 are ι-mixed.

Proof. (i) If $\mathscr{F}_0$ is punctually ι-pure of weight β, then so is every sub-quotient of it. Now suppose $\mathscr{F}_0$ is ι-mixed and $\mathscr{F}_0'$ is a subsheaf of $\mathscr{F}_0$. Let W be a finite filtration

$$0 \subset \cdots \subset \mathscr{F}_0^{i-1} \subset \mathscr{F}_0^i \subset \cdots \subset \mathscr{F}_0$$

of $\mathscr{F}_0$ such that $\mathrm{Gr}_i^W(\mathscr{F}_0) := \mathscr{F}_0^i/\mathscr{F}_0^{i-1}$ is punctually ι-pure for every i. Let W' be the induced filtration $W \cap \mathscr{F}_0'$ of $\mathscr{F}_0'$. Then $\mathrm{Gr}_i^{W'}(\mathscr{F}_0')$ is the image of

$$\mathscr{F}_0^i \cap \mathscr{F}_0' \subset \mathscr{F}_0^i \twoheadrightarrow \mathrm{Gr}_i^W(\mathscr{F}_0),$$

so it is punctually ι-pure. Let $\mathscr{F}_0''=\mathscr{F}_0/\mathscr{F}_0'$ be a quotient of $\mathscr{F}_0$, and let W'' be the induced filtration of $\mathscr{F}_0''$, namely $(\mathscr{F}_0'')^i:=\mathscr{F}_0^i/(\mathscr{F}_0^i\cap\mathscr{F}_0')$. Then $\mathrm{Gr}_i^{W''}(\mathscr{F}_0'')=\mathscr{F}_0^i/(\mathscr{F}_0^{i-1}+\mathscr{F}_0^i\cap\mathscr{F}_0')$, which is a quotient of $\mathscr{F}_0^i/\mathscr{F}_0^{i-1}=\mathrm{Gr}_i^W(\mathscr{F}_0)$, so it is punctually ι-pure. Hence every subquotient of $\mathscr{F}_0$ is ι-mixed.

(ii) Let W' and W'' be finite filtrations of $\mathscr{F}_0'$ and $\mathscr{F}_0''$ respectively, such that $\mathrm{Gr}_i^{W'}(\mathscr{F}_0')$ and $\mathrm{Gr}_i^{W''}(\mathscr{F}_0'')$ are punctually ι-pure for every i. Then W' can be regarded as a finite filtration of $\mathscr{F}_0$ such that every member of the filtration is contained in $\mathscr{F}_0'$, and W'' can be regarded as a finite filtration of $\mathscr{F}_0$ such that every member contains $\mathscr{F}_0'$. Putting these two filtrations together, we get the desired filtration for $\mathscr{F}_0$.

(iii) Being a triangulated subcategory means ([11], p.271) that, if $K_0'\to K_0\to K_0''\to K_0'[1]$ is an exact triangle in $W(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$, and two of the three complexes are ι-mixed, then so is the third. By the rotation axiom of a triangulated category, we can assume K_0' and K_0'' are ι-mixed. We have the exact sequence

$$\cdots\longrightarrow\mathscr{H}^nK_0'\longrightarrow\mathscr{H}^nK_0\longrightarrow\mathscr{H}^nK_0''\longrightarrow\cdots,$$

and by (i) and (ii) we see that $\mathscr{H}^nK_0$ is ι-mixed.

$W_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ has the induced t-structure because if K_0 is ι-mixed, then its truncations $\tau_{\le n}K_0$ and $\tau_{\ge n}K_0$ are ι-mixed.

(iv) On sheaves, f^* preserves stalks, so it is exact and preserves punctual ι-purity on sheaves. Let $f:\mathscr{X}_0\to\mathscr{Y}_0$. Given an ι-mixed sheaf $\mathscr{F}_0$ on $\mathscr{Y}_0$, let W be a finite filtration of $\mathscr{F}_0$ such that each $\mathrm{Gr}_i^W(\mathscr{F}_0)$ is punctually ι-pure. Then f^*W gives a finite filtration of $f^*\mathscr{F}_0$ and each $\mathrm{Gr}_i^{f^*W}(f^*\mathscr{F}_0)=f^*\mathrm{Gr}_i^W(\mathscr{F}_0)$ is punctually ι-pure. So $f^*\mathscr{F}_0$ is ι-mixed. For an ι-mixed complex K_0 on $\mathscr{Y}_0$, note that $\mathscr{H}^n(f^*K_0)=f^*\mathscr{H}^n(K_0)$, hence f^*K_0 is ι-mixed.

(v) One direction follows from (iv). For the other direction, note that $j_!$ and i_* are exact and preserve punctually ι-purity on sheaves. If $\mathscr{F}_0$ is an ι-mixed sheaf on $\mathscr{U}_0$, with a finite filtration W such that each $\mathrm{Gr}_i^W(\mathscr{F}_0)$ is punctually ι-pure, then for the induced filtration $j_!W$ of $j_!\mathscr{F}_0$, we see that $\mathrm{Gr}_i^{j_!W}(j_!\mathscr{F}_0)=j_!\mathrm{Gr}_i^W(\mathscr{F}_0)$ is punctually ι-pure, so $j_!\mathscr{F}_0$ is ι-mixed. For an ι-mixed complex K_0 on $\mathscr{U}_0$, use $\mathscr{H}^n(j_!K_0)=j_!\mathscr{H}^n(K_0)$. Similarly i_* also preserves ι-mixedness on complexes. Finally the result follows from (iii) and the exact triangle

$$j_!j^*K_0\longrightarrow K_0\longrightarrow i_*i^*K_0\longrightarrow\cdot$$

□

To show that ι-mixedness is stable under the six operations, we need to show that ι-mixedness of complexes on stacks can be checked locally on their presentations. To descend a filtration on a presentation to the stack, we generalize the structure theorem of ι-mixed sheaves to algebraic spaces. Recall the following theorem of Deligne ([9], 3.4.1).

Theorem 2.2.3. *Let $\mathscr{F}_0$ be an ι-mixed sheaf on a scheme X_0 over $\mathbb{F}_q$.*

(i) $\mathscr{F}_0$ has a unique decomposition $\mathscr{F}_0=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0(b)$, called the decomposition according to the weights mod $\mathbb{Z}$, *such that the punctual ι-weights of $\mathscr{F}_0(b)$ are all in the coset b. This decomposition, in which almost all the $\mathscr{F}_0(b)$ are zero, is functorial in $\mathscr{F}_0$. Note that each $\mathscr{F}_0(b)$ is deduced by torsion from an ι-mixed sheaf with integer punctual weights.*

(ii) If the punctual weights of $\mathscr{F}_0$ are integers and $\mathscr{F}_0$ is lisse, $\mathscr{F}_0$ has a unique finite increasing filtration W by lisse subsheaves, called the filtration by punctual weights, *such that* $\mathrm{Gr}_i^W(\mathscr{F}_0)$ *is punctually ι-pure of weight i. This filtration is functorial in $\mathscr{F}_0$. More precisely, any morphism between ι-mixed lisse sheaves of integer punctual weights is strictly compatible with their filtrations by punctual weights.*

(iii) If $\mathscr{F}_0$ is lisse and punctually ι-pure, and X_0 is normal, then the sheaf $\mathscr{F}$ on X is semi-simple.

Remark 2.2.3.1. (i) If $\mathscr{C}$ is an abelian category and $\mathscr{D}$ is an abelian full subcategory of $\mathscr{C}$, and C is an object in $\mathscr{D}$, then every direct summand of C in $\mathscr{C}$ lies in $\mathscr{D}$ (or isomorphic to some object in $\mathscr{D}$). This is because the kernel of the composition

$$A\oplus B\xrightarrow{pr_A}A\xhookrightarrow{i_A}A\oplus B$$

is B. So direct summands of a lisse sheaf are lisse. If $\mathscr{F}_0$ in the theorem above is lisse and ι-mixed, then each $\mathscr{F}_0(b)$ is lisse.

(ii) If the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}_0$ is defined over some finite subextension E_λ of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$, then its decomposition in (2.2.3i) and filtration in (2.2.3ii) are defined over E_λ. This is because the E_λ-action commutes with the Galois action.

(iii) In [9] Deligne made the assumption that all schemes are separated, at least in order to use Nagata compactification to define $f_!$. After the work of Laszlo and Olsson [26, 27], one can remove this assumption, and many results in [9], for instance this one and (3.3.1), remain valid. For (3.4.1) one can take a cover of a not necessarily separated scheme X_0 by open affines (which are separated), and use the functoriality to glue the decomposition or filtration on intersections.

Lemma 2.2.4. *Let $X_0/\mathbb{F}_q$ be an algebraic space, and $\mathscr{F}_0$ an ι-mixed sheaf on X_0.*

(i) $\mathscr{F}_0$ has a unique decomposition $\mathscr{F}_0=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0(b)$, the decomposition according to the weights mod $\mathbb{Z}$, *with the same property as in (2.2.3i). This decomposition is functorial in $\mathscr{F}_0$.*

(ii) If the punctual ι-weights of $\mathscr{F}_0$ are integers and $\mathscr{F}_0$ is lisse, $\mathscr{F}_0$ has a unique finite increasing filtration W by lisse subsheaves, called the filtration by punctual weights, *with the same property as in (2.2.3ii). This filtration is functorial in $\mathscr{F}_0$.*

Proof. Let $P:X_0'\to X_0$ be an étale presentation, and let $\mathscr{F}_0'=P^*\mathscr{F}_0$, which is also ι-mixed

(2.2.2iv). Let X_0'' be the fiber product

$$\begin{tikzcd} X_0''= X_0'\times_{X_0}X_0' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_0' \arrow[d, "P"] \\ X_0' \arrow[r, "P"'] & X_0. \end{tikzcd}$$

Then X_0'' is an $\mathbb{F}_q$-scheme of finite type.

(i) Applying (2.2.3i) to $\mathscr{F}_0'$ we get a decomposition $\mathscr{F}_0'=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0'(b)$. For $j=1,2$, applying p_j^* we get a decomposition

$$p_j^*\mathscr{F}_0'=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}p_j^*\mathscr{F}_0'(b).$$

Since p_j^* preserves weights, by the uniqueness in (2.2.3i), this decomposition is the decomposition of $p_j^*\mathscr{F}_0'$ according to the weights mod $\mathbb{Z}$. By the functoriality in (2.2.3i), the canonical isomorphism $\mu:p_1^*\mathscr{F}_0'\to p_2^*\mathscr{F}_0'$ takes the form $\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mu_b$, where $\mu_b:p_1^*\mathscr{F}_0'(b)\to p_2^*\mathscr{F}_0'(b)$ is an isomorphism satisfying cocycle condition as μ does. Therefore the decomposition $\mathscr{F}_0'=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0'(b)$ descends to a decomposition $\mathscr{F}_0=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0(b)$. We still need to show each direct summand $\mathscr{F}_0(b)$ is ι-mixed.

Fix a coset b and consider the summand $\mathscr{F}_0(b)$. Twisting it by a certain torsion, we can assume that its inverse image $\mathscr{F}_0'(b)$ is ι-mixed with integer punctual ι-weights. By (2.2.2v) and noetherian induction, we can shrink X_0 to a nonempty open subspace and assume $\mathscr{F}_0(b)$ is lisse. Then $\mathscr{F}_0'(b)$ is also lisse, and applying (2.2.3ii) we get a finite increasing filtration W' of $\mathscr{F}_0'(b)$ by lisse subsheaves $\mathscr{F}_0'(b)^i$, such that each $\mathrm{Gr}_i^{W'}(\mathscr{F}_0'(b))$ is punctually ι-pure of weight i. Pulling back this filtration via p_j, we get a finite increasing filtration p_j^*W' of $p_j^*\mathscr{F}_0'(b)$, and since $\mathrm{Gr}_i^{p_j^*W'}(p_j^*\mathscr{F}_0'(b))=p_j^*\mathrm{Gr}_i^{W'}(\mathscr{F}_0'(b))$ is punctually ι-pure of weight i, it is the filtration by punctual weights given by (2.2.3ii), hence functorial. So the canonical isomorphism $\mu_b:p_1^*\mathscr{F}_0'(b)\to p_2^*\mathscr{F}_0'(b)$ maps $p_1^*\mathscr{F}_0'(b)^i$ isomorphically onto $p_2^*\mathscr{F}_0'(b)^i$, satisfying cocycle condition. Therefore the filtration W' of $\mathscr{F}_0'(b)$ descends to a filtration W of $\mathscr{F}_0(b)$, and $P^*\mathrm{Gr}_i^W(\mathscr{F}_0(b))=\mathrm{Gr}_i^{W'}(\mathscr{F}_0'(b))$ is punctually ι-pure of weight i. Note that P is surjective, so every point $x\in X_0(\mathbb{F}_{q^v})$ can be lifted to a point $x'\in X_0'(\mathbb{F}_{q^{nv}})$ after some base extension $\mathbb{F}_{q^{nv}}$ of $\mathbb{F}_{q^v}$. This shows $\mathrm{Gr}_i^W(\mathscr{F}_0(b))$ is punctually ι-pure of weight i, therefore $\mathscr{F}_0(b)$ is ι-mixed. This proves the existence of the decomposition in (i).

For uniqueness, let $\mathscr{F}_0=\bigoplus\widetilde{\mathscr{F}_0}(b)$ be another decomposition with the desired property. Then their restrictions to X_0' are both equal to the decomposition of $\mathscr{F}_0'$, which is unique (2.2.3i), so they are both obtained by descending this decomposition, and so they are isomorphic, i.e. for every coset b there exists an isomorphism making the diagram commute:

$$\begin{tikzcd} \mathscr{F}_0(b) \arrow[rr, "\sim"] \arrow[dr, hook] & & \widetilde{\mathscr{F}_0}(b) \arrow[dl, hook'] \\ & \mathscr{F}_0. & \end{tikzcd}$$

For functoriality, let $\mathscr{G}_0 = \bigoplus \mathscr{G}_0(b)$ be another ι-mixed sheaf with decomposition on X_0, and let $\varphi : \mathscr{F}_0 \to \mathscr{G}_0$ be a morphism of sheaves. Pulling φ back via P we get a morphism $\varphi' : \mathscr{F}'_0 \to \mathscr{G}'_0$ on X'_0, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}'_0 \arrow[r, "\mu_{\mathscr{F}_0}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}'_0 \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}'_0 \arrow[r, "\mu_{\mathscr{G}_0}"'] & p_2^*\mathscr{G}'_0 \end{tikzcd}$$

commutes. By (2.2.3i) $\varphi' = \bigoplus \varphi'(b)$ for morphisms $\varphi'(b) : \mathscr{F}'_0(b) \to \mathscr{G}'_0(b)$, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}'_0(b) \arrow[r, "\mathrm{can}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}'_0(b) \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}'_0(b) \arrow[r, "\mathrm{can}"'] & p_2^*\mathscr{G}'_0(b) \end{tikzcd}$$

commutes for each b. Then the morphisms $\varphi'(b)$ descend to morphisms $\varphi(b) : \mathscr{F}_0(b) \to \mathscr{G}_0(b)$ such that $\varphi = \bigoplus \varphi(b)$.

(ii) The proof is similar to part (i). Applying (2.2.3ii) to $\mathscr{F}'_0$ on X'_0 we get a finite increasing filtration W' of $\mathscr{F}'_0$ by lisse subsheaves $\mathscr{F}'^i_0$ with desired property. Pulling back this filtration via $p_j : X''_0 \to X'_0$ we get the filtration by punctual weights of $p_j^*\mathscr{F}'_0$. By functoriality in (2.2.3ii), the canonical isomorphism $\mu : p_1^*\mathscr{F}'_0 \to p_2^*\mathscr{F}'_0$ maps $p_1^*\mathscr{F}'^i_0$ isomorphically onto $p_2^*\mathscr{F}'^i_0$ satisfying cocycle condition, therefore the filtration W' descends to a finite increasing filtration W of $\mathscr{F}_0$ by certain subsheaves $\mathscr{F}^i_0$. By ([35], 9.1) they are lisse subsheaves.

For uniqueness, if $\widetilde{W}$ is another filtration of $\mathscr{F}_0$ by certain subsheaves $\widetilde{\mathscr{F}^i_0}$ with desired property, then their restrictions to X'_0 are both equal to the filtration W' by punctual weights, which is unique (2.2.3ii), so they are both obtained by descending this filtration W', and therefore they are isomorphic.

For functoriality, let $\mathscr{G}_0$ be another lisse ι-mixed sheaf with integer punctual ι-weights, and let V be its filtration by punctual weights, and let $\varphi : \mathscr{F}_0 \to \mathscr{G}_0$ be a morphism. Pulling

φ back via P we get a morphism $\varphi' : \mathscr{F}'_0 \to \mathscr{G}'_0$ on X'_0, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}'_0 \arrow[r, "\mu_{\mathscr{F}_0}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}'_0 \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}'_0 \arrow[r, "\mu_{\mathscr{G}_0}"'] & p_2^*\mathscr{G}'_0 \end{tikzcd}$$

commutes. By (2.2.3ii) we have $\varphi'(\mathscr{F}'^i_0) \subset \mathscr{G}'^i_0$, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}'^i_0 \arrow[r, "\mu_{\mathscr{F}_0}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}'^i_0 \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}'^i_0 \arrow[r, "\mu_{\mathscr{G}_0}"'] & p_2^*\mathscr{G}'^i_0 \end{tikzcd}$$

commutes for each i. Let $\varphi'^i : \mathscr{F}'^i_0 \to \mathscr{G}'^i_0$ be the restriction of φ'. Then they descend to morphisms $\varphi^i : \mathscr{F}^i_0 \to \mathscr{G}^i_0$, which are restrictions of φ. □

Remark 2.2.4.1. One can prove a similar structure theorem of ι-mixed sheaves on algebraic stacks over $\mathbb{F}_q$: the proof of (2.2.4) carries over verbatim to the case of algebraic stacks, except that for a presentation $X'_0 \to \mathscr{X}_0$, the fiber product $X''_0 = X'_0 \times_{\mathscr{X}_0} X'_0$ may not be a scheme, so we use the case for algebraic spaces and replace every "(2.2.3)" in the proof by "(2.2.4)". It turns out that (2.2.3iii) also holds for algebraic stacks, as a consequence of the proof of (1.0.4). We will give the proof later (see (7.2.1)).

Proposition 2.2.5. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack, and let $P : X_0 \to \mathscr{X}_0$ be a presentation (i.e. a smooth surjection with X_0 a scheme). Then a complex $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is ι-mixed if and only if P^*K_0 (resp. $P^!K_0$) is ι-mixed.*

Proof. We consider P^*K_0 first. The "only if" part follows from (2.2.2iv). For the "if" part, since P^* is exact on sheaves and so $\mathscr{H}^i(P^*K_0) = P^*\mathscr{H}^i(K_0)$, we reduce to the case when $K_0 = \mathscr{F}_0$ is a sheaf. So we assume the sheaf $\mathscr{F}'_0 := P^*\mathscr{F}_0$ on X_0 is ι-mixed, and want to show $\mathscr{F}_0$ is also ι-mixed. The proof is similar to the argument in (2.2.4).

Let X''_0 be the fiber product

$$\begin{tikzcd} X''_0 = X_0 \times_{\mathscr{X}_0} X_0 \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_0 \arrow[d, "P"] \\ X_0 \arrow[r, "P"'] & \mathscr{X}_0. \end{tikzcd}$$

Then X''_0 is an algebraic space of finite type. Applying (2.2.3i) to $\mathscr{F}'_0$ we get a decomposition

$\mathscr{F}'_0=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}'_0(b)$. For $j=1,2$, applying p_j^* we get a decomposition

$$p_j^*\mathscr{F}'_0=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}p_j^*\mathscr{F}'_0(b),$$

which is the decomposition of $p_j^*\mathscr{F}'_0$ according to the weights mod $\mathbb{Z}$. By the functoriality in (2.2.4i), the canonical isomorphism $\mu:p_1^*\mathscr{F}'_0\to p_2^*\mathscr{F}'_0$ takes the form $\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mu_b$, where $\mu_b:p_1^*\mathscr{F}'_0(b)\to p_2^*\mathscr{F}'_0(b)$ is an isomorphism satisfying cocycle condition as μ does. Therefore the decomposition of $\mathscr{F}'_0$ descends to a decomposition $\mathscr{F}_0=\bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0(b)$. The ι-weights of the local Frobenius eigenvalues of $\mathscr{F}_0(b)$ at each point of $\mathscr{X}_0$ are in the coset b. Next we show that $\mathscr{F}_0(b)$'s are ι-mixed.

Replacing $\mathscr{F}_0$ by a direct summand $\mathscr{F}_0(b)$ and then twisting by torsion, we may assume its inverse image $\mathscr{F}'_0$ is ι-mixed with integer punctual ι-weights. By (2.2.2v) we can shrink $\mathscr{X}_0$ to a nonempty open substack and assume $\mathscr{F}_0$ is lisse. Then $\mathscr{F}'_0$ is also lisse, and applying (2.2.3ii) we get a finite increasing filtration W' of $\mathscr{F}'_0$ by lisse subsheaves $\mathscr{F}'^i_0$, such that each $\mathrm{Gr}_i^{W'}(\mathscr{F}'_0)$ is punctually ι-pure of weight i. Pulling back this filtration via p_j, we get a finite increasing filtration p_j^*W' of $p_j^*\mathscr{F}'_0$, and since $\mathrm{Gr}_i^{p_j^*W'}(p_j^*\mathscr{F}'_0)=p_j^*\mathrm{Gr}_i^{W'}(\mathscr{F}'_0)$ is punctually ι-pure of weight i, it is the filtration by punctual weights given by (2.2.4ii). By functoriality, the canonical isomorphism $\mu:p_1^*\mathscr{F}'_0\to p_2^*\mathscr{F}'_0$ maps $p_1^*\mathscr{F}'^i_0$ isomorphically onto $p_2^*\mathscr{F}'^i_0$, satisfying cocycle condition. Therefore the filtration W' of $\mathscr{F}'_0$ descends to a filtration W of $\mathscr{F}_0$, and $P^*\mathrm{Gr}_i^W(\mathscr{F}_0)=\mathrm{Gr}_i^{W'}(\mathscr{F}'_0)$ is punctually ι-pure of weight i. Since π is surjective, $\mathrm{Gr}_i^W(\mathscr{F}_0)$ is also punctually ι-pure of weight i, therefore $\mathscr{F}_0$ is ι-mixed.

Next we consider $P^!K_0$. We know that P is smooth of relative dimension d, for some function $d:\pi_0(X_0)\to\mathbb{N}$. Let X_0^0 be a connected component of X_0. Since $\pi_0(X_0)$ is finite, X_0^0 is both open and closed in X_0, so $f:X_0^0\xrightarrow{j}X_0\xrightarrow{P}\mathscr{X}_0$ is smooth of relative dimension $d(X_0^0)$. Then P^*K_0 is ι-mixed if and only if $f^*K_0=j^*P^*K_0$ is ι-mixed for the inclusion j of every connected component, if and only if $f^!K_0=f^*K_0\langle d(X_0^0)\rangle$ is ι-mixed, if and only if $P^!K_0$ is ι-mixed, since $f^!=j^!P^!=j^*P^!$. □

Remark 2.2.5.1. As a consequence of his results on the Langlands correspondence for function fields and a Ramanujan-Petersson type of result, Lafforgue was able to prove the conjecture of Deligne ([9], 1.2.10), and in particular ([9], 1.2.9) that all sheaves on any separated $\mathbb{F}_p$-scheme are ι-mixed, for any ι. See ([24], 1.3). By taking an open affine cover, one can generalize this to all $\mathbb{F}_p$-schemes and, using (2.2.5), to all $\mathbb{F}_p$-algebraic stacks. In the following, when we want to emphasize the assumption of ι-mixedness, we will still write "$W_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$", although it equals the full category $W(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$.

Next we show the stability of ι-mixedness, first for a few operations on complexes on algebraic spaces, and then for all the six operations on stacks. Denote by $D_{\mathscr{X}_0}$ or just D the dualizing functor $R\mathscr{H}om(-,K_{\mathscr{X}_0})$, where $K_{\mathscr{X}_0}$ is a dualizing complex on $\mathscr{X}_0$ ([27], §7).

2.2.6. Recall ([22], II 12.2) that, for schemes over $\mathbb{F}_q$ and bounded constructible sheaf complexes on them, the operations $f_*, f_!, f^*, f^!, D$ and $-\otimes^L-$ all preserve ι-mixedness.

Lemma 2.2.7. *Let $f: X_0 \to Y_0$ be a morphism of $\mathbb{F}_q$-algebraic spaces. Then the operations $-\otimes^L-, D_{X_0}, f_*$ and $f_!$ all preserve ι-mixedness, namely, they induce functors*

$$-\otimes^L- : W_m^-(X_0,\overline{\mathbb{Q}}_\ell)\times W_m^-(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m^-(X_0,\overline{\mathbb{Q}}_\ell),$$
$$D : W_m(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m(X_0,\overline{\mathbb{Q}}_\ell)^{\mathrm{op}},$$
$$f_* : W_m^+(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m^+(Y_0,\overline{\mathbb{Q}}_\ell)\quad\text{and}\quad f_! : W_m^-(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m^-(Y_0,\overline{\mathbb{Q}}_\ell).$$

Proof. We will reduce to the case of unbounded complexes on schemes, and then prove the scheme case. Let $P: X_0' \to X_0$ be an étale presentation.

Reduction for $\otimes^L$. For $K_0, L_0 \in W_m^-(X_0,\overline{\mathbb{Q}}_\ell)$, we have $P^*(K_0\otimes^L L_0) = (P^*K_0)\otimes^L (P^*L_0)$, and the reduction follows from (2.2.5).

Reduction for D. For $K_0 \in W_m(X_0,\overline{\mathbb{Q}}_\ell)$, we have $P^*DK_0 = DP^!K_0$, so the reduction follows from (2.2.5).

Reduction for f_* and $f_!$. By definition ([27], 9.1) we have $f_* = Df_!D$, so it suffices to prove the case for $f_!$. Let $K_0 \in W_m^-(X_0,\overline{\mathbb{Q}}_\ell)$, and let $P': Y_0' \to Y_0$ and $X_0' \to X_0 \times_{Y_0} Y_0'$ be étale presentations:

$$\begin{tikzcd}
X_0' \arrow[r] \arrow[rr, bend left, "g"] \arrow[dr, "P"'] & (X_0)_{Y_0'} \arrow[r, "f'"'] \arrow[d, "h"] & Y_0' \arrow[d, "P'"] \\
& X_0 \arrow[r, "f"] & Y_0.
\end{tikzcd}$$

By smooth base change ([27], 12.1) we have $P'^*f_!K_0 = f'_!h^*K_0$. Replacing f by f' we can assume Y_0 is a scheme. Let $j: U_0 \to X_0$ be an open dense subscheme ([21], II 6.7), with complement $i: Z_0 \to X_0$. Applying $f_!$ to the exact triangle

$$j_!j^*K_0 \longrightarrow K_0 \longrightarrow i_*i^*K_0 \longrightarrow$$

we get

$$(fj)_!j^*K_0 \longrightarrow f_!K_0 \longrightarrow (fi)_!i^*K_0 \longrightarrow \cdot$$

By (2.2.2iii) and noetherian induction, we can replace X_0 by U_0, and reduce to the case where f is a morphism between schemes.

This finishes the reduction to the case of unbounded complexes on schemes, and now we prove this case.

For the Verdier dual D_{X_0}, since the dualizing complex K_{X_0} has finite quasi-injective dimension, for every $K_0 \in W_m(X_0,\overline{\mathbb{Q}}_\ell)$ and every integer i, there exist integers a and b such that

$$\mathscr{H}^i(D_{X_0}K_0) \simeq \mathscr{H}^i(D_{X_0}\tau_{[a,b]}K_0),$$

and by (2.2.6), we see that $D_{X_0}K_0$ is ι-mixed.

Next we prove the case of $\otimes^L$. Let K_0 and $L_0 \in W_m^-(X_0, \overline{\mathbb{Q}}_\ell)$. Consider the spectral sequence

$$E_2^{rk} = \bigoplus_{i+j=k} \mathscr{H}^r(\mathscr{H}^i(K_0) \otimes \mathscr{H}^j(L_0)) \Longrightarrow \mathscr{H}^{r+k}(K_0 \otimes^L L_0).$$

The result follows from (2.2.2i, ii) and (2.2.6).

Finally we prove the case of f_* and $f_!$. Let $K_0 \in W_m^+(X_0, \overline{\mathbb{Q}}_\ell)$. Then we have the spectral sequence

$$E_1^{ij} = R^j f_*(\mathscr{H}^i K_0) \Longrightarrow R^{i+j} f_* K_0,$$

and the result follows from (2.2.6) and (2.2.2i, ii). The case for $f_! = Df_*D$ also follows. $\square$

Finally we prove the main result of this chapter. This generalizes ([3], 6.3.7).

Theorem 2.2.8. *Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$ be a morphism of $\mathbb{F}_q$-algebraic stacks. Then the operations $f_*, f_!, f^*, f^!, D_{\mathscr{X}_0}, - \otimes^L -$ and $R\mathscr{H}om(-,-)$ all preserve ι-mixedness, namely, they induce functors*

$$f_* : W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^+(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell), \qquad f_! : W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^-(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell),$$
$$f^* : W_m(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell), \qquad f^! : W_m(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell),$$
$$D : W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}},$$
$$\otimes^L : W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \times W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \quad \text{and}$$
$$R\mathscr{H}om(-,-) : W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \times W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell).$$

Proof. Recall from ([27], 9.1) that $f_! := Df_*D$ and $f^! := Df^*D$. By ([27], 6.0.12, 7.3.1), for $K_0 \in W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ and $L_0 \in W^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, we have

$$\begin{aligned} D(K_0 \otimes^L DL_0) &= R\mathscr{H}om(K_0 \otimes^L DL_0, K_{\mathscr{X}_0}) = R\mathscr{H}om(K_0, R\mathscr{H}om(DL_0, K_{\mathscr{X}_0})) \\ &= R\mathscr{H}om(K_0, DDL_0) = R\mathscr{H}om(K_0, L_0). \end{aligned}$$

Therefore it suffices to prove the result for f_*, f^*, D and $- \otimes^L -$. The case of f^* is proved in (2.2.2iv).

For D : let $P : X_0 \to \mathscr{X}_0$ be a presentation. Since $P^*D = DP^!$, the result follows from (2.2.5) and (2.2.7).

For $\otimes^L$: since $P^*(K_0 \otimes^L L_0) = P^*K_0 \otimes^L P^*L_0$, the result follows from (2.2.5) and (2.2.7).

For f_* and $f_!$: we will start with $f_!$, in order to use smooth base change to reduce to the case when $\mathscr{Y}_0$ is a scheme, and then turn to f_* in order to use cohomological descent.

Let $K_0 \in W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, and let $P : Y_0 \to \mathscr{Y}_0$ be a presentation and the following diagram be 2-Cartesian:

$$\begin{tikzcd} (\mathscr{X}_0)_{Y_0} \arrow[r, "f'"] \arrow[d, "P'"'] & Y_0 \arrow[d, "P"] \\ \mathscr{X}_0 \arrow[r, "f"] & \mathscr{Y}_0. \end{tikzcd}$$

We have ([27], 12.1) that $P^*f_!K_0 = f'_!P'^*K_0$, so by (2.2.5) we can assume $\mathscr{Y}_0 = Y_0$ is a scheme.

Now we switch to f_*, where $f : \mathscr{X}_0 \to Y_0$, and $K_0 \in W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. Let $X_0 \to \mathscr{X}_0$ be a presentation. Then it gives a smooth hypercover $X_{0,\bullet}$ of $\mathscr{X}_0$:

$$X_{0,n} := \underbrace{X_0 \times_{\mathscr{X}_0} \cdots \times_{\mathscr{X}_0} X_0}_{n+1 \text{ factors}},$$

where each $X_{0,n}$ is an $\mathbb{F}_q$-algebraic space of finite type. Let $f_n : X_{0,n} \to Y_0$ be the restriction of f to $X_{0,n}$. Then we have the spectral sequence ([27], 10.0.9)

$$E_1^{ij} = R^j f_{i*}(K_0|_{X_{0,i}}) \Longrightarrow R^{i+j} f_* K_0.$$

Since f_i's are morphisms of algebraic spaces, the result follows from (2.2.7) and (2.2.2i, ii). $\square$

Chapter 3

Stratifiable complexes.

In this chapter, we use the same notations and hypotheses in (2.1.1). Let $\mathcal{X}, \mathcal{Y} \cdots$ be S-algebraic stacks of finite type. By "sheaves" we mean "lisse-étale sheaves". "Jordan-Hölder" and "locally constant constructible" are abbreviated as "JH" and "lcc" respectively. A *stratification* $\mathscr{S}$ of an S-algebraic stack $\mathcal{X}$ is a finite set of disjoint locally closed substacks that cover $\mathcal{X}$. If $\mathscr{F}$ is a lcc $(\Lambda_n)_{\mathcal{X}}$-module, a *decomposition series* of $\mathscr{F}$ is a filtration by lcc $\Lambda_{\mathcal{X}}$-subsheaves, such that the successive quotients are simple $\Lambda_{\mathcal{X}}$-modules. Note that the filtration is always finite, and the simple successive quotients, which are $(\Lambda_0)_{\mathcal{X}}$-modules, are independent (up to order) of the decomposition series chosen. They are called the *JH components of* $\mathscr{F}$.

3.1 Basic definitions and properties.

Definition 3.1.1. *(i) A complex $K = (K_n)_n \in \mathscr{D}_c(\mathscr{A})$ is said to be* stratifiable, *if there exists a pair $(\mathscr{S}, \mathcal{L})$, where $\mathscr{S}$ is a stratification of $\mathcal{X}$, and $\mathcal{L}$ is a function that assigns to every stratum $\mathcal{U} \in \mathscr{S}$ a finite set $\mathcal{L}(\mathcal{U})$ of isomorphism classes of simple (i.e. irreducible) lcc Λ_0-modules on $\mathcal{U}_{\text{lis-ét}}$, such that for each pair (i, n) of integers, the restriction of the sheaf $\mathscr{H}^i(K_n) \in \mathrm{Mod}_c(\mathcal{X}_{\text{lis-ét}}, \Lambda_n)$ to each stratum $\mathcal{U} \in \mathscr{S}$ is lcc, with JH components (as a $\Lambda_{\mathcal{U}}$-module) contained in $\mathcal{L}(\mathcal{U})$. We say that the pair $(\mathscr{S}, \mathcal{L})$* trivializes *$K$ (or K is $(\mathscr{S}, \mathcal{L})$-*stratifiable*), and denote the full subcategory of $(\mathscr{S}, \mathcal{L})$-stratifiable complexes by $\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$. The full subcategory of stratifiable complexes in $\mathscr{D}_c(\mathscr{A})$ is denoted by $\mathscr{D}_c^{\text{stra}}(\mathscr{A})$.*

(ii) Let $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$ be the essential image of $\mathscr{D}_c^{\text{stra}}(\mathscr{A})$ in $D_c(\mathcal{X}, \Lambda)$, and we call the objects of $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$ stratifiable complexes of sheaves.

(iii) Let E_λ be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathscr{O}_\lambda$. Then the definition above applies to $\Lambda = \mathscr{O}_\lambda$. Let $D_c^{\text{stra}}(\mathcal{X}, E_\lambda)$ be the essential image of $D_c^{\text{stra}}(\mathcal{X}, \mathscr{O}_\lambda)$ in $D_c(\mathcal{X}, E_\lambda)$. Finally we define

$$D_c^{\text{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) = 2\text{-colim}_{E_\lambda} D_c^{\text{stra}}(\mathcal{X}, E_\lambda).$$

Remark 3.1.1.1. This notion is due to Beilinson, Bernstein and Deligne [4], and Behrend [3] used it to define his derived category for stacks. Many results in this section are borrowed from [3], but reformulated and reproved in terms of the derived categories defined in [27].

3.1.2. We say that the pair $(\mathscr{S}', \mathcal{L}')$ *refines the pair* $(\mathscr{S}, \mathcal{L})$, if $\mathscr{S}'$ refines $\mathscr{S}$, and for every $\mathcal{V} \in \mathscr{S}'$, $\mathcal{U} \in \mathscr{S}$ and $L \in \mathcal{L}(\mathcal{U})$, such that $\mathcal{V} \subset \mathcal{U}$, the restriction $L|_{\mathcal{V}}$ is trivialized by $\mathcal{L}'(\mathcal{V})$. Given a pair $(\mathscr{S}, \mathcal{L})$ and a refined stratification $\mathscr{S}'$ of $\mathscr{S}$, there is a canonical way to define $\mathcal{L}'$ such that $(\mathscr{S}', \mathcal{L}')$ refines $(\mathscr{S}, \mathcal{L})$: for every $\mathcal{V} \in \mathscr{S}'$, we take $\mathcal{L}'(\mathcal{V})$ to be the set of isomorphism classes of JH components of the lcc sheaves $L|_{\mathcal{V}}$ for $L \in \mathcal{L}(\mathcal{U})$, where $\mathcal{U}$ ranges over all strata in $\mathscr{S}$ that contains $\mathcal{V}$. It is clear that the set of all pairs $(\mathscr{S}, \mathcal{L})$ form a filtered direct system.

A pair $(\mathscr{S}, \mathcal{L})$ is said to be *tensor closed* if for every $\mathcal{U} \in \mathscr{S}$ and $L, M \in \mathcal{L}(\mathcal{U})$, the sheaf tensor product $L \otimes_{\Lambda_0} M$ has JH components in $\mathcal{L}(\mathcal{U})$.

For a pair $(\mathscr{S}, \mathcal{L})$, a *tensor closed hull* of this pair is a tensor closed refinement.

Lemma 3.1.3. *Every pair $(\mathscr{S}, \mathcal{L})$ can be refined to a tensor closed pair $(\mathscr{S}', \mathcal{L}')$.*

Proof. First we show that, for a lcc sheaf of sets $\mathscr{F}$ on $\mathcal{X}_{\text{lis-ét}}$, there exists a finite étale morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic S-stacks such that $f^{-1}\mathscr{F}$ is constant. Consider the total space $[\mathscr{F}]$ of the sheaf $\mathscr{F}$. Precisely, this is the category fibered in groupoids over (Aff/S) with the underlying category described as follows. Its objects are triples $(U \in \mathrm{obj}(\mathrm{Aff}/S), u \in \mathrm{obj}\ \mathcal{X}(U), s \in (u^{-1}\mathscr{F})(U))$, and morphisms from (U, u, s) to (V, v, t) are pairs $(f : U \to V, \alpha : vf \Rightarrow u)$ such that t is mapped to s under the identification $\alpha : f^{-1}v^{-1}\mathscr{F} \cong u^{-1}\mathscr{F}$. The map $(U, u, s) \mapsto (U, u)$ gives a map $g : [\mathscr{F}] \to \mathcal{X}$, which is representable finite étale (because it is so locally). The pullback sheaf $g^{-1}\mathscr{F}$ on $[\mathscr{F}]$ has a global section, so the total space breaks up into two parts, one part being mapped isomorphically onto the base $[\mathscr{F}]$. By induction on the degree of g we are done.

Next we show that, for a fixed representable finite étale morphism $\mathcal{Y} \to \mathcal{X}$, there are only finitely many isomorphism classes of simple lcc Λ_0-sheaves on $\mathcal{X}$ that become constant when pulled back to $\mathcal{Y}$. We can assume that both $\mathcal{X}$ and $\mathcal{Y}$ are connected. By the following lemma (3.1.3.1), we reduce to the case where $\mathcal{Y} \to \mathcal{X}$ is Galois with group G, for some finite group G. Then simple lcc Λ_0-sheaves on $\mathcal{X}$ that become constant on $\mathcal{Y}$ correspond to simple left $\Lambda_0[G]$-modules, which are cyclic and hence isomorphic to $\Lambda_0[G]/I$ for left maximal ideals I of $\Lambda_0[G]$. There are only finitely many such ideals since $\Lambda_0[G]$ is a finite set.

Also note that, a lcc subsheaf of a constant constructible sheaf on a connected stack is also constant. Let L be a lcc subsheaf on $\mathcal{X}$ of the constant sheaf associated to a finite set M. Consider their total spaces. We have an inclusion of substacks $i : [L] \hookrightarrow \coprod_{m \in M} \mathcal{X}_m$, where each part $\mathcal{X}_m$ is identified with $\mathcal{X}$. Then $i^{-1}(\mathcal{X}_m) \to \mathcal{X}_m$ is finite étale, and is the inclusion of a substack, hence is either an equivalence or the inclusion of the empty substack, since $\mathcal{X}$ is connected. It is clear that L is also constant, associated to the subset of those $m \in M$ for which $i^{-1}(\mathcal{X}_m) \neq \emptyset$.

Finally we prove the lemma. Refining $\mathscr{S}$ if necessary, we assume all strata are connected stacks. For each stratum $\mathcal{U} \in \mathscr{S}$, let $\mathcal{Y} \to \mathcal{U}$ be a representable finite étale morphism, such that all sheaves in $\mathcal{L}(\mathcal{U})$ become constant on $\mathcal{Y}$. Then define $\mathcal{L}'(\mathcal{U})$ to be the set of isomorphism classes of simple lcc Λ_0-sheaves on $\mathcal{U}_{\text{lis-ét}}$ which become constant on $\mathcal{Y}$. For any L and $M \in \mathcal{L}'(\mathcal{U})$, since all lcc subsheaves of $L \otimes_{\Lambda_0} M$ are constant on $\mathcal{Y}$, we see that $L \otimes_{\Lambda_0} M$ has JH components in $\mathcal{L}'(\mathcal{U})$ and hence $(\mathscr{S}, \mathcal{L}')$ is a tensor closed refinement of $(\mathscr{S}, \mathcal{L})$. $\square$

Lemma 3.1.3.1. *Let $\mathcal{Y} \to \mathcal{X}$ be a representable finite étale morphism between connected S-algebraic stacks. Then there exists a morphism $\mathcal{Z} \to \mathcal{Y}$, such that $\mathcal{Z}$ is* Galois *over $\mathcal{X}$, i.e. it is a G-torsor for some finite group G.*

Proof. Assume $\mathcal{X}$ is non-empty, and take a geometric point $\overline{x} \to \mathcal{X}$. Let $\mathscr{C}$ be the category $\text{FÉt}(\mathcal{X})$ of representable finite étale morphisms to $\mathcal{X}$, and let

$$F : \mathscr{C} \to \text{FSet}$$

be the fiber functor to the category of finite sets, namely $F(\mathcal{Y}) = Hom_{\mathcal{X}}(\overline{x}, \mathcal{Y})$. Note that this Hom, which is a priori a category, is a finite set, since $\mathcal{Y} \to \mathcal{X}$ is representable and finite. Then one can verify that $(\mathscr{C}, F : \mathscr{C} \to \text{FSet})$ satisfies the axioms of Galois formalism in ([15], Exp. V, 4), and use the consequence g) on p. 121 in loc. cit. For the reader's convenience, we follow Olsson's suggestion and explain the proof in loc. cit. briefly. Basically, we will verify certain axioms of (G1) – (G6), and deduce the conclusion as in loc. cit.

First note that $\mathscr{C}$, which is a priori a 2-category, is a 1-category. This is because for any 2-commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "f"] \arrow[dr] & & \mathcal{Z} \arrow[dl] \\ & \mathcal{X} & \end{tikzcd}$$

where $\mathcal{Y}, \mathcal{Z} \in \mathscr{C}$, the morphism f is also representable (and finite étale), so $Hom_{\mathcal{X}}(\mathcal{Y}, \mathcal{Z})$ is discrete. By definition, the functor F preserves fiber-products, and $F(\mathcal{X})$ is a one-point set.

Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism in $\mathscr{C}$, then it is finite étale. So if the degree of f is 1, then f is an isomorphism. This implies that the functor F is *conservative*, i.e. f is an isomorphism if $F(f)$ is. In particular, f is a monomorphism if and only if $F(f)$ is. This is because f is a monomorphism if and only if $p_1 : \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ is an isomorphism, and F preserves fiber-products.

Since $f : \mathcal{Y} \to \mathcal{Z}$ is finite étale, its image stack $\mathcal{Y}' \subset \mathcal{Z}$ is both open and closed, hence $\mathcal{Y}' \to \mathcal{Z}$ is a monomorphism that is an isomorphism onto a direct summand of $\mathcal{Z}$ (i.e. $\mathcal{Z} = \mathcal{Y}' \coprod \mathcal{Y}''$ for some other open and closed substack $\mathcal{Y}'' \subset \mathcal{Z}$). Also, since $\mathcal{Y} \to \mathcal{Y}'$ is epic and finite étale, it is *strictly epic*, i.e. for every $\mathcal{Z} \in \mathscr{C}$, the diagram

$$Hom(\mathcal{Y}', \mathcal{Z}) \to Hom(\mathcal{Y}, \mathcal{Z}) \rightrightarrows Hom(\mathcal{Y} \times_{\mathcal{Y}'} \mathcal{Y}, \mathcal{Z})$$

is an equalizer.

Every object $\mathcal{Y}$ in $\mathscr{C}$ is artinian: for a chain of monomorphisms

$$\cdots \to \mathcal{Y}_n \to \cdots \to \mathcal{Y}_2 \to \mathcal{Y}_1 \to \mathcal{Y},$$

we get a chain of injections

$$\cdots \to F(\mathcal{Y}_n) \to \cdots \to F(\mathcal{Y}_1) \to F(\mathcal{Y}),$$

which is stable since $F(\mathcal{Y})$ is a finite set, and so the first chain is also stable since F is conservative.

Since F is left exact and every object in $\mathscr{C}$ is artinian, by ([18], 3.1) the functor F is *strictly pro-representable*, i.e. there exists a projective system $P = \{P_i; i \in I\}$ of objects in $\mathscr{C}$ indexed by a filtered partially ordered set I, with epic transition morphisms $\varphi_{ij} : P_j \to P_i$ $(i \leq j)$, such that there is a natural isomorphism of functors

$$F \xrightarrow{\sim} Hom(P, -) := \mathrm{colim}_I Hom(P_i, -).$$

Let $\psi_i : P \to P_i$ be the canonical projection in the category $\mathrm{Pro}(\mathscr{C})$ of pro-objects of $\mathscr{C}$. We may assume that every epimorphism $P_j \to \mathcal{Z}$ in $\mathscr{C}$ is isomorphic to $P_j \overset{\varphi_{ij}}{\to} P_i$ for some $i \leq j$. This is because one can add $P_j \to \mathcal{Z}$ into the projective system P without changing the functor it represents. Also one can show that the P_i's are connected (cf. loc. cit.), and morphisms in $\mathscr{C}$ between connected stacks are strictly epic.

Given $\mathcal{Y} \in \mathscr{C}$, now we show that there exists an object $\mathcal{Z} \to \mathcal{X}$ that is Galois and factors through $\mathcal{Y}$. Since $F(\mathcal{Y})$ is a finite set, there exists an index $j \in I$ such that all maps $P \to \mathcal{Y}$ factors through $P \overset{\psi_j}{\to} P_j$. This means that the canonical map

$$P \to \mathcal{Y}^J := \underbrace{\mathcal{Y} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{Y}}_{\#J \text{ factors}}, \quad \text{where } J := F(\mathcal{Y}) = Hom_{\mathrm{Pro}(\mathscr{C})}(P, \mathcal{Y})$$

factors as

$$P \xrightarrow{\psi_j} P_j \xrightarrow{A} \mathcal{Y}^J.$$

Let $P_j \to P_i \overset{B}{\to} \mathcal{Y}^J$ be the factorization of A into a composition of an epimorphism and a monomorphism B. We claim that P_i is Galois over $\mathcal{X}$.

Since $F(P_i)$ is a finite set, there exists an index $k \in I$ such that all maps $P \to P_i$ factors through $P \overset{\psi_k}{\to} P_k$. Fix any $v : P_k \to P_i$. To show P_i is Galois, it suffices to show that $\mathrm{Aut}(P_i)$ acts on $F(P_i) = Hom(P_k, P_i)$ transitively, i.e. there exists a $\sigma \in \mathrm{Aut}(P_i)$ making the triangle commute:

$$\begin{tikzcd} P_k \arrow[r, "v"] \arrow[dr, "\varphi_{ik}"'] & P_i \arrow[d, "\sigma"] \\ & P_i. \end{tikzcd}$$

For every $u \in J = Hom(P_i, \mathcal{Y})$, we have $u \circ v \in Hom(P_k, \mathcal{Y})$, so there exists a $u' \in Hom(P_i, \mathcal{Y})$ making the diagram commute:

$$\begin{tikzcd} P_k \arrow[r, "v"] \arrow[d, "\varphi_{ik}"'] & P_i \arrow[d, "u"] \\ P_i \arrow[r, "u'"'] & \mathcal{Y}. \end{tikzcd}$$

Since v is epic, the function $u \mapsto u' : J \to J$ is injective, hence a bijection. Let $\alpha : \mathcal{Y}^J \to \mathcal{Y}^J$ be the isomorphism induced by the map $u \mapsto u'$. Then the diagram

$$\begin{tikzcd} P_k \arrow[r, "v"] \arrow[dr, "\varphi_{ik}"'] & P_i \arrow[r, "B"] & \mathcal{Y}^J \arrow[d, "\alpha"] \\ & P_i \arrow[r, "B"'] & \mathcal{Y}^J \end{tikzcd}$$

commutes. By the uniqueness of the factorization of the map $P_k \to \mathcal{Y}^J$ into the composition of an epimorphism and a monomorphism, there exists a $\sigma \in \mathrm{Aut}(P_i)$ such that $\sigma \circ v = \varphi_{ik}$. This finishes the proof. $\square$

We give some basic properties of stratifiable complexes.

Lemma 3.1.4. *(i) $\mathscr{D}_c^{\mathrm{stra}}(\mathscr{A})$ (resp. $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$) is a triangulated subcategory of $\mathscr{D}_c(\mathscr{A})$ (resp. $D_c(\mathcal{X}, \Lambda)$) with the induced t-structure.*

(ii) If $f : \mathcal{X} \to \mathcal{Y}$ is an S-morphism, then $f^* : \mathscr{D}_c(\mathscr{A}(\mathcal{Y})) \to \mathscr{D}_c(\mathscr{A}(\mathcal{X}))$ (resp. $f^* : D_c(\mathcal{Y}, \Lambda) \to D_c(\mathcal{X}, \Lambda)$) preserves stratifiability.

(iii) If $\mathscr{S}$ is a stratification of $\mathcal{X}$, then $K \in \mathscr{D}_c(\mathscr{A}(\mathcal{X}))$ is stratifiable if and only if $K|_V$ is stratifiable for every $V \in \mathscr{S}$.

(iv) Let $P : X \to \mathcal{X}$ be a presentation, and let $K = (K_n)_n \in \mathscr{D}_c(\mathscr{A}(\mathcal{X}))$. Then K is stratifiable if and only if P^*K is stratifiable.

(v) $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$ contains $D_c^b(\mathcal{X}, \Lambda)$, and the heart of $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$ is the same as that of $D_c(\mathcal{X}, \Lambda)$ (2.1.2.1i).

(vi) Let $K \in \mathscr{D}_c(\mathscr{A})$ be a normalized complex ([27], 3.0.8). Then K is trivialized by a pair $(\mathscr{S}, \mathcal{L})$ if and only if K_0 is trivialized by this pair.

(vii) Let $K \in \mathscr{D}_c^{\mathrm{stra}}(\mathscr{A})$. Then its Tate twist $K(1)$ is also stratifiable.

Proof. (i) To show $\mathscr{D}_c^{\mathrm{stra}}(\mathscr{A})$ is a triangulated subcategory, it suffices to show ([11], p.271) that for every exact triangle $K' \to K \to K'' \to K'[1]$ in $\mathscr{D}_c(\mathscr{A})$, if K' and K'' are stratifiable, so also is K.

Using refinement we may assume that K' and K'' are trivialized by the same pair $(\mathscr{S}, \mathcal{L})$. Consider the cohomology sequence of this exact triangle at level n, restricted to a stratum $\mathcal{U} \in \mathscr{S}$. By ([35], 9.1), to show that a sheaf is lcc on $\mathcal{U}$, one can pass to a presentation U

of the stack $\mathcal{U}$. Then by ([29], 20.3) and 5-lemma, we see that the $\mathscr{H}^i(K_n)$'s are lcc on $\mathcal{U}$, with JH components contained in $\mathcal{L}(\mathcal{U})$. Therefore $\mathscr{D}_c^{\mathrm{stra}}(\mathscr{A})$ (and hence $D_c^{\mathrm{stra}}(\mathcal{X},\Lambda)$) is a triangulated subcategory.

The t-structure is inherited by $\mathscr{D}_c^{\mathrm{stra}}(\mathscr{A})$ (and hence by $D_c^{\mathrm{stra}}(\mathcal{X},\Lambda)$) because, if $K \in \mathscr{D}_c(\mathscr{A})$ is stratifiable, so also are its truncations $\tau_{\leq r}K$ and $\tau_{\geq r}K$.

(ii) f^* is exact on the level of sheaves, and takes a lcc sheaf to a lcc sheaf. If $(K_n)_n \in \mathscr{D}_c(\mathscr{A}(\mathcal{Y}))$ is trivialized by $(\mathscr{S},\mathcal{L})$, then $(f^*K_n)_n$ is trivialized by $(f^*\mathscr{S}, f^*\mathcal{L})$, where $f^*\mathscr{S} = \{f^{-1}(V)|V \in \mathscr{S}\}$ and $(f^*\mathcal{L})(f^{-1}(V))$ is the set of isomorphism classes of JH components of $f^*L,\ L \in \mathcal{L}(V)$. The case of $D_c(-,\Lambda)$ follows easily.

(iii) The "only if" part follows from (ii). The "if" part is clear: if $(\mathscr{S}_V,\mathcal{L}_V)$ is a pair on V that trivializes $(K_n|_V)_n$, then the pair $(\mathscr{S}_{\mathcal{X}},\mathcal{L})$ on $\mathcal{X}$, where $\mathscr{S}_{\mathcal{X}} = \cup\mathscr{S}_V$ and $\mathcal{L} = \{\mathcal{L}_V\}_{V\in\mathscr{S}}$, trivializes $(K_n)_n$.

(iv) The "only if" part follows from (ii). For the "if" part, assume P^*K is trivialized by a pair $(\mathscr{S}_X,\mathcal{L}_X)$ on X. Let $U \in \mathscr{S}_X$ be an open stratum, and let $V \subset \mathcal{X}$ be the image of U ([25], 3.7). Recall that for every $T \in \mathrm{Aff}/S,\ V(T)$ is the full subcategory of $\mathcal{X}(T)$ consisting of objects x that are locally in the essential image of $U(T)$, i.e. such that there exists an étale surjection $T' \to T$ in Aff/S and $u' \in U(T')$, such that the image of u' in $\mathcal{X}(T')$ and $x|_{T'}$ are isomorphic. Then V is an open substack of $\mathcal{X}$ (hence also an algebraic stack) and $P|_U : U \to V$ is a presentation. Replacing $P : X \to \mathcal{X}$ by $P|_U : U \to V$ and using noetherian induction and (iii), we may assume $\mathscr{S}_X = \{X\}$. Take a pair $(\mathscr{S},\mathcal{L})$ on $\mathcal{X}$ that trivializes all R^0P_*L's, for $L \in \mathcal{L}_X$. We claim that K is trivialized by $(\mathscr{S},\mathcal{L})$.

For each sheaf $\mathscr{F}$ on $\mathcal{X}$, the natural map $\mathscr{F} \to R^0P_*P^*\mathscr{F}$ is injective. To verify this on $X_U \to U$, for any $u \in \mathcal{X}(U)$, note that the question is étale local on U, so one can assume $P : X_U \to U$ has a section $s : U \to X_U$. Then the composition $\mathscr{F}_U \to R^0P_*P^*\mathscr{F}_U \to R^0P_*R^0s_*s^*P^*\mathscr{F}_U = \mathscr{F}_U$ of the two adjunctions is the adjunction for $P \circ s = \mathrm{id}$, so the composite is an isomorphism, and the first map is injective. Taking $\mathscr{F}$ to be the cohomology sheaves $\mathscr{H}^i(K_n)$ we see that K is trivialized by $(\mathscr{S},\mathcal{L})$.

(v) It suffices to show, by (i) and (2.1.2.1i), that all adic systems $M = (M_n)_n \in \mathscr{A}$ are stratifiable. By (iv) we may assume $\mathcal{X} = X$ is an S-scheme. Since X is noetherian, there exists a stratification ([16], VI, 1.2.6) of X such that M is lisse on each stratum. By (iii) we may assume M is lisse on X.

Let $\mathcal{L}$ be the set of isomorphism classes of JH components of the Λ_0-sheaf M_0. We claim that $\mathcal{L}$ trivializes M_n for all n. Suppose it trivializes M_{n-1} for some $n \geq 1$. Consider the sub-Λ_n-modules $\lambda M_n \subset M_n[\lambda^n] \subset M_n$, where $M_n[\lambda^n]$ is the kernel of the map $\lambda^n : M_n \to M_n$. Since M is adic, we have exact sequences of Λ_X-modules

$$0 \to \lambda M_n \to M_n \to M_0 \to 0,$$

$$0 \to M_n[\lambda^n] \to M_n \to \lambda^n M_n \to 0, \quad \text{and}$$

$$0 \to \lambda^n M_n \to M_n \to M_{n-1} \to 0.$$

The natural surjection $M_n/\lambda M_n \to M_n/M_n[\lambda^n]$ implies that $\mathcal{L}$ trivializes $\lambda^n M_n$, and therefore it also trivializes M_n. By induction on n we are done.

Since $D_c^b \subset D_c^{\text{stra}} \subset D_c$, and D_c^b and D_c have the same heart, it is clear that D_c^{stra} has the same heart as them.

(vi) Applying $-\otimes^L_{\Lambda_n} K_n$ to the following exact sequence, viewed as an exact triangle in $\mathscr{D}(\mathcal{X},\Lambda_n)$

$$0 \longrightarrow \Lambda_{n-1} \xrightarrow{1\mapsto\lambda} \Lambda_n \longrightarrow \Lambda_0 \longrightarrow 0,$$

we get an exact triangle by ([27], 3.0.10)

$$K_{n-1} \longrightarrow K_n \longrightarrow K_0 \longrightarrow \cdot$$

By induction on n and (3.1.4.1) below, we see that K is trivialized by $(\mathscr{S},\mathcal{L})$ if K_0 is.

(vii) Let $K=(K_n)_n$. By definition $K(1)=(K_n(1))_n$, where $K_n(1)=K_n\otimes^L_{\Lambda_n}\Lambda_n(1)$. Note that the sheaf $\Lambda_n(1)$ is a flat Λ_n-module: to show that $-\otimes_{\Lambda_n}\Lambda_n(1)$ preserves injections, one can pass to stalks at geometric points, over which we have a trivialization $\Lambda_n\simeq\Lambda_n(1)$.

Suppose K is $(\mathscr{S},\mathcal{L})$-stratifiable. Using the degenerate spectral sequence

$$\mathscr{H}^i(K_n)\otimes_{\Lambda_n}\Lambda_n(1)=\mathscr{H}^i(K_n\otimes^L_{\Lambda_n}\Lambda_n(1)),$$

it suffices to show the existence of a pair $(\mathscr{S},\mathcal{L}')$ such that for each $\mathcal{U}\in\mathscr{S}$, the JH components of the lcc sheaves $L\otimes_{\Lambda_n}\Lambda_n(1)$ lie in $\mathcal{L}'(\mathcal{U})$, for all $L\in\mathcal{L}(\mathcal{U})$. Since L is a Λ_0-module, we have

$$L\otimes_{\Lambda_n}\Lambda_n(1)=(L\otimes_{\Lambda_n}\Lambda_0)\otimes_{\Lambda_n}\Lambda_n(1)=L\otimes_{\Lambda_n}(\Lambda_0\otimes_{\Lambda_n}\Lambda_n(1))=L\otimes_{\Lambda_n}\Lambda_0(1)=L\otimes_{\Lambda_0}\Lambda_0(1),$$

and we can take $\mathcal{L}'(\mathcal{U})$ to be a tensor closed hull of $\{\Lambda_0(1), L\in\mathcal{L}(\mathcal{U})\}$. □

Remark 3.1.4.1. In fact the proof of (3.1.4i) shows that $\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$ is a triangulated subcategory, for each fixed pair $(\mathscr{S},\mathcal{L})$. Let $D_{\mathscr{S},\mathcal{L}}(\mathcal{X},\Lambda)$ be the essential image of $\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$ in $D_c(\mathcal{X},\Lambda)$, and this is also a triangulated subcategory.

Also if $E_r^{ij}\Longrightarrow E^n$ is a spectral sequence in the category $\mathscr{A}(\mathcal{X})$, and the E_r^{ij}'s are trivialized by $(\mathscr{S},\mathcal{L})$ for all i,j, then all the E^n's are trivialized by $(\mathscr{S},\mathcal{L})$.

We denote by $D_c^{\dagger,\text{stra}}(\mathcal{X},\Lambda)$, for $\dagger=\pm,b$, the full subcategory of $\dagger$-bounded stratifiable complexes, using the induced t-structure.

3.2 Stability under the six operations.

The following is a key result for showing the stability of stratifiability under the six operations later. Recall that $M\mapsto\widehat{M}=L\pi^*R\pi_*M$ is the normalization functor, where $\pi:\mathcal{X}^{\mathbb{N}}\to\mathcal{X}$ is the morphism of topoi in ([27], 2.1), mentioned in (2.1.3).

Proposition 3.2.1. *For a pair $(\mathscr{S},\mathcal{L})$, if $M\in\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$, then $\widehat{M}\in\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$, too. In particular, if $K\in D_c(\mathcal{X},\Lambda)$, then K is stratifiable if and only if its normalization $\widehat{K}\in\mathscr{D}_c(\mathscr{A})$ is stratifiable.*

Proof. Since $\widehat{M}$ is normalized, by (3.1.4vi), it suffices to show that $(\widehat{M})_0$ is trivialized by $(\mathscr{S},\mathcal{L})$. Using projection formula and the flat resolution of Λ_0

$$0\longrightarrow\Lambda\xrightarrow{\lambda}\Lambda\xrightarrow{\epsilon}\Lambda_0\longrightarrow 0,$$

we have ([27], p.176)

$$(\widehat{M})_0=\Lambda_0\otimes^L_\Lambda R\pi_*M=R\pi_*(\pi^*\Lambda_0\otimes^L_{\Lambda_\bullet}M),$$

where $\pi^*\Lambda_0$ is the constant projective system defined by Λ_0. Let $C\in\mathscr{D}(\mathscr{A})$ be the complex of projective systems $\pi^*\Lambda_0\otimes^L_{\Lambda_\bullet}M$; it is a λ-complex, and $C_n=\Lambda_0\otimes^L_{\Lambda_n}M_n\in D_c(\mathcal{X},\Lambda_0)$.

Recall ([16], V, 3.2.3) that, a projective system $(K_n)_n$ ringed by $\Lambda_\bullet$ in an abelian category is AR-adic if and only if

• it satisfies the condition (MLAR) ([16], V, 2.1.1), hence (ML), and denote by $(N_n)_n$ the projective system of the universal images of $(K_n)_n$;

• there exists an integer $k\geq 0$ such that the projective system $(L_n)_n:=(N_{n+k}/\lambda^{n+1}N_{n+k})_n$ is adic.

Moreover, $(K_n)_n$ is AR-isomorphic to $(L_n)_n$. Now for each i, the projective system $\mathscr{H}^i(C)$ is AR-adic (2.1.2.1i). Let $N^i=(N^i_n)_n$ be the projective system of the universal images of $\mathscr{H}^i(C)$, and choose an integer $k\geq 0$ such that the system $L^i=(L^i_n)_n=(N^i_{n+k}\otimes\Lambda_n)_n$ is adic. Since $N^i_{n+k}\subset\mathscr{H}^i(C_{n+k})$ is annihilated by λ, we have $L^i_n=N^i_{n+k}$, and the transition morphism gives an isomorphism

$$L^i_n\simeq L^i_n\otimes_{\Lambda_n}\Lambda_{n-1}\xrightarrow{\sim}L^i_{n-1}.$$

This means the projective system L^i is the constant system $\pi^*L^i_0$. By ([27], 2.2.2) we have $R\pi_*\mathscr{H}^i(C)\simeq R\pi_*L^i$, which is just L^i_0 by ([27], 2.2.3).

The spectral sequence

$$R^j\pi_*\mathscr{H}^i(C)\Longrightarrow\mathscr{H}^{i+j}((\widehat{M})_0)$$

degenerates to isomorphisms $L^i_0\simeq\mathscr{H}^i((\widehat{M})_0)$, so we only need to show that L^i_0 is trivialized by $(\mathscr{S},\mathcal{L})$. Using the periodic Λ_n-flat resolution of Λ_0

$$\cdots\longrightarrow\Lambda_n\xrightarrow{\lambda}\Lambda_n\xrightarrow{\lambda^n}\Lambda_n\xrightarrow{\lambda}\Lambda_n\xrightarrow{\epsilon}\Lambda_0\longrightarrow 0,$$

we see that $\Lambda_0\otimes^L_{\Lambda_n}\mathscr{H}^j(M_n)$ is represented by the complex

$$\cdots\longrightarrow\mathscr{H}^j(M_n)\xrightarrow{\lambda^n}\mathscr{H}^j(M_n)\xrightarrow{\lambda}\mathscr{H}^j(M_n)\longrightarrow 0,$$

so $\mathscr{H}^i(\Lambda_0 \otimes^L_{\Lambda_n} \mathscr{H}^j(M_n))$ are trivialized by $(\mathscr{S}, \mathcal{L})$, for all i, j. From the spectral sequence

$$\mathscr{H}^i(\Lambda_0 \otimes^L_{\Lambda_n} \mathscr{H}^j(M_n)) \Longrightarrow \mathscr{H}^{i+j}(C_n)$$

we see (by (3.1.4.1)) that the $\mathscr{H}^i(C_n)$'s are trivialized by $(\mathscr{S}, \mathcal{L})$. The universal image N^i_n is the image of $\mathscr{H}^i(C_{n+r}) \to \mathscr{H}^i(C_n)$ for some $r \gg 0$, therefore the N^i_n's (and hence the L^i_n's) are trivialized by $(\mathscr{S}, \mathcal{L})$.

For the second claim, let $K \in D_c(\mathcal{X}, \Lambda)$. Since K is isomorphic to the image of $\widehat{K}$ under the localization $\mathscr{D}_c(\mathscr{A}) \to D_c(\mathcal{X}, \Lambda)$ ([27], 3.0.14), we see that K is stratifiable if $\widehat{K}$ is. Conversely, if K is stratifiable, which means that it is isomorphic to the image of some $M \in \mathscr{D}_c^{\mathrm{stra}}(\mathscr{A})$, then $\widehat{K} = \widehat{M}$ is also stratifiable. □

3.2.1.1. For $K \in D_c(\mathcal{X}, \Lambda)$, we say that K is $(\mathscr{S}, \mathcal{L})$-*stratifiable* if $\widehat{K}$ is, and (3.2.1) implies that $K \in D_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda)$ (cf. (3.1.4.1)) if and only if K is $(\mathscr{S}, \mathcal{L})$-stratifiable.

Corollary 3.2.2. *(i) If $\mathscr{S}$ is a stratification of $\mathcal{X}$, then $K \in D_c(\mathcal{X}, \Lambda)$ is stratifiable if and only if $K|_V$ is stratifiable for every $V \in \mathscr{S}$.*

(ii) Let $K \in D_c(\mathcal{X}, \Lambda)$. Then K is stratifiable if and only if its Tate twist $K(1)$ is.

(iii) Let $P : X \to \mathcal{X}$ be a presentation, and let $K \in D_c(\mathcal{X}, \Lambda)$. Then K is stratifiable if and only if P^*K (resp. $P^!K$) is stratifiable.

Proof. (i) The "only if" part follows from (3.1.4ii), and the "if" part follows from (3.1.4iii), (3.2.1), since $\widehat{K}|_V \simeq \widehat{(K|_V)}$.

(ii) This follows from (3.1.4vii), since $\widehat{K}(1) = \widehat{K(1)}$.

(iii) For P^*K, the "only if" part follows from (3.1.4ii), and the "if" part follows from (3.1.4iv) and (3.2.1), since $P^*\widehat{K} = \widehat{(P^*K)}$ ([27], 2.2.1, 3.0.11).

Since P is smooth of relative dimension d, for some function $d : \pi_0(X) \to \mathbb{N}$, we have $P^!K \simeq P^*K(d)[2d]$, so by (ii), P^*K is stratifiable if and only if $P^!K$ is. □

Before proving the main result of this chapter, we prove some special cases.

Lemma 3.2.3. *(i) If $f : X \to Y$ is a morphism of S-schemes, and $K \in D^+_c(X, \Lambda)$ is trivialized by $(\{X\}, \mathcal{L})$ for some $\mathcal{L}$, then f_*K is stratifiable.*

(ii) Let $K_{\mathcal{X}}$ and $K'_{\mathcal{X}}$ be two Λ-dualizing complexes on the S-algebraic stack $\mathcal{X}$, and let D and D' be the two associated dualizing functors, respectively. Let $K \in D_c(\mathcal{X}, \Lambda)$. If DK is trivialized by a pair $(\mathscr{S}, \mathcal{L})$, where all strata in $\mathscr{S}$ are connected, then $D'K$ is trivialized by $(\mathscr{S}, \mathcal{L}')$ for some other $\mathcal{L}'$. In particular, the property of the Verdier dual of K being stratifiable is independent of the choice of the dualizing complex.

(iii) Let S be a regular scheme satisfying (LO), and let $\mathcal{X}$ be a connected smooth S-algebraic stack of dimension d. If $K \in D_c(\mathcal{X}, \Lambda)$ is trivialized by a pair $(\{\mathcal{X}\}, \mathcal{L})$, then $D_{\mathcal{X}}K$ is trivialized by $(\{\mathcal{X}\}, \mathcal{L}')$ for some $\mathcal{L}'$.

Proof. (i) By definition ([27], 8), f_*K is the image of $(f_*(\widehat{K})_n)_n$. By the spectral sequence

$$R^p f_* \mathscr{H}^q((\widehat{K})_n) \Longrightarrow R^{p+q} f_*((\widehat{K})_n)$$

and (3.1.4.1), we may replace $(\widehat{K})_n$ by its cohomology sheaves $\mathscr{H}^i((\widehat{K})_n)$ and hence by $L \in \mathcal{L}$, and it suffices to show the existence of a pair $(\mathscr{S}_Y, \mathcal{L}_Y)$ on Y that trivializes $R^i f_* L$, for all $i \in \mathbb{Z}$ and $L \in \mathcal{L}$.

By a result of Gabber [19], the complex f_*L is bounded for each L. Since the set $\{R^i f_* L | i \in \mathbb{Z},\ L \in \mathcal{L}\}$ is finite, there exists a stratification $\mathscr{S}_Y$ of Y that trivializes all sheaves in this set. For each $V \in \mathscr{S}_Y$, define $\mathcal{L}_Y(V)$ to be the set of isomorphism classes of JH components of the lcc sheaves $(R^i f_* L)|_V$.

(ii) Recall that the dualizing complex $K_{\mathcal{X}}$ (resp. $K'_{\mathcal{X}}$) is defined to be the image of a normalized complex $K_{\mathcal{X},\bullet}$ (resp. $K'_{\mathcal{X},\bullet}$). See ([27], 7.2.3, 7.2.4). As in the case of schemes [36], for every n there exist an invertible Λ_n-module L_n (i.e. $L_n \otimes^L G_n \simeq \Lambda_n$ for some G_n) and an integer r_n, such that $K'_{\mathcal{X},n} \simeq K_{\mathcal{X},n} \otimes^L L_n[r_n]$, and in fact $L_n[r_n] \simeq R\mathscr{H}om(K_{\mathcal{X},n}, K'_{\mathcal{X},n})$ (this can be checked locally, hence can be reduced to the case of schemes, which is ([16], Exp. I, 2)). By ([27], 4.0.8), the system $(L_n[r_n])_n = R\mathscr{H}om((K_{\mathcal{X},n})_n, (K'_{\mathcal{X},n}))$ is normalized. Note that the stalks of L_n are free Λ_n-modules of rank 1, so L_n is flat and $L_n \otimes^L G_n = L_n \otimes G_n$.

We claim that if A is an invertible Λ_n-sheaf on $\mathcal{X}$, then it is lcc. The following proof is due to Olsson. Passing to a presentation $X \to \mathcal{X}$, we may assume $\mathcal{X} = X$ is a scheme. By ([29], 20.3), we may replace X by Spec $\mathscr{O}_{X,\overline{x}}$ with generic point η. Let B be such that $A \otimes B \simeq \Lambda_n$. Regard $H^0(\text{Spec } \mathscr{O}_{X,\overline{x}}, A) = A_{\overline{x}} \simeq \Lambda_n$ as a constant sheaf on Spec $\mathscr{O}_{X,\overline{x}}$, and consider the natural morphism $A_{\overline{x}} \to A$. It suffices to show that this is an isomorphism, since taking stalks at $\overline{\eta}$ gives the cospecialization map. Let K be the kernel of $A_{\overline{x}} \to A$. Since B is flat, we get a diagram where the row is exact:

$$\begin{tikzcd}
 & & A_{\overline{x}} \otimes B_{\overline{x}} \arrow[r, "\sim"] \arrow[d] & (A \otimes B)_{\overline{x}} \arrow[d, "(1)"] \\
0 \arrow[r] & K \otimes B \arrow[r] & A_{\overline{x}} \otimes B \arrow[r, "(2)"'] & A \otimes B \simeq \Lambda_n.
\end{tikzcd}$$

Since (1) is an isomorphism, we see that (2) is surjective, hence an isomorphism (all stalks are finite sets). This implies $K \otimes B$, and hence K, are zero. So $A_{\overline{x}} \to A$ is injective, hence an isomorphism.

For every stratum $\mathcal{U} \in \mathscr{S}$, let $\mathcal{L}_0(\mathcal{U})$ be the union of $\mathcal{L}(\mathcal{U})$ and the set of isomorphism classes of the JH components of the lcc sheaf $L_0|_{\mathcal{U}}$. By (3.1.4vi), the complex $D'K_{\mathcal{X}}$ is trivialized by $(\mathscr{S}, \mathcal{L}_0)$. Since all strata in $\mathscr{S}$ are connected, there exists a tensor closed hull of $(\mathscr{S}, \mathcal{L}_0)$ of the form $(\mathscr{S}, \mathcal{L}')$, i.e. they have the same stratification $\mathscr{S}$. Since $D'K_{\mathcal{X}}$ and DK are both trivialized by $(\mathscr{S}, \mathcal{L}')$, so also is $D'K \simeq DK \otimes^L D'K$.

(iii) Note that by (ii), the question is independent of the choice of the dualizing complex. By a result of Gabber ([36], 0.2), one can take the constant sheaf Λ to be the dualizing complex on S, so $K_{\mathcal{X}} = \Lambda\langle d\rangle$. By definition ([27], 7.3.1), DK is the image of $R\mathscr{H}om_{\Lambda_\bullet}(\widehat{K}, \Lambda_\bullet\langle d\rangle)$ in $D_c(\mathcal{X}, \Lambda)$, which is a normalized complex ([27], 4.0.8). Therefore, by (3.1.4vi) it suffices to show that $R\mathscr{H}om_{\Lambda_0}(\widehat{K}_0, \Lambda_0\langle d\rangle)$ is trivialized by $(\{\mathcal{X}\}, \mathcal{L}')$ for some $\mathcal{L}'$. Replacing $\widehat{K}_0$ by its cohomology sheaves $\mathscr{H}^i(\widehat{K}_0)$, and hence by their JH components, we see that all the $\mathscr{H}^n(R\mathscr{H}om_{\Lambda_0}(\widehat{K}_0, \Lambda_0\langle d\rangle))$'s are trivialized by $(\{\mathcal{X}\}, \mathcal{L}')$, where $\mathcal{L}'$ is a tensor-closed hull of $\{\Lambda_0(1), L^\vee | L \in \mathcal{L}\}$. Here $L^\vee = \mathscr{H}om_{\Lambda_0}(L, \Lambda_0)$. □

Next we prove the main result of this chapter.

Theorem 3.2.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks. Then the operations $f_*, f_!, f^*, f^!, D_{\mathcal{X}}, - \otimes^L -$ and $R\mathscr{H}om(-,-)$ all preserve stratifiability, namely, they induce functors*

$$f_* : D_c^{+,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{+,\mathrm{stra}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell), \qquad f_! : D_c^{-,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{-,\mathrm{stra}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell),$$
$$f^* : D_c^{\mathrm{stra}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell), \qquad f^! : D_c^{\mathrm{stra}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell),$$
$$D : D_c^{\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}},$$
$$\otimes^L : D_c^{-,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \times D_c^{-,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{-,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \quad \text{and}$$
$$R\mathscr{H}om(-,-) : D_c^{-,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \times D_c^{+,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{+,\mathrm{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell).$$

Proof. It suffices to show stability of $D_c^{\dagger,\mathrm{stra}}(-, \Lambda)$, for every finite extension Λ of $\mathbb{Z}_\ell$.

We consider the Verdier dual functor D first. Let $P : X \to \mathcal{X}$ be a presentation. Since $P^*D = DP^!$, by (3.2.2ii) we can assume $\mathcal{X} = X$ is a scheme. Let K be a complex on X trivialized by $(\mathscr{S}, \mathcal{L})$. Let $S' \subset S$ be a non-empty open affine regular subscheme. It is clear that S' satisfies the condition (LO). Let $X_{S'} \subset X$ be the inverse image of S' in X. Refining if necessary, we may assume all strata in $\mathscr{S}$ are connected and smooth over S, and are contained in either $X_{S'}$ or $X - X_{S'}$. Let $j : U \to X$ be the immersion of an open stratum in $\mathscr{S}$ which is contained in $X_{S'}$, with complement $i : Z \to X$. Consider the exact triangle

$$i_*D_Z(K|_Z) \longrightarrow D_X K \longrightarrow j_*D_U(K|_U) \longrightarrow \cdot$$

Since U is connected and smooth over S', by (3.2.3iii) we see that $D_U(K|_U)$ is trivialized by $(\{U\}, \mathcal{L}')$ for some $\mathcal{L}'$, so $j_*D_U(K|_U)$ is stratifiable (3.2.3i). By noetherian induction we may assume $D_Z(K|_Z)$ is stratifiable, and it is clear that i_* preserves stratifiability. Therefore by (3.1.4i), $D_X K$ is stratifiable.

The case of f^* (and hence $f^!$) is proved in (3.1.4ii).

Next we prove the case of $\otimes^L$. For $i = 1, 2$, let $K_i \in D_c^-(\mathcal{X}, \Lambda)$, trivialized by $(\mathscr{S}_i, \mathcal{L}_i)$. Let $(\mathscr{S}, \mathcal{L})$ be a common tensor closed refinement (by (3.1.3)) of $(\mathscr{S}_i, \mathcal{L}_i)$, $i = 1, 2$. The derived tensor product $K_1 \otimes^L K_2$ is defined to be the image in $D_c(\mathcal{X}, \Lambda)$ of $\widehat{K}_1 \otimes^L_{\Lambda_\bullet} \widehat{K}_2$, which we claim to be stratifiable. By ([27], 3.0.10) it is normalized, so it suffices to show (by (3.1.4vi)) that $\widehat{K}_{1,0} \otimes^L_{\Lambda_0} \widehat{K}_{2,0}$ is trivialized by $(\mathscr{S}, \mathcal{L})$. This follows from the spectral sequence

$$\bigoplus_{i+j=q} \mathscr{H}^p(\mathscr{H}^i(\widehat{K}_{1,0}) \otimes_{\Lambda_0} \mathscr{H}^j(\widehat{K}_{2,0})) \Longrightarrow \mathscr{H}^{p+q}(\widehat{K}_{1,0} \otimes^L_{\Lambda_0} \widehat{K}_{2,0})$$

and the assumption that $(\mathscr{S}, \mathcal{L})$ is tensor closed.

The case of $R\mathscr{H}om(K_1, K_2) = D(K_1 \otimes^L DK_2)$ follows.

Finally we prove the case of f_* and $f_!$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an S-morphism, and let K be a bounded above stratifiable complex on $\mathcal{X}$, trivialized by a tensor closed pair $(\mathscr{S}, \mathcal{L})$. We want to show $f_!K$ is stratifiable. Let $j : \mathcal{U} \to \mathcal{X}$ be the immersion of an open stratum in $\mathscr{S}$, with complement $i : \mathcal{Z} \to \mathcal{X}$. From the exact triangle

$$(fj)_!j^*K \longrightarrow f_!K \longrightarrow (fi)_!i^*K \longrightarrow$$

we see that it suffices to prove the claim for fj and fi. By noetherian induction we can replace $\mathcal{X}$ by $\mathcal{U}$. By (3.2.2ii) and smooth base change ([27], 12.1), we can replace $\mathcal{Y}$ by a presentation Y, and by (3.2.2i) and ([27], 12.3) we can shrink Y to an open subscheme. So we assume that $\mathcal{Y} = Y$ is a smooth affine S-scheme, that K is trivialized by $(\{\mathcal{X}\}, \mathcal{L})$, and that the relative inertia stack $\mathcal{I}_f := \mathcal{X} \times_{\Delta, \mathcal{X} \times_Y \mathcal{X}, \Delta} \mathcal{X}$ is flat and has components over $\mathcal{X}$ ([3], 5.1.14). Therefore by ([3], 5.1.13), f factors as $\mathcal{X} \xrightarrow{g} \mathcal{Z} \xrightarrow{h} Y$, where g is gerbe-like and h is representable (cf. ([3], 5.1.3-5.1.6) for relevant notions). So we reduce to two cases: f is representable, or f is gerbe-like.

Case when f is representable. By shrinking the S-algebraic space $\mathcal{X}$ we can assume $\mathcal{X} = X$ is a connected scheme, smooth over an affine open regular subscheme $S' \subset S$. By (3.2.3iii) we see that DK is trivialized by some $(\{X\}, \mathcal{L}')$, and by (3.2.3i), f_*DK is stratifiable. Therefore $f_!K = Df_*DK$ is also stratifiable.

Case when f is gerbe-like. In this case f is smooth ([3], 5.1.5), hence étale locally on Y it has a section. Replacing Y by an étale cover, we may assume that f is a neutral gerbe, so $f : B(G/Y) \to Y$ is the structural map, for some flat group space G of finite type over Y ([25], 3.21). By ([3], 5.1.1) and (3.2.2i) we may assume G is a Y-group scheme. Next we reduce to the case when G is smooth over Y.

Let $k(Y)$ be the function field of Y and $\overline{k(Y)}$ an algebraic closure. Then $G_{\overline{k(Y)},\mathrm{red}}$ is smooth over $\overline{k(Y)}$, so there exists a finite extension L over $k(Y)$ such that $G_{L,\mathrm{red}}$ is smooth over L. Let Y' be the normalization of Y in L, which is a scheme of finite type over S, and the natural map $Y' \to Y$ is finite surjective. It factors through $Y' \to Z \to Y$, where Z is the normalization of Y in the separable closure of $k(Y)$ in $L = k(Y')$. So $Z \to Y$ is generically étale, and $Y' \to Z$ is purely inseparable, hence a universal homeomorphism, so Y' and Z have equivalent étale sites. Replacing Y' by Z and shrinking Y we can assume $Y' \to Y$ is finite étale. Replacing Y by Y' (by (3.2.2ii)) we assume G_{red} over Y has smooth generic fiber, and by shrinking Y we assume G_{red} is smooth over Y.

G_{red} is a subgroup scheme of G ([12], $\mathrm{VI_A}$, 0.2). $h : G_{\mathrm{red}} \hookrightarrow G$ is a closed immersion, so

$Bh : B(G_{\mathrm{red}}/Y) \to B(G/Y)$ is faithful and hence representable. It is also radicial: consider the diagram where the square is 2-Cartesian

$$\begin{tikzcd} Y \arrow[r, hook, "i"] & G/G_{\mathrm{red}} \arrow[r, "g"] \arrow[d] & Y \arrow[d, "P"] \\ & B(G_{\mathrm{red}}/Y) \arrow[r, "Bh"'] & B(G/Y). \end{tikzcd}$$

The map i is a nilpotent closed embedding, so g is radicial. Since P is flat and surjective, Bh is also radicial. This shows that

$$(Bh)^* : D_c^-(B(G/Y), \Lambda) \to D_c^-(B(G_{\mathrm{red}}/Y), \Lambda)$$

is an equivalence of categories. Replacing G by G_{red} we assume G is smooth over Y, and hence $P : Y \to B(G/Y)$ is a presentation.

Shrinking Y we can assume that the relative dimension of G over Y is a constant, say d. Consider the associated smooth hypercover, and let $f_i : G^i \to Y$ be the structural map. To show $f_! K$ is stratifiable is equivalent to showing $Df_!K = f_*DK$ is stratifiable. By (3.2.3iii), we see that DK is also trivialized by a pair of the form $(\{\mathcal{X}\}, \mathcal{L}')$, so replacing K by DK, it suffices to show that f_*K is stratifiable, where $K \in D_c^+(\mathcal{X}, \Lambda)$ is trivialized by $(\{\mathcal{X}\}, \mathcal{L})$ for some $\mathcal{L}$. As in the proof of (3.2.3i), it suffices to show the existence of a pair $(\mathscr{S}_Y, \mathcal{L}_Y)$ on Y that trivializes $R^n f_* L$, for all $L \in \mathcal{L}$ and $n \in \mathbb{Z}$.

From the spectral sequence ([27], 10.0.9)

$$E_1^{ij} = R^j f_{i*} f_i^* P^* L \Longrightarrow R^{i+j} f_* L,$$

we see that it suffices for the pair $(\mathscr{S}_Y, \mathcal{L}_Y)$ to trivialize all the E_1^{ij} terms. Assume $i \geq 1$. If we regard the map $f_i : G^i \to Y$ as the product map

$$\prod_i f_1 : \prod_i G \to \prod_i Y,$$

where the products are fiber products over Y, then we can write $f_i^* P^* L$ as

$$f_1^* P^* L \boxtimes^L_{\Lambda_0} \Lambda_0 \boxtimes^L_{\Lambda_0} \cdots \boxtimes^L_{\Lambda_0} \Lambda_0.$$

By Künneth formula ([27], 11.0.14) we have

$$f_{i*} f_i^* P^* L = f_{1*} f_1^* P^* L \otimes^L_{\Lambda_0} f_{1*}\Lambda_0 \otimes^L_{\Lambda_0} \cdots \otimes^L_{\Lambda_0} f_{1*}\Lambda_0.$$

Since $f_{1*} f_1^* P^* L$ and $f_{1*}\Lambda_0$ are bounded complexes, there exists a tensor closed pair $(\mathscr{S}_Y, \mathcal{L}_Y)$ that trivializes them, for all $L \in \mathcal{L}$. The proof is finished. $\square$

Finally we give a lemma which will be used in the next chapter. This will play the same role as ([3], 6.3.16).

Lemma 3.2.5. *Let X be a connected variety over an algebraically closed field k of characteristic not equal to ℓ, and let $\mathcal{L}$ be a finite set of isomorphism classes of simple lcc Λ_0-sheaves on X. Then there exists an integer d (depending only on $\mathcal{L}$) such that, for every lisse Λ-adic sheaf $\mathscr{F}$ on X trivialized by $\mathcal{L}$, and for every integer i, we have*

$$\dim_E H_c^i(X, \mathscr{F} \otimes_\Lambda E) \leq d \cdot \operatorname{rank}_E(\mathscr{F} \otimes_\Lambda E),$$

where E is the fraction field of Λ.

Proof. Since $\mathcal{L}$ is finite and $0 \leq i \leq 2\dim X$, there exists an integer $d > 0$ such that $\dim_{\Lambda_0} H_c^i(X, L) \leq d \cdot \operatorname{rank}_{\Lambda_0} L$, for every i and every $L \in \mathcal{L}$. For a short exact sequence of lcc Λ_0-sheaves

$$0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$$

on X, the cohomological sequence

$$\cdots \longrightarrow H_c^i(X, \mathscr{G}') \longrightarrow H_c^i(X, \mathscr{G}) \longrightarrow H_c^i(X, \mathscr{G}'') \longrightarrow \cdots$$

implies that $\dim_{\Lambda_0} H_c^i(X, \mathscr{G}) \leq \dim_{\Lambda_0} H_c^i(X, \mathscr{G}') + \dim_{\Lambda_0} H_c^i(X, \mathscr{G}'')$. So it is clear that if $\mathscr{G}$ is trivialized by $\mathcal{L}$, then $\dim_{\Lambda_0} H_c^i(X, \mathscr{G}) \leq d \cdot \operatorname{rank}_{\Lambda_0} \mathscr{G}$, for every i.

Since we only consider $\mathscr{F} \otimes_\Lambda E$, we may assume $\mathscr{F} = (\mathscr{F}_n)_n$ is torsion-free, of some constant rank over Λ (since X is connected), and this Λ-rank is equal to

$$\operatorname{rank}_{\Lambda_0} \mathscr{F}_0 = \operatorname{rank}_E(\mathscr{F} \otimes_\Lambda E).$$

$H_c^i(X, \mathscr{F})$ is a finitely generated Λ-module ([16], VI, 2.2.2), so by Nakayama's lemma, the minimal number of generators is at most $\dim_{\Lambda_0}(\Lambda_0 \otimes_\Lambda H_c^i(X, \mathscr{F}))$. Similar to ordinary cohomology groups ([29], 19.2), we have an injection

$$\Lambda_0 \otimes_\Lambda H_c^i(X, \mathscr{F}) \hookrightarrow H_c^i(X, \mathscr{F}_0)$$

of Λ_0-vector spaces. Therefore, $\dim_E H_c^i(X, \mathscr{F} \otimes_\Lambda E)$ is less than or equal to the minimal number of generators of $H_c^i(X, \mathscr{F})$ over Λ, which is at most

$$\dim_{\Lambda_0}(\Lambda_0 \otimes_\Lambda H_c^i(X, \mathscr{F})) \leq \dim_{\Lambda_0} H_c^i(X, \mathscr{F}_0) \leq d \cdot \operatorname{rank}_{\Lambda_0} \mathscr{F}_0 = d \cdot \operatorname{rank}_E(\mathscr{F} \otimes_\Lambda E).$$

$\square$

Chapter 4

Convergent complexes, finiteness and trace formula.

We return to $\mathbb{F}_q$-algebraic stacks $\mathscr{X}_0, \mathscr{Y}_0, \cdots$ of finite type. A complex $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be *stratifiable* if K on $\mathscr{X}$ is stratifiable, and we denote by $W^{\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ the full subcategory of such complexes. Note that if K_0 is a lisse-étale complex, and it is stratifiable on $\mathscr{X}_0$, then it is geometrically stratifiable (i.e. K on $\mathscr{X}$ is stratifiable). In turns out that in order for the trace formula to hold, it suffices to make this weaker assumption of geometric stratifiability. So we will only discuss stratifiable Weil complexes. Again, by a sheaf we mean a Weil $\overline{\mathbb{Q}}_\ell$-sheaf.

4.1 Convergent complexes.

Definition 4.1.1. *(i) Let $K \in D_c(\overline{\mathbb{Q}}_\ell)$ and $\varphi : K \to K$ an endomorphism. The pair (K, φ) is said to be an ι-*convergent complex *(or just a convergent complex, since we fixed ι) if the complex series in two directions*

$$\sum_{n \in \mathbb{Z}} \sum_{H^n(K), H^n(\varphi)} |\alpha|^s$$

is convergent, for every real number $s > 0$. In this case let $\mathrm{Tr}(\varphi, K)$ be the absolutely convergent complex series

$$\sum_n (-1)^n \iota \mathrm{Tr}(H^n(\varphi), H^n(K))$$

or its limit.

*(ii) Let $K_0 \in W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. We call K_0 an ι-*convergent complex of sheaves *(or just a convergent complex of sheaves), if for every integer $v \geq 1$ and every point $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, the pair $(K_{\overline{x}}, F_x)$ is a convergent complex. In particular, all bounded complexes are convergent.*

(iii) Let $K_0 \in W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Define

$$c_v(\mathscr{X}_0, K_0) = \sum_{x \in [\mathscr{X}_0(\mathbb{F}_{q^v})]} \frac{1}{\#\mathrm{Aut}_x(\mathbb{F}_{q^v})} \mathrm{Tr}(F_x, K_{\overline{x}}) \in \mathbb{C},$$

and define the L-series *of K_0 to be the formal power series*

$$L(\mathscr{X}_0, K_0, t) = \exp\Big(\sum_{v \geq 1} c_v(\mathscr{X}_0, K_0) \frac{t^v}{v} \Big) \in \mathbb{C}[[t]].$$

The zeta function $Z(\mathscr{X}_0, t)$ in (1.0.2) is a special case: $Z(\mathscr{X}_0, t) = L(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell, t)$. It has rational coefficients.

Notation 4.1.1.1. We sometimes write $c_v(K_0)$ for $c_v(\mathscr{X}_0, K_0)$, if it is clear that K_0 is on $\mathscr{X}_0$. We also write $c_v(\mathscr{X}_0)$ for $c_v(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$.

Remark 4.1.1.2. (i) Behrend defined convergent complexes with respect to arithmetic Frobenius elements ([3], 6.2.3), and our definition is for geometric Frobenius, and it is essentially the same as Behrend's definition. It is a little more general though, in the sense that we fixed an (arbitrary) isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and work with ι-mixed Weil sheaves (in fact *all* Weil sheaves, by (2.2.5.1)), while [3] works with pure or mixed lisse-étale sheaves with integer weights. Also our definition is a little different from that in [33]; the condition there is weaker.

(ii) Recall that Aut_x is defined to be the fiber over x of the inertia stack $\mathscr{I}_0 \to \mathscr{X}_0$. It is a group scheme of finite type ([25], 4.2) over $k(x)$, so $\mathrm{Aut}_x(k(x))$ is a finite group.

(iii) If we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{F}_{q^{vd}} \arrow[r] \arrow[dr, "x'"'] & \mathrm{Spec}\ \mathbb{F}_{q^v} \arrow[d, "x"] \\ & \mathscr{X}_0, \end{tikzcd}$$

then $(K_{\overline{x}}, F_x)$ is convergent if and only if $(K_{\overline{x'}}, F_{x'})$ is convergent, because $F_{x'} = F_x^d$ and $s \mapsto sd : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ is a bijection. In particular, for a lisse-étale complex of sheaves, the property of being a convergent complex is independent of q and the structural morphism $\mathscr{X}_0 \to \mathrm{Spec}\ \mathbb{F}_q$. Also note that, for every integer $v \geq 1$, a complex K_0 on $\mathscr{X}_0$ is convergent if and only if $K_0 \otimes \mathbb{F}_{q^v}$ on $\mathscr{X}_0 \otimes \mathbb{F}_{q^v}$ is convergent.

We restate the main theorem in [3] using compactly supported cohomology as follows. It generalizes (1.0.1). We will prove this theorem in this chapter.

Theorem 4.1.2. *Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$ be a morphism of $\mathbb{F}_q$-algebraic stacks, and let $K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Then*

(i) (Finiteness) $f_!K_0$ *is a convergent complex of sheaves on* $\mathscr{Y}_0$, *and*
(ii) (Trace formula) $c_v(\mathscr{X}_0, K_0) = c_v(\mathscr{Y}_0, f_!K_0)$ *for every integer* $v \geq 1$.

Before proving this theorem, we give a few preliminary lemmas and prove a few special cases.

Lemma 4.1.3. *Let*

$$\begin{array}{ccccccc} K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & K'[1] \\ \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \downarrow{\scriptstyle\varphi'[1]} \\ K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & K'[1] \end{array}$$

be an endomorphism of an exact triangle $K' \to K \to K'' \to K'[1]$ *in* $D_c^-(\overline{\mathbb{Q}}_\ell)$. *If any two of the three pairs* $(K', \varphi'), (K'', \varphi'')$ *and* (K, φ) *are convergent, then so is the third, and*

$$\mathrm{Tr}(\varphi, K) = \mathrm{Tr}(\varphi', K') + \mathrm{Tr}(\varphi'', K'').$$

Proof. By the rotation axiom we can assume (K', φ') and (K'', φ'') are convergent. From the exact sequence

$$\cdots \longrightarrow H^n(K') \longrightarrow H^n(K) \longrightarrow H^n(K'') \longrightarrow H^{n+1}(K') \longrightarrow \cdots$$

we see that

$$\sum_{H^n(K),\varphi} |\alpha|^s \leq \sum_{H^n(K'),\varphi'} |\alpha|^s + \sum_{H^n(K''),\varphi''} |\alpha|^s$$

for every real $s > 0$, so (K, φ) is convergent.

Since the series $\sum_{n\in\mathbb{Z}}(-1)^n \sum_{H^n(K),\varphi} \iota\alpha$ converges absolutely, we can change the order of summation, and the second assertion is clear. □

Corollary 4.1.4. *If* $K_0' \to K_0 \to K_0'' \to K_0'[1]$ *is an exact triangle in* $W_c^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, *and two of the three complexes* K_0', K_0'' *and* K_0 *are convergent complexes, then so is the third, and* $c_v(K_0) = c_v(K_0') + c_v(K_0'')$.

Proof. For every $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, we have an exact triangle

$$K'_{\overline{x}} \longrightarrow K_{\overline{x}} \longrightarrow K''_{\overline{x}} \longrightarrow$$

in $D_c^-(\overline{\mathbb{Q}}_\ell)$, equivariant under the action of F_x. Then apply (4.1.3). □

Lemma 4.1.5. *(4.1.2) holds for* $f : \mathrm{Spec}\ \mathbb{F}_{q^d} \to \mathrm{Spec}\ \mathbb{F}_q$.

Proof. We have an equivalence of triangulated categories

$$W^-(\mathrm{Spec}\ \mathbb{F}_q, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)),$$

where G is the Weil group $F^{\mathbb{Z}}$ of $\mathbb{F}_q$. Let H be the subgroup $F^{d\mathbb{Z}}$, the Weil group of $\mathbb{F}_{q^d}$. Since $f : \operatorname{Spec} \mathbb{F}_{q^d} \to \operatorname{Spec} \mathbb{F}_q$ is finite, we have $f_! = f_*$, and it is the induced-module functor

$$\operatorname{Hom}_{\overline{\mathbb{Q}}_\ell[H]}\big(\overline{\mathbb{Q}}_\ell[G], -\big) : D_c^-\big(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(H)\big) \longrightarrow D_c^-\big(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G)\big),$$

which is isomorphic to the coinduced-module functor $\overline{\mathbb{Q}}_\ell[G] \otimes_{\overline{\mathbb{Q}}_\ell[H]} -$. In particular, $f_!$ is exact on the level of sheaves.

Let A be a $\overline{\mathbb{Q}}_\ell$-representation of H, and $B = \overline{\mathbb{Q}}_\ell[G] \otimes_{\overline{\mathbb{Q}}_\ell[H]} A$. Let $x_1, \cdots, x_m$ be an ordered basis for A with respect to which F^d is an upper triangular matrix

$$\begin{bmatrix} \alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_m \end{bmatrix}$$

with eigenvalues $\alpha_1, \cdots, \alpha_m$. Then B has an ordered basis

$$\begin{gathered} 1 \otimes x_1,\ F \otimes x_1,\ \cdots,\ F^{d-1} \otimes x_1, \\ 1 \otimes x_2,\ F \otimes x_2,\ \cdots,\ F^{d-1} \otimes x_2, \\ \cdots\ \cdots \\ 1 \otimes x_m,\ F \otimes x_m,\ \cdots,\ F^{d-1} \otimes x_m, \end{gathered}$$

with respect to which F is the matrix $\begin{bmatrix} M_1 & * & * \\ & \ddots & * \\ & & M_m \end{bmatrix}$, where $M_i = \begin{bmatrix} 0 & \cdots & 0 & \alpha_i \\ 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}$.

The characteristic polynomial of F on B is $\prod_{i=1}^m (t^d - \alpha_i)$.

Let K_0 be a complex of sheaves on $\operatorname{Spec} \mathbb{F}_{q^d}$. The eigenvalues of F on $\mathscr{H}^n(f_!K) = f_!\mathscr{H}^n(K)$ are all the d-th roots of the eigenvalues of F^d on $\mathscr{H}^n(K)$, so for every $s > 0$ we have

$$\sum_n \sum_{\mathscr{H}^n(f_!K),F} |\alpha|^s = d \sum_n \sum_{\mathscr{H}^n(K),F^d} |\alpha|^{s/d}.$$

This shows that $f_!K_0$ is a convergent complex on $\operatorname{Spec} \mathbb{F}_q$ if and only if K_0 is a convergent complex on $\operatorname{Spec} \mathbb{F}_{q^d}$.

Next we prove

$$c_v(\operatorname{Spec} \mathbb{F}_{q^d}, K_0) = c_v(\operatorname{Spec} \mathbb{F}_q, f_!K_0)$$

for every $v \geq 1$. Since $H^n(f_!K) = f_!H^n(K)$, and both sides are absolutely convergent series so that one can change the order of summation without changing the limit, it suffices to prove it when $K = A$ is a single representation concentrated in degree 0. Let us review this classical calculation. Use the notation as above. For each i, fix a d-th root $\alpha_i^{1/d}$ of α_i, and let ζ_d be a primitive d-th root of unity. Then the eigenvalues of F on B are $\zeta_d^k\alpha_i^{1/d}$, for $i = 1, \cdots, m$ and $k = 0, \cdots, d-1$.

If $d \nmid v$, then $\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^v}) = \emptyset$, so $c_v(\mathrm{Spec}\ \mathbb{F}_{q^d}, A) = 0$. On the other hand,

$$c_v(\mathrm{Spec}\ \mathbb{F}_q, f_!A) = \mathrm{Tr}(F^v, B) = \sum_{i,k} \zeta_d^{vk}\alpha_i^{v/d} = \sum_i \alpha_i^{v/d} \sum_{k=0}^{d-1} \zeta_d^{vk} = 0.$$

If $d|v$, then $\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^v}) = \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^d}) = \mathbb{Z}/d\mathbb{Z}$. So

$$c_v(\mathbb{F}_{q^d}, A) = d\mathrm{Tr}(F^v, A) = d\sum_i \alpha_i^{v/d}.$$

On the other hand,

$$c_v(\mathbb{F}_q, B) = \mathrm{Tr}(F^v, B) = \sum_{i,k} \zeta_d^{vk}\alpha_i^{v/d} = \sum_{i,k} \alpha_i^{v/d} = d\sum_i \alpha_i^{v/d}.$$

□

Next, we consider BG_0, for a finite group scheme G_0 over $\mathbb{F}_q$.

Lemma 4.1.6. *Let G_0 be a finite $\mathbb{F}_q$-group scheme, and let $\mathscr{F}_0$ be a sheaf on BG_0. Then $H_c^r(BG, \mathscr{F}) = 0$ for all $r \neq 0$, and $H_c^0(BG, \mathscr{F}) \simeq H^0(BG, \mathscr{F})$ has dimension at most* $\mathrm{rank}(\mathscr{F})$*. If $\mathscr{F}_0$ is ι-mixed, then the set of weights of $H_c^0(BG, \mathscr{F})$ is a subset of the weights of $\mathscr{F}_0$.*

Proof. By ([33], 7.12-7.14) we can replace G_0 by its maximal reduced closed subscheme, and assume G_0 is reduced, hence étale. Then G_0 is the same as a finite group $G(\mathbb{F})$ with a continuous action of $\mathrm{Gal}(\mathbb{F}_q)$ ([30], 7.8). We will also write G for the group $G(\mathbb{F})$, if there is no confusion. Since $\mathrm{Spec}\ \mathbb{F} \to BG$ is surjective, we see that there is no non-trivial stratification on BG. In particular, all sheaves on BG are lisse, and they are just $\overline{\mathbb{Q}}_\ell$-representations of G.

BG is quasi-finite and proper over $\mathbb{F}$, with finite diagonal, so by ([33], 5.8), $H_c^r(BG, \mathscr{F}) = 0$ for all $r \neq 0$. From ([33], 5.1), we see that if $\mathscr{F}$ is a sheaf on BG corresponding to the representation V of G, then $H_c^0(BG, \mathscr{F}) = V_G$ and $H^0(BG, \mathscr{F}) = V^G$, and there is a natural isomorphism

$$v \mapsto \sum_{g \in G} gv : V_G \longrightarrow V^G.$$

Therefore

$$h_c^0(BG, \mathscr{F}) = \dim V_G \leq \dim V = \mathrm{rank}(\mathscr{F}),$$

and if $\mathscr{F}_0$ is ι-mixed, then the weights of V_G form a subset of the weights of V. □

4.1.7. (i) If k is a field, by a *k-algebraic group G* we mean a smooth k-group scheme of finite type. If G is connected, then it is geometrically connected ([12], $\mathrm{VI_A}$, 2.1.1).

(ii) For a connected k-algebraic group G, let $a : BG \to \mathrm{Spec}\ k$ be the structural map. Then

$$a^* : \Lambda\text{-Sh}(\mathrm{Spec}\ k) \longrightarrow \Lambda\text{-Sh}(BG)$$

is an equivalence of categories. This is because

•BG has no non-trivial stratifications (it is covered by Spec k which has no non-trivial stratifications), and therefore

• any constructible Λ-adic sheaf on BG is lisse, given by an adic system $(M_n)_n$ of sheaves on Spec k with G-actions, and these actions are trivial since G is connected. See ([3], 5.2.9).

(iii) Let G_0 be a connected $\mathbb{F}_q$-algebraic group. By a theorem of Serge Lang ([23], Th. 2), every G_0-torsor over Spec $\mathbb{F}_q$ is trivial, with automorphism group G_0, therefore

$$c_v(BG_0) = \frac{1}{c_v(G_0)} = \frac{1}{\#G_0(\mathbb{F}_{q^v})}.$$

Recall the following theorem of Borel as in ([3], 6.1.6).

Theorem 4.1.8. *Let k be a field and G a connected k-algebraic group. Consider the Leray spectral sequence given by the projection $f : \mathrm{Spec}\ k \to BG$,*

$$E_2^{rs} = H^r(BG_{\overline{k}}) \otimes H^s(G_{\overline{k}}) \Longrightarrow \overline{\mathbb{Q}}_\ell.$$

Let $N^s = E_{s+1}^{0,s} \subset H^s(G_{\overline{k}})$ be the transgressive subspaces, for $s \geq 1$, and let N be the graded $\overline{\mathbb{Q}}_\ell$-vector space $\bigoplus_{s\geq 1} N^s$. We have

(a). $N^s = 0$ if s is even,

(b). the canonical map $\bigwedge N \longrightarrow H^*(G_{\overline{k}})$ is an isomorphism of graded $\overline{\mathbb{Q}}_\ell$-algebras.

(c). The spectral sequence above induces an epimorphism of graded $\overline{\mathbb{Q}}_\ell$-vector spaces $H^*(BG_{\overline{k}}) \twoheadrightarrow N[-1]$. Any section induces an isomorphism

$$\mathrm{Sym}^*(N[-1]) \xrightarrow{\sim} H^*(BG_{\overline{k}}).$$

Remark 4.1.8.1. (i) The E_2^{rs}-term of the Leray spectral sequence of f should be $H^r(BG_{\overline{k}}, R^sf_*\overline{\mathbb{Q}}_\ell)$, and $R^sf_*\overline{\mathbb{Q}}_\ell$ is a constructible sheaf on BG. By (4.1.7ii), the sheaf $R^sf_*\overline{\mathbb{Q}}_\ell$ is isomorphic to $a^*f^*R^sf_*\overline{\mathbb{Q}}_\ell = a^*H^s(G_{\overline{k}})$, where $a : BG \to \mathrm{Spec}\ k$ is the structural map and $H^s(G_{\overline{k}})$ is the $\mathrm{Gal}(k)$-module regarded as a sheaf on Spec k. Therefore by projection formula, $E_2^{rs} = H^r(BG_{\overline{k}}) \otimes H^s(G_{\overline{k}})$.

(ii) Since the spectral sequence converges to $\overline{\mathbb{Q}}_\ell$ sitting in degree 0, all E_∞^{rs} are zero, except E_∞^{00}. For each $s \geq 1$, consider the differential map $d_{s+1}^{0,s} : E_{s+1}^{0,s} \to E_{s+1}^{s+1,0}$ on the $(s+1)$st page. This map must be injective (resp. surjective) because it is the last possibly non-zero map from $E_*^{0,s}$ (resp. into $E_*^{s+1,0}$). Therefore, it is an isomorphism. Note that $N^s = E_{s+1}^{0,s}$

is a subspace of $E_2^{0,s} = H^s(G_{\overline{k}})$, and $E_{s+1}^{s+1,0}$ is a quotient of $E_2^{s+1,0} = H^{s+1}(BG_{\overline{k}})$. Using the isomorphism $d_{s+1}^{0,s}$ we get the surjection $H^{s+1}(BG_{\overline{k}}) \to N^s$.

4.1.8.2. Let G_0 be a connected $\mathbb{F}_q$-algebraic group of dimension d. We apply (4.1.8) to investigate the compact support cohomology groups $H_c^*(BG)$.

We have graded Galois-invariant subspaces $N = \bigoplus_{r\geq 1} N^r \subset \bigoplus_{r \geq 0} H^r(G)$ concentrated in odd degrees, such that the induced map

$$\bigwedge N \longrightarrow H^*(G)$$

is an isomorphism, and $H^*(BG) \cong \mathrm{Sym}^* N[-1]$. Let $n_r = \dim N^r$, and let $v_{r1}, \cdots, v_{rn_r}$ be a basis for N^r with respect to which the Frobenius acting on N^r is upper triangular

$$\begin{bmatrix} \alpha_{r1} & * & * \\ & \ddots & * \\ & & \alpha_{rn_r} \end{bmatrix}$$

with eigenvalues $\alpha_{r1}, \cdots, \alpha_{rn_r}$. By ([9], 3.3.5), the weights of $H^r(G)$ are $\geq r$, so $|\alpha_{ri}| \geq q^{r/2} > 1$. We have

$$H^*(BG) = \mathrm{Sym}^* \overline{\mathbb{Q}}_\ell \langle v_{ij} | \text{for all } i, j \rangle = \overline{\mathbb{Q}}_\ell [v_{ij}],$$

with $\deg(v_{ij}) = i + 1$. Note that all $i + 1$ are even. In particular, $H^{2r-1}(BG) = 0$ and

$$\begin{aligned} H^{2r}(BG) &= \{\text{homogeneous polynomials of degree } 2r \text{ in } v_{ij}\} \\ &= \overline{\mathbb{Q}}_\ell \langle \prod_{i,j} v_{ij}^{m_{ij}} ; \sum_{i,j} m_{ij}(i+1) = 2r \rangle. \end{aligned}$$

With respect to an appropriate order of the basis, the matrix representing F acting on $H^{2r}(BG)$ is upper triangular, with eigenvalues

$$\prod_{i,j} \alpha_{ij}^{m_{ij}}, \qquad \text{for } \sum_{i,j} m_{ij}(i+1) = 2r.$$

By Poincaré duality, the eigenvalues of F acting on $H_c^{-2r-2d}(BG)$ are $q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}}$, for tuples of non-negative integers $(m_{ij})_{i,j}$ such that $\sum_{i,j} m_{ij}(i+1) = 2r$. Therefore the reciprocal characteristic polynomial of F on $H_c^{-2r-2d}(BG)$ is

$$P_{-2r-2d}(BG_0, t) = \prod_{\substack{m_{ij} \geq 0 \\ \sum_{i,j} m_{ij}(i+1) = 2r}} \Big(1 - q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \cdot t \Big).$$

4.2 Finiteness theorem: Proof of (4.1.2i).

In the following two lemmas we prove two key cases of (4.1.2i).

Lemma 4.2.1. *Let G_0 be an $\mathbb{F}_q$-group scheme of finite type. Then (4.1.2i) holds for the structural map $f : BG_0 \to \operatorname{Spec} \mathbb{F}_q$.*

Proof. In fact, we will show that (4.1.2i) holds for all convergent complexes $K_0 \in W^-(BG_0, \overline{\mathbb{Q}}_\ell)$, without assuming it is ι-mixed or stratifiable.

By ([33], 7.12-7.14) we may assume G_0 is reduced, and hence smooth. Let G_0^0 be the identity component of G_0 and consider the exact sequence of algebraic groups

$$1 \longrightarrow G_0^0 \longrightarrow G_0 \longrightarrow \pi_0(G_0) \longrightarrow 1.$$

The fibers of the induced map $BG_0 \to B\pi_0(G_0)$ are isomorphic to BG_0^0, so we reduce to prove two cases: G_0 is finite étale (or even a finite constant group scheme, by (4.1.1.2iii)), or G_0 is connected and smooth.

Case of G_0 finite constant. Let $G_0/\mathbb{F}_q$ be the finite constant group scheme associated with a finite group G, and let $K_0 \in W_c^-(BG_0, \overline{\mathbb{Q}}_\ell)$. Again we denote by G both the group scheme $G_0 \otimes \mathbb{F}$ and the finite group $G_0(\mathbb{F})$, if no confusion arises. Let y be the unique point in $\operatorname{Spec} \mathbb{F}_q$,

$$\begin{tikzcd} BG \arrow[r] \arrow[d, "f_{\overline{y}}"'] & BG_0 \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{F} \arrow[r, "\overline{y}"] & \operatorname{Spec} \mathbb{F}_q. \end{tikzcd}$$

Then $D_c^-(BG, \overline{\mathbb{Q}}_\ell)$ is equivalent to $D_c^-(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$, and the functor

$$(f_{\overline{y}})_! : D_c^-(BG/\mathbb{F}, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^-(\operatorname{Spec} \mathbb{F}, \overline{\mathbb{Q}}_\ell)$$

is identified with the coinvariance functor

$$(\ )_G : D_c^-(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G)) \longrightarrow D_c^-(\overline{\mathbb{Q}}_\ell),$$

which is exact on the level of modules, since the category $\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ is semisimple. So $(f_!K_0)_{\overline{y}} = (f_{\overline{y}})_!K = K_G$ and $\mathscr{H}^n(K_G) = \mathscr{H}^n(K)_G$. Therefore

$$\sum_{\mathscr{H}^n((f_{\overline{y}})_!K),F} |\alpha|^s \leq \sum_{\mathscr{H}^n(K),F} |\alpha|^s$$

for every $n \in \mathbb{Z}$ and $s > 0$. Therefore, if K_0 is a convergent complex, so is $f_!K_0$.

Case of G_0 smooth and connected. In this case

$$f^* : \overline{\mathbb{Q}}_\ell\text{-Sh}(\operatorname{Spec} \mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell\text{-Sh}(BG_0)$$

is an equivalence of categories (4.1.7ii). Let $d = \dim G_0$, and let $\mathscr{F}_0$ be a sheaf on BG_0, corresponding to a representation V of the Weil group $W(\mathbb{F}_q)$, with $\beta_1, \cdots, \beta_m$ as eigenvalues of F. By the projection formula ([27], 9.1.i) we have $H^n_c(BG,\mathscr{F}) \simeq H^n_c(BG) \otimes V$, and by (4.1.8.2) the eigenvalues of F on $H^{-2r-2d}_c(BG) \otimes V$ are (using the notation in (4.1.8.2))

$$q^{-d}\beta_k \prod_{i,j} \alpha_{ij}^{-m_{ij}},$$

for $k = 1, \cdots, m$ and tuples (m_{ij}) such that $\sum_{i,j} m_{ij}(i+1) = 2r$. For every $s > 0$,

$$\sum_{n\in\mathbb{Z}} \sum_{H^n_c(BG)\otimes V,F} |\alpha|^s = \sum_{m_{ij},k} q^{-ds}|\beta_k|^s \prod_{i,j} |\alpha_{ij}^{-m_{ij}}|^s = \big(\sum_{k=1}^m |\beta_k|^s\big)\big(\sum_{m_{ij}} q^{-ds} \prod_{i,j} |\alpha_{ij}|^{-m_{ij}s}\big),$$

which converges to

$$q^{-ds}\Big(\sum_{k=1}^m |\beta_k|^s\Big) \prod_{i,j} \frac{1}{1-|\alpha_{ij}|^{-s}},$$

since $|\alpha_{ij}|^{-s} < 1$ and the product above is taken over finitely many indices.

Let K_0 be a convergent complex on BG_0, and for each $k \in \mathbb{Z}$, let V_k be a $W(\mathbb{F}_q)$-module corresponding to $\mathscr{H}^k K_0$. For every $x \in BG_0(\mathbb{F}_q)$ (for instance the trivial G_0-torsor), the pair $(\mathscr{H}^k(K)_{\overline{x}}, F_x)$ is isomorphic to (V_k, F). Consider the $W(\mathbb{F}_q)$-equivariant spectral sequence

$$H^r_c(BG, \mathscr{H}^k(K)) \Longrightarrow H^{r+k}_c(BG, K).$$

We have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \sum_{H^n_c(BG,K),F} |\alpha|^s &\le \sum_{n\in\mathbb{Z}} \sum_{r+k=n} \sum_{H^r_c(BG,\mathscr{H}^k K),F} |\alpha|^s = \sum_{r,k\in\mathbb{Z}} \sum_{H^r_c(BG)\otimes V_k,F} |\alpha|^s \\ &= \sum_{k\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \sum_{H^r_c(BG)\otimes V_k,F} |\alpha|^s = \sum_{k\in\mathbb{Z}} q^{-ds}\big(\sum_{V_k,F} |\alpha|^s\big) \prod_{i,j} \frac{1}{1-|\alpha_{ij}|^{-s}} \\ &= \big(\sum_{k\in\mathbb{Z}} \sum_{V_k,F} |\alpha|^s\big)\big(q^{-ds} \prod_{i,j} \frac{1}{1-|\alpha_{ij}|^{-s}}\big),\end{aligned}$$

where the first factor is convergent by assumption, and the product in the second factor is taken over finitely many indices. This shows that $f_!K_0$ is a convergent complex. □

Let E_λ be a finite subextension of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with ring of integers Λ and residue field Λ_0, and let $(\mathscr{S}, \mathcal{L})$ be a pair on $\mathscr{X}$ defined by simple lcc Λ_0-sheaves on strata. A complex $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be $(\mathscr{S}, \mathcal{L})$-*stratifiable* (or *trivialized by* $(\mathscr{S}, \mathcal{L})$), if K is defined over E_λ, with an integral model over Λ trivialized by $(\mathscr{S}, \mathcal{L})$.

Lemma 4.2.2. *Let $X_0/\mathbb{F}_q$ be a geometrically connected variety, E_λ a finite subextension of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with ring of integers Λ, and let $\mathcal{L}$ be a finite set of simple lcc Λ_0-sheaves on X. Then (4.1.2i) holds for the structural map $f : X_0 \to \operatorname{Spec} \mathbb{F}_q$ and all lisse ι-mixed convergent complexes K_0 on X_0 that are trivialized by $(\{X\}, \mathcal{L})$.*

Proof. Let $N = \dim X_0$. From the spectral sequence

$$E_1^{rk} = H_c^r(X, \mathscr{H}^k K) \Longrightarrow H_c^{r+k}(X, K)$$

we see that

$$\sum_{n\in\mathbb{Z}} \sum_{H_c^n(X,K),F} |\alpha|^s \le \sum_{n\in\mathbb{Z}} \sum_{r+k=n} \sum_{H_c^r(X,\mathscr{H}^k K),F} |\alpha|^s = \sum_{\substack{0\le r\le 2N\\ k\in\mathbb{Z}}} \sum_{H_c^r(X,\mathscr{H}^k K),F} |\alpha|^s,$$

therefore it suffices to show that, for each $0 \le r \le 2N$, the series $\sum_{k\in\mathbb{Z}} \sum_{H_c^r(X,\mathscr{H}^k K),F} |\alpha|^s$ converges.

Let d be the number in (3.2.5) for $\mathcal{L}$. Each cohomology sheaf $\mathscr{H}^n K_0$ is ι-mixed and lisse on X_0, so by (2.2.3i) we have the decomposition

$$\mathscr{H}^n K_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} (\mathscr{H}^n K_0)(b)$$

according to the weights mod $\mathbb{Z}$, defined over E_λ (2.2.3.1ii). For each coset b, we choose a representative $b_0 \in b$, and take a $b_1 \in \overline{\mathbb{Q}}_\ell^*$ such that $w_q(b_1) = -b_0$. Then the sheaf $(\mathscr{H}^n K_0)(b)^{(b_1)}$ deduced by torsion is lisse with integer punctual weights. Let W be the filtration by punctual weights (2.2.3ii) of $(\mathscr{H}^n K_0)(b)^{(b_1)}$. For every $v \ge 1$ and $x \in X_0(\mathbb{F}_{q^v})$, and every real $s > 0$, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \sum_{(\mathscr{H}^n K_0)_{\overline{x}},F_x} |\alpha|^{s/v} &= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{(\mathscr{H}^n K_0)(b)_{\overline{x}},F_x} |\alpha|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{(\mathscr{H}^n K_0)(b)_{\overline{x}}^{(b_1)},F_x} |\alpha/b_1^v|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{(\mathscr{H}^n K_0)(b)_{\overline{x}}^{(b_1)},F_x} |\alpha|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} \sum_{\operatorname{Gr}_i^W((\mathscr{H}^n K_0)(b)^{(b_1)})_{\overline{x}},F_x} |\alpha|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} q^{is/2} \cdot \operatorname{rank}(\operatorname{Gr}_i^W((\mathscr{H}^n K_0)(b)^{(b_1)})).
\end{aligned}$$

Since K_0 is a convergent complex, this series is convergent.

For each $n \in \mathbb{Z}$, every direct summand $(\mathscr{H}^n K_0)(b)$ of $\mathscr{H}^n K_0$ is trivialized by $(\{X\}, \mathcal{L})$. The filtration W of each $(\mathscr{H}^n K_0)(b)^{(b_1)}$ gives a filtration of $(\mathscr{H}^n K_0)(b)$ (also denoted W) by twisting back, and it is clear that this latter filtration is defined over E_λ. We have $\mathrm{Gr}_i^W((\mathscr{H}^n K_0)(b)^{(b_1)}) = (\mathrm{Gr}_i^W(\mathscr{H}^n K_0)(b))^{(b_1)}$, and each $\mathrm{Gr}_i^W((\mathscr{H}^n K_0)(b))$ is trivialized by $(\{X\}, \mathcal{L})$. By (3.2.5),

$$\begin{aligned} h_c^r(X, \mathrm{Gr}_i^W((\mathscr{H}^n K)(b)^{(b_1)})) &= h_c^r(X, \mathrm{Gr}_i^W((\mathscr{H}^n K)(b))) \qquad ([27], 9.1.\mathrm{i}) \\ &\leq d \cdot \mathrm{rank}(\mathrm{Gr}_i^W((\mathscr{H}^n K)(b))) \\ &= d \cdot \mathrm{rank}(\mathrm{Gr}_i^W((\mathscr{H}^n K)(b)^{(b_1)})). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \sum_{H_c^r(X,\mathscr{H}^n K),F} |\alpha|^s &= \sum_{\substack{n\in\mathbb{Z} \\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{H_c^r(X,(\mathscr{H}^n K)(b)),F} |\alpha|^s \\ &= \sum_{\substack{n\in\mathbb{Z} \\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{H_c^r(X,(\mathscr{H}^n K)(b)^{(b_1)}),F} |b_1^{-1}\alpha|^s \\ &\leq \sum_{\substack{n\in\mathbb{Z} \\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} \sum_{H_c^r(X,\mathrm{Gr}_i^W((\mathscr{H}^n K)(b)^{(b_1)})),F} |\alpha|^s \\ &\leq \sum_{\substack{n\in\mathbb{Z} \\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} q^{(i+r)s/2} \cdot h_c^r(X, \mathrm{Gr}_i^W((\mathscr{H}^n K)(b)^{(b_1)})) \\ &\leq q^{rs/2} d \sum_{\substack{n\in\mathbb{Z} \\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} q^{is/2} \cdot \mathrm{rank}(\mathrm{Gr}_i^W((\mathscr{H}^n K)(b)^{(b_1)})), \end{aligned}$$

and it converges. □

Now we prove (4.1.2i) in general.

Proof. We may assume all stacks involved are reduced. From (2.2.8) and (3.2.4) we know that $f_! K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell)$.

Let $y : \mathrm{Spec}\ \mathbb{F}_{q^d} \to \mathscr{Y}_0$ be an $\mathbb{F}_q$-morphism, and we want to show that $((f_! K_0)_{\overline{y}}, F_y)$ is a convergent complex. Since the property of being convergent depends only on the cohomology sheaves, by ([27], 12.5.3) we reduce to the case when $\mathscr{Y}_0 = \mathrm{Spec}\ \mathbb{F}_{q^d}$. Replacing q by q^d, we may assume $d = 1$. By (4.1.1.2iii) we only need to show that $(R\Gamma_c(\mathscr{X}, K), F)$ is convergent.

If $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ is an open substack with complement $i : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$, then we have an exact triangle

$$j_! j^* K_0 \longrightarrow K_0 \longrightarrow i_* i^* K_0 \longrightarrow$$

in $W_c^-(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$, which induces an exact triangle

$$R\Gamma_c(\mathscr{U}_0,j^*K_0)\longrightarrow R\Gamma_c(\mathscr{X}_0,K_0)\longrightarrow R\Gamma_c(\mathscr{Z}_0,i^*K_0)\longrightarrow$$

in $W^-(\mathrm{Spec}\ \mathbb{F}_q,\overline{\mathbb{Q}}_\ell)$. So by (4.1.4) and noetherian induction, it suffices to prove (4.1.2i) for a nonempty open substack. By ([3], 5.1.14) we may assume that the inertia stack $\mathscr{I}_0$ is flat over $\mathscr{X}_0$. Then we can form the rigidification $\pi:\mathscr{X}_0\to X_0$ with respect to $\mathscr{I}_0$ ([34], §1.5), where X_0 is an $\mathbb{F}_q$-algebraic space of quasi-compact diagonal. X_0 contains an open dense subscheme ([21], II, 6.7). Replacing $\mathscr{X}_0$ by the inverse image of this scheme, we can assume X_0 is a scheme.

If (4.1.2i) holds for two composable morphisms f and g, then it holds for their composition $g\circ f$. Since $R\Gamma_c(\mathscr{X}_0,-)=R\Gamma_c(X_0,-)\circ\pi_!$, we reduce to prove (4.1.2i) for these two morphisms. For every $x\in X_0(\mathbb{F}_{q^v})$, the fiber of π over x is a gerbe over Spec $k(x)$. Extending the base $k(x)$ (4.1.1.2iii) one can assume it is a neutral gerbe (in fact all gerbes over a finite field are neutral; see ([3], 6.4.2)). This means the following diagram is 2-Cartesian:

$$\begin{tikzcd} B\mathrm{Aut}_x \arrow[r] \arrow[d] & \mathscr{X}_0 \arrow[d, "\pi"] \\ \mathrm{Spec}\ \mathbb{F}_{q^v} \arrow[r, "x"] & X_0. \end{tikzcd}$$

So we reduce to two cases: $\mathscr{X}_0=BG_0$ for an algebraic group $G_0/\mathbb{F}_q$, or $\mathscr{X}_0=X_0$ is an $\mathbb{F}_q$-scheme. The first case is proved in (4.2.1).

For the second case, given a convergent complex $K_0\in W_m^{-,\mathrm{stra}}(X_0,\overline{\mathbb{Q}}_\ell)$, defined over some E_λ with ring of integers Λ, and trivialized by a pair $(\mathscr{S},\mathcal{L})$ ($\mathcal{L}$ being defined over Λ_0) on X, we can refine this pair so that every stratum is connected, and replace X_0 by models of the strata over some finite extension of $\mathbb{F}_q$ (4.1.1.2iii). This is proved in (4.2.2). □

4.3 Trace formula for stacks.

4.3.1. Before proving (4.1.2ii), we give some notations that will be used in this section.

For a finite étale $\mathbb{F}_q$-scheme X_0, we will denote by X both the scheme $X_0\otimes\mathbb{F}$ and the finite set $X_0(\mathbb{F})$, with the map $F_{X_0}(\mathbb{F})=\sigma_{X_0}(\mathbb{F})$ on it written as $\sigma_X:X\to X$, if there is no confusion.

Every action of the group $\widehat{\mathbb{Z}}$ on a finite set X is continuous (since any subgroup of finite index in $\widehat{\mathbb{Z}}$, for instance the kernel of $\widehat{\mathbb{Z}}\to\mathrm{Aut}(X)$, is open). By descent theory (for instance ([30], 7.8)), the functor

$$X_0\mapsto(X,\sigma_X)$$

is an equivalence from the category $\mathrm{FEt}/\mathbb{F}_q$ (resp. $\mathrm{Gp}(\mathrm{FEt}/\mathbb{F}_q)$) of finite étale $\mathbb{F}_q$-schemes (resp. finite étale $\mathbb{F}_q$-group schemes) to the category of finite sets with a $\widehat{\mathbb{Z}}$-action (resp.

finite groups with a $\widehat{\mathbb{Z}}$-action by group homomorphisms), which is equivalent to the category FSet^σ (resp. FGp^σ) of pairs (X, σ), where X is a finite set (resp. a finite group) and σ is a permutation (resp. a group automorphism) on X.

Lemma 4.3.2. *Let G_0 be an $\mathbb{F}_q$-group scheme. Then $F_{G_0} : G_0 \to G_0$ is a homomorphism of group schemes, and the induced morphism $BF_{G_0} : BG_0 \to BG_0$ is 2-isomorphic to the morphism $F_{BG_0} : BG_0 \to BG_0$.*

Proof. For any $\mathbb{F}_q$-scheme T and any $g \in G_0(T)$, we have a commutative diagram

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "F_T"'] & G_0 \arrow[d, "F_{G_0}"] \\ T \arrow[r, "g"] & G_0. \end{tikzcd}$$

So $F_{G_0}(T)(g) := F_{G_0} \circ g = g \circ F_T =: G_0(F_T)(g)$, and $F_{G_0}(T)$ is a group homomorphism, since $G_0(F_T)$ is. This shows that F_{G_0} is a homomorphism of group schemes.

Next we compare BF_{G_0} with F_{BG_0}. Let $a \in BG_0(T)$ correspond to the G_0-torsor $P \xrightarrow{a} T$ over T with G_0-action

$$\begin{tikzcd} P \times_{\mathbb{F}_q} G_0 \arrow[r, "\sim"] \arrow[d, "\mathrm{pr}_1"'] & P \times_T (T \times_{\mathbb{F}_q} G_0) \arrow[r, "\text{action}"] \arrow[dr] & P \arrow[d, "a"] \\ P \arrow[rr, "a"] & & T \end{tikzcd}.$$

Then $(BF_{G_0})(T)(a)$ is the G_0-torsor

$$P \times^{G_0, F_{G_0}} G_0 \longrightarrow T,$$

and $F_{BG_0}(T)(a)$ is the fiber product

$$\begin{tikzcd} P \times_{T, F_T} T \arrow[r] \arrow[d, "\mathrm{pr}_2"'] & P \arrow[d, "a"] \\ T \arrow[r, "F_T"'] & T. \end{tikzcd}$$

We shall define a natural map of G_0-torsors

$$P \times^{G_0, F_{G_0}} G_0 \longrightarrow P \times_{T, F_T} T,$$

which will automatically be an isomorphism. By the universal property of the pushed forward torsor, it suffices to define a G_0-equivariant map $P \to P \times_{T,F_T} T$. Define it to be (F_P, a) :

$$\begin{tikzcd} P \arrow[r, dashed, "{(F_P,a)}"] \arrow[rrr, bend left, "F_P"] \arrow[dr, "a"'] & P \times_{T,F_T} T \arrow[rr, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & & P \arrow[d, "a"] \\ & T \arrow[rr, "F_T"] & & T. \end{tikzcd}$$

Note that it is functorial in T and P. The G_0-equivariance of this map is equivalent to the commutativity of the following diagram

$$\begin{tikzcd} P \times G_0 \arrow[r, "\text{action}"] \arrow[d, "{(F_P,a)\times F_{G_0}}"'] & P \arrow[d, "{(F_P,a)}"] \\ (P \times_{T,F_T} T) \times G_0 \arrow[r, "\text{action}"] & P \times_{T,F_T} T, \end{tikzcd}$$

which can be verified on the two factors of $P \times_{T,F_T} T$.

The commutativities of

$$\begin{tikzcd} P \times G_0 \arrow[r, "\text{action}"] \arrow[d, "\mathrm{pr}_1"'] & P \arrow[d, "a"] \\ P \arrow[r, "a"'] & T \end{tikzcd}$$

and

$$\begin{tikzcd} P \times G_0 \arrow[r, "{(F_P,a)\times F_{G_0}}"] \arrow[d, "\mathrm{pr}_1"] & (P \times_{T,F_T} T) \times G_0 \arrow[r, "\text{action}"] \arrow[dr, "\mathrm{pr}_2"] & P \times_{T,F_T} T \arrow[d, "\mathrm{pr}_2"] \\ P \arrow[rr, "a"'] & & T, \end{tikzcd}$$

shows the commutativity when projected to the second factor T, and the commutativities of

$$\begin{tikzcd} P \times G_0 \arrow[r, "\text{action}"] \arrow[d, "F_P \times F_{G_0}"'] & P \arrow[r, "{(F_P,a)}"] \arrow[d, "F_P"'] & P \times_{T,F_T} T \arrow[dl, "\mathrm{pr}_1"] \\ P \times G_0 \arrow[r, "\text{action}"] & P & \end{tikzcd}$$

(since the action of G_0 on P is defined over $\mathbb{F}_q$ and $F_P \times F_{G_0} = F_{P \times G_0}$) and

$$\begin{tikzcd} P \times G_0 \arrow[r, "{(F_P,a)\times F_{G_0}}"] \arrow[dr, "F_P \times F_{G_0}"'] & G_0 \times (P \times_{T,F_T} T) \arrow[r, "\text{action}"] \arrow[d, "\mathrm{pr}_{13}"] & P \times_{T,F_T} T \arrow[d, "\mathrm{pr}_1"] \\ & P \times G_0 \arrow[r, "\text{action}"'] & P, \end{tikzcd}$$

gives the commutativity when projected to the first factor P. This finishes the proof that $BF_{G_0} \cong F_{BG_0}$. $\square$

We prove two special cases of (4.1.2ii) in the following two lemmas.

Proposition 4.3.3. *Let G_0 be a finite étale group scheme over $\mathbb{F}_q$, and $\mathscr{F}_0$ a sheaf on BG_0. Then*

$$c_1(BG_0, \mathscr{F}_0) = c_1(\mathrm{Spec}\ \mathbb{F}_q, R\Gamma_c(BG_0, \mathscr{F}_0)).$$

Proof. This is a special case of ([33], 8.6), on correspondences given by group homomorphisms, due to Olsson. For the reader's convenience, we apply Olsson's idea and prove it again.

Let $(G, \sigma) \in \mathrm{FGp}^\sigma$ correspond to G_0. Let $x_0 \in BG_0(\mathbb{F}_q)$ correspond to the trivial G_0-torsor on Spec $\mathbb{F}_q$. Let $V = \mathscr{F}_{\overline{x}_0}$ be the representation of G corresponding to the sheaf $\mathscr{F}$ on BG, and let $\tau : V \to V$ be the map F_{x_0}, which gives a $W(\mathbb{F}_q)$-module structure on V. The action of G on V is semi-linear for the action of $W(\mathbb{F}_q)$, i.e. if $\rho : G \to \mathrm{GL}(V)$ is the G-action, then

$$\tau(\rho(g) \cdot v) = \rho(\sigma g) \cdot \tau(v), \text{ for } g \in G,\ v \in V.$$

In particular, $\tau(V^G) \subset V^G$. By (4.1.6) we see that $H^0_c(BG, \mathscr{F}) = V_G = V^G$ is a $W(\mathbb{F}_q)$-submodule of V, and $\tau|_{V_G}$ is just the global Frobenius action F. So

$$c_1(\mathrm{Spec}\ \mathbb{F}_q, R\Gamma_c(BG_0, \mathscr{F}_0)) = \mathrm{Tr}(\tau|_{V_G}, V_G).$$

To compute $c_1(BG_0, \mathscr{F}_0)$, we shall realize the groupoid of rational points $BG_0(\mathbb{F}_q)$ as the $\mathbb{F}$-points of the fixed point stack. Let $\mathrm{Fix}(B\sigma)$ be the fixed point stack of the endomorphism $B\sigma$ on BG ([33], 1.4), i.e. the 2-fiber product over $\mathbb{F}$:

$$\begin{tikzcd} \mathrm{Fix}(B\sigma) \arrow[r] \arrow[d] & BG \arrow[d, "{(1,B\sigma)}"] \\ BG \arrow[r, "\Delta"] & BG \times BG. \end{tikzcd}$$

Lemma 4.3.3.1. *There is a natural equivalence of groupoids $BG_0(\mathbb{F}_q) \to \mathrm{Fix}(B\sigma)(\mathbb{F})$.*

Proof. The groupoid $BG_0(\mathbb{F}_q)$ consists of G_0-torsors over Spec $\mathbb{F}_q$ and isomorphisms between them. The groupoid $\mathrm{Fix}(B\sigma)(\mathbb{F})$ has pairs (P, μ) as objects, where P is a G-torsor over Spec $\mathbb{F}$ (which can be identified with a set-theoretic torsor) and $\mu : P \to B\sigma(P)$ is an isomorphism of G-torsors, with morphisms of torsors compatible with μ.

$B\sigma(P) = P \times^{G,\sigma} G$, the G-torsor with underlying set the quotient of $P \times G$ by the equivalence relation $(p, h) \sim (pg^{-1}, \sigma(g)h)$, and the G-action given by $(p, h)g = (p, hg)$. It is isomorphic to the G-torsor P^σ with underlying set P and G-action $p * g = p\sigma^{-1}(g)$, via the isomorphism

$$[p, h] \mapsto p * h : P \times^{G,\sigma} G \longrightarrow P^\sigma,$$

which is easily verified to be well-defined.

Define the functor $BG_0(\mathbb{F}_q) \to \mathrm{Fix}(B\sigma)(\mathbb{F})$ as follows. Given a G_0-torsor P_0, let P be the G-torsor on Spec $\mathbb{F}$ obtained by base extension, regarded as a set-theoretic torsor, and define $\mu : P \to P^\sigma$ to be $\mu(p) = \sigma_P^{-1}(p)$. Since the action of G_0 on P_0 is defined over $\mathbb{F}_q$, the diagram

$$\begin{tikzcd} P \times G \arrow[r, "\text{action}"] \arrow[d, "\sigma_P \times \sigma_G"'] & P \arrow[d, "\sigma_P"] \\ P \times G \arrow[r, "\text{action}"] & P \end{tikzcd}$$

commutes, and so μ is G-equivariant:

$$\mu(pg) = \sigma_P^{-1}(pg) = \sigma_P^{-1}(p)\sigma^{-1}(g) = \sigma_P^{-1}(p) * g = \mu(p) * g.$$

If $f_0 : P_0 \to P_0'$ is an isomorphism of G_0-torsors, the commutativity of

$$\begin{tikzcd} P_0 \arrow[r, "f_0"] \arrow[d, "F_{P_0}"'] & P_0' \arrow[d, "F_{P_0'}"] \\ P_0 \arrow[r, "f_0"] & P_0' \end{tikzcd}$$

gives the commutativity of

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\mu"'] & P' \arrow[d, "\mu'"] \\ P^\sigma \arrow[r, "f^\sigma"] & P'^\sigma. \end{tikzcd}$$

This defines a functor $BG_0(\mathbb{F}_q) \to \mathrm{Fix}(B\sigma)(\mathbb{F})$. We will show that it is fully faithful and essentially surjective.

Let P_0 and $P_0' \in BG_0(\mathbb{F}_q)$, with images (P, μ) and $(P', \mu') \in \mathrm{Fix}(B\sigma)(\mathbb{F})$. Then the Hom set $\mathrm{Hom}_{BG_0(\mathbb{F}_q)}(P_0, P_0')$ consists of morphisms $f_0 : P_0 \to P_0'$ of $\mathbb{F}_q$-schemes such that the diagrams

$$\begin{tikzcd} P_0 \times G_0 \arrow[r, "\text{action}"] \arrow[d, "f_0 \times 1"'] & P_0 \arrow[d, "f_0"] \\ P_0' \times G_0 \arrow[r, "\text{action}"'] & P_0' \end{tikzcd}$$

commute. This Hom set is in natural bijection with the set of $\mathbb{F}_q$-morphisms f_0 such that the diagrams of finite sets

$$\begin{tikzcd} P \times G \arrow[r, "\text{action}"] \arrow[d, "f \times 1"'] & P \arrow[d, "f"] \\ P' \times G \arrow[r, "\text{action}"'] & P' \end{tikzcd}$$

commute, because the functor

$$\mathrm{FEt}/\mathbb{F}_q \xrightarrow{\sim} \mathrm{FSet}^\sigma \xrightarrow{\text{forget}} \mathrm{FSet}$$

sending $X_0 \mapsto X$ is faithful. Hence it is in natural bijection with the set of G-torsor homomorphisms $f : P \to P'$ such that

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\mu"'] & P' \arrow[d, "\mu'"] \\ P^\sigma \arrow[r, "f^\sigma"'] & P'^\sigma \end{tikzcd}$$

commutes. This set is $\mathrm{Hom}_{\mathrm{Fix}(B\sigma)(\mathbb{F})}((P,\mu),(P',\mu'))$. Therefore, the functor is fully faithful.

Given a set-theoretic G-torsor P with an isomorphism $\mu : P \to P^\sigma$ of G-torsors, we regard μ as a permutation of the finite set P such that $\mu(pg) = \mu(p)\sigma^{-1}(g)$, for $p \in P, g \in G$. By descent theory, the pair (P, μ^{-1}) descends to a (unique up to isomorphism) finite étale $\mathbb{F}_q$-scheme P_0. We will show that the G-torsor structure on P also descends to a G_0-torsor structure on P_0.

The condition $\mu(pg) = \mu(p)\sigma^{-1}(g)$ is equivalent to the commutativity of the diagram

$$\begin{tikzcd} P \times G \arrow[r, "\text{action}"] \arrow[d, "\sigma_P \times \sigma"'] & P \arrow[d, "\sigma_P"] \\ P \times G \arrow[r, "\text{action}"'] & P, \end{tikzcd}$$

since σ_P is identified with μ^{-1}. This implies that the action map

$$P \times G \longrightarrow P$$

descends to an $\mathbb{F}_q$-morphism

$$P_0 \times_{\mathbb{F}_q} G_0 \longrightarrow P_0.$$

The facts that this defines an action of G_0 on P_0 and that this action makes P_0 into a G_0-torsor over $\mathrm{Spec}\ \mathbb{F}_q$, i.e. the commutativity of certain diagrams and the fact that

$$(\mathrm{pr}_1, \text{action}) : P_0 \times_{\mathbb{F}_q} G_0 \longrightarrow P_0 \times_{\mathbb{F}_q} P_0$$

is an isomorphism, can be easily verified in FSet^σ. This shows the essential surjectivity of the functor $BG_0(\mathbb{F}_q) \to \mathrm{Fix}(B\sigma)(\mathbb{F})$. $\square$

Let $(P,\mu) \in \mathrm{Fix}(B\sigma)(\mathbb{F})$. Then P is always a trivial torsor, and let us take a section $e \in P$. Then μ is determined by some $g \in G$ such that $\mu(e) = e * g$. If we choose another section $eh \in P$, then

$$\begin{aligned}\mu(eh) = \mu(e) * h = e * (gh) &= (eh)h^{-1}\sigma^{-1}(gh) \\ &= (eh)\sigma^{-1}(\sigma(h^{-1})gh) = (eh) * (\sigma(h^{-1})gh).\end{aligned}$$

This shows the category $\mathrm{Fix}(B\sigma)(\mathbb{F})$ is equivalent to the quotient $G/\rho^{(1)}$, where $\rho^{(1)}$ is the right action of G on G given by $h : g \mapsto \sigma(h^{-1})gh$. Here we regard the quotient $G/\rho^{(1)}$, or more accurately, the action $\rho^{(1)}$, as a groupoid whose objects are elements in G and

$$\mathrm{Hom}(g_1, g_2) = \{h \in G | \rho^{(1)}(h)(g_1) = g_2\}.$$

See ([33], 7.9, 7.10).

Fix an element $g \in G$. Its orbit $[g]$ in the quotient $G/\rho^{(1)}$ determines a pair $(P_g, \mu_g) \in \mathrm{Fix}(B\sigma)(\mathbb{F})$, where $P_g = G$ is the trivial G-torsor with right multiplication action, $e \in G$ is the identity element, and $\mu_g(e) = e * g = \mu^{-1}(g)$. Let $x_g \in BG_0(\mathbb{F}_q)$ be the corresponding rational point, determined up to isomorphism. Note that x_e is the trivial torsor $x_0 \in BG_0(\mathbb{F}_q)$ we had before.

Let us denote by $\mathscr{F}_g$ the stalk of $\mathscr{F}_0$ at the geometric point $\overline{x}_g$, with the local Frobenius automorphism F_{x_g}. Then we claim that the eigenvalues of F_{x_g} on $\mathscr{F}_g$ are the same as that of τg on V (recall that $V := \mathscr{F}_{\overline{x}_0}$ and $\tau := F_{x_0}$).

The map $F_{x_g} : \mathscr{F}_g \to \mathscr{F}_g$ is induced by the 2-isomorphism

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{F} \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{x}_g"'] & \mathrm{Spec}\ \mathbb{F} \arrow[d, "\overline{x}_g"] \arrow[dl, Rightarrow, "\mu_g"] \\ BG \arrow[r, "B\sigma"'] & BG. \end{tikzcd}$$

The automorphism group of the trivial torsor P_e is G, so g gives a 2-isomorphism

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{F} \arrow[r, bend left, "\overline{x}_0"] \arrow[r, bend right, "\overline{x}_0"'] \arrow[r, phantom, "\Downarrow g"] & BG. \end{tikzcd}$$

Regard the identity map $G \to G$ as a fixed isomorphism $\lambda : P_g \to P_e$ of trivial G-torsors. This is a 2-isomorphism

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{F} \arrow[r, bend left, "\overline{x}_g"] \arrow[r, bend right, "\overline{x}_0"'] \arrow[r, phantom, "\Downarrow \lambda"] & BG. \end{tikzcd}$$

Since $\mu_e = \sigma^{-1}$, the following diagram of G-torsors commutes:

$$\begin{tikzcd}
P_g \arrow[r, "\mu_g"] & P_g^\sigma \arrow[dr, "\lambda^\sigma"] & \\
P_e \arrow[u, "\lambda^{-1}"] \arrow[r, "g"'] & P_e \arrow[r, "\mu_e"'] & P_e^\sigma .
\end{tikzcd}$$

This means the following two composite 2-isomorphisms are the same:

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{F} \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{x}_0"'] & \operatorname{Spec} \mathbb{F} \arrow[dl, Rightarrow, "\mu_e"'] \arrow[d, "\overline{x}_0"'] \arrow[d, bend left=60, "\overline{x}_0"] \arrow[d, phantom, shift left=3, "\Leftarrow g"] \\
BG \arrow[r, "B\sigma"'] & BG
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\operatorname{Spec} \mathbb{F} \arrow[r, "\mathrm{id}"] \arrow[d, bend right=60, "\overline{x}_0"'] \arrow[d, phantom, shift right=3, "\Leftarrow \lambda"] \arrow[d, "\overline{x}_g"] & \operatorname{Spec} \mathbb{F} \arrow[dl, Rightarrow, "\mu_g"] \arrow[d, "\overline{x}_g"'] \arrow[d, bend left=60, "\overline{x}_0"] \arrow[d, phantom, shift left=3, "\Leftarrow \lambda^{-1}"] \\
BG \arrow[r, "B\sigma"'] & BG .
\end{tikzcd}$$

Conjugate matrices have the same eigenvalues, so by pulling back along these 2-isomorphisms, we conclude that the eigenvalues of F_{x_g} on $\mathscr{F}_g$ are the same as the eigenvalues of τg on V. In particular,

$$\operatorname{Tr}(F_{x_g}, \mathscr{F}_g) = \operatorname{Tr}(\tau g, V).$$

We have

$$\begin{aligned}
c_1(BG_0, \mathscr{F}_0) &= \sum_{x \in [BG_0(\mathbb{F}_q)]} \frac{1}{\#\mathrm{Aut}_x(\mathbb{F}_q)} \operatorname{Tr}(F_x, \mathscr{F}_{\overline{x}}) = \frac{1}{\#G} \sum_{[g] \in G/\rho^{(1)}} \frac{\#G}{\#\mathrm{Stab}(g)} \operatorname{Tr}(F_{x_g}, \mathscr{F}_g) \\
&= \frac{1}{\#G} \sum_{[g] \in G/\rho^{(1)}} \#\mathrm{Orb}(g) \operatorname{Tr}(F_{x_g}, \mathscr{F}_g) = \frac{1}{\#G} \sum_{g \in G} \operatorname{Tr}(F_{x_g}, \mathscr{F}_g) \\
&= \frac{1}{\#G} \sum_{g \in G} \operatorname{Tr}(\tau g, V) = \operatorname{Tr}(\tau \circ \frac{1}{\#G} \sum_{g \in G} g, V).
\end{aligned}$$

Consider the map

$$\gamma := \frac{1}{\#G} \sum_{g \in G} g : \ V \longrightarrow V.$$

Its image is V^G, and the natural inclusion $V^G \hookrightarrow V$ gives a splitting of the surjection $\gamma : V \to V^G$, so $V = V^G \oplus W$ and with this identification, γ is the projection to V^G followed by the inclusion. Since τ preserves V^G, the map $\tau \circ \gamma : V \to V$ can be written as the block matrix

$$\begin{bmatrix} \tau|_{V^G} & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $V = V^G \oplus W$. Its trace is obviously equal to $\operatorname{Tr}(\tau|_{V^G}, V^G)$.

This shows
$$c_1(BG_0,\mathscr{F}_0)=c_1(\text{Spec }\mathbb{F}_q,R\Gamma_c(BG_0,\mathscr{F}_0)).$$
$\square$

Proposition 4.3.4. *Let G_0 be a connected $\mathbb{F}_q$-algebraic group, and let $\mathscr{F}_0$ be a sheaf on BG_0. Then*
$$c_1(BG_0,\mathscr{F}_0)=c_1(\text{Spec }\mathbb{F}_q,R\Gamma_c(BG_0,\mathscr{F}_0)).$$

Proof. Let $f:BG_0\to\text{Spec }\mathbb{F}_q$ be the structural map and $d=\dim G_0$. By (4.1.7ii), the sheaf $\mathscr{F}_0$ on BG_0 takes the form f^*V, for some sheaf V on Spec $\mathbb{F}_q$. By (4.1.7iii), we have
$$c_1(BG_0,\mathscr{F}_0)=\frac{1}{\#G_0(\mathbb{F}_q)}\text{Tr}(F_x,\mathscr{F}_{\overline{x}})=\frac{\text{Tr}(F,V)}{\#G_0(\mathbb{F}_q)}.$$
By the projection formula we have $H^n_c(BG,\mathscr{F})\simeq H^n_c(BG)\otimes V$, so
$$\text{Tr}(F,H^n_c(BG,\mathscr{F}))=\text{Tr}(F,H^n_c(BG))\cdot\text{Tr}(F,V).$$
Then
$$\begin{aligned}c_1(\text{Spec }\mathbb{F}_q,R\Gamma_c(BG_0,\mathscr{F}_0))&=\sum_n(-1)^n\text{Tr}(F,H^n_c(BG,\mathscr{F}))\\&=\text{Tr}(F,V)\sum_n(-1)^n\text{Tr}(F,H^n_c(BG)),\end{aligned}$$
so we can assume $\mathscr{F}_0=\overline{\mathbb{Q}}_\ell$. Using the notations in (4.1.8.2) we have
$$\begin{aligned}\sum_n(-1)^n\text{Tr}(F,H^n_c(BG))=\sum_{r\geq0}\text{Tr}(F,H_c^{-2r-2d}(BG))&=\sum_{\substack{\sum m_{ij}(i+1)=2r\\ m_{ij}\geq0}}q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}}\\&=q^{-d}\sum_{m_{ij}\geq0}\prod_{i,j}\alpha_{ij}^{-m_{ij}}=q^{-d}\prod_{i,j}(1+\alpha_{ij}^{-1}+\alpha_{ij}^{-2}+\cdots)\\&=q^{-d}\prod_{i,j}\frac{1}{1-\alpha_{ij}^{-1}}.\end{aligned}$$
It remains to show
$$\#G_0(\mathbb{F}_q)=q^d\prod_{i,j}(1-\alpha_{ij}^{-1}).$$

In (4.1.8.2), we saw that if each N^i has an ordered basis $v_{i1},\cdots,v_{in_i}$ with respect to which F is upper triangular, then since $H^*(G)=\bigwedge N,\ H^i(G)$ has basis
$$v_{i_1j_1}\wedge v_{i_2j_2}\wedge\cdots\wedge v_{i_mj_m},$$

such that $\sum_{r=1}^m i_r = i,\ i_r \leq i_{r+1}$, and if $i_r = i_{r+1}$, then $j_r < j_{r+1}$. The eigenvalues of F on $H^i(G)$ are $\alpha_{i_1j_1}\cdots\alpha_{i_mj_m}$ for such indices. By Poincaré duality, the eigenvalues of F on $H_c^{2d-i}(G)$ are $q^d(\alpha_{i_1j_1}\cdots\alpha_{i_mj_m})^{-1}$. Note that all the i_r are odd, so

$$2d - i \equiv i = \sum_{r=1}^m i_r \equiv m \mod 2.$$

Applying the classical trace formula to G_0, we have

$$\#G_0(\mathbb{F}_q) = \sum (-1)^m q^d \alpha_{i_1j_1}^{-1}\cdots\alpha_{i_mj_m}^{-1} = q^d \prod_{i,j}(1-\alpha_{ij}^{-1}).$$

This finishes the proof. $\square$

4.3.4.1. Note that, in (4.3.3) and (4.3.4) we did not assume the sheaf $\mathscr{F}_0$ to be ι-mixed.

Now we prove (4.1.2ii) in general.

Proof. Since $c_v(\mathscr{X}_0, K_0) = c_1(\mathscr{X}_0 \otimes \mathbb{F}_{q^v}, K_0 \otimes \mathbb{F}_{q^v})$, we can assume $v = 1$. We shall reduce to proving (4.1.2ii) for all fibers of f over $\mathbb{F}_q$-points of $\mathscr{Y}_0$, following the idea of Behrend ([3], 6.4.9).

Let $y \in \mathscr{Y}_0(\mathbb{F}_q)$ and $(\mathscr{X}_0)_y$ the fiber over y. Then $(\mathscr{X}_0)_y(\mathbb{F}_q)$ is the groupoid of pairs (x, α), where $x \in \mathscr{X}_0(\mathbb{F}_q)$ and $\alpha : f(x) \to y$ is an isomorphism in $\mathscr{Y}_0(\mathbb{F}_q)$. Fix an $x \in \mathscr{X}_0(\mathbb{F}_q)$ such that $f(x) \cong y$ over $\mathbb{F}_q$, then $\mathrm{Isom}(f(x), y)(\mathbb{F}_q)$ is a trivial left $\mathrm{Aut}_y(\mathbb{F}_q)$-torsor. There is also a natural right action of $\mathrm{Aut}_x(\mathbb{F}_q)$ on $\mathrm{Isom}(f(x), y)(\mathbb{F}_q)$, namely $\varphi \in \mathrm{Aut}_x(\mathbb{F}_q)$ takes α to $\alpha \circ f(\varphi)$. Under this action, for α and α' to be in the same orbit, there should be a $\varphi \in \mathrm{Aut}_x(\mathbb{F}_q)$ such that the diagram

$$\begin{tikzcd} f(x) \arrow[rr, "f(\varphi)"] \arrow[dr, "\alpha'"'] & & f(x) \arrow[dl, "\alpha"] \\ & y & \end{tikzcd}$$

commutes, and this is the definition for (x, α) to be isomorphic to (x, α') in $(\mathscr{X}_0)_y(\mathbb{F}_q)$. So the set of orbits $\mathrm{Isom}(f(x), y)(\mathbb{F}_q)/\mathrm{Aut}_x(\mathbb{F}_q)$ is identified with the inverse image of the class of x along the map $[(\mathscr{X}_0)_y(\mathbb{F}_q)] \to [\mathscr{X}_0(\mathbb{F}_q)]$. The stabilizer group of $\alpha \in \mathrm{Isom}(f(x), y)(\mathbb{F}_q)$ is $\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)$, the automorphism group of (x, α) in $(\mathscr{X}_0)_y(\mathbb{F}_q)$.

In general, if a finite group G acts on a finite set S, then we have

$$\sum_{[x]\in S/G} \frac{\#G}{\#\mathrm{Stab}_G(x)} = \sum_{[x]\in S/G} \#\mathrm{Orb}_G(x) = \#S.$$

Now $S=\mathrm{Isom}(f(x),y)(\mathbb{F}_q)$ and $G=\mathrm{Aut}_x(\mathbb{F}_q)$, so we have

$$\sum_{\substack{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]\\(x,\alpha)\mapsto x}}\frac{\#\mathrm{Aut}_x(\mathbb{F}_q)}{\#\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)}=\#\mathrm{Isom}(f(x),y)(\mathbb{F}_q)=\#\mathrm{Aut}_y(\mathbb{F}_q);$$

the last equality follows from the fact that S is a trivial $\mathrm{Aut}_y(\mathbb{F}_q)$-torsor.

If we assume (4.1.2ii) holds for the fibers $f_y:(\mathscr{X}_0)_y\to\mathrm{Spec}\,\mathbb{F}_q$ of f, for all $y\in\mathscr{Y}_0(\mathbb{F}_q)$, then

$$\begin{aligned}
c_1(\mathscr{Y}_0,f_!K_0)&=\sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]}\frac{\mathrm{Tr}(F_y^*,(f_!K)_{\overline{y}})}{\#\mathrm{Aut}_y(\mathbb{F}_q)}\\
&=\sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]}\frac{\mathrm{Tr}(F_y^*,(f_{y!}K)_{\overline{y}})}{\#\mathrm{Aut}_y(\mathbb{F}_q)}\\
&=\sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]}\frac{1}{\#\mathrm{Aut}_y(\mathbb{F}_q)}\sum_{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]}\frac{\mathrm{Tr}(F_x,K_{\overline{x}})}{\#\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)}\\
&=\sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]}\frac{1}{\#\mathrm{Aut}_y(\mathbb{F}_q)}\sum_{\substack{x\in[\mathscr{X}_0(\mathbb{F}_q)]\\x\mapsto y}}\Big(\sum_{\substack{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]\\(x,\alpha)\mapsto x}}\frac{\mathrm{Tr}(F_x,K_{\overline{x}})}{\#\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)}\Big)\\
&=\sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]}\frac{1}{\#\mathrm{Aut}_y(\mathbb{F}_q)}\sum_{\substack{x\in[\mathscr{X}_0(\mathbb{F}_q)]\\x\mapsto y}}\frac{1}{\#\mathrm{Aut}_x(\mathbb{F}_q)}\\
&\qquad\qquad\Big(\sum_{\substack{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]\\(x,\alpha)\mapsto x}}\frac{\#\mathrm{Aut}_x(\mathbb{F}_q)}{\#\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)}\Big)\mathrm{Tr}(F_x,K_{\overline{x}})\\
&=\sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]}\frac{1}{\#\mathrm{Aut}_y(\mathbb{F}_q)}\sum_{\substack{x\in[\mathscr{X}_0(\mathbb{F}_q)]\\x\mapsto y}}\frac{\mathrm{Tr}(F_x,K_{\overline{x}})}{\#\mathrm{Aut}_x(\mathbb{F}_q)}\#\mathrm{Aut}_y(\mathbb{F}_q)\\
&=\sum_{x\in[\mathscr{X}_0(\mathbb{F}_q)]}\frac{\mathrm{Tr}(F_x,K_{\overline{x}})}{\#\mathrm{Aut}_x(\mathbb{F}_q)}=:c_1(\mathscr{X}_0,K_0).
\end{aligned}$$

So we reduce to the case when $\mathscr{Y}_0=\mathrm{Spec}\,\mathbb{F}_q$.

If $K_0'\to K_0\to K_0''\to K_0'[1]$ is an exact triangle of convergent complexes in $W_m^{-,\mathrm{stra}}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$, then by (4.1.4) and (4.1.2i) we have

$$c_1(\mathscr{X}_0,K_0)=c_1(\mathscr{X}_0,K_0')+c_1(\mathscr{X}_0,K_0'')$$

and

$$c_1(\mathscr{Y}_0, f_!K_0) = c_1(\mathscr{Y}_0, f_!K_0') + c_1(\mathscr{Y}_0, f_!K_0'').$$

If $j : \mathscr{U}_0 \to \mathscr{X}_0$ is an open substack with complement $i : \mathscr{Z}_0 \to \mathscr{X}_0$, then

$$c_1(\mathscr{X}_0, j_!j^*K_0) = c_1(\mathscr{U}_0, j^*K_0) \quad \text{and} \quad c_1(\mathscr{X}_0, i_*i^*K_0) = c_1(\mathscr{Z}_0, i^*K_0).$$

By noetherian induction we can shrink $\mathscr{X}_0$ to a nonempty open substack. So as before we may assume the inertia stack $\mathscr{I}_0$ is flat over $\mathscr{X}_0$, with rigidification $\pi : \mathscr{X}_0 \to X_0$, where X_0 is a scheme. If (4.1.2ii) holds for two composable morphisms f and g, then it holds for $g \circ f$. So we reduce to two cases as before: $\mathscr{X}_0 = X_0$ is a scheme, or $\mathscr{X}_0 = BG_0$, where G_0 is either a connected algebraic group, or a finite étale algebraic group over $\mathbb{F}_q$. We may assume X_0 is separated, by further shrinking (for instance to an affine open subscheme).

For a sheaf complex K_0 and an integer n, we have an exact triangle

$$\tau_{<n}K_0 \longrightarrow \tau_{<n+1}K_0 \longrightarrow \mathscr{H}^n(K_0)[-n] \longrightarrow ,$$

so

$$\begin{aligned} c_1(\tau_{<n+1}K_0) &= c_1(\tau_{<n}K_0) + c_1(\mathscr{H}^n(K_0)[-n]) \\ &= c_1(\tau_{<n}K_0) + (-1)^n c_1(\mathscr{H}^n(K_0)). \end{aligned}$$

Since K_0 is bounded from above, $\tau_{<N}K_0 \simeq K_0$ for $N \gg 0$. Since K_0 is convergent, $c_1(\tau_{<n}K_0) \to 0$ absolutely as $n \to -\infty$, so the series $\sum_{n\in\mathbb{Z}}(-1)^n c_1(\mathscr{H}^n(K_0))$ converges absolutely to $c_1(K_0)$.

Applying $R\Gamma_c$ we get an exact triangle

$$R\Gamma_c(\mathscr{X}_0, \tau_{<n}K_0) \longrightarrow R\Gamma_c(\mathscr{X}_0, \tau_{<n+1}K_0) \longrightarrow R\Gamma_c(\mathscr{X}_0, \mathscr{H}^nK_0)[-n] \longrightarrow$$

in $W_c^-(\mathrm{Spec}\ \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$. We claim that, for $\mathscr{X}_0 = X_0$ a scheme, or BG_0, we have

$$\lim_{n\to-\infty} c_1(\mathrm{Spec}\ \mathbb{F}_q, R\Gamma_c(\mathscr{X}_0, \tau_{<n}K_0)) = 0$$

absolutely. Recall that $c_1(R\Gamma_c(\tau_{<n}K_0)) = \sum_{i\in\mathbb{Z}}(-1)^i\mathrm{Tr}(F, H_c^i(\mathscr{X}, \tau_{<n}K))$, so we need to show

$$\sum_{i\in\mathbb{Z}} \sum_{H_c^i(\mathscr{X},\tau_{<n}K),F} |\alpha| \to 0 \text{ as } n \to -\infty.$$

From the spectral sequence

$$E_1^{rk} = H_c^r(\mathscr{X}, \mathscr{H}^k\tau_{<n}K) \Longrightarrow H_c^{r+k}(\mathscr{X}, \tau_{<n}K)$$

we see that

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\sum_{H_c^i(\mathscr{X},\tau_{<n}K),F}|\alpha| &\leq \sum_{i\in\mathbb{Z}}\sum_{r+k=i}\sum_{H_c^r(\mathscr{X},\mathscr{H}^k\tau_{<n}K),F}|\alpha| \\ &= \sum_{i\in\mathbb{Z}}\sum_{\substack{r+k=i\\k<n}}\sum_{H_c^r(\mathscr{X},\mathscr{H}^kK),F}|\alpha|.\end{aligned}$$

Let $d=\dim\mathscr{X}_0$ (cf. 6.0.3). In the cases where $\mathscr{X}_0$ is a scheme or BG_0, we have $H_c^r(\mathscr{X},\mathscr{F})=0$ for every sheaf $\mathscr{F}$ unless $r\leq 2d$ (cf. (4.1.8.2) and (4.1.6)). Therefore

$$\sum_{i\in\mathbb{Z}}\sum_{\substack{r+k=i\\k<n}}\sum_{H_c^r(\mathscr{X},\mathscr{H}^kK),F}|\alpha| \leq \sum_{i<n+2d}\sum_{r+k=i}\sum_{H_c^r(\mathscr{X},\mathscr{H}^kK),F}|\alpha|,$$

and it suffices to show that the series

$$\sum_{i\in\mathbb{Z}}\sum_{r+k=i}\sum_{H_c^r(\mathscr{X},\mathscr{H}^kK),F}|\alpha|$$

converges. This is proved for BG_0 in (4.2.1), and for schemes X_0 in (4.2.2) (we may shrink X_0 so that the assumption in (4.2.2) is satisfied).

Note that in the two cases $\mathscr{X}_0=X_0$ or BG_0 we are considering, (4.1.2ii) holds when K_0 is a sheaf concentrated in degree 0. For separated schemes X_0, this is a classical result of Grothendieck and Verdier [17, 38]; for BG_0, this is done in (4.3.3) and (4.3.4). Therefore, for a general convergent complex K_0, we have

$$\begin{aligned}c_1(R\Gamma_c(\tau_{<n+1}K_0)) &= c_1(R\Gamma_c(\tau_{<n}K_0))+c_1(R\Gamma_c(\mathscr{H}^nK_0)[-n]) \\ &= c_1(R\Gamma_c(\tau_{<n}K_0))+(-1)^nc_1(\mathscr{H}^nK_0),\end{aligned}$$

and so

$$c_1(R\Gamma_c(K_0))=\sum_{n\in\mathbb{Z}}(-1)^nc_1(\mathscr{H}^nK_0)+\lim_{n\to-\infty}c_1(R\Gamma_c(\tau_{<n}K_0))=c_1(K_0).$$

□

Corollary 4.3.5. *Let $f:\mathscr{X}_0\to\mathscr{Y}_0$ be a morphism of $\mathbb{F}_q$-algebraic stacks, and let $K_0\in W_m^{-,\mathrm{stra}}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Then*

$$L(\mathscr{X}_0,K_0,t)=L(\mathscr{Y}_0,f_!K_0,t).$$

Chapter 5

Meromorphic continuation of L-series.

In this chapter, we will show that the L-series of every ι-mixed stratifiable convergent complex of sheaves on an algebraic $\mathbb{F}_q$-stack has a meromorphic continuation to the whole complex plane. First of all, we give some basic results on infinite products, or more generally, sequences that are convergent term-by-term. Then we give some examples of zeta functions, and finally use (4.1.2) to prove the meromorphic continuation.

5.1 Infinite products

For a convergent complex K_0 on $\mathscr{X}_0$, the series $\sum_{v\geq 1} c_v(K_0)t^v/v$ (and hence the L-series $L(\mathscr{X}_0, K_0, t)$) usually has a finite radius of convergence. For instance, we have the following lemma.

Lemma 5.1.1. *Let $X_0/\mathbb{F}_q$ be a variety of dimension d. Then the radius of convergence of $\sum_{v\geq 1} c_v(X_0)t^v/v$ is $1/q^d$.*

Proof. Let $f_{X_0}(t) = \sum_{v\geq 1} c_v(X_0)t^v/v$. Let Y_0 be an irreducible component of X_0 with complement U_0. Then $c_v(X_0) = c_v(Y_0) + c_v(U_0)$, and since all the c_v-terms are non-negative, we see that the radius of convergence of $f_{X_0}(t)$ is the minimum of that of $f_{Y_0}(t)$ and that of $f_{U_0}(t)$. Since $\max\{\dim(Y_0), \dim(U_0)\} = d$, and U_0 has fewer irreducible component than X_0, by induction we can assume X_0 is irreducible.

Then there exists an open dense subscheme $U_0 \subset X_0$ that is smooth over Spec $\mathbb{F}_q$. Let $Z_0 = X_0 - U_0$, then $\dim(Z_0) < \dim(X_0) = d$. From the cohomology sequence

$$H_c^{2d-1}(Z) \longrightarrow H_c^{2d}(U) \longrightarrow H_c^{2d}(X) \longrightarrow H_c^{2d}(Z)$$

we see that $H_c^{2d}(X) = H_c^{2d}(U) = \overline{\mathbb{Q}}_\ell(-d)$. The Frobenius eigenvalues $\{\alpha_{ij}\}_j$ on $H_c^i(X)$ have

ι-weights $\leq i$, for $0 \leq i < 2d$ ([9], 3.3.4). By the fixed point formula,

$$\frac{c_v(X_0)}{c_{v+1}(X_0)} = \frac{q^{vd} + \sum_{0 \leq i < 2d}(-1)^i \sum_j \alpha_{ij}^v}{q^{(v+1)d} + \sum_{0 \leq i < 2d}(-1)^i \sum_j \alpha_{ij}^{v+1}} = \frac{\frac{1}{q^d} + \frac{1}{q^d}\sum_{0 \leq i < 2d}(-1)^i \sum_j (\frac{\alpha_{ij}}{q^d})^v}{1 + \sum_{0 \leq i < 2d}(-1)^i \sum_j (\frac{\alpha_{ij}}{q^d})^{v+1}},$$

which converges to $1/q^d$ as $v \to \infty$, therefore the radius of convergence of $f_{X_0}(t)$ is

$$\lim_{v \to \infty} \frac{c_v(X_0)/v}{c_{v+1}(X_0)/(v+1)} = \frac{1}{q^d}.$$

□

In order to prove the meromorphic continuation (5.3.1), we want to express the L-series as a possibly infinite product. For schemes, if we consider only bounded complexes, the L-series can be expressed as a finite alternating product of polynomials $P_n(X_0, K_0, t)$, so it is rational [17]. In the stack case, even for the sheaf $\overline{\mathbb{Q}}_\ell$, there might be infinitely many nonzero compact cohomology groups, and we need to consider the issue of convergence of the coefficients in an infinite products.

Definition 5.1.2. *Let $f_n(t) = \sum_{k \geq 0} a_{nk} t^k \in \mathbb{C}[[t]]$ be a sequence of power series over $\mathbb{C}$. The sequence is said to be* convergent term by term*, if for each k, the sequence $(a_{nk})_n$ converges, and the series*

$$\lim_{n \to \infty} f_n(t) := \sum_{k \geq 0} t^k \lim_{n \to \infty} a_{nk}$$

is called the limit of the sequence $\big(f_n(t)\big)_n$.

5.1.2.1. Strictly speaking, a series (resp. infinite product) is defined to be a sequence $(a_n)_n$, usually written as an "infinite sum" (resp. "infinite product") so that $(a_n)_n$ is the sequence of finite partial sums (resp. finite partial products) of it. So the definition above applies to series and infinite products.

Recall that $\log(1+g) = \sum_{m \geq 1} (-1)^{m+1} g^m/m$ for $g \in t\mathbb{C}[[t]]$.

Lemma 5.1.3. *(i) Let $f_n(t) = 1 + \sum_{k \geq 1} a_{nk} t^k \in \mathbb{C}[[t]]$ be a sequence of power series. Then $\big(f_n(t)\big)_n$ is convergent term by term if and only if $\big(\log f_n(t)\big)_n$ is convergent term by term, and*

$$\lim_{n \to \infty} \log f_n(t) = \log \lim_{n \to \infty} f_n(t).$$

(ii) Let f and g be two power series with constant term 1. Then

$$\log(fg) = \log(f) + \log(g).$$

(iii) Let $f_n(t) \in 1 + t\mathbb{C}[[t]]$ be a sequence as in (i). Then the infinite product $\prod_{n\geq 1} f_n(t)$ converges term by term if and only if the series $\sum_{n\geq 1} \log f_n(t)$ converges term by term, and

$$\sum_{n\geq 1} \log f_n(t) = \log \prod_{n\geq 1} f_n(t).$$

Proof. (i) We have

$$\begin{aligned}
\log f_n(t) &= \sum_{m\geq 1} (-1)^{m+1} \big(\sum_{k\geq 1} a_{nk} t^k\big)^m / m \\
&= t \cdot a_{n1} + t^2(a_{n2} - \frac{a_{n1}^2}{2}) + t^3(a_{n3} - a_{n1}a_{n2} + \frac{a_{n1}^3}{3}) \\
&\quad + t^4(a_{n4} - a_{n1}a_{n3} - \frac{a_{n2}^2}{2} + a_{n1}^2 a_{n2}) + \cdots \\
&=: \sum_{k\geq 1} A_{nk} t^k.
\end{aligned}$$

In particular, for each k, $A_{nk} - a_{nk} = h(a_{n1}, \cdots, a_{n,k-1})$ is a polynomial in $a_{n1}, \cdots, a_{n,k-1}$ with rational coefficients. So if $(a_{nk})_n$ converges for each k, then $(A_{nk})_n$ also converges, and by induction the converse also holds. If $\lim_{n\to\infty} a_{nk} = a_k$, then $\lim_{n\to\infty} A_{nk} = a_k + h(a_1, \cdots, a_{k-1})$, and

$$\log \lim_{n\to\infty} f_n(t) = \log(1 + \sum_{k\geq 1} a_k t^k) = \sum_{k\geq 1} (a_k + h(a_1, \cdots, a_{k-1}))t^k = \lim_{n\to\infty} \log f_n(t).$$

(ii) log and exp are inverse to each other on power series, so it suffices to prove that for f and $g \in t\mathbb{C}[[t]]$, we have

$$\exp(f+g) = \exp(f)\exp(g).$$

This follows from the binomial formula:

$$\begin{aligned}
\exp(f+g) &= \sum_{n\geq 0} (f+g)^n/n! = \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k} = \sum_{n\geq 0} \quad \sum_{k=0}^{n} \frac{f^k}{k!} \cdot \frac{g^{n-k}}{(n-k)!} \\
&= \sum_{i,j\geq 0} \frac{f^i}{i!} \cdot \frac{g^j}{j!} = \big(\sum_{i\geq 0} f^i/i!\big)\big(\sum_{j\geq 0} g^j/j!\big) = \exp(f)\exp(g).
\end{aligned}$$

(iii) Let $F_N(t) = \prod_{n=1}^{N} f_n(t)$. Applying (i) to the sequence $(F_N(t))_N$, we see that the infinite product $\prod_{n\geq 1} f_n(t)$ converges term by term if and only if (by definition) $\big(F_N(t)\big)_N$ converges term by term, if and only if the sequence $\big(\log F_N(t)\big)_N$ converges term by term, if and only if (by definition) the series $\sum_{n\geq 1}\log f_n(t)$ converges term by term, since by (ii)

$$\log\prod_{n=1}^{N} f_n(t)=\sum_{n=1}^{N}\log f_n(t)$$

Also

$$\log\prod_{n\geq 1} f_n(t)=\log\lim_{N\to\infty}F_N(t)=\lim_{N\to\infty}\log F_N(t)=\lim_{N\to\infty}\sum_{n=1}^{N}\log f_n(t)=:\sum_{n\geq 1}\log f_n(t).$$

$\square$

5.1.4. For a complex of sheaves K_0 on $\mathscr{X}_0$ and $n\in\mathbb{Z}$, define

$$P_n(\mathscr{X}_0,K_0,t):=\det(1-Ft,H^n_c(\mathscr{X},K)).$$

We regard $P_n(\mathscr{X}_0,K_0,t)^{\pm 1}$ as a complex power series with constant term 1 via ι.

Proposition 5.1.5. *For every convergent complex of sheaves $K_0\in W_m^{-,\mathrm{stra}}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$, the infinite product*

$$\prod_{n\in\mathbb{Z}}P_n(\mathscr{X}_0,K_0,t)^{(-1)^{n+1}}$$

is convergent term by term to the L-series $L(\mathscr{X}_0,K_0,t)$.

Proof. The complex $R\Gamma_c(\mathscr{X},K)$ is bounded above, so $P_n(\mathscr{X}_0,K_0,t)=1$ for $n\gg 0$, and the infinite product is a one-direction limit, namely $n\to-\infty$.

Let $\alpha_{n1},\cdots,\alpha_{nm_n}$ be the eigenvalues (counted without multiplicity) of F on $H^n_c(\mathscr{X},K)$, regarded as complex numbers via ι, so that

$$P_n(t)=P_n(\mathscr{X}_0,K_0,t)=(1-\alpha_{n1}t)\cdots(1-\alpha_{nm_n}t).$$

By (5.1.3iii) it suffices to show that the series

$$\sum_{n\in\mathbb{Z}}(-1)^{n+1}\log P_n(t)$$

converges term by term to $\sum_{v\geq 1}c_v(K_0)t^v/v$.

We have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(-1)^{n+1}\log P_n(t)&=\sum_{n\in\mathbb{Z}}(-1)^{n+1}\log\prod_i(1-\alpha_{ni}t)=\sum_{n\in\mathbb{Z}}(-1)^n\sum_i\sum_{v\geq 1}\frac{\alpha_{ni}^v t^v}{v}\\&=\sum_{v\geq 1}\frac{t^v}{v}\sum_{n\in\mathbb{Z}}(-1)^n\sum_i\alpha_{ni}^v=\sum_{v\geq 1}\frac{t^v}{v}c_v(R\Gamma_c(K_0)),\end{aligned}$$

which converges term by term (4.1.2i), and is equal (4.1.2ii) to $\sum_{v\geq 1} c_v(K_0)t^v/v$. $\square$

Remark 5.1.5.1. By (5.1.5) we have

$$Z(\mathscr{X}_0,t)=\prod_{n\in\mathbb{Z}}P_n(\mathscr{X}_0,t)^{(-1)^{n+1}},$$

where $P_n(\mathscr{X}_0,t)=\det(1-Ft,H^n_c(\mathscr{X}))$. When we want to emphasize the dependence on the prime ℓ, we will write $P_{n,\ell}(\mathscr{X}_0,t)$. This generalizes the classical result for schemes ([17], 5.1).

If G_0 is a connected $\mathbb{F}_q$-algebraic group, (4.1.8.2) shows that the zeta function of BG_0 is given by

$$\begin{aligned}Z(BG_0,t)&=\prod_{r\geq 0}\prod_{\substack{m_{ij}\geq 0\\ \sum_{i,j}m_{ij}(i+1)=2r}}\Big(1-q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}}\cdot t\Big)^{-1}\\ &=\prod_{m_{ij}\geq 0}\Big(1-q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}}\cdot t\Big)^{-1}.\end{aligned}$$

5.2 Examples of zeta functions

In this section we compute the zeta functions of some stacks, and in each example we do it in two ways: counting rational points and computing cohomology groups. Also we investigate some analytic properties.

Example 5.2.1. $B\mathbb{G}_m$. By ([**?**]iii) we have $c_v(B\mathbb{G}_m)=1/c_v(\mathbb{G}_m)$, so the zeta function is

$$Z(B\mathbb{G}_m,t)=\exp\Big(\sum_{v\geq 1}c_v(B\mathbb{G}_m)\frac{t^v}{v}\Big)=\exp\Big(\sum_{v\geq 1}\frac{1}{q^v-1}\frac{t^v}{v}\Big).$$

Using Borel's theorem (4.1.8) one can show (or see ([25], 19.3.2)) that the cohomology ring $H^*(B\mathbb{G}_m)$ is a polynomial ring $\overline{\mathbb{Q}}_\ell[T]$, generated by a variable T in degree 2, and the global Frobenius action is given by $FT^n=q^nT^n$. So by Poincaré duality, we have

$$\begin{aligned}\mathrm{Tr}(F,H_c^{-2n-2}(B\mathbb{G}_m))&=\mathrm{Tr}(F,H_c^{-2n-2}(B\mathbb{G}_m,\overline{\mathbb{Q}}_\ell(-1)))/q\\ &=\mathrm{Tr}(F^{-1},H^{2n}(B\mathbb{G}_m))/q=q^{-n-1}.\end{aligned}$$

This gives

$$\prod_{n\in\mathbb{Z}}P_n(B\mathbb{G}_m,t)^{(-1)^{n+1}}=\prod_{n\geq 1}(1-q^{-n}t)^{-1}.$$

It is easy to verify (5.1.5.1) directly:

$$\exp\Big(\sum_{v\geq 1}\frac{1}{q^v-1}\frac{t^v}{v}\Big)=\exp\Big(\sum_{v\geq 1}\frac{1/q^v}{1-1/q^v}\frac{t^v}{v}\Big)=\exp\Big(\sum_{v\geq 1}\frac{t^v}{v}\sum_{n\geq 1}\frac{1}{q^{nv}}\Big)$$
$$=\prod_{n\geq 1}\exp\Big(\sum_{v\geq 1}\frac{(t/q^n)^v}{v}\Big)=\prod_{n\geq 1}(1-t/q^n)^{-1}.$$

There is also a functional equation

$$Z(B\mathbb{G}_m,qt)=\frac{1}{1-t}Z(B\mathbb{G}_m,t),$$

which implies that $Z(B\mathbb{G}_m,t)$ has a meromorphic continuation to the whole complex plane, with simple poles at $t=q^n$, for $n\geq 1$.

$H_c^{-2n-2}(B\mathbb{G}_m)$ is pure of weight $-2n-2$. A natural question is whether Deligne's theorem of weights ([9], 3.3.4) still holds for algebraic stacks. Olsson told me that it does not hold in general, as the following example shows.

Example 5.2.2. BE, where E is an elliptic curve over $\mathbb{F}_q$. Again by (4.1.7iii) we have

$$c_v(BE)=\frac{1}{\#E(\mathbb{F}_{q^v})}.$$

Let α and β be the roots of the reciprocal characteristic polynomial of the Frobenius on $H^1(E)$:

$$x^2-(1+q-c_1(E))x+q=0. \tag{5.1}$$

Then for every $v\geq 1$, we have $c_v(E)=1-\alpha^v-\beta^v+q^v=(1-\alpha^v)(1-\beta^v)$. So

$$c_v(BE)=\frac{1}{(1-\alpha^v)(1-\beta^v)}=\frac{\alpha^{-v}}{1-\alpha^{-v}}\cdot\frac{\beta^{-v}}{1-\beta^{-v}}$$
$$=\big(\sum_{n\geq 1}\alpha^{-nv}\big)\big(\sum_{m\geq 1}\beta^{-nv}\big)=\sum_{n,m\geq 1}\Big(\frac{1}{\alpha^n\beta^m}\Big)^v,$$

and the zeta function is

$$Z(BE,t)=\exp\Big(\sum_{v\geq 1}c_v(BE)\frac{t^v}{v}\Big)=\exp\Big(\sum_{\substack{n,m\geq 1\\ v\geq 1}}\big(\frac{t}{\alpha^n\beta^m}\big)^v/v\Big)=\prod_{n,m\geq 1}\big(1-\frac{t}{\alpha^n\beta^m}\big)^{-1}.$$

To compute its cohomology, one can apply Borel's theorem (4.1.8) to E, and we have $N=N^1=H^1(E)$, so $N[-1]$ is a 2-dimensional vector space sitting in degree 2, on which F has eigenvalues α and β. Then $H^*(BE)$ is a polynomial ring $\overline{\mathbb{Q}}_\ell[a,b]$ in two variables, both sitting in degree 2, and the basis a, b can be chosen so that the Frobenius action F on $H^2(BE)$ is upper triangular (or even diagonal)

$$\begin{bmatrix} \alpha & \gamma \\ & \beta \end{bmatrix}.$$

Then F acting on

$$H^{2n}(BE) = \mathrm{Sym}^n N[-1] = \overline{\mathbb{Q}}_\ell \langle a^n, a^{n-1}b, \cdots, b^n \rangle$$

can be represented by

$$\begin{bmatrix} \alpha^n & * & * & * \\ & \alpha^{n-1}\beta & * & * \\ & & \ddots & * \\ & & & \beta^n \end{bmatrix},$$

with eigenvalues $\alpha^n, \alpha^{n-1}\beta, \cdots, \beta^n$. So the eigenvalues of F on $H_c^{-2-2n}(BE)$ are

$$q^{-1}\alpha^{-n}, q^{-1}\alpha^{1-n}\beta^{-1}, \cdots, q^{-1}\beta^{-n},$$

and $\prod_{n \in \mathbb{Z}} P_n(BE, t)^{(-1)^{n+1}}$ is

$$\frac{1}{(1 - q^{-1}t)[(1 - q^{-1}\alpha^{-1}t)(1 - q^{-1}\beta^{-1}t)][(1 - q^{-1}\alpha^{-2}t)(1 - q^{-1}\alpha^{-1}\beta^{-1}t)(1 - q^{-1}\beta^{-2}t)] \cdots},$$

which is the same as $Z(BE, t)$ above (since $\alpha\beta = q$).

Let $Z_1(t) := Z(BE, qt)$. Its radius of convergence is 1, since by (5.1.1)

$$\lim_{v \to \infty} \frac{c_v(BE)}{c_{v+1}(BE)} = \lim_{v \to \infty} \frac{c_{v+1}(E)}{c_v(E)} = q.$$

There is also a functional equation

$$Z_1(\alpha t) = \frac{1}{1 - \alpha t} Z_1(t) Z_2(t),$$

where

$$Z_2(t) = \frac{1}{(1 - \alpha\beta^{-1}t)(1 - \alpha\beta^{-2}t)(1 - \alpha\beta^{-3}t) \cdots}.$$

$Z_2(t)$ is holomorphic in the open unit disk and satisfies the functional equation

$$Z_2(\beta t) = \frac{1}{1 - \alpha t} Z_2(t).$$

Therefore $Z_2(t)$, and hence $Z(BE,t)$, has a meromorphic continuation to the whole complex t-plane with the obvious poles.

Remark 5.2.2.1. $H_c^{-2-2n}(BE)$ is pure of weight $-2-n$, which is not $\leq -2-2n$ unless $n=0$. So the upper bound of weights for schemes fails for BE. Also note that, the equation (5.1) is independent of ℓ, so all the eigenvalues are independent of ℓ.

Example 5.2.3. BG_0, where G_0 is a finite étale $\mathbb{F}_q$-group scheme. In the proof of (4.3.3) we see that $BG_0(\mathbb{F}_{q^v}) \cong \mathrm{Fix}(B\sigma^v)(\mathbb{F}) \cong G/\rho^{(v)}$, where $\rho^{(v)}$ is the right action of G on the set G given by $h : g \mapsto \sigma^v(h^{-1})gh$. So

$$c_v(BG_0) = \sum_{[g]\in G/\rho^{(v)}} \frac{1}{\#\mathrm{Stab}_{\rho^{(v)}}(g)} = \frac{\#G}{\#G} = 1,$$

and the zeta function is

$$Z(BG_0,t) = \frac{1}{1-t}.$$

Its cohomology groups are given in (4.1.6): $H_c^0(BG) = \overline{\mathbb{Q}}_\ell$, and other $H_c^i = 0$. This verifies (5.1.5.1).

Note that $Z(BG_0,t)$ is the same as the zeta function of its coarse moduli space Spec $\mathbb{F}_q$. As a consequence, for every $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$, with finite inertia $\mathscr{I}_0 \to \mathscr{X}_0$ and coarse moduli space $\pi : \mathscr{X}_0 \to X_0$ ([5], 1.1), we have $Z(\mathscr{X}_0,t) = Z(X_0,t)$, and hence it is a rational function. This is because for every $x \in X_0(\mathbb{F}_{q^v})$, the fiber $\pi^{-1}(x)$ is a neutral gerbe over Spec $k(x)$, and from the above we see that $c_v(\pi^{-1}(x)) = 1$, and hence $c_v(\mathscr{X}_0) = c_v(X_0)$. The fact that $Z(X_0,t)$ is a rational function follows from ([21], II, 6.7) and noetherian induction. More generally, we have the following.

Proposition 5.2.3.1. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack. Suppose that $\mathscr{X}_0$ either has finite inertia, or is Deligne-Mumford (not necessarily separated). Then for every $K_0 \in W^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, its L-series $L(\mathscr{X}_0, K_0, t)$ is a rational function.*

Proof. Note that we do not assume K_0 to be ι-mixed. We will show that (4.1.2) holds for the structural map $\mathscr{X}_0 \to$ Spec $\mathbb{F}_q$ and $K_0 \in W^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ in these two cases, without using (2.2.5.1).

Case when $\mathscr{X}_0$ has finite inertia. Let $\pi : \mathscr{X}_0 \to X_0$ be its coarse moduli space. For any sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$, by (4.1.6) we have isomorphisms $H_c^r(X, R^0\pi_!\mathscr{F}) \simeq H_c^r(\mathscr{X}, \mathscr{F})$, so $R\Gamma_c(\mathscr{X}_0, \mathscr{F}_0)$ is a bounded complex, hence a convergent complex. To prove the trace formula for $\mathscr{X}_0 \to$ Spec $\mathbb{F}_q$ and the sheaf $\mathscr{F}_0$, it suffices to prove it for $\mathscr{X}_0 \to X_0$ and $X_0 \to$ Spec $\mathbb{F}_q$. The first case, when passing to fibers, is reduced to BG_0, and when passing to fibers again, it is reduced to the two subcases: when G_0 is finite, or when G_0 is connected. In both of these two cases as well as the case of algebraic spaces $X_0 \to$ Spec $\mathbb{F}_q$, the trace formula holds without the assumption of ι-mixedness (4.3.4.1). Therefore, (4.1.2) holds for $\mathscr{X}_0 \to$ Spec $\mathbb{F}_q$ and any sheaf, and hence any bounded complex, on $\mathscr{X}_0$.

The trace formula is equivalent to the equality of power series

$$L(\mathscr{X}_0,K_0,t)=\prod_{i\in\mathbb{Z}}P_i(\mathscr{X}_0,K_0,t)^{(-1)^{i+1}},$$

and the right-hand side is a finite product, because from the argument above, $R\Gamma_c(\mathscr{X}_0,K_0)$ is a bounded complex. Therefore, $L(\mathscr{X}_0,K_0,t)$ is rational.

Case when $\mathscr{X}_0$ is Deligne-Mumford. For both (i) and (ii) of (4.1.2), we may replace $\mathscr{X}_0$ by a non-empty open substack, hence by ([25], 6.1.1) we may assume $\mathscr{X}_0$ is the quotient stack $[X_0'/G]$, where X_0' is an affine $\mathbb{F}_q$-scheme of finite type and G is a finite group acting on X_0'. This stack has finite diagonal, and hence finite inertia, so by the previous case we are done. Also, we know that $R\Gamma_c(\mathscr{X}_0,K_0)$ is bounded, therefore $L(\mathscr{X}_0,K_0,t)$ is rational. □

If one wants to use Poincaré duality to get a functional equation for the zeta function, ([33], 5.17) and ([27], 9.1.2) suggest that we should assume $\mathscr{X}_0$ to be proper smooth and of finite diagonal. Under these assumptions, one gets the expected functional equation for the zeta function, as well as the independence of ℓ for the coarse moduli space, which is proper but possibly singular. Examples of such stacks include $\overline{\mathscr{M}}_{g,n}$ over $\mathbb{F}_q$.

Proposition 5.2.3.2. *Let $\mathscr{X}_0$ be a proper smooth $\mathbb{F}_q$-algebraic stack of equidimension d, with finite diagonal, and let $\pi:\mathscr{X}_0\to X_0$ be its coarse moduli space. Then $Z(X_0,t)$ satisfies the usual functional equation*

$$Z(X_0,\frac{1}{q^dt})=\pm q^{d\chi/2}t^{\chi}Z(X_0,t),$$

where $\chi:=\sum_{i=0}^{2d}(-1)^i\deg P_{i,\ell}(X_0,t)$. Moreover, $H^i(X)$ is pure of weight i, for every $0\le i\le 2d$, and the reciprocal roots of each $P_{i,\ell}(X_0,t)$ are algebraic integers independent of ℓ.

Proof. First we show that the adjunction map $\overline{\mathbb{Q}}_\ell\to\pi_*\pi^*\overline{\mathbb{Q}}_\ell=\pi_*\overline{\mathbb{Q}}_\ell$ is an isomorphism. Since π is quasi-finite and proper ([5], 1.1), we have $\pi_*=\pi_!$ ([33], 5.1) and $R^r\pi_!\overline{\mathbb{Q}}_\ell=0$ for $r\ne 0$ ([33], 5.8). The natural map $\overline{\mathbb{Q}}_\ell\to R^0\pi_*\overline{\mathbb{Q}}_\ell$ is an isomorphism, since the geometric fibers of π are connected.

Therefore $R\Gamma(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)=R\Gamma(X_0,\pi_*\overline{\mathbb{Q}}_\ell)=R\Gamma(X_0,\overline{\mathbb{Q}}_\ell)$, and hence ([33], 5.17) $H^i(\mathscr{X})\simeq H^i_c(\mathscr{X})\simeq H^i(X)\simeq H^i_c(X)$ for all i. Let $P_i(t)=P_i(\mathscr{X}_0,t)=P_i(X_0,t)$. Since X_0 is an algebraic space of dimension d, $P_i(t)=1$ if $i\notin[0,2d]$. Since $\mathscr{X}_0$ is proper and smooth, Poincaré duality gives a perfect pairing

$$H^i(\mathscr{X})\times H^{2d-i}(\mathscr{X})\longrightarrow\overline{\mathbb{Q}}_\ell(-d).$$

Following the standard proof for proper smooth varieties (as in ([29], 27.12)) we get the expected functional equation for $Z(\mathscr{X}_0,t)=Z(X_0,t)$.

$H^i(X)$ is mixed of weights $\leq i$ ([9], 3.3.4), so by Poincaré duality, it is pure of weight i. Following the proof in ([8], p.276), this purity implies that the characteristic polynomials

$$P_{i,\ell}(X_0, t) = \det(1 - Ft, H^i(X, \mathbb{Q}_\ell))$$

have integer coefficients independent of ℓ. □

Remark 5.2.3.3. Weizhe Zheng suggested (5.2.3.1) to me. He also suggested that we give a functional equation relating $L(\mathscr{X}_0, DK_0, t)$ and $L(\mathscr{X}_0, K_0, t)$, for $K_0 \in W^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, where $\mathscr{X}_0$ is a proper $\mathbb{F}_q$-algebraic stack with finite diagonal, of equidimension d, but not necessarily smooth. Here is the functional equation:

$$L(\mathscr{X}_0, K_0, t^{-1}) = t^{\chi_c} \cdot Q \cdot L(\mathscr{X}_0, DK_0, t),$$

where $\chi_c = \sum_{i=0}^{2d} h_c^i(\mathscr{X}, K)$ and $Q = (t^{\chi_c} L(\mathscr{X}_0, K_0, t))|_{t=\infty}$. Note that the rational function $L(\mathscr{X}_0, K_0, t)$ has degree $-\chi_c$, hence Q is well-defined. The proof is similar to the above.

Example 5.2.4. BGL_N. We have $\#GL_N(\mathbb{F}_{q^v}) = (q^{vN} - 1)(q^{vN} - q^v) \cdots (q^{vN} - q^{v(N-1)})$, so one can use $c_v(BGL_N) = 1/c_v(GL_N)$ to compute $Z(BGL_N, t)$. One can also compute the cohomology groups of BGL_N using Borel's theorem (4.1.8). We refer to ([2], 2.3.2) for the result. Let us consider the case $N = 2$ only. The general case is similar.

We have

$$c_v(BGL_2) = \frac{1}{q^{4v}}\left(1 + \frac{1}{q^v} + \frac{2}{q^{2v}} + \frac{2}{q^{3v}} + \frac{3}{q^{4v}} + \frac{3}{q^{5v}} + \cdots\right),$$

and therefore

$$\begin{aligned} Z(BGL_2, t) &= \exp\Big(\sum_v \frac{(t/q^4)^v}{v}\Big) \cdot \exp\Big(\sum_v \frac{(t/q^5)^v}{v}\Big) \cdot \exp\Big(\sum_v \frac{2(t/q^6)^v}{v}\Big) \cdots \\ &= \frac{1}{1 - t/q^4} \cdot \frac{1}{1 - t/q^5} \cdot \Big(\frac{1}{1 - t/q^6}\Big)^2 \cdot \Big(\frac{1}{1 - t/q^7}\Big)^2 \cdot \Big(\frac{1}{1 - t/q^8}\Big)^3 \cdots . \end{aligned}$$

So $Z(BGL_2, qt) = Z(BGL_2, t) \cdot Z_1(t)$, where

$$Z_1(t) = \frac{1}{(1 - t/q^3)(1 - t/q^5)(1 - t/q^7)(1 - t/q^9) \cdots}.$$

$Z_1(t)$ satisfies the functional equation

$$Z_1(q^2 t) = \frac{1}{1 - t/q} \cdot Z_1(t),$$

So $Z_1(t)$, and hence $Z(BGL_2, t)$, has a meromorphic continuation with the obvious poles.

The non-zero compactly supported cohomology groups of BGL_2 are given as follows:

$$H_c^{-8-2n}(BGL_2) = \overline{\mathbb{Q}}_\ell(n+4)^{\oplus\left(\lfloor\frac{n}{2}\rfloor+1\right)},\ n \geq 0.$$

This gives

$$\prod_{n\in\mathbb{Z}} P_n(BGL_2,t)^{(-1)^{n+1}} = \frac{1}{(1-t/q^4)(1-t/q^5)(1-t/q^6)^2(1-t/q^7)^2\cdots},$$

and (5.1.5.1) is verified. Note that the eigenvalues are $1/q^{n+4}$, which are independent of ℓ.

5.3 Meromorphic continuation.

We state and prove a generalized version of (1.0.3).

Theorem 5.3.1. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack, and let $K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ be a convergent complex. Then $L(\mathscr{X}_0,K_0,t)$ has a meromorphic continuation to the whole complex t-plane, and its poles can only be zeros of the polynomials $P_{2n}(\mathscr{X}_0,K_0,t)$ for some integers n.*

We need a preliminary lemma. For an open subset $U \subset \mathbb{C}$, let $\mathscr{O}(U)$ be the set of analytic functions on U. There exists a sequence $\{K_n\}_{n\geq 1}$ of compact subsets of U such that $U=\bigcup_n K_n$ and $K_n \subset (K_{n+1})^\circ$. For f and g in $\mathscr{O}(U)$, define

$$\rho_n(f,g)=\sup\{|f(z)-g(z)|; z\in K_n\} \quad\text{and}\quad \rho(f,g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

Then ρ is a metric on $\mathscr{O}(U)$ and the topology is independent of the subsets $\{K_n\}_n$ chosen (cf. ([6], VII, §1)).

The following lemma is ([6], p.167, 5.9).

Lemma 5.3.2. *Let $U \subset \mathbb{C}$ be connected and open and let $(f_n)_n$ be a sequence in $\mathscr{O}(U)$ such that no f_n is identically zero. If $\sum_n(f_n(z)-1)$ converges absolutely and uniformly on compact subsets of U, then $\prod_{n\geq1}f_n(z)$ converges in $\mathscr{O}(U)$ to an analytic function $f(z)$. If z_0 is a zero of f, then z_0 is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at z_0 is the sum of the multiplicities of the zeros of the functions f_n at z_0.*

Now we prove (5.3.1).

Proof. Factorize $P_n(\mathscr{X}_0,K_0,t)$ as $\prod_{j=1}^{m_n}(1-\alpha_{nj}t)$ in $\mathbb{C}$. Since $R\Gamma_c(\mathscr{X}_0,K_0)$ is a convergent complex (4.1.2i), the series $\sum_{n,j}|\alpha_{nj}|$ converges.

By (5.1.5) we have

$$L(\mathscr{X}_0, K_0, t) = \prod_{n\in\mathbb{Z}} \Big(\prod_{j=1}^{m_n}(1-\alpha_{nj}t)\Big)^{(-1)^{n+1}}$$

as formal power series. To apply (5.3.2), take U to be the region $\mathbb{C} - \{\alpha_{nj}^{-1}; n \text{ even}\}$. Take the lexicographical order on the set of all factors

$$1-\alpha_{nj}t, \text{ for } n \text{ odd}; \quad \frac{1}{1-\alpha_{nj}t}, \text{ for } n \text{ even}.$$

Each factor is an analytic function on U. The sum "$\sum_n (f_n(z)-1)$" here is equal to

$$\sum_{n \text{ odd},j} (-\alpha_{nj}t) + \sum_{n \text{ even},j} \frac{\alpha_{nj}t}{1-\alpha_{nj}t}.$$

Let

$$g_n(t) = \begin{cases} \sum_{j=1}^{m_n} |\alpha_{nj}t|, & n \text{ odd}, \\ \sum_{j=1}^{m_n} \frac{|\alpha_{nj}t|}{|1-\alpha_{nj}t|}, & n \text{ even}. \end{cases}$$

We need to show that $\sum_n g_n(t)$ is pointwise convergent, uniformly on compact subsets of U. Precisely, we want to show that for any compact subset $B \subset U$, and for any $\varepsilon > 0$, there exists a constant $N_B \in \mathbb{Z}$ such that

$$\sum_{n\leq N} g_n(t) < \varepsilon$$

for all $N \leq N_B$ and $t \in B$. Since $g_n(t)$ are non-negative, it suffices to do this for $N = N_B$. There exists a constant M_B such that $|t| < M_B$ for all $t \in B$. Since $\sum_{n,j} |\alpha_{nj}|$ converges, $|\alpha_{nj}| \to 0$ as $n \to -\infty$, and there exists a constant $L_B \in \mathbb{Z}$ such that $|\alpha_{nj}| < 1/(2M_B)$ for all $n < L_B$. So

$$g_n(t) \leq 2\sum_{j=1}^{m_n} |\alpha_{nj}t|$$

for all $n < L_B$ and $t \in B$. There exists a constant $N_B < L_B$ such that

$$\sum_{n\leq N_B} \sum_j |\alpha_{nj}| < \varepsilon/(2M_B)$$

and hence

$$\sum_{n\leq N_B} g_n(t) \leq 2\sum_{n\leq N_B}\sum_j |\alpha_{nj}t| \leq 2M_B \sum_{n\leq N_B}\sum_j |\alpha_{nj}| < \varepsilon.$$

By (5.3.2), $L(\mathscr{X}_0, K_0, t)$ extends to an analytic function on U. By the second part of (5.3.2), the α_{nj}^{-1}'s, for n even, are at worst poles rather than essential singularities, therefore the L-series is meromorphic on $\mathbb{C}$. $\square$

Now $L(\mathscr{X}_0, K_0, t)$ can be called an "L-function".

Chapter 6

Weight theorem for algebraic stacks, and first applications to independence of ℓ.

6.0.3. We prove (1.0.4) in this chapter. For the reader's convenience, we briefly review the definition of the *dimension* of a locally noetherian S-algebraic stack $\mathcal{X}$ from ([25], chapter 11).

If X is a locally noetherian S-algebraic space and x is a point of X, the dimension $\dim_x(X)$ of X at x is defined to be $\dim_{x_0}(X_0)$, for any pair (X_0, x_0), where X_0 is an S-scheme étale over X and $x_0 \in X_0$ maps to x. This is independent of the choice of the pair. If $f : X \to Y$ is a morphism of algebraic S-spaces, locally of finite type, and x is a point of X with image y in Y, then the relative dimension $\dim_x(f)$ of f at x is defined to be $\dim_x(X_y)$.

Let $P : X \to \mathcal{X}$ be a presentation of an S-algebraic stack $\mathcal{X}$, and let x be a point of X. Then the relative dimension $\dim_x(P)$ of P at x is defined to be the relative dimension at (x, x) of the smooth morphism of S-algebraic spaces $\mathrm{pr}_1 : X \times_{\mathcal{X}} X \to X$.

If $\mathcal{X}$ is a locally noetherian S-algebraic stack and if ξ is a point of $\mathcal{X}$, we define the dimension of $\mathcal{X}$ at ξ to be $\dim_\xi(\mathcal{X}) = \dim_x(X) - \dim_x(P)$, where $P : X \to \mathcal{X}$ is an arbitrary presentation of $\mathcal{X}$ and x is an arbitrary point of X lying over ξ. This definition is independent of all the choices made. At last we define $\dim \mathcal{X} = \sup_\xi \dim_\xi \mathcal{X}$. For quotient stacks we have $\dim[X/G] = \dim X - \dim G$.

6.1 Weight theorem.

Now we prove (1.0.4).

Proof. If $j : \mathscr{U}_0 \to \mathscr{X}_0$ is an open substack with complement $i : \mathscr{Z}_0 \to \mathscr{X}_0$, then we have an exact sequence

$$\cdots \longrightarrow H_c^n(\mathscr{U}, j^*\mathscr{F}) \longrightarrow H_c^n(\mathscr{X}, \mathscr{F}) \longrightarrow H_c^n(\mathscr{Z}, i^*\mathscr{F}) \longrightarrow \cdots.$$

If both $H_c^n(\mathscr{U}, j^*\mathscr{F})$ and $H_c^n(\mathscr{Z}, i^*\mathscr{F})$ are zero (resp. have all punctual ι-weights $\leq m$ for some number m), then so is $H_c^n(\mathscr{X}, \mathscr{F})$. Since the dimensions of $\mathscr{U}_0$ and $\mathscr{Z}_0$ are no more than that of $\mathscr{X}_0$, and the set of punctual ι-weights of $i^*\mathscr{F}_0$ and $j^*\mathscr{F}_0$ is the same as the set of punctual ι-weights of $\mathscr{F}_0$, we may shrink $\mathscr{X}_0$ to a non-empty open substack. We can also make any finite base change on $\mathbb{F}_q$. As before, we reduce to the case when $\mathscr{X}_0$ is geometrically connected, and the inertia $f : \mathscr{I}_0 \to \mathscr{X}_0$ is flat, with rigidification $\pi : \mathscr{X}_0 \to X_0$, where X_0 is a scheme. The squares in the following diagram are 2-Cartesian:

$$\begin{tikzcd}
 & & \mathscr{I}_0 \arrow[d, "f"'] & \mathrm{Aut}_y \arrow[l] \arrow[d] \\
B\mathrm{Aut}_{\overline{x}} \arrow[r] \arrow[d, "\pi_{\overline{x}}"'] & B\mathrm{Aut}_x \arrow[r] \arrow[d, "\pi_x"'] & \mathscr{X}_0 \arrow[d, "\pi"] & \mathrm{Spec}\ \mathbb{F}_{q^v} \arrow[l, "y"'] \\
\mathrm{Spec}\ \mathbb{F} \arrow[r, "\overline{x}"'] & \mathrm{Spec}\ \mathbb{F}_{q^v} \arrow[r, "x"'] & X_0 & .
\end{tikzcd}$$

We have $(R^k\pi_!\mathscr{F}_0)_{\overline{x}} = H_c^k(B\mathrm{Aut}_{\overline{x}}, \mathscr{F})$. Since f is representable and flat, and $\mathscr{X}_0$ is connected, all automorphism groups Aut_x have the same dimension, say d.

Assume (1.0.4) holds for all BG_0, where G_0 are $\mathbb{F}_q$-algebraic groups. Then $R^k\pi_!\mathscr{F}_0 = 0$ for $k > -2d$, and for $k \leq -2d$, the punctual ι-weights of $R^k\pi_!\mathscr{F}_0$ are $\leq \frac{k}{2} - d + w$, hence by ([9], 3.3.4), the punctual ι-weights of $H_c^r(X, R^k\pi_!\mathscr{F})$ are $\leq \frac{k}{2} - d + w + r$. Consider the Leray spectral sequence

$$E_2^{rk} = H_c^r(X, R^k\pi_!\mathscr{F}) \Longrightarrow H_c^{r+k}(\mathscr{X}, \mathscr{F}).$$

If we maximize $\frac{k}{2} - d + w + r$ under the constraints

$$r + k = n,\ 0 \leq r \leq 2\dim X_0,\ \text{and}\ k \leq -2d,$$

we find that $H_c^n(\mathscr{X}, \mathscr{F}) = 0$ for $n > 2\dim X_0 - 2d = 2\dim \mathscr{X}_0$, and for $n \leq 2\dim \mathscr{X}_0$, the punctual ι-weights of $H_c^n(\mathscr{X}, \mathscr{F})$ are $\leq \dim X_0 + \frac{n}{2} + w - d = \dim \mathscr{X}_0 + \frac{n}{2} + w$.

So we reduce to the case $\mathscr{X}_0 = BG_0$. The Leray spectral sequence for $h : BG_0 \to B\pi_0(G_0)$ degenerates (by (4.1.6)) to isomorphisms

$$H_c^0(B\pi_0(G), R^nh_!\mathscr{F}) \simeq H_c^n(BG, \mathscr{F}).$$

The fibers of h are isomorphic to BG_0^0, so by base change and (4.1.6) we reduce to the case when G_0 is connected. Let $d = \dim G_0$ and $f : BG_0 \to \mathrm{Spec}\ \mathbb{F}_q$ be the structural map. In this case, $\mathscr{F}_0 \cong f^*V$ for some $\overline{\mathbb{Q}}_\ell$-representation V of $W(\mathbb{F}_q)$, and hence $\mathscr{F}_0$ and V have the same punctual ι-weights. Using the natural isomorphism $H^n_c(BG) \otimes V \simeq H^n_c(BG, \mathscr{F})$, we reduce to the case when $\mathscr{F}_0 = \overline{\mathbb{Q}}_\ell$. In (4.1.8.2) we see that, if $\alpha_{i1}, \cdots, \alpha_{in_i}$ are the eigenvalues of F on N^i, $i \geq 1$ odd, then the eigenvalues of F on $H^{-2k-2d}_c(BG)$ are

$$q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}}, \text{ where } \sum_{i,j} m_{ij}(i+1) = 2k.$$

Since $i \geq 1$, we have $\sum im_{ij} \geq k$; together with $|\alpha_{ij}| \geq q^{i/2}$ one deduces

$$|q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}}| \leq q^{\frac{-k-2d}{2}},$$

so the punctual ι-weights of $H^{-2k-2d}_c(BG)$ are $\leq -k-2d$ for $k \geq 0$, and the other compactly supported cohomology groups are zero.

It is clear from the proof and ([9], 3.3.10) that the weights of $H^n_c(\mathscr{X}, \mathscr{F})$ differ from the weights of $\mathscr{F}_0$ by integers.

Recall that $H^{2k}(BG)$ is pure of weight $2k$, for a linear algebraic group G_0 over $\mathbb{F}_q$ ([10], 9.1.4). Therefore, $H^{-2k-2d}_c(BG)$ is pure of weight $-2k-2d$, and following the same proof as above, we are done. $\square$

Remark 6.1.1. When $\mathscr{X}_0 = X_0$ is a scheme, and $n \leq 2\dim X_0$, we have $\dim X_0 + \frac{n}{2} + w \geq n + w$, so our bound for weights is worse than the bound in ([9], 3.3.4). For an abelian variety $A/\mathbb{F}_q$, our bound for the weights of $H^n_c(BA)$ is sharp: the weights are exactly $\dim(BA) + \frac{n}{2}$, whenever the cohomology group is non-zero.

In the following chapter, as an application of the weight theorem, we will generalize Gabber's decomposition theorem to stacks with affine diagonal. We also expect (1.0.4) to be useful when studying the Hasse-Weil zeta functions of Artin stacks over number fields in future. For instance, it implies that the Hasse-Weil zeta function is analytic in some right half complex s-plane.

6.2 Some examples on the existence of rational points.

Using (1.0.4) we can show certain stacks have $\mathbb{F}_q$-points.

Example 6.2.1. Let $\mathscr{X}_0$ be a form of $B\mathbb{G}_m$, i.e., $\mathscr{X} \cong B\mathbb{G}_m$ over $\mathbb{F}$. Then all the automorphism group schemes in $\mathscr{X}_0$ are affine, and $h^{-2-2n}_c(\mathscr{X}) = h^{-2-2n}_c(B\mathbb{G}_m) = 1$, for all $n \geq 0$. Let α_{-2-2n} be the eigenvalue of F on $H^{-2-2n}_c(\mathscr{X})$. Then by (1.0.4) we have $|\alpha_{-2-2n}| \leq q^{-1-n}$. Smoothness is fppf local on the base, so $\mathscr{X}_0$ is smooth, hence $H^{-2}_c(\mathscr{X}) = \overline{\mathbb{Q}}_\ell(1)$ and

$\alpha_{-2} = q^{-1}$. So

$$\begin{aligned}\#\mathscr{X}_0(\mathbb{F}_q) &= \sum_{n\geq 0} \mathrm{Tr}(F, H_c^{-2-2n}(\mathscr{X})) = q^{-1} + \alpha_{-4} + \alpha_{-6} + \cdots \\ &\geq q^{-1} - q^{-2} - q^{-3} + \cdots = q^{-1} - \frac{q^{-1}}{q-1} > 0\end{aligned}$$

when $q \neq 2$. In fact, since there exists an integer $r \geq 1$ such that $\mathscr{X}_0 \otimes \mathbb{F}_{q^r} \cong B\mathbb{G}_{m,/\mathbb{F}_{q^r}}$, we see that all cohomology groups $H_c^{-2-2n}(\mathscr{X})$ are pure, i.e. $|\alpha_{-2-2n}| = q^{-1-n}$.

In fact, one can classify the forms of $B\mathbb{G}_m/\mathbb{F}_q$ as follows. If $\mathscr{X}_0$ is a form, then it is also a gerbe over Spec $\mathbb{F}_q$, hence a neutral gerbe BG_0 for some algebraic group G_0 by ([3], 6.4.2). By comparing the automorphism groups, we see that G_0 is a form of $\mathbb{G}_m/\mathbb{F}_q$. There is only one nontrivial form of $\mathbb{G}_m/\mathbb{F}_q$, because

$$H^1(\mathbb{F}_q, \mathrm{Aut}(\mathbb{G}_m)) = H^1(\mathbb{F}_q, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$

and this form is the kernel of the following norm map

$$1 \longrightarrow R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m \longrightarrow R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m \xrightarrow{\mathrm{Nm}} \mathbb{G}_m \longrightarrow 1,$$

where $R_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is the operation of Weil's restriction of scalars. Therefore, the only non-trivial form of $B\mathbb{G}_m/\mathbb{F}_q$ is $B(R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)$. In particular, they all have $\mathbb{F}_q$-points, even when $q = 2$.

Example 6.2.2. Consider the projective line $\mathbb{P}^1$ with the following action of $\mathbb{G}_m$: it acts by multiplication on the open $\mathbb{G}_m \subset \mathbb{P}^1$, and acts trivially on the other two points $0, \infty$. So we get a quotient stack $[\mathbb{P}^1/\mathbb{G}_m]$ over $\mathbb{F}_q$. Let $\mathscr{X}_0$ be a form of $[\mathbb{P}^1/\mathbb{G}_m]$ over $\mathbb{F}_q$. We want to find an $\mathbb{F}_q$-point on $\mathscr{X}_0$, or even better, an $\mathbb{F}_q$-point on $\mathscr{X}_0$ which, when considered as a point in $\mathscr{X}(\mathbb{F}) \cong [\mathbb{P}^1/\mathbb{G}_m](\mathbb{F})$, lies in the open dense orbit $[\mathbb{G}_m/\mathbb{G}_m](\mathbb{F})$.

6.2.2.1. Consider the following general situation. Let G_0 be a connected $\mathbb{F}_q$-algebraic group, and let X_0 be a proper smooth variety with a G_0-action over $\mathbb{F}_q$. Let

$$[X_0/G_0] \xrightarrow{f} BG_0 \xrightarrow{g} \mathrm{Spec}\ \mathbb{F}_q$$

be the natural maps, and let $\mathscr{X}_0$ be a form of $[X_0/G_0]$. Then f is representable and proper. For every k, $R^kf_*\overline{\mathbb{Q}}_\ell$ is a lisse sheaf, and takes the form g^*V_k for some sheaf V_k on Spec $\mathbb{F}_q$. Consider the Leray spectral sequence

$$E_2^{rk} = R^rg_!R^kf_*\overline{\mathbb{Q}}_\ell \Longrightarrow R^{r+k}(gf)_!\overline{\mathbb{Q}}_\ell.$$

Since $R^r g_! R^k f_* \overline{\mathbb{Q}}_\ell = R^r g_!(g^* V_k) = (R^r g_! \overline{\mathbb{Q}}_\ell) \otimes V_k$, we have

$$h_c^n(\mathscr{X}) = h_c^n([X/G]) \leq \sum_{r+k=n} h_c^r(BG) \cdot \dim V_k = \sum_{r+k=n} h_c^r(BG) \cdot h^k(X).$$

Return to $[\mathbb{P}^1/\mathbb{G}_m]$. Since $h^0(\mathbb{P}^1) = h^2(\mathbb{P}^1) = 1$ and $h_c^{-2i}(B\mathbb{G}_m) = 1$ for $i \geq 1$, we see that $h_c^n(\mathscr{X}) = 0$ for n odd and

$$h_c^{2n}(\mathscr{X}) \leq h^0(\mathbb{P}^1) h_c^{2n}(B\mathbb{G}_m) + h^2(\mathbb{P}^1) h_c^{2n-2}(B\mathbb{G}_m) = \begin{cases} 0, & n \geq 1, \\ 1, & n = 0, \\ 2, & n < 0. \end{cases}$$

Since $\mathscr{X}_0$ is smooth of dimension 0, we have $H_c^0(\mathscr{X}) = \overline{\mathbb{Q}}_\ell$. By (1.0.4), the punctual ι-weights of $H_c^{2n}(\mathscr{X})$ are $\leq 2n$. The trace formula gives

$$\#\mathscr{X}_0(\mathbb{F}_q) = \sum_{n \leq 0} \mathrm{Tr}(F, H_c^{2n}(\mathscr{X})) = 1 + \sum_{n<0} \mathrm{Tr}(F, H_c^{2n}(\mathscr{X})) \geq 1 - 2\sum_{n<0} q^n = 1 - \frac{2}{q-1} > 0$$

when $q \geq 4$.

In order for the rational point to be in the open dense orbit, we need an upper bound for the number of $\mathbb{F}_q$-points on the closed orbits. When passing to $\mathbb{F}$, there are 2 closed orbits, both having stabilizer $\mathbb{G}_m$. So in $[\mathscr{X}_0(\mathbb{F}_q)]$ there are at most 2 points whose automorphism groups are forms of the algebraic group $\mathbb{G}_m$. Consider the non-split torus $R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m$ in (6.2.1). By the cohomology sequence

$$1 \longrightarrow (R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)(\mathbb{F}_q) \longrightarrow \mathbb{F}_{q^2}^* \xrightarrow{\mathrm{Nm}} \mathbb{F}_q^*$$

we see that $\#(R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m)(\mathbb{F}_q) = q+1$. Since $\frac{1}{q+1} \leq \frac{1}{q-1}$, the space that the closed orbits can take is at most $\frac{2}{q-1}$, and equality holds only when the two closed orbits are both defined over $\mathbb{F}_q$ with stabilizer $\mathbb{G}_m$. In order for there to exist an $\mathbb{F}_q$-point in the open dense orbit, we need

$$1 - \frac{2}{q-1} > \frac{2}{q-1},$$

and this is so when $q \geq 7$.

6.3 About independence of ℓ.

The coefficients of the expansion of the infinite product

$$Z(\mathscr{X}_0,t)=\prod_{i\in\mathbb{Z}}P_{i,\ell}(\mathscr{X}_0,t)^{(-1)^{i+1}}$$

are rational numbers and are independent of ℓ, because the $c_v(\mathscr{X}_0)$'s are rational numbers and are independent of ℓ. We want to know if this is true for each $P_{i,\ell}(\mathscr{X}_0,t)$. First we show that the roots of $P_{i,\ell}(\mathscr{X}_0,t)$ are Weil q-numbers. Note that $P_{i,\ell}(\mathscr{X}_0,t)\in\mathbb{Q}_\ell[t]$.

Definition 6.3.1. *An algebraic number is called a* Weil q-number *if all of its conjugates have the same weight relative to q, and this weight is a rational integer. It is called a* Weil q-integer *if in addition it is an algebraic integer. A number in $\overline{\mathbb{Q}}_\ell$ is called a* Weil q-number *if it is a Weil q-number via ι.*

For $\alpha\in\overline{\mathbb{Q}}_\ell$, being a Weil q-number or not is independent of ι; in fact the images in $\mathbb{C}$ under various ι's are conjugate.

This definition is different from the classical definition (for instance ([31], II.2)). The classical notion can be called Weil q-numbers of weight 1, according to our definition.

For an $\mathbb{F}_q$-variety X_0, not necessarily smooth or proper, ([9], 3.3.4) implies all Frobenius eigenvalues of $H^i_c(X)$ are Weil q-integers. The following lemma generalizes this.

Lemma 6.3.2. *For every $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$, and a prime number $\ell\neq p$, the roots of each $P_{i,\ell}(\mathscr{X}_0,t)$ are Weil q-numbers. In particular, the coefficients of $P_{i,\ell}(\mathscr{X}_0,t)$ are algebraic numbers in $\mathbb{Q}_\ell$ (i.e. algebraic over $\mathbb{Q}$).*

Proof. For an open immersion $j:\mathscr{U}_0\to\mathscr{X}_0$ with complement $i:\mathscr{Z}_0\to\mathscr{X}_0$, we have an exact sequence

$$\cdots\longrightarrow H^i_c(\mathscr{U})\longrightarrow H^i_c(\mathscr{X})\longrightarrow H^i_c(\mathscr{Z})\longrightarrow\cdots,$$

so the set of Frobenius eigenvalues of $H^i_c(\mathscr{X})$ is a subset of the union of the Frobenius eigenvalues of $H^i_c(\mathscr{U})$ and $H^i_c(\mathscr{Z})$. Thus we may shrink to a non-empty open substack. In particular, (6.3.2) holds for algebraic spaces, by ([21], II 6.7) and ([9], 3.3.4).

By dévissage we can assume $\mathscr{X}_0$ is smooth and connected. By Poincaré duality, it suffices to show that the Frobenius eigenvalues of $H^i(\mathscr{X})$ are Weil q-numbers, for all i. Take a presentation $X_0\to\mathscr{X}_0$ and consider the associated smooth simplicial covering $X_0^\bullet\to\mathscr{X}_0$ by algebraic spaces. Then there is a spectral sequence ([27], 10.0.9)

$$E_1^{rk}=H^k(X^r)\Longrightarrow H^{r+k}(\mathscr{X}),$$

and the assertion for $\mathscr{X}_0$ follows from the assertion for algebraic spaces. $\square$

Problem 6.3.3. Is each

$$P_{i,\ell}(\mathscr{X}_0,t)=\det(1-Ft,H^i_c(\mathscr{X},\mathbb{Q}_\ell))$$

a polynomial with coefficients in $\mathbb{Q}$, and the coefficients are independent of ℓ?

Remark 6.3.3.1. (i) Note that, unlike the case for varieties, we cannot expect the coefficients to be integers (for instance, for $B\mathbb{G}_m$, the coefficients are $1/q^i$).

(ii) (6.3.3) is known to be true for smooth proper varieties ([9], 3.3.9), and (coarse moduli spaces of) proper smooth algebraic stacks of finite diagonal (5.2.3.2). It remains open for general varieties. Even the Betti numbers are not known to be independent of ℓ for a general variety. See [20].

Let us give positive answer to (6.3.3) in some special cases of algebraic stacks. In §7 we see that it holds for BE and BGL_N. We can generalize these two cases as follows.

Lemma 6.3.4. *(i) (6.3.3) holds for BA, where A is an abelian variety over $\mathbb{F}_q$.*

(ii) (6.3.3) holds for BG_0, where G_0 is a linear algebraic group over $\mathbb{F}_q$.

Proof. (i) Let $g=\dim A$. Then $N=H^1(A)$ is a $2g$-dimensional vector space, with eigenvalues $\alpha_1,\cdots,\alpha_{2g}$ for the Frobenius action F, and N is pure of weight 1. Let $a_1,\cdots,a_{2g}$ be a basis for N so that F is upper-triangular

$$\begin{bmatrix}\alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_{2g}\end{bmatrix}.$$

Then $H^*(BA)=\mathrm{Sym}^*N[-1]=\overline{\mathbb{Q}}_\ell[a_1,\cdots,a_{2g}]$, where each a_i sits in degree 2. In degree $2n$, $H^{2n}(BA)=\overline{\mathbb{Q}}_\ell\langle a_{i_1}\cdots a_{i_n}|1\leq i_1,\cdots,i_n\leq 2g\rangle$, and the eigenvalues are $\alpha_{i_1}\cdots\alpha_{i_n}$. By Poincaré duality

$$H_c^{-2n-2g}(BA)=H^{2n}(BA)^\vee\otimes\overline{\mathbb{Q}}_\ell(g)$$

we see that the eigenvalues of F on $H_c^{-2g-2n}(BA)$ are

$$q^{-g}\cdot\alpha_{i_1}^{-1}\cdots\alpha_{i_n}^{-1}.$$

Each factor

$$P_{-2g-2n}(q^gt)=\prod_{1\leq i_1,\cdots,i_n\leq 2g}\big(1-(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}t\big)$$

stays unchanged if we permute the α_i's arbitrarily, so the coefficients are symmetric polynomials in the α_i^{-1}'s with integer coefficients, hence are polynomials in the elementary symmetric functions, which are coefficients of $\prod_{i=1}^{2g}(t-\alpha_i^{-1})$. The polynomial

$$\prod_{i=1}^{2g}(1-\alpha_i t)=\det\big(1-Ft,H^1(A,\mathbb{Q}_\ell)\big)$$

also has roots α_i^{-1}, and this is a polynomial with integer coefficients, independent of ℓ, since A is smooth and proper. Let $m=\pm q^g$ be leading coefficient of it. Then

$$\prod_{i=1}^{2g}(t-\alpha_i^{-1})=\frac{1}{m}\prod_{i=1}^{2g}(1-\alpha_i t).$$

Therefore (6.3.3) holds for BA.

(ii) Let $d=\dim G_0$. For every $k\geq 0$, $H^{2k}(BG)$ is pure of weight $2k$ ([10], 9.1.4), hence by Poincaré duality, $H_c^{-2d-2k}(BG)$ is pure of weight $-2d-2k$. The entire function

$$\frac{1}{Z(BG_0,t)}=\prod_{k\geq 0}P_{-2d-2k}(BG_0,t)\in\mathbb{Q}[[t]]$$

is independent of ℓ, and invariant under the action of $\mathrm{Gal}(\mathbb{Q})$ on the coefficients of the Taylor expansion. Therefore the roots of $P_{-2d-2k}(BG_0,t)$ can be described as

$$\text{“zeros of } \frac{1}{Z(BG_0,t)} \text{ that have weight } 2d+2k \text{ relative to } q\text{”},$$

which is a description independent of ℓ, and these roots (which are algebraic numbers) are permuted under $\mathrm{Gal}(\mathbb{Q})$. Hence $P_{-2d-2k}(BG_0,t)$ has rational coefficients. □

The following proposition generalizes both (5.2.3.2) and (6.3.4ii).

Proposition 6.3.5. *Let X_0 be the coarse moduli space of a proper smooth $\mathbb{F}_q$-algebraic stack of finite diagonal, and let G_0 be a linear $\mathbb{F}_q$-algebraic group that acts on X_0, and let $\mathscr{X}_0$ be the quotient stack $[X_0/G_0]$. Then (6.3.3) holds for $\mathscr{X}_0$.*

Proof. Let

$$\mathscr{X}_0\xrightarrow{f}BG_0\xrightarrow{h}B\pi_0(G_0)$$

be the natural maps. It suffices to show that $H_c^n(\mathscr{X})$ is pure of weight n, for every n.

Let $d=\dim G_0$. Consider the spectral sequence

$$H_c^{-2d-2r}(BG,R^kf_!\overline{\mathbb{Q}}_\ell)\Longrightarrow H_c^{-2d-2r+k}(\mathscr{X}).$$

The E_2-terms can be computed from the degenerate Leray spectral sequence for h :

$$H_c^{-2d-2r}(BG, R^k f_!\overline{\mathbb{Q}}_\ell) \simeq H_c^0(B\pi_0(G), R^{-2d-2r}h_! R^k f_!\overline{\mathbb{Q}}_\ell).$$

The fibers of $R^{-2d-2r}h_! R^k f_!\overline{\mathbb{Q}}_\ell$ are isomorphic to $H_c^{-2d-2r}(BG^0, R^k f_!\overline{\mathbb{Q}}_\ell)$, and since G^0 is connected, $R^k f_!\overline{\mathbb{Q}}_\ell$ is the inverse image of some sheaf via the structural map $BG_0^0 \to \operatorname{Spec} \mathbb{F}_q$. By projection formula we have

$$H_c^{-2d-2r}(BG^0, R^k f_!\overline{\mathbb{Q}}_\ell) \simeq H_c^{-2d-2r}(BG^0) \otimes H^k(X)$$

as representations of $\mathrm{Gal}(\mathbb{F}_q)$, and by (5.2.3.2), the right hand side is pure of weight $-2d-2r+k$. By (4.1.6), $H_c^{-2d-2r}(BG, R^k f_!\overline{\mathbb{Q}}_\ell)$ is also pure of weight $-2d-2r+k$, therefore $H_c^n(\mathscr{X})$ is pure of weight n, for every n. $\square$

6.3.6. Finally, let us consider the following much weaker version of independence of ℓ. For $\mathscr{X}_0$ and $i \in \mathbb{Z}$, let $\Psi(\mathscr{X}_0, i)$ be the following property: the Frobenius eigenvalues of $H_c^i(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$, counted with multiplicity, for all $\ell \neq p$, are contained in a finite set of algebraic numbers with multiplicities assigned, and this set together with the assignment of multiplicity, depends only on $\mathscr{X}_0$ and i. In particular it is independent of ℓ. In other words, there is a finite decomposition of the set of all prime numbers $\ell \in \mathbb{N} - \{p\}$ into disjoint union of some subsets, such that the Frobenius eigenvalues of $H_c^i(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ depends only on the subset that ℓ belongs to. If this property holds, we also denote such a finite set of algebraic numbers (which is not unique) by $\Psi(\mathscr{X}_0, i)$, if there is no confusion.

Proposition 6.3.6.1. *The property $\Psi(\mathscr{X}_0, i)$ holds for every $\mathscr{X}_0$ and i.*

Proof. If $\mathscr{U}_0$ is an open substack of $\mathscr{X}_0$ with complement $\mathscr{Z}_0$, and properties $\Psi(\mathscr{U}_0, i)$ and $\Psi(\mathscr{Z}_0, i)$ hold, then $\Psi(\mathscr{X}_0, i)$ also holds, and the finite set $\Psi(\mathscr{X}_0, i)$ a subset of $\Psi(\mathscr{U}_0, i) \cup \Psi(\mathscr{Z}_0, i)$.

Firstly we prove this for schemes X_0. By shrinking X_0 we can assume it is a connected smooth variety. By Poincaré duality it suffices to prove the similar statement $\Psi^*(X_0, i)$ for ordinary cohomology, i.e. with H_c^i replaced by H^i, for all i. This follows from [7] and ([9], 3.3.9). Therefore it also holds for all algebraic spaces.

For a general algebraic stack $\mathscr{X}_0$, by shrinking it we can assume it is connected smooth. By Poincaré duality, it suffices to prove $\Psi^*(\mathscr{X}_0, i)$ for all i. This can be done by taking a hypercover by simplicial algebraic spaces, and considering the associated spectral sequence. $\square$

Chapter 7

Decomposition theorem for stacks over $\mathbb{F}$.

Firstly, we study a counter-example given by Drinfeld, to see why the assumption on the diagonal is necessary. Then we give the structure theorem of ι-mixed sheaves on stacks, as the prototype for the analogous results for perverse sheaves. Finally we prove the stack version of some results in ([4], 5), and deduce the decomposition theorem.

7.1 A counter-example: BE.

Let E be a complex elliptic curve, and let f : pt=Spec $\mathbb{C} \to BE$ be the natural projection; this is a representable proper map. There is a natural non-zero morphism $\underline{\mathbb{C}}_{BE} \to Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}$, adjoint to the isomorphism $f^*\underline{\mathbb{C}}_{BE} \simeq \underline{\mathbb{C}}_{\mathrm{pt}}$, but there is no non-zero morphism in the other direction, because

$$Hom(Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{BE}) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, f^!\underline{\mathbb{C}}_{BE}) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}[2]) = 0.$$

Here the Hom's are taken in the derived categories. Similarly, the non-zero natural map $Rf_*\underline{\mathbb{C}}_{\mathrm{pt}} \to R^2f_*\underline{\mathbb{C}}_{\mathrm{pt}}[-2] = \underline{\mathbb{C}}_{BE}[-2]$ lies in

$$Hom(Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{BE}[-2]) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, f^!\underline{\mathbb{C}}_{BE}[-2]) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}) = \mathbb{C},$$

but the Hom set in the other direction is zero:

$$Hom(\underline{\mathbb{C}}_{BE}[-2], Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}) = Hom(f^*\underline{\mathbb{C}}_{BE}[-2], \underline{\mathbb{C}}_{\mathrm{pt}}) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}[-2], \underline{\mathbb{C}}_{\mathrm{pt}}) = 0.$$

Therefore, $Rf_*\mathbb{C}$ is not semi-simple of geometric origin (since it is not a direct sum of the ${}^p\mathscr{H}^i(f_*\underline{\mathbb{C}})[-i]$'s). The same argument applies to finite fields, with $\underline{\mathbb{C}}$ replaced by $\overline{\mathbb{Q}}_\ell$.

Remark 7.1.1. This example was first given by Drinfeld, who asked for the reason of the failure of the usual argument for schemes. Later, it was communicated by J. Bernstein to Y. Varshavsky, who asked M. Olsson in an email correspondence. Olsson kindly shared this email with me, and explained to me that the reason is the failure of the upper bound of weights in [9] for stacks.

In the following we explain why the usual proof (as in [4]) fails for f. The proof in [4] of the decomposition theorem over $\mathbb{C}$ relies on the decomposition theorems over finite fields ([4], 5.3.8, 5.4.5), so it suffices to explain why the proof of ([4], 5.4.5) fails for f, for an elliptic curve $E/\mathbb{F}_q$.

Let $K_0 = Rf_*\overline{\mathbb{Q}}_\ell$. The perverse t-structure agrees with the trivial t-structure on Spec $\mathbb{F}_q$, and by definition ([28], 4), we have ${}^p\mathscr{H}^iK_0 = \mathscr{H}^{i+1}(K_0)[-1]$ on BE, and so

$$\bigoplus_i ({}^p\mathscr{H}^iK)[-i] = \bigoplus_i (\mathscr{H}^iK)[-i].$$

Each $R^if_*\overline{\mathbb{Q}}_\ell[-i]$ is pure of weight 0. In the proof of ([4], 5.4.5), the exact triangles

$$\tau_{<i}K_0 \longrightarrow \tau_{\leq i}K_0 \longrightarrow (\mathscr{H}^iK_0)[-i] \longrightarrow$$

split geometrically, because $Ext^1((\mathscr{H}^iK)[-i], \tau_{<i}K)$ has weights > 0. We will see that for $f : \text{Spec } \mathbb{F}_q \to BE$, this group is pure of weight 0, and in fact has 1 as a Frobenius eigenvalue.

Let $\pi : BE \to \text{Spec } \mathbb{F}_q$ be the structural map; then $\pi \circ f = \text{id}$. Since E is connected, the sheaves $R^if_*\overline{\mathbb{Q}}_\ell$ are inverse images of some sheaves on Spec $\mathbb{F}_q$, namely $f^*R^if_*\overline{\mathbb{Q}}_\ell$. By smooth base change, they are isomorphic to $\pi^*H^i(E)$ as $\text{Gal}(\mathbb{F}_q)$-modules. In particular, $R^0f_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell$, $R^1f_*\overline{\mathbb{Q}}_\ell \cong \pi^*H^1(E)$ and $R^2f_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell(-1)$. Then the exact triangle above becomes

$$\begin{aligned} i = 2: &\qquad \tau_{\leq 1}K_0 \longrightarrow K_0 \longrightarrow \overline{\mathbb{Q}}_\ell(-1)[-2] \longrightarrow \\ i = 1: &\qquad \overline{\mathbb{Q}}_\ell \longrightarrow \tau_{\leq 1}K_0 \longrightarrow \pi^*H^1(E)[-1] \longrightarrow \cdot \end{aligned}$$

Apply $Ext^*(\overline{\mathbb{Q}}_\ell(-1)[-2], -)$ to the second triangle. From (5.2.2) we see that $H^{2i-1}(BE) = 0$, and $H^{2i}(BE) = \text{Sym}^iH^1(E)$. Let α and β be the eigenvalues of the Frobenius F on $H^1(E)$. We have

$$Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2], \overline{\mathbb{Q}}_\ell) = Ext^3(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell(1)) = H^3(BE)(1) = 0,$$

and

$$Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2], \pi^*H^1(E)[-1]) = H^2(BE)\otimes H^1(E)(1) = H^1(E)\otimes H^1(E)(1) = End(H^1(E)),$$

which is 4-dimensional with eigenvalues $\alpha/\beta, \beta/\alpha, 1, 1$, and

$$Ext^2(\overline{\mathbb{Q}}_\ell(-1)[-2], \overline{\mathbb{Q}}_\ell) = H^4(BE)(1),$$

which is 3-dimensional with eigenvalues $\alpha/\beta, \beta/\alpha, 1$. This implies that the kernel

$$Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2], \tau_{\leq 1}K) = \\ \mathrm{Ker}\Big(Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2], \pi^*H^1(E)[-1]) \to Ext^2(\overline{\mathbb{Q}}_\ell(-1)[-2], \overline{\mathbb{Q}}_\ell)\Big)$$

is non-zero, pure of weight 0, and has 1 as a Frobenius eigenvalue. So the first exact triangle above does not necessarily (in fact does not, as the argument in the beginning shows) split geometrically. Also

$$Ext^1(\pi^*H^1(E)[-1], \overline{\mathbb{Q}}_\ell) = Ext^2(\overline{\mathbb{Q}}_\ell, \pi^*H^1(E)^\vee) = H^1(E) \otimes H^1(E)^\vee = End(H^1(E))$$

is 4-dimensional and has eigenvalues $\alpha/\beta, \beta/\alpha, 1, 1$, hence the proof for the geometric splitting of the second exact triangle fails too.

In [28], Laszlo and Olsson generalized the theory of perverse sheaves to Artin stacks locally of finite type over some field. In §6.1, we proved that for Artin stacks of finite type over a finite field, with affine automorphism groups (defined below (8.1.5.1)), Deligne's upper bound of weights for the compactly supported cohomology groups still applies. In this chapter, we will show that for such stacks, similar argument as in [4] gives the decomposition theorem.

7.2 The prototype: the structure theorem of mixed sheaves on stacks.

We generalize the structure theorem of ι-mixed sheaves ([9], 3.4.1) to stacks. This result is independent from other results in this chapter, but it is the prototype, in some sense I think, of the corresponding results (e.g. weight filtrations and the decomposition theorem) for perverse sheaves.

Theorem 7.2.1. *(stack version of ([9], 3.4.1)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack.*

(i) Every ι-mixed sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ has a unique decomposition $\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$, called the decomposition according to the weights mod $\mathbb{Z}$, *such that the punctual ι-weights of $\mathscr{F}_0(b)$ are all in the coset b. This decomposition, in which almost all the $\mathscr{F}_0(b)$'s are zero, is functorial in $\mathscr{F}_0$.*

(ii) Every ι-mixed lisse sheaf $\mathscr{F}_0$ with integer punctual ι-weights on $\mathscr{X}_0$ has a unique finite increasing filtration W by lisse subsheaves, called the weight filtration, *such that* Gr_i^W

is punctually ι-pure of weight i. Every morphism between such sheaves on $\mathscr{X}_0$ is strictly compatible with their weight filtrations.

(iii) If $\mathscr{X}_0$ is a normal algebraic stack, and $\mathscr{F}_0$ is a lisse and punctually ι-pure sheaf on $\mathscr{X}_0$, then $\mathscr{F}$ on $\mathscr{X}$ is semi-simple.

Proof. (i) and (ii) are proved in (2.2.4.1), where (iii) is claimed to hold without giving a detailed proof. Here we complete the proof of (iii).

First of all, note that we may replace $\mathscr{X}_0$ and $\mathscr{F}_0$ by $\mathscr{X}_0 \otimes \mathbb{F}_{q^v}$ and $\mathscr{F}_0 \otimes \mathbb{F}_{q^v}$, for any finite base change $\mathbb{F}_{q^v}/\mathbb{F}_q$.

From the proof of ([28], 8.3), we see that if $\mathscr{U} \subset \mathscr{X}$ is an open substack, and $\mathscr{G}_{\mathscr{U}}$ is a subsheaf of $\mathscr{F}|_{\mathscr{U}}$, then it extends to a unique subsheaf $\mathscr{G} \subset \mathscr{F}$. Therefore, we may shrink $\mathscr{X}$ to a dense open substack $\mathscr{U}$, and replace $\mathscr{X}_0$ by some model of $\mathscr{U}$ over a finite extension $\mathbb{F}_{q^v}$. We can assume $\mathscr{X}_0$ is smooth and geometrically connected.

Following the proof ([9], 3.4.5), it suffices to show ([9], 3.4.3) for stacks. We claim that, if $\mathscr{F}_0$ is lisse and punctually ι-pure of weight w, then $H^1(\mathscr{X},\mathscr{F})$ is ι-mixed of weights $\geq 1+w$. The conclusion follows from this claim.

Let $D = \dim \mathscr{X}_0$. By Poincaré duality, it suffices to show that, for every lisse sheaf $\mathscr{F}_0$, punctually ι-pure of weight w, $H_c^{2D-1}(\mathscr{X},\mathscr{F})$ is ι-mixed of weights $\leq 2D-1+w$. To show this, we may shrink $\mathscr{X}_0$ to open substacks, and hence we may assume that the inertia $\mathscr{I}_0 \to \mathscr{X}_0$ is flat. As in the proof of (1.0.4), we have the spectral sequence

$$H_c^r(X, R^k\pi_!\mathscr{F}) \Longrightarrow H_c^{r+k}(\mathscr{X},\mathscr{F}),$$

so let $r+k = 2D-1$. Note that k can only be of the form $-2i-2d$, for $i \geq 0$, where $d = \text{rel.dim}(\mathscr{I}_0/\mathscr{X}_0)$. So we have $r = 2\dim X_0 + 2i - 1$, and in order for $H_c^r(X,-)$ to be non-zero, $i = 0$. Then

$$H_c^{2D-1}(\mathscr{X},\mathscr{F}) = H_c^{2\dim X-1}(X, R^{-2d}\pi_!\mathscr{F}).$$

It suffices to show that $H_c^{-2d}(BG,\mathscr{F})$ has weights $\leq w-2d$, where G_0 is an algebraic group of dimension d, and $\mathscr{F}_0$ is a lisse punctually ι-pure sheaf on BG_0 of weight w. In fact, $R^{-2d}\pi_!\mathscr{F}$ is punctually ι-pure of weight $w-2d$. We reduce to the case where G_0 is connected, and the claim is clear. □

7.3 Decomposition theorem for stacks over $\mathbb{F}_q$.

For an algebraic stack $\mathscr{X}_0/\mathbb{F}_q$, let $D_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ be the full subcategory of ι-mixed sheaf complexes in $D_c(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ as before (2.2.2iii). It is stable under the perverse truncations ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$. This can be checked smooth locally, and hence follows from (2.2.5) and ([4], 5.1.6). The core of $D_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ with respect to this induced perverse t-structure is called the category of ι-mixed perverse sheaves on $\mathscr{X}_0$, denoted $\mathrm{Perv}_m(\mathscr{X}_0)$. This is a Serre subcategory of $\mathrm{Perv}(\mathscr{X}_0)$. Again, using (2.2.5.1), all of these are trivial.

Definition 7.3.1. *Let $K_0 \in D_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$.*

(i) We say that K_0 has ι-weights $\leq w$ if for each $i \in \mathbb{Z}$, the ι-weights of $\mathscr{H}^i K_0$ are $\leq i + w$, and we denote by $D_{\leq w}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ the subcategory of such complexes. We say that K_0 has ι-weights $\geq w$ if its Verdier dual DK_0 has ι-weights $\leq -w$, and denote by $D_{\geq w}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ the subcategory of such complexes.

(ii) For a coset $b \in \mathbb{R}/\mathbb{Z}$, we say that K_0 has ι-weights in b if the ι-weights of $\mathscr{H}^i K_0$ are in b, for all $i \in \mathbb{Z}$.

Lemma 7.3.2. *Let $P : \mathscr{X}_0' \to \mathscr{X}_0$ be a representable surjection of $\mathbb{F}_q$-algebraic stacks, and $K_0 \in D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. Then K_0 is ι-mixed of weights $\leq w$ (resp. $\geq w$) if and only if P^*K_0 (resp. $P^!K_0$) is so.*

Proof. It suffices to consider only the case where K_0 has weights $\leq w$, since the other statement is dual to this one. The "only if" part is obvious. The "if" part for ι-mixedness follows from (2.2.5), and the "if" part for the weights follows from the surjectivity of P. $\square$

In particular, this applies to the case where P is a presentation.

We say that an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ has *affine automorphism groups* if for every integer $v \geq 1$ and every $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, the automorphism group scheme Aut_x over $k(x)$ is affine. In the following, some results require the stack to have affine automorphism groups. We will first give results that apply to all stacks, and then give those that require this condition.

The following lemma is the perverse sheaf version of (7.2.1i).

Lemma 7.3.3. *Every ι-mixed perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ has a unique decomposition*

$$\mathscr{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$$

into perverse subsheaves, called the decomposition according to the weights mod $\mathbb{Z}$, *such that for each coset b, the ι-weights of $\mathscr{F}_0(b)$ belong to b. This decomposition, in which almost all the $\mathscr{F}_0(b)$'s are zero, is functorial in $\mathscr{F}_0$.*

Proof. By descent theory ([28], 7.1) we reduce to the case where $\mathscr{X}_0 = X_0$ is a scheme. One can further replace X_0 by the disjoint union of finitely many open affines, and assume X_0 is separated. We want to reduce to the case where X_0 is proper.

Let $j : X_0 \hookrightarrow Y_0$ be a Nagata compactification, i.e. an open dense immersion into a proper scheme Y_0, and assume we have the existence and uniqueness of the decomposition of any ι-mixed perverse sheaf on Y_0 according to the weights mod $\mathbb{Z}$, and the decomposition is functorial. Let

$$j_{!*}\mathscr{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} G_0(b)$$

be the decomposition for $j_{!*}\mathscr{F}_0$. Applying j^* we get a decomposition

$$\mathscr{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} j^* G_0(b).$$

Note that j^* takes a perverse sheaf to a perverse sheaf. This shows the existence. For uniqueness, let $\mathscr{F}_0 = \bigoplus_b \mathscr{F}_0(b)$ be another such decomposition. Then we have

$$j_{!*}\mathscr{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} j_{!*}\mathscr{F}_0(b).$$

Following the proof in [4] we see that $j_{!*}\mathscr{F}_0(b)$ is ι-mixed of weights in b (by twisting, we may assume $\mathscr{F}_0(b)$ is ι-mixed of integer weights, then follow the proof in ([4], 5.3.1) to show $j_{!*}$ preserves ι-mixedness with integer weights, and finally twist back). By uniqueness of the decomposition for $j_{!*}\mathscr{F}_0$ we have $j_{!*}\mathscr{F}_0(b) = G_0(b)$, and so $\mathscr{F}_0(b) = j^* G_0(b)$. For functoriality, given a morphism $\mathscr{F}_0 \to \mathscr{G}_0$ between ι-mixed perverse sheaves on X_0, we get a morphism $j_{!*}\mathscr{F}_0 \to j_{!*}\mathscr{G}_0$ of ι-mixed sheaves on Y_0, which respects their decompositions by assumption, and then apply j^*.

So we may assume that $X_0/\mathbb{F}_q$ is proper. Let a be the structural map of $X_0/\mathbb{F}_q$. Let K_0 and L_0 in $D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$ be ι-pure complexes of ι-weights w and w', respectively, and assume $w - w' \notin \mathbb{Z}$. Then we claim that $Ext^1(K_0, L_0) = 0$. From the exact sequence ([4], 5.1.2.5)

$$0 \longrightarrow Ext^{i-1}(K, L)_F \longrightarrow Ext^i(K_0, L_0) \longrightarrow Ext^i(K, L)^F \longrightarrow 0$$

we see it suffices to show that 1 cannot be a Frobenius eigenvalue on $Ext^i(K, L)$, for every i. Note that $R\mathscr{H}om(K_0, L_0) = D(K_0 \otimes^L DL_0)$ is ι-pure of weight $w' - w$, by the spectral sequence

$$\mathscr{H}^i(K_0 \otimes^L \mathscr{H}^j DL_0) \Longrightarrow \mathscr{H}^{i+j}(K_0 \otimes^L DL_0)$$

and the similar one for the first factor K_0. Consider the spectral sequence

$$R^i a_* R^j \mathscr{H}om(K_0, L_0) \Longrightarrow R^{i+j}(a_* \mathscr{H}om)(K_0, L_0).$$

Since $a_* = a_!$, by ([9], 3.3.10) we see that the ι-weights of $Ext^i(K, L)$ cannot be integers. Therefore $Ext^1(K_0, L_0) = 0$.

For every $b \in \mathbb{R}/\mathbb{Z}$, we apply ([4], 5.3.6) to $\mathrm{Perv}_m(X_0)$, taking S^+ (resp. S^-) to be the set of isomorphism classes of simple ι-mixed perverse sheaves (and hence ι-pure (7.3.5)) of weight not in b (resp. in b). Then for every ι-mixed perverse sheaf $\mathscr{F}_0$, we get a unique subobject $\mathscr{F}_0(b)$ with ι-weights in b, such that $\mathscr{F}_0/\mathscr{F}_0(b)$ has ι-weights not in b, and $\mathscr{F}_0(b)$ is functorial in $\mathscr{F}_0$. As we see from the argument above, this extension splits: $\mathscr{F}_0 = \mathscr{F}_0(b) \oplus \mathscr{F}_0/\mathscr{F}_0(b)$, so by induction we get the decomposition, which is unique and functorial. $\square$

Lemma 7.3.4. *(stack version of ([4], 5.3.2)) Let $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ be an immersion of algebraic* *stacks. Then for any real number w, the intermediate extension $j_{!*}$ ([28], 6) respects* $\mathrm{Perv}_{\geq w}$ *and* $\mathrm{Perv}_{\leq w}$. *In particular, if $\mathscr{F}_0$ is an ι-pure perverse sheaf on $\mathscr{U}_0$, then $j_{!*}\mathscr{F}_0$ is ι-pure of the same weight.*

Proof. For a closed immersion i, we see that i_* respects $D_{\geq w}$ and $D_{\leq w}$, so we may assume that j is an open immersion. We only need to consider the case for $\mathrm{Perv}_{\leq w}$, since the case for $\mathrm{Perv}_{\geq w}$ follows from $j_{!*}D = Dj_{!*}$.

Let $P : X_0 \to \mathscr{X}_0$ be a presentation, and let the following diagram be 2-Cartesian:

$$\begin{tikzcd} U_0 \arrow[r, "j'"] \arrow[d, "P'"'] & X_0 \arrow[d, "P"] \\ \mathscr{U}_0 \arrow[r, "j"'] & \mathscr{X}_0. \end{tikzcd}$$

For $\mathscr{F}_0 \in \mathrm{Perv}_{\leq w}(\mathscr{U}_0)$, by (7.3.2) it suffices to show that $P^*j_{!*}\mathscr{F}_0 \in D_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$. Let d be the relative dimension of P. By ([28], 6.2) we have

$$P^*j_{!*}\mathscr{F}_0 = (P^*(j_{!*}\mathscr{F}_0)[d])[-d] = j'_{!*}(P'^*\mathscr{F}_0[d])[-d].$$

Since $P'^*\mathscr{F}_0 \in D_{\leq w}$, $P'^*\mathscr{F}_0[d] \in D_{\leq w+d}$, and by ([4], 5.3.2), $j'_{!*}(P'^*\mathscr{F}_0[d]) \in \mathrm{Perv}_{\leq w+d}$, and by definition $P^*j_{!*}\mathscr{F}_0 \in D_{\leq w}$. $\square$

Corollary 7.3.5. *(stack version of ([4], 5.3.4)) Every ι-mixed simple perverse sheaf $\mathscr{F}_0$ on an algebraic stack $\mathscr{X}_0$ is ι-pure.*

Proof. By ([28], 8.2ii), there exists a d-dimensional irreducible substack $j : \mathscr{V}_0 \hookrightarrow \mathscr{X}_0$ such that $\mathscr{V}_{\mathrm{red}}$ is smooth, and a simple ι-mixed (hence ι-pure) lisse sheaf L_0 on $\mathscr{V}_0$ such that $\mathscr{F}_0 \cong j_{!*}L_0[d]$. The result follows from (7.3.4). $\square$

The stack version of ([4], 5.3.5) is given in ([28], 9.2), and the following is a version for ι-mixed perverse sheaves with integer weights (7.3.1ii), which is the perverse sheaf version of (7.2.1ii).

Theorem 7.3.6. *Let $\mathscr{F}_0$ be an ι-mixed perverse sheaf on $\mathscr{X}_0$ with integer weights. Then there exists a unique finite increasing filtration W of $\mathscr{F}_0$ by perverse subsheaves, called the* weight filtration, *such that $\mathrm{Gr}_i^W\mathscr{F}_0$ is ι-pure of weight i, for each i. Every morphism between such perverse sheaves on $\mathscr{X}_0$ is strictly compatible with their weight filtrations.*

Proof. As in ([28], 9.2), we may assume $\mathscr{X}_0 = X_0$ is a scheme. The proof in ([4], 5.3.5) still applies. Namely, by (7.3.9ii), if $\mathscr{F}_0$ and $\mathscr{G}_0$ are ι-pure simple perverse sheaves on X_0, of ι-weights f and g respectively, and $f > g$, then $Ext^1(\mathscr{G}_0, \mathscr{F}_0) = 0$. Then take S^+ (resp. S^-) to be the set of isomorphism classes of ι-pure simple perverse sheaves on X_0 of ι-weights $> i$ (resp. $\leq i$) for each integer i, and apply ([4], 5.3.6). $\square$

Theorem 7.3.7. *(stack version of ([4], 5.4.1, 5.4.4)) Let $K_0 \in D_m^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. Then K_0 has ι-weights $\leq w$ (resp. $\geq w$) if and only if ${}^p\mathscr{H}^i K_0$ has ι-weights $\leq w+i$ (resp. $\geq w+i$), for each $i \in \mathbb{Z}$. In particular, K_0 is ι-pure of weight w if and only if each ${}^p\mathscr{H}^i K_0$ is ι-pure of weight $w+i$.*

Proof. The case of " $\geq$ " follows from the case of " $\leq$ " and ${}^p\mathscr{H}^i \circ D = D \circ {}^p\mathscr{H}^{-i}$. So we only need to show the case of " $\geq$ ".

Let $P : X_0 \to \mathscr{X}_0$ be a presentation of relative dimension d. Then K_0 has ι-weights $\leq w$ if and only if (7.3.2) P^*K_0 has ι-weights $\leq w$, if and only if ([4], 5.4.1) each ${}^p\mathscr{H}^i(P^*K_0)$ has ι-weights $\leq w+i$. We have ${}^p\mathscr{H}^i(P^*K_0) = {}^p\mathscr{H}^i(P^*(K_0[-d])[d]) = P^{*p}\mathscr{H}^i(K_0[-d])[d] = P^*({}^p\mathscr{H}^{i-d}K_0)[d]$, so $P^*({}^p\mathscr{H}^{i-d}K_0)$, and hence ${}^p\mathscr{H}^{i-d}K_0$, has ι-weights $\leq w+i-d$. $\square$

In the following results, except (7.3.8i, ii, iv, v), we will need the assumption of affine automorphism groups.

Proposition 7.3.8. *(stack version of ([4], 5.1.14)) (i) The Verdier dual D interchanges $D_{\leq w}$ and $D_{\geq -w}$.*

(ii) For every morphism f of $\mathbb{F}_q$-algebraic stacks, f^* respects $D_{\leq w}$ and $f^!$ respects $D_{\geq w}$.

(iii) For every morphism $f : \mathscr{X}_0 \to \mathscr{Y}_0$, where $\mathscr{X}_0$ is an $\mathbb{F}_q$-algebraic stack with affine automorphism groups, $f_!$ respects $D^{-,\mathrm{stra}}_{\leq w}$ and f_* respects $D^{+,\mathrm{stra}}_{\geq w}$.

(iv) $\otimes^L$ takes $D^-_{\leq w} \times D^-_{\leq w'}$ into $D^-_{\leq w+w'}$.

(v) $R\mathscr{H}om$ takes $D^-_{\leq w} \times D^+_{\geq w'}$ into $D^+_{\geq w'-w}$.

Proof. (i), (ii) and (iv) are clear, and (v) follows from (iv). For (iii), if $\mathscr{X}_0$ has affine automorphism groups, so are all fibers $f^{-1}(y)$, for $y \in \mathscr{Y}_0(\mathbb{F}_{q^v})$, and the claim for $f_!$ follows from the spectral sequence

$$H^i_c(f^{-1}(\overline{y}), \mathscr{H}^j K) \Longrightarrow H^{i+j}_c(f^{-1}(\overline{y}), K)$$

and (1.0.4), and the claim for f_* follows from (2.2.8, 3.2.4) and the claim for $f_!$. $\square$

Corollary 7.3.9. *(stack version of ([4], 5.1.15)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack with affine automorphism groups, and let $a : \mathscr{X}_0 \to \mathrm{Spec}\ \mathbb{F}_q$ be the structural map. Let K_0 (resp. L_0) be in $D^-_{\leq w}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (resp. $D^+_{>w}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$) for some real number w. Then*

(i) $a_*R\mathscr{H}om(K_0, L_0)$ is in $D^+_{>0}(\mathrm{Spec}\ \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$.

(ii) $Ext^i(K_0, L_0) = 0$ for $i > 0$.

If $L_0 \in D^+_{\geq w}$, then $a_*R\mathscr{H}om(K_0, L_0)$ is in $D^+_{\geq 0}$, and we have

(iii) $Ext^i(K, L)^F = 0$ for $i > 0$. In particular, for $i > 0$, the morphism $Ext^i(K_0, L_0) \to Ext^i(K, L)$ is zero.

The proof is the same as ([4], 5.1.15), using the above stability result for stacks with affine automorphism groups.

The following is a perverse sheaf version of (7.2.1iii), the decomposition theorem.

Theorem 7.3.10. *(stack version of ([4], 5.3.8)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack with affine automorphism groups. Then every ι-pure perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ is geometrically semi-simple (i.e. $\mathscr{F}$ is semi-simple), hence $\mathscr{F}$ is a direct sum of perverse sheaves of the form $j_{!*}L[d_{\mathscr{U}}]$, for inclusions $j : \mathscr{U} \hookrightarrow \mathscr{X}$ of $d_{\mathscr{U}}$-dimensional irreducible substacks that are essentially smooth, and for simple ι-pure lisse sheaves L on $\mathscr{U}$.*

Proof. Let $\mathscr{F}'$ be the sum in $\mathscr{F}$ of simple perverse subsheaves; it is a direct sum, and is the largest semi-simple perverse subsheaf of $\mathscr{F}$. Then $\mathscr{F}'$ is stable under Frobenius, hence descends to a perverse subsheaf $\mathscr{F}'_0 \subset \mathscr{F}_0$ (([4], 5.1.2) holds for stacks also). Let $\mathscr{F}''_0 = \mathscr{F}_0/\mathscr{F}'_0$. By (7.3.9iii), the extension

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

splits, because $\mathscr{F}'_0$ and $\mathscr{F}''_0$ have the same weight ([28], 9.3). Then $\mathscr{F}''$ must be zero, since otherwise it contains a simple perverse subsheaf, and this contradicts the maximality of $\mathscr{F}'$. Therefore $\mathscr{F} = \mathscr{F}'$ is semi-simple. The other claim follows from ([28], 8.2ii). $\square$

Theorem 7.3.11. *(stack version of ([4], 5.4.5)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack with affine automorphism groups, and let $K_0 \in D^b_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be ι-pure. Then K on $\mathscr{X}$ is isomorphic to the direct sum of the shifted perverse cohomology sheaves $({}^p\mathscr{H}^i K)[-i]$.*

Proof. By (7.3.7), both ${}^p\tau_{<i}K_0$ and $({}^p\mathscr{H}^i K_0)[-i]$ are ι-pure of the same weight as that of K_0. Therefore, by (7.3.9iii), the exact triangle

$${}^p\tau_{<i}K_0 \longrightarrow {}^p\tau_{\leq i}K_0 \longrightarrow ({}^p\mathscr{H}^i K_0)[-i] \longrightarrow$$

is geometrically split, i.e. we have

$${}^p\tau_{\leq i}K \simeq {}^p\tau_{<i}K \oplus ({}^p\mathscr{H}^i K)[-i],$$

and the result follows by induction. $\square$

Chapter 8

Decomposition theorem over the complex numbers.

In this chapter, we prove the decomposition theorem for complex algebraic stacks with affine automorphism groups. To mimic the proof in [4], there are several things to generalize.

• To use the weight technique and results over finite fields, we need to prove a comparison between the derived categories with prescribed stratification over $\mathbb{C}$ and over $\mathbb{F}$. For that, we generalize the generic base change theorem to stacks.

• Since the statement of the decomposition theorem will be about the analytic stack associated to a complex algebraic stack, we need to prove a comparison between the derived categories of the lisse-étale topos and of the lisse-analytic topos. Note that the lisse-analytic derived category will also be defined using the usual adic formalism, since the comparison is proved over the torsion level first.

• One can define another lisse-analytic derived category, using the ordinary topology, and the statement of the decomposition theorem will be about this derived category. Therefore, we need to prove a comparison between this one and the one defined using adic formalism.

8.1 Generic base change.

We prove a stack version of the generic base change theorem ([11], Th. finitude) in this section.

8.1.1. Let S be a scheme satisfying the condition (LO) as in (2.1.1): it is a noetherian affine excellent finite-dimensional scheme in which ℓ is invertible, and all S-schemes of finite type have finite ℓ-cohomological dimension. As before, let $(\Lambda, \mathfrak{m})$ be a complete DVR of mixed characteristic, with finite residue field Λ_0 of characteristic ℓ and uniformizer λ. Let $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$. Let $\mathscr{A} = \mathscr{A}(\mathcal{X}) = \mathrm{Mod}(\mathcal{X}^{\mathbb{N}}_{\text{lis-ét}}, \Lambda_\bullet)$.

For a pair $(\mathscr{S}, \mathcal{L})$, where $\mathscr{S}$ is a stratification of the stack $\mathcal{X}$, and $\mathcal{L}$ assigns to every stratum $\mathcal{U} \in \mathscr{S}$ a finite set $\mathcal{L}(\mathcal{U})$ of isomorphism classes of simple lcc Λ_0-sheaves on $\mathcal{U}$, let

$\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$ be the full subcategory of $\mathscr{D}_c(\mathscr{A})$ consisting of the complexes of projective systems $K = (K_n)_n$ trivialized by $(\mathscr{S}, \mathcal{L})$, and define $D_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda)$ to be its essential image under the localization $\mathscr{D}_c(\mathscr{A}) \to D_c(\mathcal{X}, \Lambda)$; in other words, it is the quotient of $\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$ by the thick subcategory of AR-null complexes. It is a triangulated category.

8.1.2. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of S-algebraic stacks and $K \in D_c^+(\mathcal{X}, \Lambda_n)$ (resp. $D_c^+(\mathcal{X}, \Lambda)$), we say that *the formation of f_*K commutes with generic base change,* if there exists an open dense subset $U \subset S$ such that for any morphism $g : S' \to U \subset S$ with S' satisfying (LO), the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ is an isomorphism. The base change morphism is defined to be the one corresponding by adjunction (g'^*, g'_*) to $f_* K \to g'_* f_{S'*} g''^* K \simeq f_* g''_* g''^* K$, obtained by applying f_* to the adjunction map $K \to g''_* g''^* K$.

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "f"'] & \mathcal{X}_{S'} \arrow[l, "g''"'] \arrow[d, "f_{S'}"] \\
\mathcal{Y} \arrow[d] & \mathcal{Y}_{S'} \arrow[l, "g'"'] \arrow[d] \\
S & U \arrow[l, hook'] & S' \arrow[l, "g"']
\end{tikzcd}$$

We give some basic results on generic base change.

Lemma 8.1.3. *(i) Let $P : Y \to \mathcal{Y}$ be a presentation, and let the following diagram be 2-Cartesian:*

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "f"'] & \mathcal{X}_Y \arrow[l, "P'"'] \arrow[d, "f'"] \\
\mathcal{Y} & Y \arrow[l, "P"]
\end{tikzcd} .$$

Then for $K \in D_c^+(\mathcal{X}, \Lambda_n)$ (resp. $K \in D_c^+(\mathcal{X}, \Lambda)$), the formation of f_*K commutes with generic base change if and only if the formation of $f'_* P'^* K$ commutes with generic base change.

(ii) Let $K' \to K \to K'' \to K'[1]$ be an exact triangle in $D_c^+(\mathcal{X}, \Lambda_n)$ (resp. $D_c^+(\mathcal{X}, \Lambda)$), and let $f : \mathcal{X} \to \mathcal{Y}$ be an S-morphism. If the formations of f_*K' and f_*K'' commute with generic base change, then so is the formation of f_*K.

(iii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a schematic morphism, and let $K \in D^+_{\{\mathcal{X}\},\mathcal{L}}(\mathcal{X}, \Lambda)$ for some finite set $\mathcal{L}$ of isomorphism classes of simple Λ_0-modules on $\mathcal{X}$. Then the formation of f_*K commutes with generic base change.

(iv) Let $K \in D_c^+(\mathcal{X}, \Lambda)$, and let $j : \mathcal{U} \to \mathcal{X}$ be an open immersion with complement

$i : \mathcal{Z} \to \mathcal{X}$. *For* $g : S' \to S$, *consider the following diagram obtained by base change:*

$$\begin{tikzcd}
 & & \mathcal{U}_{S'} \arrow[r, hook, "j_{S'}"] \arrow[dll, "g_{\mathcal{U}}"'] & \mathcal{X}_{S'} \arrow[dll, "g''"'] \arrow[d, "f_{S'}"] & \mathcal{Z}_{S'} \arrow[l, hook', "i_{S'}"'] \arrow[dll, "g_{\mathcal{Z}}"] . \\
\mathcal{U} \arrow[r, hook, "j"'] & \mathcal{X} \arrow[d, "f"] & \mathcal{Z} \arrow[l, hook', "i"] & \mathcal{Y}_{S'} \arrow[dll, "g'"] & \\
 & \mathcal{Y} & & &
\end{tikzcd}$$

Suppose the base change morphisms

$$g'^*(fj)_*j^*K \longrightarrow (f'j_{S'})_*g^*_{\mathcal{U}}j^*K,$$
$$g'^*(fi)_*i^!K \longrightarrow (f'i_{S'})_*g^*_{\mathcal{Z}}i^!K \qquad \text{and}$$
$$g''^*j_*j^*K \longrightarrow j_{S'*}g^*_{\mathcal{U}}j^*K$$

are isomorphisms, then the base change morphism $g'^*f_*K \to f_{S'*}g''^*K$ *is also an isomorphism.*

(v) Let $f : \mathcal{X} \to \mathcal{Y}$ *be a schematic morphism of* S*-Artin stacks, and let* $K \in D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda)$. *Then the formation of* f_*K *commutes with generic base change on* S.

(vi) Let $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism of* S*-Artin stacks, and let* $j : \mathcal{U} \to \mathcal{Y}$ *be an open immersion with complement* $i : \mathcal{Z} \to \mathcal{Y}$. *Let* $K \in D_c^+(\mathcal{X}, \Lambda)$ *(or* $D_c^+(\mathcal{X}, \Lambda_n)$*). For a map* $g : S' \to S$, *consider the following diagram, in which the squares are 2-Cartesian:*

$$\begin{tikzcd}
 & \mathcal{X}_{\mathcal{U},S'} \arrow[rr, hook, "j'_{S'}"] \arrow[dl, "g''_{\mathcal{U}}"'] \arrow[dd, "f_{\mathcal{U}_{S'}}"] & & \mathcal{X}_{S'} \arrow[dl, "g''"'] \arrow[dd, "f_{S'}"] & & \mathcal{X}_{\mathcal{Z},S'} \arrow[ll, hook', "i'_{S'}"'] \arrow[dl, "g''_{\mathcal{Z}}"'] \arrow[dd, "f_{\mathcal{Z}_{S'}}"] \\
\mathcal{X}_{\mathcal{U}} \arrow[rr, hook, "j'"'] \arrow[dd, "f_{\mathcal{U}}"'] & & \mathcal{X} \arrow[dd, "f"] & & \mathcal{X}_{\mathcal{Z}} \arrow[ll, hook', "i'"] \arrow[dd, "f_{\mathcal{Z}}"'] & \\
 & \mathcal{U}_{S'} \arrow[rr, hook, "j_{S'}"'] \arrow[dl, "g'_{\mathcal{U}}"'] & & \mathcal{Y}_{S'} \arrow[dl, "g'"] & & \mathcal{Z}_{S'}, \arrow[ll, hook', "i_{S'}"] \arrow[dl, "g'_{\mathcal{Z}}"] \\
\mathcal{U} \arrow[rr, hook, "j"'] & & \mathcal{Y} & & \mathcal{Z} \arrow[ll, hook', "i"] &
\end{tikzcd}$$

and assume that the base change morphisms

$$g'^*_{\mathcal{U}}f_{\mathcal{U}*}j'^*K \to f_{\mathcal{U}_{S'}*}g''^*_{\mathcal{U}}j'^*K \quad \textit{and} \quad g'^*_{\mathcal{Z}}f_{\mathcal{Z}*}i'^!K \to f_{\mathcal{Z}_{S'}*}g''^*_{\mathcal{Z}}i'^!K$$

are isomorphisms. Then after shrinking S, *the base change morphism* $g'^*f_*K \to f_{S'*}g''^*K$ *is an isomorphism.*

Proof. (i) Given a map $g : S' \to S$, consider the following diagram

$$\begin{tikzcd}
& \mathcal{X}_Y \arrow[dl, "P'"'] \arrow[dd, "f'"' near start] & & \mathcal{X}_{Y,S'} \arrow[ll, "g''_Y"'] \arrow[dl, "P'_{S'}"] \arrow[dd, "f'_{S'}"] \\
\mathcal{X} \arrow[dd, "f"'] & & \mathcal{X}_{S'} \arrow[ll, "g''"' near end] \arrow[dd, "f_{S'}" near start] & \\
& Y \arrow[dl, "P"] & & Y_{S'} \arrow[ll, "g'_Y" near start] \arrow[dl, "P_{S'}"] \\
\mathcal{Y} & & \mathcal{Y}_{S'} \arrow[ll, "g'"] &
\end{tikzcd},$$

where all squares are 2-Cartesian. For the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ to be an isomorphism on $\mathcal{Y}_{S'}$, it suffices for it to be an isomorphism locally on $Y_{S'}$. In the following commutative diagram

$$\begin{tikzcd}
P^*_{S'} g'^* f_* K \arrow[r, "(0)"'] \arrow[d, equal, "(1)"'] & P^*_{S'} f_{S'*} g''^* K \arrow[d, "(2)"] \\
g'^*_Y P^* f_* K \arrow[d, "(3)"'] & f'_{S'*} P'^*_{S'} g''^* K \arrow[d, equal, "(4)"] \\
g'^*_Y f'_* P'^* K \arrow[r, "(5)"] & f'_{S'*} g''^*_Y P'^* K,
\end{tikzcd}$$

(1) and (4) are canonical isomorphisms given by "$P^* g^* \simeq g^* P^*$", (2) and (3) are canonical isomorphisms given by "$P^* f_* = f_* P^*$", which follows from the definition of f_* on the lisse-étale site. Therefore, (0) is an isomorphism if and only if (5) is an isomorphism.

(ii) This follows easily from the axioms of a triangulated category (or 5-lemma):

$$\begin{tikzcd}
g'^* f_* K' \arrow[r] \arrow[d, "\sim"] & g'^* f_* K \arrow[r] \arrow[d] & g'^* f_* K'' \arrow[r] \arrow[d, "\sim"] & {} \\
f_{S'*} g''^* K' \arrow[r] & f_{S'*} g''^* K \arrow[r] & f_{S'*} g''^* K'' \arrow[r] & {}
\end{tikzcd}.$$

(iii) By (i) we may assume that $f : X \to Y$ is a morphism of S-schemes. Note that the property of being trivialized by a pair of the form $(\{\mathcal{X}\}, \mathcal{L})$ is preserved when passing to a presentation. By definition $f_* K$ is the class of the system $(f_* \widehat{K}_n)_n$, so it suffices to show that there exists a nonempty open subscheme of S over which the formation of $f_* \widehat{K}_n$ commutes with base change, for every n. By the spectral sequence

$$R^p f_* \mathscr{H}^q(\widehat{K}_n) \Longrightarrow R^{p+q} f_* \widehat{K}_n$$

and (ii), it suffices to show the existence of a nonempty open subset of S, over which the formations of $f_* L$ commute with generic base change, for all $L \in \mathcal{L}$. This follows from ([11], Th. finitude).

(iv) Consider the commutative diagram

$$\begin{tikzcd} g'^*f_*i_*i^!K \arrow[r, "(2)"'] \arrow[d, equal, "(1)"'] & f_{S'*}g''^*i_*i^!K \arrow[r, "(3)"'] & f_{S'*}i_{S'*}g_{\mathcal{Z}}^*i^!K \arrow[d, equal, "(4)"] \\ g'^*(fi)_*i^!K \arrow[rr, "(5)"] & & (f_{S'}i_{S'})_*g_{\mathcal{Z}}^*i^!K. \end{tikzcd}$$

(1) and (4) are canonical isomorphisms, (5) is an isomorphism by assumption, and (3) is the base change morphism for i_*, which is an isomorphism by ([27], 12.5.3), since $i_* = i_!$ (note that i_* has finite cohomological dimension, so it is defined on complexes unbounded in both directions). Therefore, (2) is an isomorphism. Similarly, consider the commutative diagram

$$\begin{tikzcd} g'^*f_*j_*j^*K \arrow[r, "(2)"'] \arrow[d, equal, "(1)"'] & f_{S'*}g''^*j_*j^*K \arrow[r, "(3)"'] & f_{S'*}j_{S'*}g_{\mathcal{U}}^*j^*K \arrow[d, equal, "(4)"] \\ g'^*(fj)_*j^*K \arrow[rr, "(5)"] & & (f_{S'}j_{S'})_*g_{\mathcal{U}}^*j^*K. \end{tikzcd}$$

(1) and (4) are canonical isomorphisms, and (3) and (5) are isomorphisms by assumption, so (2) is an isomorphism. Then applying (ii) to the exact triangle $i_*i^!K \to K \to j_*j^*K \to$, we are done.

(v) By (i), we may assume that $f : X \to Y$ is a morphism of S-schemes. Assume K is trivialized by $(\mathscr{S}, \mathcal{L})$, and let $j : U \to X$ be the immersion of an open stratum in $\mathscr{S}$ with complement $i : Z \to X$. Then $j^*K \in D^+_{\{U\},\mathcal{L}(U)}(U, \Lambda)$, so by (iii), the formation of $j_*(K|_U)$ commutes with generic base change. This is the third base change isomorphism in the assumption of (iv). By noetherian induction and (iv), we replace X by U and assume that $\mathscr{S} = \{X\}$. The result follows from (iii).

(vi) In the commutative diagrams

$$\begin{tikzcd} g'^*j_*f_{\mathcal{U}*}j'^*K \arrow[rr, "(1)"] \arrow[d, "(2)"'] & & j_{S'*}g'^*_{\mathcal{U}}f_{\mathcal{U}*}j'^*K \arrow[d, "(3)"] \\ g'^*(fj')_*j'^*K \arrow[r, "(4)"'] & (f_{S'}j'_{S'})_*g''^*_{\mathcal{U}}j'^*K \arrow[r, "(5)"'] & j_{S'*}f_{\mathcal{U}_{S'}*}g''^*_{\mathcal{U}}j'^*K \end{tikzcd}$$

and

$$\begin{tikzcd} g'^*i_*f_{\mathcal{Z}*}i'^!K \arrow[rr, "(6)"] \arrow[d, "(7)"'] & & i_{S'*}g'^*_{\mathcal{Z}}f_{\mathcal{Z}*}i'^!K \arrow[d, "(8)"] \\ g'^*(fi')_*i'^!K \arrow[r, "(9)"'] & (f_{S'}i'_{S'})_*g''^*_{\mathcal{Z}}i'^!K \arrow[r, "(10)"'] & i_{S'*}f_{\mathcal{Z}_{S'}*}g''^*_{\mathcal{Z}}i'^!K, \end{tikzcd}$$

(2), (5), (7) and (10) are canonical isomorphisms, (3) and (8) are isomorphisms by assumption, (6) is an isomorphism by proper base change, and (1) is an isomorphism after shrinking

S by (v). Therefore, (4) and (9) are isomorphisms. Also by (iii), the base change morphism $g''^* j'_* j'^* K \to j'_{S'*} g''^*_{\mathcal{U}} j'^* K$ becomes an isomorphism after shrinking S. Hence by (iv), the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ is an isomorphism after shrinking S. $\square$

8.1.4. For $K \in D_c^-(\mathcal{X}, \Lambda)$ and $L \in D_c^+(\mathcal{X}, \Lambda)$, and for a morphism $g : \mathcal{Y} \to \mathcal{X}$, the base change morphism $g^* R\mathscr{H}om_{\mathcal{X}}(K, L) \to R\mathscr{H}om_{\mathcal{Y}}(g^*K, g^*L)$ is defined as follows. By adjunction (g^*, g_*), it corresponds to the morphism

$$R\mathscr{H}om_{\mathcal{X}}(K, L) \to g_* R\mathscr{H}om_{\mathcal{Y}}(g^*K, g^*L) \simeq R\mathscr{H}om_{\mathcal{X}}(K, g_*g^*L)$$

obtained by applying $R\mathscr{H}om_{\mathcal{X}}(K, -)$ to the adjunction morphism $L \to g_*g^*L$.

The following is the main result of this section.

Theorem 8.1.5. *(i) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks. For every $K \in D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda)$ (resp. $D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda_n)$), the formation of f_*K commutes with generic base change on S.*

(ii) For every $K, L \in D_c^b(\mathcal{X}, \Lambda)$ (resp. $D_c^b(\mathcal{X}, \Lambda_n)$), the formation of $R\mathscr{H}om_{\mathcal{X}}(K, L)$ commutes with generic base change on S.

Proof. (i) We can always replace a stack by its maximal reduced closed substack, so we will assume all stacks in the proof are reduced.

Suppose K is $(\mathscr{S}, \mathcal{L})$-stratifiable for some pair $(\mathscr{S}, \mathcal{L})$. By (8.1.3i,iii,iv), we can replace $\mathcal{Y}$ by a presentation and replace $\mathcal{X}$ by an open stratum in $\mathscr{S}$, to assume that $\mathcal{Y} = Y$ is a scheme, that $\mathscr{S} = \{\mathcal{X}\}$, that the relative inertia $\mathcal{I}_f$ is flat and has components over $\mathcal{X}$ ([3], 5.1.14), and let

$$\mathcal{X} \xrightarrow{\pi} X \xrightarrow{b} Y$$

be the rigidification with respect to $\mathcal{I}_f$. Replacing $\mathcal{X}$ by the inverse image of an open dense subscheme of the S-algebraic space X, we may assume X is a scheme. Let $\mathscr{F} = \pi_*K$, which is stratifiable (3.2.4). By (8.1.3v), the formation of $b_*\mathscr{F}$ commutes with generic base change. To finish the proof, we shall show that the formation of π_*K commutes with generic base change. As in the proof of (8.1.3iii), it suffices to show that there exists an open dense subscheme of S, over which the formations of π_*L commute with any base change $g : S' \to U$, for all $L \in \mathcal{L}$.

By ([3], 5.1.5), π is smooth, so étale locally it has a section. By (8.1.3i) we may assume that $\pi : BG \to X$ is a neutral gerbe, associated to a flat group space G/X. By (8.1.3vi) we can use dévissage and shrink X to an open subscheme. Using the same technique as the proof of (3.2.4), we can reduce to the case where G/X is smooth. For the reader's convenience, we briefly recall this reduction. Shrinking X, we may assume X is an integral scheme with function field $k(X)$, and G/X is a group scheme. There exists a finite field extension $L/k(X)$ such that G_{red} is smooth over Spec L. Factor $L/k(X)$ as a separable extension $L'/k(X)$ and a purely inseparable extension L/L'. Purely inseparable morphisms are universal homeomorphisms. By taking the normalization of X in these field extensions, we get a finite generically étale surjection $X' \to X$, such that G_{red} is generically smooth over X'. Shrinking X and X' we may assume $X' \to X$ is an étale surjection, and replacing X by X' (8.1.3i) we may assume G_{red} is generically smooth over X, and shrinking X we may assume G_{red} is smooth over X. Replacing G by G_{red} (since the morphism $BG_{\text{red}} \to BG$ is representable and radicial) one can assume G/X is smooth.

Now $P : X \to BG$ is a presentation. Consider the associated smooth hypercover, and let $f_i : G^i \to X$ be the structural maps. We have the spectral sequence ([27], 10.0.9)

$$R^j f_{i*} f_i^* P^* L \Longrightarrow R^{i+j} \pi_* L.$$

As in the proof of (3.2.4), we can regard the map f_i as a product $\prod_i f_1$ and apply Künneth formula (shrinking X we can assume X is affine, so X satisfies the condition (LO), and we can apply ([27], 11.0.14))

$$f_{i*} f_i^* P^* L = f_{1*} f_1^* P^* L \otimes^L f_{1*}\Lambda_0 \otimes^L \cdots \otimes^L f_{1*}\Lambda_0.$$

Shrink S so that the formations of $f_{1*} f_1^* P^* L$ and $f_{1*}\Lambda_0$ commute with any base change on S. From the base change morphism of the spectral sequences

$$\begin{tikzcd}
g^* R^j f_{i*} f_i^* P^* L \arrow[r, Rightarrow] \arrow[d, equal] & g^* R^{i+j} \pi_* L \arrow[dddddd, "(1)"] \\
\mathscr{H}^j g^* (f_{1*} f_1^* P^* L \otimes^L_{\Lambda_0} f_{1*}\Lambda_0 \otimes^L_{\Lambda_0} \cdots \otimes^L_{\Lambda_0} f_{1*}\Lambda_0) \arrow[d, equal] & \\
\mathscr{H}^j \big((g^* f_{1*} f_1^* P^* L) \otimes^L_{\Lambda_0} (g^* f_{1*}\Lambda_0) \otimes^L_{\Lambda_0} \cdots \otimes^L_{\Lambda_0} (g^* f_{1*}\Lambda_0) \big) \arrow[d, "\sim"] & \\
\mathscr{H}^j \big((f_{1*} g^* f_1^* P^* L) \otimes^L_{\Lambda_0} (f_{1*} g^* \Lambda_0) \otimes^L_{\Lambda_0} \cdots \otimes^L_{\Lambda_0} (f_{1*} g^* \Lambda_0) \big) \arrow[d, equal] & \\
\mathscr{H}^j \big((f_{1*} f_1^* P^* g^* L) \otimes^L_{\Lambda_0} (f_{1*}\Lambda_0) \otimes^L_{\Lambda_0} \cdots \otimes^L_{\Lambda_0} (f_{1*}\Lambda_0) \big) \arrow[d, equal] & \\
R^j f_{i*} f_i^* P^* g^* L \arrow[r, Rightarrow] & R^{i+j} \pi_* g^* L
\end{tikzcd}$$

we see that the base change morphism (1) is an isomorphism.

(ii) For K and $L \in D_c^b(\mathcal{X}, \Lambda)$, the complex $R\mathscr{H}om(K, L)$ is defined to be the image in $D_c(\mathcal{X}, \Lambda)$ of the projective system $R\mathscr{H}om_{\Lambda_\bullet}(\widehat{K}, \widehat{L})$, so we only need to prove the case where K and L are in $D_c^b(\mathcal{X}, \Lambda_n)$.

Note that for an algebraic stack $\mathcal{X}$, $R\mathscr{H}om_{\mathcal{X}}$ takes $D_c^{b,\text{op}} \times D_c^b$ into D_c^b. To see this, take a presentation $P : X \to \mathcal{X}$ of relative dimension d, for some locally constant function d on X. For bounded complexes K and L on $\mathcal{X}$, to show $R\mathscr{H}om_{\mathcal{X}}(K, L)$ is bounded, it suffices to show that $P^*R\mathscr{H}om_{\mathcal{X}}(K,L)$ is bounded. We have

$$\begin{aligned} P^*R\mathscr{H}om_{\mathcal{X}}(K,L) &= P^!R\mathscr{H}om_{\mathcal{X}}(K,L)\langle -d\rangle = R\mathscr{H}om_X(P^*K,P^!L)\langle -d\rangle \\ &= R\mathscr{H}om_X(P^*K,P^*L), \end{aligned}$$

which is bounded on X.

Let $g : S' \to S$ be any morphism, and consider the 2-Cartesian diagrams

$$\begin{tikzcd} X_{S'} \arrow[r, "P'"] \arrow[d, "g''"'] & \mathcal{X}_{S'} \arrow[r] \arrow[d, "g'"'] & S' \arrow[d, "g"] \\ X \arrow[r, "P"'] & \mathcal{X} \arrow[r] & S. \end{tikzcd}$$

For the base change morphism

$$g'^*R\mathscr{H}om_{\mathcal{X}}(K,L) \to R\mathscr{H}om_{\mathcal{X}_{S'}}(g'^*K,g'^*L)$$

to be an isomorphism, we can check it locally on $X_{S'}$. Consider the commutative diagram

$$\begin{tikzcd} P'^*g'^*R\mathscr{H}om_{\mathcal{X}}(K,L) \arrow[r, "(1)"] \arrow[d, "(2)"'] & P'^*R\mathscr{H}om_{\mathcal{X}_{S'}}(g'^*K,g'^*L) \arrow[d, "(3)"] \\ g''^*P^*R\mathscr{H}om_{\mathcal{X}}(K,L) \arrow[d, "(4)"'] & R\mathscr{H}om_{X_{S'}}(P'^*g'^*K,P'^*g'^*L) \arrow[d, "(5)"] \\ g''^*R\mathscr{H}om_X(P^*K,P^*L) \arrow[r, "(6)"] & R\mathscr{H}om_{X_{S'}}(g''^*P^*K,g''^*P^*L), \end{tikzcd}$$

where (2) and (5) are canonical isomorphisms, (3) and (4) are proved to be isomorphisms above, and (6) is an isomorphism after shrinking S ([11], Th. finitude, 2.10). Therefore (1) is an isomorphism after shrinking S. $\square$

Remark 8.1.5.1. This result generalizes ([35], 9.10ii), in that the open subscheme in S can be chosen to be independent of the index i as in R^if_*F.

8.2 Complex analytic stacks.

In this section, we give some fundamental results on constructible sheaves and derived categories on the lisse-analytic topos of the analytification of a complex algebraic stack, and prove a comparison between the lisse-étale topos and the lisse-analytic topos of the stack.

8.2.1 Lisse-analytic topos.

For the definition of analytic stacks, we follow [32, 37]. Strictly speaking, Toen only discussed analytic Deligne-Mumford stacks in [37], and Noohi only discussed topological stacks in [32] (and mentioned analytic stacks briefly). I believe that they could have done the theory of analytic stacks in their papers. For completeness, we give a definition as follows.

Definition 8.2.1.1. *Let* **Ana-Sp** *be the site of complex analytic spaces with the analytic topology. A stack $\mathfrak{X}$ over this site is called an* analytic stack*, if the following hold:*

(i) the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representible (by analytic spaces) and quasi-compact,

(ii) there exists an analytic smooth surjection $P : X \to \mathfrak{X}$, where X is an analytic space.

8.2.1.2. Similar to the lisse-étale topos of an algebraic stack, one can define the *lisse-analytic topos* $\mathfrak{X}_{\text{lis-an}}$ of an analytic stack $\mathfrak{X}$ to be the topos associated to the *lisse-analytic site* Lis-an($\mathfrak{X}$) defined as follows:

• Objects: pairs $(U, u : U \to \mathfrak{X})$, where U is an complex analytic space and u is a smooth morphism (or an analytic *submersion*, in the topological terminology);

• Morphisms: a morphism $(U, u \in \mathfrak{X}(U)) \to (V, v \in \mathfrak{X}(V))$ is given by a pair (f, α), where $f : U \to V$ is a morphism of analytic spaces and $\alpha : vf \cong u$ is a 2-isomorphism in $\mathfrak{X}(U)$;

• Open coverings: $\{(j_i, \alpha_i) : (U_i, u_i \in \mathfrak{X}(U_i)) \to (U, u \in \mathfrak{X}(U))\}_{i\in I}$ is an open covering if the maps j_i's are open immersions of analytic subspaces and their images cover U.

To give a sheaf $F \in \mathfrak{X}_{\text{lis-an}}$ is equivalent to giving the data

• for every $(U, u) \in$ Lis-an($\mathfrak{X}$), a sheaf F_u in the analytic topos U_{an} of U, and

• for every morphism $(f, \alpha) : (U, u) \to (V, v)$, a morphism $f^* : f^{-1}F_v \to F_u$.

The sheaf F is *Cartesian* if f^* is an isomorphism, for every (f, α). By abuse of notation, we will also denote "F_u" by "F_U", if there is no confusion about the reference to u.

This topos is equivalent to the "lisse-étale" topos $\mathfrak{X}_{\text{lis-ét}}$ associated to the site Lis-ét($\mathfrak{X}$) with the same underlying category as that of Lis-an($\mathfrak{X}$), but the open coverings are surjective families of local isomorphisms. This is because the two topologies are cofinal: for a local isomorphism $V \to U$ of analytic spaces, there exists an open covering $\{V_i \subset V\}_i$ of V by analytic subspaces, such that for each i, the composition $V_i \subset V \to U$ is isomorphic to the natural map from a disjoint union of open analytic subspaces of U to U.

8.2.2 Locally constant sheaves and constructible sheaves.

For a sheaf on the analytic site of an analytic space, we say that the sheaf is *locally constant constructible*, abbreviated as *lcc*, if it is locally constant with respect to the analytic topology, and has finite stalks.

Let $\mathfrak{X}$ be an analytic stack. For a Cartesian sheaf $F \in \mathfrak{X}_{\text{lis-an}}$, we say that F is *locally constant* (resp. *lcc*) if the conditions in the following (8.2.2.1) hold. This lemma is an analytic version of ([35], 9.1).

Lemma 8.2.2.1. *Let $F \in \mathfrak{X}_{\text{lis-an}}$ be a Cartesian sheaf. Then the following are equivalent.*

(i) For every $(U, u) \in \text{Lis-an}(\mathfrak{X})$, the sheaf F_U is locally constant (resp. lcc).

(ii) There exists an analytic presentation $P : X \to \mathfrak{X}$ such that F_X is locally constant (resp. lcc).

Proof. It suffices to show that (ii)⇒(i), which is similar to that of ([35], 9.1). There exists an open covering $U = \cup U_i$, such that over each U_i, the smooth surjection $X \times_{P,\mathfrak{X},u} U \to U$ has a section s_i :

$$
\begin{tikzcd}
 & X \times_{\mathfrak{X}} U \arrow[r] \arrow[d] & X \arrow[d, "P"] \\
U_i \arrow[ur, dashed, "s_i"] \arrow[r, hook] & U \arrow[r, "u"] & \mathfrak{X}.
\end{tikzcd}
$$

Therefore $F_{U_i} \simeq s_i^* F_{X \times_{\mathfrak{X}} U}$, which is locally constant (resp. lcc). □

8.2.2.2. Let $\mathcal{X}$ be a complex algebraic stack. Following ([32], 20), one can define its associated analytic stack $\mathcal{X}^{\text{an}}$ as follows. If $X_1 \rightrightarrows X_0 \to \mathcal{X}$ is a smooth groupoid presentation, then $\mathcal{X}^{\text{an}}$ is defined to be the analytic stack given by the presentation $X_1^{\text{an}} \rightrightarrows X_0^{\text{an}}$, and it can be proved that this is independent of the choice of the presentation, up to an isomorphism that is unique up to 2-isomorphism. Similarly, for a morphism $f : \mathcal{X} \to \mathcal{Y}$ of complex algebraic stacks, one can choose their presentations so that f lifts to a morphism of groupoids, hence induces a morphism of their analytifications, denoted $f^{\text{an}} : \mathcal{X}^{\text{an}} \to \mathcal{Y}^{\text{an}}$. The analytification functor preserves 2-Cartesian products.

8.2.2.3. Let $\mathfrak{X} = \mathcal{X}^{\text{an}}$ for a complex algebraic stack $\mathcal{X}$, and let $P : X \to \mathcal{X}$ be a presentation. For a Cartesian sheaf $F \in \mathfrak{X}_{\text{lis-an}}$, we say that F is *constructible*, if for every $(U, u) \in \text{Lis-ét}(\mathcal{X})$, the sheaf $F_{U^{\text{an}}}$ is constructible, i.e. lcc on each stratum in an *algebraic* stratification of the analytic space U^{an}.

One could also define a notion of *analytic constructibility*, using analytic stratifications rather than algebraic ones, but this notion will not give us a comparison between the constructible derived categories of the lisse-étale topos and of the lisse-analytic topos.

Lemma 8.2.2.4. *Let $F \in \mathfrak{X}_{\text{lis-an}}$ be a Cartesian sheaf. Then the following are equivalent.*

(i) F is constructible.

(ii) $F_{X^{\text{an}}}$ is constructible on X^{an} (in the algebraic sense above).

(iii) There exists an algebraic stratification $\mathscr{S}^{\text{an}}$ on $\mathfrak{X}$, such that for each stratum U^{an}, the sheaf $F_{U^{\text{an}}}$ is lcc.

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii). Let $\mathscr{S}_X$ be a stratification of the scheme X, such that for each $U \in \mathscr{S}_X$, the sheaf $F_{U^{\text{an}}}$ is lcc. Let U be an open stratum, and let V be the image of U under the map P; then V is an open substack of $\mathcal{X}$, and $P_U : U \to V$ is a presentation. Let $V' \to V^{\text{an}}$ be an analytic presentation. There exists an analytic open covering $V' = \cup V_i'$, over which P_U^{an} has a section:

$$\begin{tikzcd} & & U^{\mathrm{an}} \arrow[r, hook] \arrow[d, "P_U^{an}"] & X^{\mathrm{an}} \arrow[d, "P^{an}"] \\ V_i' \arrow[r, hook] \arrow[urr, dashed, "s_i"] & V' \arrow[r] & V^{\mathrm{an}} \arrow[r, hook] & \mathfrak{X}, \end{tikzcd}$$

so $F_{V_i'} \simeq s_i^{-1}F_{U^{\mathrm{an}}}$ is lcc, therefore $F_{V'}$ (and hence $F_{V^{\mathrm{an}}}$, by (8.2.2.1)) is lcc. Note that $X - P^{-1}(V) \to \mathcal{X} - V$ gives an algebraic presentation of $(\mathcal{X} - V)^{\mathrm{an}} = \mathfrak{X} - V^{\mathrm{an}}$, and

$$(F|_{\mathfrak{X}-V^{\mathrm{an}}})_{(X-P^{-1}(V))^{\mathrm{an}}} \simeq F_{X^{\mathrm{an}}}|_{(X-P^{-1}(V))^{\mathrm{an}}}$$

is still constructible, so by noetherian induction we are done.

(iii)$\Rightarrow$(i). Let $(U,u) \in \text{Lis-ét}(\mathcal{X})$. Then $u^{\mathrm{an},*}\mathscr{S}^{\mathrm{an}} = (u^*\mathscr{S})^{\mathrm{an}}$ is an algebraic stratification of U^{an}, and it is clear that $F_{U^{\mathrm{an}}}$ is lcc on each stratum of this stratification. $\square$

8.2.2.5. A *constructible* Λ_n*-module* on $\mathfrak{X}_{\text{lis-an}}$ is a Λ_n-sheaf, which is constructible as a sheaf of sets. They form a full subcategory of $\mathrm{Mod}(\mathfrak{X}, \Lambda_n)$ that is closed under kernels, cokernels and extensions (i.e. it is a Serre subcategory). It suffices to show that Cartesian sheaves form a Serre subcategory, because lcc Λ_n-modules form a Serre subcategory, and one can use (8.2.2.4iii).

Let $(f,\alpha) : (U,u) \to (V,v)$ be a morphism in Lis-an$(\mathfrak{X})$. Note that the functor $F \mapsto f^*F$: $\mathrm{Mod}(V_{\mathrm{an}}, \Lambda_n) \to \mathrm{Mod}(U_{\mathrm{an}}, \Lambda_n)$ is exact, because $f^*F = \Lambda_{n,U} \otimes_{f^{-1}\Lambda_{n,V}} f^{-1}F = f^{-1}F$. Let a : $F \to G$ be a morphism of Cartesian sheaves. Then $\mathrm{Ker}(f^*a_V : f^*F_V \to f^*G_V) = f^*\mathrm{Ker}(a_V)$, and it is clear that the induced morphism $f^*\mathrm{Ker}(a_V) \to \mathrm{Ker}(a_U)$ is an isomorphism:

$$\begin{tikzcd} f^*\mathrm{Ker}(a_V) \arrow[r] \arrow[d] & f^*F_V \arrow[r, "f^*a_V"] \arrow[d, "\sim"] & f^*G_V \arrow[d, "\sim"] \\ \mathrm{Ker}(a_U) \arrow[r] & F_U \arrow[r, "a_U"] & G_U. \end{tikzcd}$$

The proof for cokernels and extensions (using 5-lemma) is similar. One can also mimic the proof in ([35], 3.8, 3.9) to prove a similar statement for analytic stacks, in the more general situation where the coefficient ring is a *flat sheaf*. In this chapter, we will only need the case of a constant coefficient ring.

8.2.3 Derived categories.

8.2.3.1. Again assume $\mathfrak{X} = \mathcal{X}^{\mathrm{an}}$. Let $D(\mathfrak{X}_{\text{lis-an}}, \Lambda_n)$ be the ordinary derived category of Λ_n-modules on $\mathfrak{X}$. By (8.2.2.5), we have the triangulated subcategory $D_c(\mathfrak{X}_{\text{lis-an}}, \Lambda_n)$ of complexes with constructible cohomology sheaves. We follow [27] and define the *derived category* $D_c(\mathfrak{X}_{\textit{lis-an}}, \Lambda)$ *of constructible* Λ*-adic sheaves* (by abuse of language, as usual) as follows. A complex of projective systems K in the ordinary derived category $\mathscr{D}(\mathfrak{X}^{\mathbb{N}}_{\text{lis-an}}, \Lambda_\bullet)$ of the simplicial topos $\mathfrak{X}^{\mathbb{N}}_{\text{lis-an}}$ ringed by $\Lambda_\bullet = (\Lambda_n)_n$, is called a λ-*complex* if for every i and n, the sheaf $\mathscr{H}^i(K_n)$ is constructible and the cohomology system $\mathscr{H}^i(K)$ is AR-adic. A λ-*module* is a λ-complex concentrated in degree 0. Then we define $D_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ to be the quotient of the full subcategory $\mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}_{\text{lis-an}}, \Lambda_\bullet)$ of λ-complexes by the full subcategory of AR-null complexes (i.e. those with AR-null cohomology systems).

This quotient has a natural t-structure, and we define the category $\Lambda\text{-Sh}_c(\mathfrak{X})$ of *constructible* Λ-*adic sheaves* on $\mathfrak{X}_{\text{lis-an}}$ to be its core, namely the quotient of the AR-adic projective systems with constructible components by the thick full subcategory of AR-null systems. By ([16], p.234), this is equivalent to the category of adic systems, i.e. those projective systems $F = (F_n)_n$, such that for each n, F_n is a constructible Λ_n-module on $\mathfrak{X}_{\text{lis-an}}$, and the induced morphism $F_n \otimes_{\Lambda_n} \Lambda_{n-1} \to F_{n-1}$ is an isomorphism.

Using localization and 2-colimit, one can also define the categories $D_c(\mathfrak{X}_{\text{lis-an}}, E_\lambda)$ and $D_c(\mathfrak{X}_{\text{lis-an}}, \overline{\mathbb{Q}}_\ell)$, and their cores, the categories of constructible E_λ or $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathfrak{X}_{\text{lis-an}}$.

8.2.3.2. Let $\text{Mod}(\mathfrak{X}, \mathbb{C})$ be the category of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}_{\text{lis-an}}$, with $\mathbb{C}$-linear morphisms, and define the category $\text{Mod}_c(\mathfrak{X}, \mathbb{C})$ of *constructible* $\mathbb{C}_{\mathfrak{X}}$-*modules* to be the full subcategory of $\text{Mod}(\mathfrak{X}, \mathbb{C})$ consisting of those sheaves M, such that there exists an *algebraic stratification* $\mathscr{S}$ of $\mathfrak{X}$, over each stratum of which M is locally constant, and stalks of M are finite dimensional $\mathbb{C}$-vector spaces. Note that, in order for $M|_U$ to be constant, we may have to refine $\mathscr{S}$ to an analytic stratification that is not necessarily algebraic. Then we define $D_c(\mathfrak{X}_{\text{lis-an}}, \mathbb{C})$ to be the full subcategory of the ordinary derived category of $\mathbb{C}_{\mathfrak{X}}$-modules, consisting of those sheaf complexes with constructible cohomology sheaves. The core of the natural t-structure on $D_c(\mathfrak{X}_{\text{lis-an}}, \mathbb{C})$ is $\text{Mod}_c(\mathfrak{X}, \mathbb{C})$.

Similarly, one can also define the category $\mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$ of *constructible* $\Lambda_{\mathfrak{X}}$-*modules*, i.e. $\Lambda_{\mathfrak{X}}$-sheaves for which there exists an algebraic stratification of $\mathfrak{X}$, such that over each stratum the sheaf is locally constant, and stalks are finitely generated Λ-modules. Then we define $\mathscr{D}_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ (and also with E_λ- and $\overline{\mathbb{Q}}_\ell$-coefficients) to be the full subcategory of the ordinary derived category (denoted $\mathscr{D}(\mathfrak{X}_{\text{lis-an}}, \Lambda)$) of $\Lambda_{\mathfrak{X}}$-modules, consisting of those with constructible cohomology. In (8.2.5.4), we will show that the two derived categories $D_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ and $\mathscr{D}_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ are equivalent.

For simplicity, for any coefficient Ω, we will usually drop "lis-an" in $D_c(\mathfrak{X}_{\text{lis-an}}, \Omega)$, if there is no confusion. Also we will drop "lis-ét" in $D_c(\mathcal{X}_{\text{lis-ét}}, \Omega)$.

In the following lemma, we show that the category $\Lambda\text{-Sh}_c(\mathfrak{X})$ admits a similar description as $\text{Mod}_c(\mathfrak{X}, \mathbb{C})$.

Lemma 8.2.3.3. *There is a natural equivalence between* $\Lambda\text{-Sh}_c(\mathfrak{X})$ *and* $\mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$.

Proof. Firstly, we define the functor $\phi : \Lambda\text{-Sh}_c(\mathfrak{X}) \to \mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$. Let $F = (F_n)_n$ be an adic sheaf on $\mathfrak{X}_{\text{lis-an}}$, and define $\phi(F)$ to be $\varprojlim_n (F_n)_n$. For a morphism $b : F \to G$ of adic sheaves, define $\phi(b)$ to be the induced morphism on their inverse limits.

Then we show it is well-defined. Let $P : X \to \mathcal{X}$ be a presentation. Then $F_{X^{\text{an}}} := (F_{n,X^{\text{an}}})_n$ is an adic sheaf on X^{an}. By the comparison ([4], 6.1.2, (A'')), $F_{X^{\text{an}}}$ is algebraic, i.e. it comes from a constructible Λ-adic sheaf G on $X_{\text{ét}}$. Since X is noetherian, G is lisse over the strata of a stratification $\mathscr{S}_X$ of X. Let $U \in \mathscr{S}_X$ be an open stratum, and let V be its image under P. Then $V \subset \mathcal{X}$ is an open substack and $P_U : U \to V$ is a presentation. We have $(G_U)^{\text{an}} = \phi(F)_{U^{\text{an}}} \cong P_U^{\text{an},*}(\phi(F)_{V^{\text{an}}})$, and it is the Λ-local system on $U(\mathbb{C})$ obtained by restricting the continuous representation ρ_{G_U} of $\pi_1^{\text{ét}}(U, \overline{u})$ corresponding to the lisse sheaf G_U to $\pi_1^{\text{top}}(U^{\text{an}}, \overline{u})$:

$$\pi_1^{\text{top}}(U^{\text{an}}, \overline{u}) \longrightarrow \pi_1^{\text{ét}}(U, \overline{u}) \xrightarrow{\rho_{G_U}} \mathrm{GL}(G_{U,\overline{u}}).$$

The sheaf $\phi(F)_{U^{\text{an}}}$ is locally constant because U^{an} is covered by contractible analytic open subspaces.

As in the proof of (8.2.2.4), one can take an analytic presentation $V' \to V^{\text{an}}$, and cover the analytic space V' by analytic open subspaces V_i', such that P_U^{an} has a section s_i over each V_i', and so $\phi(F)_{V_i'}$ is locally constant with stalks finitely generated Λ-modules, and the same is true for $\phi(F)_{V^{\text{an}}}$. Finally apply noetherian induction to the complement $\mathfrak{X} - V^{\text{an}}$ to finish the proof that ϕ is well-defined.

Then we define a functor $\psi : \mathfrak{Mod}_c(\Lambda_{\mathfrak{X}}) \to \Lambda\text{-Sh}_c(\mathfrak{X})$. Given a $\Lambda_{\mathfrak{X}}$-module M, let $M_n = M \otimes_\Lambda \Lambda_n$, and define $\psi(M)$ to be the adic system $(M_n)_n$. For a morphism $c : M \to N$ of constructible $\Lambda_{\mathfrak{X}}$-modules, define $\psi(c)_n$ to be $c \otimes \Lambda_n$.

We need to show $\psi(M)$ gives a constructible Λ-adic sheaf on $\mathfrak{X}_{\text{lis-an}}$. It is clearly adic. To show each M_n is constructible, by (8.2.2.4), it suffices to show that there exists an algebraic stratification of $\mathfrak{X}$, such that over each stratum M_n is lcc. This follows from the definition of $\mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$.

Finally, note that ϕ and ψ are quasi-inverse to each other. □

8.2.4 Comparison between the derived categories of lisse-étale and lisse-analytic topoi.

Given an algebraic stack $\mathcal{X}/\mathbb{C}$, let $\mathfrak{X} = \mathcal{X}^{\text{an}}$, and let $P : X \to \mathcal{X}$ be a presentation, with analytification $P^{\text{an}} : X^{\text{an}} \to \mathfrak{X}$. Let $\epsilon : X_\bullet \to \mathcal{X}$ be the associated strictly simplicial smooth hypercover, and let $\epsilon^{\text{an}} : X_\bullet^{\text{an}} \to \mathfrak{X}$ be the analytification. They induce morphisms of topoi, denoted by the same symbol. Consider the following morphisms of topoi:

$$\begin{tikzcd}
\mathfrak{X}_{\text{lis-an}} & \mathfrak{X}_{\text{lis-an}}|_{X_\bullet^{\text{an}}} \arrow[l, "\gamma^{\text{an}}"'] \arrow[r, "\delta_\bullet^{\text{an}}"] & X^{\text{an}}_{\bullet,\text{an}} \arrow[ll, bend left, "\epsilon^{\text{an}}"] \arrow[d, "\xi_\bullet"] \\
\mathcal{X}_{\text{lis-ét}} & \mathcal{X}_{\text{lis-ét}}|_{X_\bullet} \arrow[l, "\gamma"'] \arrow[r, "\delta_\bullet"] & X_{\bullet,\text{ét}}. \arrow[ll, bend left, "\epsilon"]
\end{tikzcd}$$

Following ([27], 10.0.6), we define the derived category $D_c(X_\bullet^{\mathrm{an}},\Lambda)$ as follows. A sheaf $F\in \mathrm{Mod}(X_\bullet^{\mathrm{an},\mathbb{N}},\Lambda_\bullet)$ is *AR-adic* if it is Cartesian (in the $\bullet$-direction) and $F|_{X_{n,\mathrm{an}}^{\mathrm{an},\mathbb{N}}}$ is AR-adic for every n. A complex $C\in\mathscr{D}(X_\bullet^{\mathrm{an},\mathbb{N}},\Lambda_\bullet)$ is a λ*-complex* (resp. an *AR-null complex*) if the cohomology sheaf $\mathscr{H}^i(C)$ is AR-adic and $\mathscr{H}^i(C_m)|_{X_n^{\mathrm{an}}}$ is constructible, for every i,m,n (resp. $C|_{X_n^{\mathrm{an}}}$ is AR-null, for every n). Finally we define $D_c(X_\bullet^{\mathrm{an}},\Lambda)$ to be the quotient of the full subcategory $\mathscr{D}_c(X_\bullet^{\mathrm{an},\mathbb{N}},\Lambda_\bullet)\subset\mathscr{D}(X_\bullet^{\mathrm{an},\mathbb{N}},\Lambda_\bullet)$ consisting of all λ-complexes by the full subcategory of AR-null complexes.

Using the diagram above, we will show that $R\epsilon_*\circ R\xi_{\bullet,*}\circ\epsilon^{\mathrm{an},*}$ gives an equivalence between $D_c(\mathfrak{X},\Lambda)$ and $D_c(\mathcal{X},\Lambda)$, and it is compatible with pushforwards. It is proved in ([27], 10.0.8) that, $(\epsilon^*,R\epsilon_*)$ induce an equivalence between the triangulated categories $D_c(\mathcal{X},\Lambda)$ and $D_c(X_\bullet,\Lambda)$. We mimic the proof to give a proof of the analytic analogue.

Proposition 8.2.4.1. *(i) The functors $(\epsilon^{\mathrm{an},*},R\epsilon_*^{\mathrm{an}})$ induce an equivalence between the triangulated categories $D_c(\mathfrak{X},\Lambda)$ and $D_c(X_\bullet^{\mathrm{an}},\Lambda)$.*

(ii) Let X be a $\mathbb{C}$-scheme, and let $\xi:X^{\mathrm{an}}\to X_{\text{ét}}$ be the natural morphism of topoi. Then $R\xi_*$ is defined on the unbounded derived category, and the functors $(\xi^*,R\xi_*)$ induce an equivalence between $D_c(X^{\mathrm{an}},\Lambda)$ and $D_c(X,\Lambda)$.

(iii) Let $f:X\to Y$ be a morphism of $\mathbb{C}$-schemes, and let ξ_X,ξ_Y be as in (ii). Then for every $F\in D_c^+(X,\Lambda)$, the natural morphism

$$\xi_Y^*f_*F\to f_*^{\mathrm{an}}(\xi_X^*F)$$

is an isomorphism.

Proof. (i) Firstly, note that $\delta_{\bullet,*}^{\mathrm{an}}:\mathrm{Ab}(\mathfrak{X}_{\text{lis-an}}|_{X_\bullet^{\mathrm{an}}})\to\mathrm{Ab}(X_{\bullet,\mathrm{an}}^{\mathrm{an}})$ is exact, since the topologies are the same. So in fact, $R\delta_{\bullet,*}^{\mathrm{an}}=\delta_{\bullet,*}^{\mathrm{an}}$. The functor $\delta_{n,*}^{\mathrm{an}}$ is the restriction functor, and $\delta_n^{\mathrm{an},*}$ takes a sheaf $F\in X_{n,\mathrm{an}}^{\mathrm{an}}$ to the sheaf $\delta_n^{\mathrm{an},*}F$ that assigns to the object

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[dr] & X_n^{\mathrm{an}} \arrow[d] \\ & \mathfrak{X} \end{tikzcd}$$

the sheaf u^*F on U_{an}. It is clear that $(\delta_n^{\mathrm{an},*},\delta_{n,*}^{\mathrm{an}})$ induce an equivalence between the category $\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}|_{X_n^{\mathrm{an}}},\Lambda_m)$ of Cartesian sheaves on the localized topos $\mathfrak{X}|_{X_n^{\mathrm{an}}}$ and $\mathrm{Mod}(X_n^{\mathrm{an}},\Lambda_m)$. For $K\in D(X_n^{\mathrm{an}},\Lambda_m)$, we see that the adjunction morphism

$$K\to\delta_{n,*}^{\mathrm{an}}\delta_n^{\mathrm{an},*}K$$

is an isomorphism by applying $\mathscr{H}^i$:

$$\mathscr{H}^iK\to\mathscr{H}^i(\delta_{n,*}^{\mathrm{an}}\delta_n^{\mathrm{an},*}K)\simeq\delta_{n,*}^{\mathrm{an}}\delta_n^{\mathrm{an},*}\mathscr{H}^i(K),$$

noting that $\delta^{\mathrm{an}}_{n,*}$ is exact. Similarly, if $K \in D(\mathfrak{X}|_{X^{\mathrm{an}}_n}, \Lambda_m)$ has Cartesian cohomology sheaves, the coadjunction morphism

$$\delta^{\mathrm{an},*}_n \delta^{\mathrm{an}}_{n,*} K \to K$$

is an isomorphism. Hence $(\delta^{\mathrm{an},*}_\bullet, \delta^{\mathrm{an}}_{\bullet,*})$ induce an equivalence

$$D_{\mathrm{cart}}(\mathfrak{X}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m) \leftrightarrow D(X^{\mathrm{an}}_\bullet, \Lambda_m).$$

We will show later that constructible sheaves form a Serre subcategory in $\mathrm{Mod}(\mathfrak{X}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m)$, and then it is also clear that $(\delta^{\mathrm{an},*}_\bullet, \delta^{\mathrm{an}}_{\bullet,*})$ gives an equivalence

$$D_c(\mathfrak{X}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m) \leftrightarrow D_c(X^{\mathrm{an}}_\bullet, \Lambda_m).$$

To show $\gamma^{\mathrm{an},*}$ induces an equivalence on the torsion level Λ_m, we will apply ([26], 2.2.3). For the morphism $\gamma^{\mathrm{an}} : (\mathfrak{X}_{\text{lis-an}}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m) \to (\mathfrak{X}_{\text{lis-an}}, \Lambda_m)$, all the transition morphisms of topoi in the strictly simplicial ringed topos $(\mathfrak{X}_{\text{lis-an}}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m)$ as well as γ^{an} are flat. Let $\mathscr{C}$ be the category of constructible Λ_m-modules on $\mathfrak{X}_{\text{lis-an}}$, which is a Serre subcategory (8.2.2.5). We need to verify the assumption ([26], 2.2.1), which has two parts.

• ([26], 2.1.2) for the ringed site $(\text{Lis-an}(\mathfrak{X})|_{X^{\mathrm{an}}_n}, \Lambda_m)$ with $\mathscr{C} =$ constructible Λ_m-modules. This means, for every object U in this site, we need to show that there exist an analytic open covering $U = \cup U_i$ and an integer n_0, such that for every constructible Λ_m-module F on this site and $n \geq n_0$, we have $H^n(U_i, F) = 0$. This follows from ([13], 3.1.5, 3.4.1).

• $\gamma^{\mathrm{an},*} : \mathscr{C} \to \mathscr{C}_\bullet$ is an equivalence with quasi-inverse $R\gamma^{\mathrm{an}}_*$. Here $\mathscr{C}_\bullet$ is the essential image of $\mathscr{C}$ under $\gamma^{\mathrm{an},*} : \mathrm{Mod}(\mathfrak{X}, \Lambda_m) \to \mathrm{Mod}(\mathfrak{X}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m)$, called the *category of constructible sheaves* in the target. Recall that, an object in $\mathrm{Mod}(\mathfrak{X}|_{X^{\mathrm{an}}_\bullet}, \Lambda_m)$ is given by a family of objects $F_i \in \mathrm{Mod}(\mathfrak{X}|_{X^{\mathrm{an}}_i}, \Lambda_m)$ indexed by i, together with transition morphisms $a^*F_j \to F_i$ for each $a : i \to j$ in the strictly simplicial set $\Delta^{+,\mathrm{op}}$. Consider the commutative diagram

$$\begin{tikzcd} X^{\mathrm{an}}_i \arrow[r, "a"] \arrow[dr] & X^{\mathrm{an}}_j \arrow[d] \\ & \mathfrak{X}. \end{tikzcd}$$

For $F \in \mathrm{Mod}(\mathfrak{X}, \Lambda_m)$, its image $\gamma^{\mathrm{an},*}F$ is given by $F_i = F_{X^{\mathrm{an}}_i} \in \mathrm{Mod}(X^{\mathrm{an}}_i, \Lambda_m) \simeq \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}|_{X^{\mathrm{an}}_i}, \Lambda_m)$, and the transition morphisms $a^*F_j \to F_i$ are part of the data in the definition of F. One can prove the analytic version of ([35], 4.4, 4.5) stated as follows.

Let $\mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m)$ be the category of pairs (F, α), where $F \in \mathrm{Mod}(X^{\mathrm{an}}, \Lambda_m)$, and $\alpha : p_1^*F \to p_2^*F$ is an isomorphism on the analytic topos $X^{\mathrm{an}}_{1,\mathrm{an}}$ (where p_1 and p_2 are the natural projections $X^{\mathrm{an}}_1 \rightrightarrows X^{\mathrm{an}}_0 = X^{\mathrm{an}}$), such that $p^*_{13}(\alpha) = p^*_{23}(\alpha) \circ p^*_{12}(\alpha) : \bar{p}^*_1 F \to \bar{p}^*_3 F$ on X^{an}_2. Here $\bar{p}_i : X_2 \to X_0$ are the natural projections. There is a natural functor A : $\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}, \Lambda_m) \to \mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m)$, sending M to (F, α), where $F = M_{X^{\mathrm{an}}}$ and α is the composite

$$p_1^* F \xrightarrow{p_1^*} M_{X_1^{\mathrm{an}}} \xrightarrow{(p_2^*)^{-1}} p_2^* F.$$

There is also a natural functor $B : \mathrm{Mod}_{\mathrm{cart}}(X_\bullet^{\mathrm{an},+}, \Lambda_m) \to \mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m)$ sending $F = (F_i)_i$ to (F_0, α), where α is the composite

$$p_1^* F_0 \xrightarrow{\mathrm{can}} F_1 \xrightarrow{\mathrm{can}^{-1}} p_2^* F_0,$$

and the cocycle condition is verified as in ([35], 4.5.4).

Lemma 8.2.4.2. *The natural functors in the diagram*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{cart}}(X_\bullet^{\mathrm{an}}, \Lambda_m) & \xrightarrow{\mathrm{res}} & \mathrm{Mod}_{\mathrm{cart}}(X_\bullet^{\mathrm{an},+}, \Lambda_m) \\ {\scriptstyle \epsilon^{\mathrm{an},*}}\uparrow & & \downarrow{\scriptstyle B} \\ \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}, \Lambda_m) & \xrightarrow{A} & \mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m) \end{array}$$

are all equivalences, and the diagram is commutative up to natural isomorphism.

The proof in ([35], 4.4, 4.5) carries verbatim to analytic stacks, so we do not write down the proof again. This finishes the verification of ([26], 2.2.1). In particular, $\mathscr{C}_\bullet = \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m)$ is a Serre subcategory ([26], 2.2.2), so as we mentioned before, $(\delta_\bullet^{\mathrm{an},*}, \delta_{\bullet,*}^{\mathrm{an}})$ give an equivalence

$$D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \leftrightarrow D_c(X_\bullet^{\mathrm{an}}, \Lambda_m).$$

By ([26], 2.2.3), the functors $(\gamma^{\mathrm{an},*}, R\gamma_*^{\mathrm{an}})$ induce an equivalence

$$D_c(\mathfrak{X}, \Lambda_m) \leftrightarrow D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m).$$

It is clear that the composition of equivalences

$$D_c(\mathfrak{X}, \Lambda_m) \xrightarrow{\gamma^{\mathrm{an},*}} D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \xrightarrow{\delta_{\bullet,*}^{\mathrm{an}}} D_c(X_\bullet^{\mathrm{an}}, \Lambda_m)$$

is just $\epsilon^{\mathrm{an},*}$ (they are both restrictions). Since $\delta_{\bullet,*}^{\mathrm{an}}$ is the quasi-inverse of $\delta_\bullet^{\mathrm{an},*}$, it is both a left adjoint and a right adjoint of $\delta_\bullet^{\mathrm{an},*}$. This implies that $R\epsilon_*^{\mathrm{an}} = R\gamma_*^{\mathrm{an}} \circ \delta_\bullet^{\mathrm{an},*}$ and it is a quasi-inverse of the equivalence

$$\epsilon^{\mathrm{an},*} : D_c(\mathfrak{X}, \Lambda_m) \to D_c(X_\bullet^{\mathrm{an}}, \Lambda_m).$$

Note that for $M \in \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$ (resp. $\mathscr{D}_c(X_\bullet^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet)$), each level M_m is in $D_c(\mathfrak{X}, \Lambda_m)$ (resp. $D_c(X_\bullet^{\mathrm{an}}, \Lambda_m)$), and the property of M being AR-adic (resp. AR-null) is intrinsic ([16], V, 3.2.3). So the notions of AR-adic (resp. AR-null) on the two sides correspond under this equivalence. Therefore, we get equivalences

$$(\epsilon^{\mathrm{an},*}, R\epsilon^{\mathrm{an}}_*) : \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet) \leftrightarrow \mathscr{D}_c(X^{\mathrm{an},\mathbb{N}}_\bullet, \Lambda_\bullet)$$

and

$$(\epsilon^{\mathrm{an},*}, R\epsilon^{\mathrm{an}}_*) : D_c(\mathfrak{X}, \Lambda) \leftrightarrow D_c(X^{\mathrm{an}}_\bullet, \Lambda).$$

(ii) This is a generalization of ([4], 6.1.2 (B'')), which says that $\xi^* : D^b_c(X, \Lambda) \to D^b_c(X^{\mathrm{an}}, \Lambda)$ is an equivalence. We prove it on the torsion level first.

For a Λ_n-module G on X^{an}, the sheaf $R^i\xi_*G$ on $X_{\text{ét}}$ is the sheafification of the presheaf

$$(U \to X) \mapsto H^i(U^{\mathrm{an}}, G).$$

By ([13], 3.1.5, 3.4.1), $R^i\xi_*G = 0$ for all sheaves G and all $i > 1 + 2\dim_{\mathbb{C}} X$, so $R\xi_*$ has finite cohomological dimension, hence it extends to a functor

$$R\xi_* : D(X^{\mathrm{an}}, \Lambda_n) \to D(X, \Lambda_n).$$

It takes the full subcategory $D_c(X^{\mathrm{an}}, \Lambda_n)$ into $D_c(X, \Lambda_n)$, since for any i there exist integers a and b such that $R^i\xi_*G = R^i\xi_*\tau_{[a,b]}G$.

Given $F \in D_c(X, \Lambda_n)$, we want to show that the adjunction morphism $F \to R\xi_*\xi^*F$ is an isomorphism. Recall that ξ^* is the analytification functor, which is exact. For each $i \in \mathbb{N}$, we want to show the morphism

$$\mathscr{H}^iF \to R^i\xi_*\xi^*F$$

is an isomorphism. Consider the spectral sequence

$$R^p\xi_*\xi^*\mathscr{H}^qF \Longrightarrow R^{p+q}\xi_*\xi^*F,$$

where $R^p\xi_*\xi^*\mathscr{H}^qF$ is the sheafification of the functor

$$(U \to X) \mapsto H^p(U^{\mathrm{an}}, \xi^*\mathscr{H}^qF) = H^p(U^{\mathrm{an}}, (\mathscr{H}^qF)^{\mathrm{an}}).$$

By the comparison theorem of Artin ([1], XVI, 4.1), we have $H^p(U^{\mathrm{an}}, (\mathscr{H}^qF)^{\mathrm{an}}) = H^p(U, \mathscr{H}^qF)$, and this presheaf sheafifies to zero if $p > 0$ ([29], 10.4). When $p = 0$, the sheafification is obviously $\mathscr{H}^qF$. Therefore, the spectral sequence degenerates to isomorphisms

$$\mathscr{H}^iF = \xi_*\xi^*\mathscr{H}^iF \simeq R^i\xi_*\xi^*F,$$

and the adjunction morphism is an isomorphism.

Given $G \in D_c(X^{\mathrm{an}}, \Lambda_n)$, we want to show that the coadjunction morphism $\xi^*R\xi_*G \to G$ is an isomorphism. Consider the spectral sequence

$$\xi^*R^p\xi_*\mathscr{H}^qG \Longrightarrow \xi^*R^{p+q}\xi_*G,$$

where $\xi^* R^p \xi_* \mathscr{H}^q G$ is the analytification of the sheafification of the presheaf on $\acute{\mathrm{E}}\mathrm{t}(X)$

$$(U \to X) \mapsto H^p(U^{\mathrm{an}}, \mathscr{H}^q G).$$

By the comparison ([4], 6.1.2 (A')), the constructible Λ_n-sheaf $\mathscr{H}^q G$ is algebraic, therefore by Artin's comparison theorem ([1], XVI, 4.1) again, $\xi^* R^p \xi_* \mathscr{H}^q G = 0$ for $p > 0$, and the spectral sequence degenerates to isomorphisms

$$\mathscr{H}^i G = \xi^* \xi_* \mathscr{H}^i G \simeq \xi^* R^i \xi_* G.$$

This proves that we have an equivalence

$$(\xi^*, R\xi_*) : D_c(X^{\mathrm{an}}, \Lambda_n) \leftrightarrow D_c(X, \Lambda_n)$$

for each n. As in the proof of (i), the notions of being AR-adic (resp. AR-null) for complexes in $\mathscr{D}(X^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet)$ and in $\mathscr{D}(X^{\mathbb{N}}, \Lambda_\bullet)$ are the same, therefore, we have equivalences

$$(\xi^*, R\xi_*) : \mathscr{D}_c(X^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet) \leftrightarrow \mathscr{D}_c(X^{\mathbb{N}}, \Lambda_\bullet)$$

and

$$(\xi^*, R\xi_*) : D_c(X^{\mathrm{an}}, \Lambda) \leftrightarrow D_c(X, \Lambda).$$

(iii) Applying $\mathscr{H}^i$ on both sides, we should show that

$$\xi_Y^* R^i f_* F \to R^i f_*^{\mathrm{an}}(\xi_X^* F)$$

is an isomorphism. Replacing F by various levels $\widehat{F}_n$ of its normalization, we reduce to the case where $F \in D_c^+(X, \Lambda_n)$. We know that f_* and f_*^{an} have finite cohomological dimension (for instance by generic base change), so one can replace F by $\tau_{[a,b]} F$ and reduce to the case where F is bounded. Taking truncations again and using 5-lemma, we reduce to the case where F is a constructible Λ_n-sheaf, and this follows from Artin's comparison ([1], XVI, 4.1). $\square$

8.2.4.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of $\mathbb{C}$-algebraic stacks. Choose a commutative diagram

$$\begin{tikzcd} X_\bullet \arrow[r, "\widetilde{f}"] \arrow[d, "\epsilon_{\mathcal{X}}"'] & Y_\bullet \arrow[d, "\epsilon_{\mathcal{Y}}"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y}. \end{tikzcd}$$

Then by construction, the diagram

$$\begin{tikzcd} D_c^+(\mathfrak{X},\Lambda) \arrow[r, "\epsilon_{\mathcal{X}}^{\mathrm{an},*}"] \arrow[d, "f_*^{\mathrm{an}}"'] & D_c^+(X_\bullet^{\mathrm{an}},\Lambda) \arrow[d, "\widetilde{f}_*^{\mathrm{an}}"] \\ D_c^+(\mathfrak{Y},\Lambda) \arrow[r, "\epsilon_{\mathcal{Y}}^{\mathrm{an},*}"] & D_c^+(Y_\bullet^{\mathrm{an}},\Lambda) \end{tikzcd}$$

commutes. On the algebraic side, the equivalence ϵ^* is also compatible with taking cohomology (cf. [27], p.202). As a summary, we have the following commutative diagram

$$\begin{tikzcd} D_c^+(\mathfrak{X},\Lambda) \arrow[r, "\epsilon_{\mathcal{X}}^{\mathrm{an},*}"] \arrow[d, "f_*^{\mathrm{an}}"] & D_c^+(X_\bullet^{\mathrm{an}},\Lambda) \arrow[d, "\widetilde{f}_*^{\mathrm{an}}"] & D_c^+(X_\bullet,\Lambda) \arrow[l, "\xi_{X_\bullet}^*"'] \arrow[d, "\widetilde{f}_*"] & D_c^+(\mathcal{X},\Lambda) \arrow[l, "\epsilon_{\mathcal{X}}^*"'] \arrow[d, "f_*"] \\ D_c^+(\mathfrak{Y},\Lambda) \arrow[r, "\epsilon_{\mathcal{Y}}^{\mathrm{an},*}"'] & D_c^+(Y_\bullet^{\mathrm{an}},\Lambda) & D_c^+(Y_\bullet,\Lambda) \arrow[l, "\xi_{Y_\bullet}^*"] & D_c^+(\mathcal{Y},\Lambda), \arrow[l, "\epsilon_{\mathcal{Y}}^*"] \end{tikzcd}$$

where the horizontal arrows are all equivalences of triangulated categories.

8.2.5 Comparison between the two derived categories on the lisse-analytic topos.

In (8.2.3.1) and (8.2.3.2), we defined two derived categories, denoted $D_c(\mathfrak{X},\Lambda)$ and $\mathscr{D}_c(\mathfrak{X},\Lambda)$ respectively. Before proving they are equivalent, we give some preparation on the analytic analogues of some concepts and results in [27].

8.2.5.1. As in [14], let $\pi : \mathfrak{X}^{\mathbb{N}} \to \mathfrak{X}$ be the morphism of topoi, with $\pi_* = \varprojlim$. We have derived functors $R\pi_*$ and $L\pi^*$ between $\mathscr{D}(\mathfrak{X}^{\mathbb{N}},\Lambda_\bullet)$ and $\mathscr{D}(\mathfrak{X},\Lambda)$. Denote $\mathrm{Mod}(\mathfrak{X}^{\mathbb{N}},\Lambda_\bullet)$ by $\mathscr{A}(\mathfrak{X})$ or just $\mathscr{A}$.

Lemma 8.2.5.2. *Let M be an AR-null complex in $\mathscr{D}(\mathscr{A})$. Then $R\pi_*M = 0$.*

Proof. Each of $\mathscr{H}^i(M)$ and $\tau_{>i}M$ is AR-null, so by ([14], 1.1) we have $R\pi_*\mathscr{H}^i(M) \cong R\pi_*\tau_{>i}M = 0$. By ([13], 3.1.5, 3.4.1), the assumption ([26], 2.1.7) for the ringed topoi $(\mathfrak{X}_{\text{lis-an}},\Lambda_n)$ with $\mathscr{C}_n =$ all Λ_n-sheaves is satisfied, so by ([26], 2.1.10) we have $R\pi_*M = 0$. $\square$

Therefore the functor $R\pi_* : \mathscr{D}_c(\mathscr{A}) \to \mathscr{D}_c(\mathfrak{X},\Lambda)$ factors through the quotient category $D_c(\mathfrak{X},\Lambda)$:

$$\mathscr{D}_c(\mathscr{A}) \xrightarrow{Q} D_c(\mathfrak{X},\Lambda) \xrightarrow{\overline{R\pi_*}} \mathscr{D}_c(\mathfrak{X},\Lambda) \xrightarrow{L\pi^*} \mathscr{D}(\mathscr{A}).$$

One can also define the *normalization functor* to be $K \mapsto \widehat{K} := L\pi^*\overline{R\pi_*}K$. For $M \in \mathscr{D}_c(\mathscr{A})$, we will also write $\widehat{M}$ for $\widehat{Q(M)}$, if there is no confusion. A complex M is *normalized* if the natural map $\widehat{M} \to M$ is an isomorphism. The analytic versions of ([27], 2.2.1, 3.0.11, 3.0.10) hold, as we state in the following.

Proposition 8.2.5.3. *(i) For $U \to \mathfrak{X}$ in* $\text{Lis-an}(\mathfrak{X})$ *and $M \in \mathscr{D}(\mathscr{A}(\mathfrak{X}))$, we have $R\pi_*(M_U) = (R\pi_*M)_U$ in $\mathscr{D}(U_{\text{an}}, \Lambda)$.*

(ii) For $U \to \mathfrak{X}$ in $\text{Lis-an}(\mathfrak{X})$ *and $M \in \mathscr{D}(\mathfrak{X}, \Lambda)$, we have $L\pi^*(M_U) = (L\pi^*M)_U$ in $\mathscr{D}(\mathscr{A}(U_{\text{an}}))$.*

(iii) For $M \in \mathscr{D}(\mathscr{A}(\mathfrak{X}))$, it is normalized if and only if the natural map

$$M_n \otimes^L_{\Lambda_n} \Lambda_{n-1} \to M_{n-1}$$

is an isomorphism for each n.

They can be proved in the same way as in [27], and we do not repeat the proof here.

Proposition 8.2.5.4. *(i) The functors $(Q \circ L\pi^*, \overline{R\pi_*})$ induce an equivalence $D_c(\mathfrak{X}, \Lambda) \leftrightarrow \mathscr{D}_c(\mathfrak{X}, \Lambda)$.*

(ii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of complex algebraic stacks, and let $f^{\text{an}} : \mathfrak{X} \to \mathfrak{Y}$ be its analytification. Then the following diagram commutes:

$$\begin{tikzcd}
D_c^+(\mathfrak{X}, \Lambda) \arrow[r, "\overline{R\pi}_{\mathcal{X},*}"] \arrow[d, "f_*^{\text{an}}"'] & \mathscr{D}_c^+(\mathfrak{X}, \Lambda) \arrow[d, "f_*^{\text{an}}"] \\
D_c^+(\mathfrak{Y}, \Lambda) \arrow[r, "\overline{R\pi}_{\mathcal{Y},*}"] & \mathscr{D}_c^+(\mathfrak{Y}, \Lambda).
\end{tikzcd}$$

Proof. (i) We will show that the adjunction and coadjunction maps are isomorphisms. For coadjunction maps, this is an analogue of ([27], 3.0.14).

Lemma 8.2.5.5. *Let $M \in \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$. Then $\widehat{M}$ is constructible and the coadjunction map $\widehat{M} \to M$ has an AR-null cone. In particular, $\widehat{M} \in \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$.*

Proof. It can be proved in the same way as ([27], 3.0.14). We go over the proof briefly.

Let $P : X \to \mathfrak{X}$ be an algebraic presentation, i.e. the analytification of a presentation of the algebraic stack $\mathcal{X}$. By (8.2.5.3), the restriction of the natural map $\widehat{M} \to M$ to X gives the natural map $\widehat{N} \to N$ in $\mathscr{D}(\mathscr{A}(X))$, where $N = M|_X$. It suffices to show the cone of $\widehat{N} \to N$ is AR-null, and $\widehat{M}$ is Cartesian.

1. By ([13], 3.1.5, 3.4.1) and ([27], 2.1.i), the cohomological dimension of $R\pi_*$ on X_{an} is finite. Since $L\pi^*$ also has finite cohomological dimension, the same is true for the normalization functor, namely there exists an integer d, such that for every a and $N \in \mathscr{D}^{\geq a}(X^{\mathbb{N}})$ (resp. $\mathscr{D}^{\leq a}(X^{\mathbb{N}})$), we have $\widehat{N} \in \mathscr{D}^{\geq a-d}(X^{\mathbb{N}})$ (resp. $\mathscr{D}^{\leq a+d}(X^{\mathbb{N}})$).

2. One reduces to the case where N is a λ-module. This is because

$$\mathscr{H}^i(\widehat{N}) = \mathscr{H}^i(\widehat{\tau_{\geq i-d}\tau_{\leq i+d}N})$$

and hence one can assume N is bounded, and then a λ-module.

3. One reduces to the case where N is an adic system. There exists an adic system K with an AR-isomorphism $K \to N$, whose normalization $\widehat{K} \to \widehat{N}$ is an isomorphism (8.2.5.2).

4. By comparison ([4], 6.1.2 (A'')), the adic system N on X is *algebraic*, so by ([11], Rapport sur la formule des traces, 2.8) there exists an n_0 such that $N/\mathrm{Ker}(\lambda^{n_0})$ is torsion-free. Hence one reduces to two cases: N is torsion-free, or $\lambda^{n_0}N = 0$.

5. Assume N is torsion-free and adic. Then the component N_n is flat over Λ_n, for each n, and the natural map

$$N_n \otimes^L_{\Lambda_n} \Lambda_{n-1} \simeq N_n \otimes_{\Lambda_n} \Lambda_{n-1} \xrightarrow{\sim} N_{n-1}$$

is an isomorphism, i.e. N is normalized. Then the cone of $\widehat{N} \to N$ is zero.

6. Assume $\lambda^{n_0}N = 0$. One reduces to the case where $n_0 = 1$ by considering the λ-filtration. This means the map

$$(N_n)_n \to (N_n/\lambda N_n)_n = (N_0)_n$$

is an AR-isomorphism, so N and $\pi^*N_0 = (N_0)_n$ have the same normalization. Note that $R\pi_*(N_0)_n = N_0$ ([27], 2.2.3), hence $\widehat{\pi^*N_0} = L\pi^*N_0$. By (8.2.2.4), N_0 is lcc on each stratum of an algebraic stratification of X, and one can check if $L\pi^*N_0 \to \pi^*N_0$ is an isomorphism on each stratum. By ([4], 6.1.2 (A')), N_0 is algebraic, and one can replace each stratum by an étale cover on which N_0 is constant. Finally by additivity we reduce to the case $N_0 = \Lambda_0$, which is proved by computing $L\pi^*\Lambda_0$ via the 2-term flat Λ-resolution of Λ_0 (cf. ([27], 3.0.10)).

The proof for $\widehat{M}$ being Cartesian is also the same as ([27], 3.0.14) (note that the analytic version of ([27], 3.0.13) holds). Let us not to repeat it here.

In particular, $\widehat{M} \in \mathscr{D}_c(\mathscr{A}(\mathfrak{X}))$, since the cone (which is AR-null) is AR-adic, and λ-complexes form a triangulated subcategory. □

We prove that the adjunction map is an isomorphism in the following lemma. This will be the crucial step; it only holds in the analytic category.

Lemma 8.2.5.6. *Let $M \in \mathscr{D}_c(\mathfrak{X}, \Lambda)$. Then the adjunction map $M \to R\pi_*L\pi^*M$ is an isomorphism.*

Proof. For simplicity, let us denote $R\pi_*L\pi^*M$ by $\check{M}$. Note that if $M' \to M \to M'' \to M'[1]$ is an exact triangle, and the adjunction maps for M' and M'' are isomorphisms, then the same holds for M, since $\check{M}' \to \check{M} \to \check{M}'' \to \check{M}'[1]$ is also an exact triangle.

1. That the map $M \to \check{M}$ is an isomorphism is a local property, since it is equivalent to the vanishing of all the cohomology sheaves of the cone, which can be checked locally. So we may replace $\mathfrak{X}$ by the algebraic presentation X.

2. On the analytic topos X_{an}, the functor $R\pi_*$ has finite cohomological dimension ([27], 2.1.i). Then as explained in (8.2.5.5), since the functor $M \mapsto \check{M}$ has finite cohomological

dimension, one reduces to the case where M is a constructible Λ_X-module. This case follows from ([4], 6.1.2 (B'')), but for the reader's convenience, we continue to finish the proof.

By (8.2.3.3), M is the limit of some adic sheaf $F \in \Lambda\text{-Sh}_c(X_{\text{an}})$, and by comparison ([4], 6.1.2 (A'')) we see that F is algebraic. Therefore by ([11], Rapport sur la formule des traces, 2.8), we reduce to two cases: M is torsion-free (i.e. stalks are free Λ-modules of finite type), or M is killed by λ. The second case follows from ([27], 2.2.3).

3. Assume M is a torsion-free constructible sheaf. We want to use noetherian induction to reduce to the case where M is locally constant. Let $j : U \hookrightarrow X$ be the open immersion of a Zariski open subspace over which M is locally constant (by definition; see (8.2.3.3)), and let $i : Z \hookrightarrow X$ be the complement. Consider the exact triangle

$$i_*F \longrightarrow M \longrightarrow Rj_*M_U \longrightarrow ,$$

where $F = Ri^!M \in \mathscr{D}_c(Z_{\text{an}}, \Lambda)$. It suffices to show that the adjunction maps for i_*F and Rj_*M_U are isomorphisms.

We have the following commutative diagram of topoi

$$\begin{array}{ccc} Z_{\text{an}}^{\mathbb{N}} & \xrightarrow{i^{\mathbb{N}}} & X_{\text{an}}^{\mathbb{N}} \\ {\scriptstyle \pi_Z}\big\downarrow & & \big\downarrow{\scriptstyle \pi_X} \\ Z_{\text{an}} & \xrightarrow{i} & X_{\text{an}}, \end{array}$$

so $R\pi_{X,*} \circ i^{\mathbb{N}}_* \simeq i_* \circ R\pi_{Z,*}$. Also $L\pi_X^* \circ i_* \simeq i^{\mathbb{N}}_* \circ L\pi_Z^*$, since i_* is just extension by zero, and $i_*(F \otimes^L_\Lambda \Lambda_n) \simeq i_*F \otimes^L_\Lambda \Lambda_n$. Therefore, the adjunction map for i_*F on X is obtained by applying i_* to the adjunction map for F on Z :

$$i_*F \to R\pi_{X,*}L\pi_X^* i_*F \simeq i_*R\pi_{Z,*}L\pi_Z^*F,$$

which is an isomorphism by noetherian hypothesis.

We have the commutative diagram of topoi

$$\begin{array}{ccc} U_{\text{an}}^{\mathbb{N}} & \xrightarrow{j^{\mathbb{N}}} & X_{\text{an}}^{\mathbb{N}} \\ {\scriptstyle \pi_U}\big\downarrow & & \big\downarrow{\scriptstyle \pi_X} \\ U_{\text{an}} & \xrightarrow{j} & X_{\text{an}}, \end{array}$$

so $R\pi_{X,*} \circ Rj^{\mathbb{N}}_* \simeq Rj_* \circ R\pi_{U,*}$. Also $Rj^{\mathbb{N}}_* \circ L\pi_U^* \simeq L\pi_X^* \circ Rj_*$. For each n we have a natural morphism $\Lambda_n \otimes^L_\Lambda Rj_*F \to Rj_*(F \otimes^L_\Lambda \Lambda_n)$. Let $P^\bullet$ be the flat Λ-resolution $0 \to \Lambda \xrightarrow{\lambda^{n+1}} \Lambda \to \Lambda_n$ of Λ_n, and let $F \to I^\bullet$ be an injective resolution of the sheaf F. Then $I^\bullet \otimes_\Lambda P^\bullet$ is also a complex of injectives, and it is clear that $j_*(I^\bullet \otimes P^\bullet) = j_*(I^\bullet) \otimes P^\bullet$. Therefore, the adjunction map for Rj_*M_U on X is obtained by applying Rj_* to the adjunction map for M_U on U. Hence we reduce to the case where M is a locally constant sheaf on X.

4. Since the question is local for the analytic topology, we may cover X by analytic open subspaces over which M is constant, and hence reduce to the case where M is constant, defined by a free module Λ^r of finite rank. By additivity we may assume $M = \Lambda$. Then $L\pi^*\Lambda = (\Lambda_n)_n$, and $\pi_*(\Lambda_n)_n = \varprojlim(\Lambda_n)_n = \Lambda$. To finish the proof, we shall show $R^i\pi_*(\Lambda_n)_n = 0$ for $i \neq 0$.

Recall that $R^i\pi_*(\Lambda_n)_n$ is the sheafification of the presheaf on X_{an}

$$U \mapsto H^i(\pi^*U, (\Lambda_n)_n).$$

Consider the exact sequence ([27], 2.1.i)

$$0 \longrightarrow R^1 \varprojlim H^{i-1}(U, \Lambda_n) \longrightarrow H^i(\pi^*U, \Lambda_\bullet) \longrightarrow \varprojlim H^i(U, \Lambda_n) \longrightarrow 0.$$

Since X is locally contractible, and $R^1 \varprojlim H^0(U, \Lambda_n) = R^1 \varprojlim \Lambda_\bullet = 0$ for U connected, we see that the sheafification $R^i\pi_*\Lambda_\bullet$ is zero for $i \neq 0$. This proves that the adjunction morphism $M \to \check{M}$ is an isomorphism. $\square$

Therefore, $(Q \circ L\pi^*, \overline{R\pi_*})$ induce an equivalence between $D_c(\mathfrak{X}, \Lambda)$ and $\mathscr{D}_c(\mathfrak{X}, \Lambda)$.

(ii) If $X_\bullet \to \mathfrak{X}$ is a strictly simplicial algebraic smooth hypercover, we have $D_c(\mathfrak{X}, \Lambda) \simeq D_c(X_\bullet, \Lambda)$ by (8.2.4.1i). Similarly, $\mathscr{D}_c(\mathfrak{X}, \Lambda)$ is naturally equivalent to $\mathscr{D}_c(X_\bullet, \Lambda)$. This can be proved in the same way as we prove "$D_c(\mathfrak{X}, \Lambda_m) \simeq D_c(X_\bullet^{\mathrm{an}}, \Lambda_m)$" in (8.2.4.1).

So we may assume that $\mathfrak{X} = X$ is the analytic space associated to an algebraic scheme. By definition of $\overline{R\pi_*}$, it suffices to show the following diagram commutes

$$\begin{tikzcd} \mathscr{D}_c^+(\mathscr{A}(X)) \arrow[r, "R\pi_{X,*}"] \arrow[d, "f_*^{\mathbb{N}}"'] & \mathscr{D}_c^+(X, \Lambda) \arrow[d, "f_*"] \\ \mathscr{D}_c^+(\mathscr{A}(Y)) \arrow[r, "R\pi_{Y,*}"] & \mathscr{D}_c^+(Y, \Lambda), \end{tikzcd}$$

and this follows from the commutativity of the diagram of topoi

$$\begin{tikzcd} X_{\mathrm{an}}^{\mathbb{N}} \arrow[r, "\pi_X"] \arrow[d, "f^{\mathbb{N}}"'] & X_{\mathrm{an}} \arrow[d, "f"] \\ Y_{\mathrm{an}}^{\mathbb{N}} \arrow[r, "\pi_Y"] & Y_{\mathrm{an}}. \end{tikzcd}$$

Note that the corresponding diagram for $f : \mathfrak{X} \to \mathfrak{Y}$ does not even make sense, since if f is not smooth, it does not necessarily induce a morphism of their lisse-analytic topoi. $\square$

Similarly, $\overline{R\pi_{\mathcal{X},*}}$ induces a fully faithful functor $D_c(\mathfrak{X}, \Lambda) \to \mathscr{D}_c(\mathfrak{X}, \Lambda)$, which is compatible with f_*.

8.3 Over $\mathbb{C}$.

Let $(\Lambda, \mathfrak{m})$ be a complete DVR as before, with residue characteristic $\ell \neq 2$. Let $\mathcal{X}$ be an algebraic stack over Spec $\mathbb{C}$. We first prove a comparison theorem between the lisse-étale topoi over $\mathbb{C}$ and over $\mathbb{F}$, and then use this together with (8.2.4.3) to deduce the decomposition theorem for $\mathbb{C}$-algebraic stacks with affine diagonal.

8.3.1 Comparison between the lisse-étale topoi over $\mathbb{C}$ and over $\mathbb{F}$.

Let $(\mathscr{S}, \mathcal{L})$ be a pair on $\mathcal{X}$ defined on the level of Λ. By refining we may assume all strata in $\mathscr{S}$ are essentially smooth (i.e. their maximal reduced substack is smooth) and connected. Let $A \subset \mathbb{C}$ be a subring of finite type over $\mathbb{Z}$, large enough so that there exists a triple $(\mathcal{X}_S, \mathscr{S}_S, \mathcal{L}_S)$ over $S :=$ Spec A giving rise to $(\mathcal{X}, \mathscr{S}, \mathcal{L})$ by base change, and $1/\ell \in A$. Then S satisfies the condition (LO); the hypothesis on ℓ-cohomological dimension follows from ([1], X, 6.2). We may shrink S to assume that strata in $\mathscr{S}_S$ are smooth over S with geometrically connected fibers, which is possible because one can take a presentation $P : X_S \to \mathscr{X}_S$ and shrink S so that the strata in $P^*\mathscr{S}_S$ are smooth over S with geometrically connected fibers. Let $a : \mathcal{X}_S \to S$ be the structural map.

Let $A \subset V \subset \mathbb{C}$, where V is a strictly henselian DVR whose residue field is an algebraic closure of a finite residue field of A. Let $(\mathcal{X}_V, \mathscr{S}_V, \mathcal{L}_V)$ be the triple obtained by base change to V, and let $(\mathcal{X}_s, \mathscr{S}_s, \mathcal{L}_s)$ be its special fiber. Then we have morphisms

$$\mathcal{X} \xrightarrow{u} \mathcal{X}_V \xleftarrow{i} \mathcal{X}_s.$$

Proposition 8.3.1.1. *(stack version of ([4], 6.1.9)) For S small enough, the functors*

$$D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda_n) \xleftarrow{u_n^*} D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V, \Lambda_n) \xrightarrow{i_n^*} D^b_{\mathscr{S}_s,\mathcal{L}_s}(\mathcal{X}_s, \Lambda_n)$$

and

$$D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda) \xleftarrow{u^*} D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V, \Lambda) \xrightarrow{i^*} D^b_{\mathscr{S}_s,\mathcal{L}_s}(\mathcal{X}_s, \Lambda)$$

are equivalences of triangulated categories.

Proof. Clearly, they are all triangulated functors.

By (8.1.5), we can shrink $S =$ Spec A so that for any F and G of the form $j_!L$, where $j : \mathcal{U}_S \to \mathcal{X}_S$ in $\mathscr{S}_S$ and $L \in \mathcal{L}_S(\mathcal{U}_S)$, the formations of $R\mathscr{H}om_{\mathcal{X}_S}(F, G)$ commute with base change on S, and the complexes $a_*\mathscr{E}xt^q_{\mathcal{X}_S}(F, G)$ on S are lcc and of formation compatible with base change, i.e. the cohomology sheaves are lcc, and for any $g : S' \to S$, the base change morphism for a_* :

$$g^*a_*\mathscr{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{S'*}g'^*\mathscr{E}xt^q_{\mathcal{X}_S}(F, G)$$

is an isomorphism. Then using the same argument as in [4], the claim for u_n^* and i_n^* follows. For the reader's convenience, we explain the proof in [4] in more detail.

Note that the spectra of $V, \mathbb{C}$ and s have no non-trivial étale surjections mapping to them, so their small étale topoi are the same as Sets. In particular, Ra_{V*} (resp. $Ra_{\mathbb{C}*}$ and Ra_{s*}) is just $R\Gamma$. Let us show the full faithfulness of u_n^* and i_n^* first. For $K, L \in D^b_{\mathscr{S}_V, \mathcal{L}_V}(\mathcal{X}_V, \Lambda_n)$, let $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ (resp. K_s and L_s) be their images under u_n^* (resp. i_n^*). Then the full faithfulness follows from the more general claim that, the maps

$$Ext^i_{\mathcal{X}}(K_{\mathbb{C}}, L_{\mathbb{C}}) \xleftarrow{u_n^*} Ext^i_{\mathcal{X}_V}(K, L) \xrightarrow{i_n^*} Ext^i_{\mathcal{X}_s}(K_s, L_s)$$

are bijective for all i.

Since $Hom_{D_c(\mathcal{X}, \Lambda_n)}(K, -)$ and $Hom_{D_c(\mathcal{X}, \Lambda_n)}(-, L)$ are cohomological functors, by 5-lemma we may assume that $K = F$ and $L = G$ are Λ_n-sheaves. Let $j : \mathcal{U}_S \to \mathcal{X}_S$ be the immersion of an open stratum in $\mathscr{S}_S$, with complement $i : \mathcal{Z}_S \to \mathcal{X}_S$. Using the short exact sequence

$$0 \longrightarrow j_{V!} j_V^* F \longrightarrow F \longrightarrow i_{V*} i_V^* F \longrightarrow 0$$

and noetherian induction on the support of F and G, we may assume that they take the form $j_{V!}L$, where j is the immersion of some stratum in $\mathscr{S}_S$, and L is a sheaf in $\mathcal{L}_V$. The spectral sequence

$$R^p a_{\square,*} \mathscr{E}xt^q_{\mathcal{X}_\square}(F_\square, G_\square) \Longrightarrow Ext^{p+q}_{\mathcal{X}_\square}(F_\square, G_\square)$$

is natural in the base $\square$, which can be $V, \mathbb{C}$ or s. The assumption on S made before implies that the composite base change morphism

$$g^* a_* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{S'*} g'^* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{S'*} \mathscr{E}xt^q_{\mathcal{X}_{S'}}(g'^* F, g'^* G)$$

is an isomorphism, for all $g : S' \to S$. Therefore, the maps

$$Ext^i_{\mathcal{X}}(F_{\mathbb{C}}, G_{\mathbb{C}}) \xleftarrow{u_n^*} Ext^i_{\mathcal{X}_V}(F, G) \xrightarrow{i_n^*} Ext^i_{\mathcal{X}_s}(F_s, G_s)$$

are bijective for all i. The claim (hence the full faithfulness of u_n^* and i_n^*) follows.

This claim also implies their essential surjectivity. To see this, let us give a lemma first.

Lemma 8.3.1.2. *Let $F : \mathscr{C} \to \mathscr{D}$ be a triangulated functor between triangulated categories. Let $A, B \in \text{Obj } \mathscr{C}$, and let $F(A) \xrightarrow{v} F(B) \to C \to F(A)[1]$ be an exact triangle in $\mathscr{D}$. If the map*

$$F : Hom_{\mathscr{C}}(A, B) \to Hom_{\mathscr{D}}(F(A), F(B))$$

is surjective, then C is in the essential image of F.

Proof. Let $u : A \to B$ be a morphism such that $F(u) = v$. Let C' be the mapping cone of u, i.e. let the triangle $A \xrightarrow{u} B \to C' \to A[1]$ be exact. Then its image

$$F(A) \xrightarrow{v} F(B) \longrightarrow F(C') \longrightarrow F(A)[1]$$

is also an exact triangle. This implies that $C \simeq F(C')$. $\square$

Now we can show the essential surjectivity of u_n^* and i_n^*. For $K \in D^b_{\mathscr{S}_{\mathbb{C}},\mathcal{L}_{\mathbb{C}}}(\mathcal{X}, \Lambda_n)$, to show that K lies in the essential image of u_n^*, using the truncation exact triangles and (8.3.1.2), we reduce to the case where K is a sheaf. Using noetherian induction on the support of K, we reduce to the case where $K = j_!L$, where $j : \mathcal{U} \to \mathcal{X}$ is the immersion of a stratum in $\mathscr{S}$, and $L \in \mathcal{L}(\mathcal{U})$. This is the image of the corresponding $j_{V!}L_V$, since they are all defined over S. Similarly, i_n^* is also essentially surjective.

Next, we prove that u^* and i^* are equivalences.

We claim that for $K, L \in D^b_c(\mathcal{X}_V, \Lambda)$, if the morphisms

$$u_n^* : Hom_{D_c(\mathcal{X}_V,\Lambda_n)}(\widehat{K}_n, \widehat{L}_n) \to Hom_{D_c(\mathcal{X},\Lambda_n)}(\widehat{K}_{n,\mathbb{C}}, \widehat{L}_{n,\mathbb{C}})$$

and

$$i_n^* : Hom_{D_c(\mathcal{X}_V,\Lambda_n)}(\widehat{K}_n, \widehat{L}_n) \to Hom_{D_c(\mathcal{X}_s,\Lambda_n)}(\widehat{K}_{n,s}, \widehat{L}_{n,s})$$

are bijective for all n, then the morphisms

$$u^* : Hom_{D_c(\mathcal{X}_V,\Lambda)}(K, L) \to Hom_{D_c(\mathcal{X},\Lambda)}(K_{\mathbb{C}}, L_{\mathbb{C}})$$

and

$$i^* : Hom_{D_c(\mathcal{X}_V,\Lambda)}(K, L) \to Hom_{D_c(\mathcal{X}_s,\Lambda)}(K_s, L_s)$$

are bijective. Let $\square$ be one of the bases $V, \mathbb{C}$ or s. Since K and L are bounded, we see from the spectral sequence

$$R^p a_{\square,*}\mathscr{E}xt^q_{\mathcal{X}_\square}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square}) \Longrightarrow Ext^{p+q}_{\mathcal{X}_\square}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square})$$

and the finiteness of $R\mathscr{H}om$ and $Ra_{\square,*}$ ([26], 4.2.2, 4.1) that, the groups $Ext^{-1}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square})$ are finite for all n, hence they form a projective system satisfying the condition (ML). By ([27], 3.1.3), we have an isomorphism

$$Hom_{D_c(\mathcal{X}_\square,\Lambda)}(K_\square, L_\square) \xrightarrow{\sim} \varprojlim_n Hom_{D_c(\mathcal{X}_\square,\Lambda_n)}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square}),$$

natural in the base $\square$, and the claim follows.

Since when restricted to $D^b_{\mathscr{S}_\square,\mathcal{L}_\square}$, the functors u_n^* and i_n^* are fully faithful for all n, we deduce that u^* and i^* are also fully faithful.

Finally we prove the essential surjectivity of u^* and i^*. In the following 2-commutative diagram

$$\begin{array}{ccc}\mathscr{D}^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathscr{A}(\mathcal{X}_V)) & \xrightarrow{Q_V} & D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V,\Lambda) \\ {\scriptstyle u'^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathscr{D}^b_{\mathscr{S},\mathcal{L}}(\mathscr{A}(\mathcal{X})) & \xrightarrow{Q} & D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X},\Lambda),\end{array}$$

the localization functors Q and Q_V are essentially surjective. Given $K \in D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X},\Lambda)$, to show that K lies in the essential image of u^*, it suffices to show that $\widehat{K} \in \mathscr{D}^b_{\mathscr{S},\mathcal{L}}(\mathscr{A}(\mathcal{X}))$ lies in the essential image of u'^*.

Let $M = \widehat{K} = (M_n)_n$; it is a normalized complex ([27], 3.0.8). Let $\rho_n : M_n \to M_{n-1}$ be the transition maps. Since u_n^* is an equivalence, there exists (uniquely up to isomorphism) an $M_{n,V} \in D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V,\Lambda_n)$ such that $u_n^*(M_{n,V}) \simeq M_n$, for each n. Also there exists $\rho_{n,V} \in Hom_{D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V,\Lambda_n)}(M_{n,V}, M_{n-1,V})$, which is mapped to ρ_n via u_n^*. We see that the induced morphism $\overline{\rho}_{n,V} : M_{n,V} \otimes^L_{\Lambda_n} \Lambda_{n-1} \to M_{n-1,V}$ is an isomorphism, since its image under the equivalence u_{n-1}^* is an isomorphism ([27], 3.0.10).

For $M_V = (M_{n,V})_n$ to be an object of $\mathscr{D}^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathscr{A}(\mathcal{X}_V))$, we need to show it is an object of $\mathscr{D}_c(\mathscr{A}(\mathcal{X}_V))$, i.e. the cohomology systems $\mathscr{H}^i(M_V)$ are AR-adic ([27], 3.0.6).

Let $N^i = (N^i_n)_n$ be the universal image of the projective system $\mathscr{H}^i(M)$. Recall ([16], V, 3.2.3) that, since $\mathscr{H}^i(M)$ is AR-adic, it satisfies the condition (MLAR) (so that it makes sense to talk about its system of the universal images), and there exists an integer $k \geq 0$ such that $l_k(N^i) := (N^i_{n+k}/\lambda^{n+1}N^i_{n+k})_n$ is an adic system. Let $r \geq 0$ be an integer such that N^i_n is the image of $\mathscr{H}^i(M_{n+r}) \to \mathscr{H}^i(M_n)$, for each n. Then for every $s \geq r$, we have

$$\frac{\mathrm{Im}(\mathscr{H}^i(M_{n+r,V}) \to \mathscr{H}^i(M_{n,V}))}{\mathrm{Im}(\mathscr{H}^i(M_{n+s,V}) \to \mathscr{H}^i(M_{n,V}))} = 0,$$

since its image under the equivalence u_n^* is zero. This shows that $\mathscr{H}^i(M_V)$ also satisfies the condition (MLAR), with universal images $N^i_{n,V} = \mathrm{Im}(\mathscr{H}^i(M_{n+r,V}) \to \mathscr{H}^i(M_{n,V}))$. Also, the projective system $l_k(N^i_V)$ is adic, since the image under u_n^* of the transition map

$$\left(N^i_{n+k+1,V}/\lambda^{n+2}N^i_{n+k+1,V}\right) \otimes_{\Lambda_{n+1}} \Lambda_n \to N^i_{n+k,V}/\lambda^{n+1}N^i_{n+k,V}$$

is an isomorphism. By ([16], V, 3.2.3) again, the system $\mathscr{H}^i(M_V)$ is AR-adic. This finishes the proof that u^* (and similarly, i^*) is essentially surjective. □

8.3.2 The proof.

Let $P : X \to \mathcal{X}$ be a presentation and let $\mathfrak{X} = \mathcal{X}^{\mathrm{an}}$ be the associated analytic stack.

8.3.2.1. Following [28], one can define Ω-perverse sheaves (for $\Omega = \mathbb{C}, E_\lambda$ or $\overline{\mathbb{Q}}_\ell$) on $\mathfrak{X}$ as follows. Let $p = p_{1/2}$ be the middle perversity on X^{an}. Let $d : \pi_0(X) \to \mathbb{N}$ be the dimension of the smooth map P. Define ${}^pD^{\leq 0}_c(\mathfrak{X},\Omega)$ (resp. ${}^pD^{\geq 0}_c(\mathfrak{X},\Omega)$) to be the full subcategory of objects $K \in D_c(\mathfrak{X}, \Omega)$ such that $P^{\mathrm{an},*}K[d]$ is in ${}^pD_c^{\leq 0}(X^{\mathrm{an}}, \Omega)$ (resp. ${}^pD_c^{\geq 0}(X^{\mathrm{an}}, \Omega)$). As in ([28], 4.1, 4.2), one can show that these subcategories do not depend on the choice of the presentation P, and they define a t-structure, called the (middle) perverse t-structure on $\mathfrak{X}$.

8.3.2.2. Following ([4], 6.2.4), one can define the sheaf complexes *of geometric origin* as follows. Let $\mathscr{F}$ be a Ω-perverse sheaf on $\mathfrak{X}$ (resp. a $\overline{\mathbb{Q}}_\ell$-perverse sheaf on $\mathcal{X}$). We say that $\mathscr{F}$ is *semi-simple of geometric origin* if it is semi-simple, and every simple direct summand belongs to the smallest family of simple perverse sheaves on complex analytic stacks (resp. lisse-étale sites of $\mathbb{C}$-algebraic stacks) that

(a) contains the constant sheaf $\underline{\Omega}$ over a point, and is stable under the following operations:

(b) taking the constituents of ${}^p\mathscr{H}^iT$, for $T = f_*, f_!, f^*, f^!, R\mathscr{H}om(-,-)$ and $- \otimes^L -$, where f is an arbitrary *algebraic* morphism between stacks.

A complex $K \in D_c^b(\mathfrak{X}, \Omega)$ (resp. $D_c^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$) is said to be *semi-simple of geometric origin* if it is isomorphic to the direct sum of the $({}^p\mathscr{H}^iK)[-i]$'s, and each ${}^p\mathscr{H}^iK$ is semi-simple of geometric origin.

One can replace the constant sheaf E_λ by its ring of integers $\mathscr{O}_\lambda$, and deduce that every complex $K \in \mathscr{D}_c^b(\mathfrak{X}, \overline{\mathbb{Q}}_\ell)$ that is semi-simple of geometric origin has an integral structure. Then we can apply (8.2.5.4).

Lemma 8.3.2.3. *(stack version of ([4], 6.2.6)) Let $\mathscr{F}$ be a simple $\overline{\mathbb{Q}}_\ell$-perverse sheaf of geometric origin on $\mathcal{X}$. For $A \subset \mathbb{C}$ large enough, the equivalence (8.3.1.1)*

$$D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \leftrightarrow D^b_{\mathscr{S}_s,\mathcal{L}_s}(\mathcal{X}_s, \overline{\mathbb{Q}}_\ell)$$

takes $\mathscr{F}$ to a simple perverse sheaf $\mathscr{F}_s$ on $\mathcal{X}_s$, such that $(\mathcal{X}_s, \mathscr{F}_s)$ is deduced by base extension from a pair $(\mathcal{X}_0, \mathscr{F}_0)$ defined over a finite field $\mathbb{F}_q$, and $\mathscr{F}_0$ is ι-pure.

Proof. $\mathscr{F}_s$ is obtained by base extension from some simple perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$, so it suffices to show $\mathscr{F}_0$ is ι-mixed (7.3.5). This is clear, since the six operations, the perverse truncation functors and taking subquotients in the category of perverse sheaves all preserve ι-mixedness, and the constant sheaf $\overline{\mathbb{Q}}_\ell$ on a point is punctually pure. $\square$

Finally, we are ready to prove the stack version of the decomposition theorem over $\mathbb{C}$.

Theorem 8.3.2.4. *(stack version of ([4], 6.2.5)) Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism with finite diagonal of $\mathbb{C}$-algebraic stacks with affine automorphism groups. If $K \in D_c^b(\mathfrak{X}, \mathbb{C})$ is semi-simple of geometric origin, then $f_*^{\mathrm{an}}K$ is also bounded, and is semi-simple of geometric origin on $\mathfrak{Y}$.*

Proof. By (8.2.5.4) we can replace $D_c^b(\mathfrak{X}, \mathbb{C})$ by $D_c^b(\mathfrak{X}, \overline{\mathbb{Q}}_\ell)$, and by (8.2.4.3) we can replace this by $D_c^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$.

From ([33], 5.17) we know that there is a canonical isomorphism $f_! \simeq f_*$ on $D_c^-(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$. For $K \in D_c^b$, we have $f_!K \in D_c^-$ and $f_*K \in D_c^+$, hence $f_*K \in D_c^b$.

Lemma 8.3.2.5. *We can reduce to the case where K is a simple perverse sheaf $\mathscr{F}$.*

Proof. There are two steps: firstly, we show that the statement for simple perverse sheaves of geometric origin implies the statement for semi-simple perverse sheaves of geometric origin. This is clear:

$$f_*(\bigoplus_i \mathscr{F}_i) = \bigoplus_i f_*\mathscr{F}_i = \bigoplus_i \bigoplus_j {}^p\mathscr{H}^j(f_*\mathscr{F}_i)[-j] = \bigoplus_j {}^p\mathscr{H}^j(f_*(\bigoplus_i \mathscr{F}_i))[-j].$$

Then we show that the statement for semi-simple perverse sheaves implies the general statement. If K is semi-simple of geometric origin, we have

$$f_*K = \bigoplus_i f_*{}^p\mathscr{H}^i(K)[-i] = \bigoplus_i \bigoplus_j {}^p\mathscr{H}^j f_*{}^p\mathscr{H}^i(K)[-i-j].$$

Taking ${}^p\mathscr{H}^n$ on both sides, we get

$${}^p\mathscr{H}^n(f_*K) = \bigoplus_{i+j=n} {}^p\mathscr{H}^j f_*{}^p\mathscr{H}^i(K),$$

therefore $f_*K = \bigoplus_n {}^p\mathscr{H}^n(f_*K)$ and each summand is semi-simple of geometric origin. □

Now assume K is a simple perverse sheaf $\mathscr{F}$. By (3.1.4v) every bounded complex is stratifiable. By (8.3.2.3), $\mathscr{F}$ corresponds to a simple perverse sheaf $\mathscr{F}_s$ which is induced from an ι-pure perverse sheaf $\mathscr{F}_0$ by base change. By (8.1.5), the formation of f_* over $\mathbb{C}$ is the same as the formation of $f_{s,*}$ over $\mathbb{F}$ or $f_{0,*}$ over a finite field. By (7.3.8), $f_{0,*}\mathscr{F}_0$ is also ι-pure. By (7.3.10, 7.3.11), we have

$$f_{s,*}\mathscr{F}_s \simeq \bigoplus_{i\in\mathbb{Z}} {}^p\mathscr{H}^i(f_{s,*}\mathscr{F}_s)[-i],$$

and each ${}^p\mathscr{H}^i(f_{s,*}\mathscr{F}_s)$ is semi-simple of geometric origin. Therefore $f_*\mathscr{F}$ is semi-simple of geometric origin. □

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