Optimal Placement and Dynamics of Order Positions with Related Queues in the Limit Order Book

By

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A dissertation submitted in partial satisfaction of the requirements for the degree of

Doctor of Philosophy

in

Engineering – Industrial Engineering and Operations Research

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2015
Abstract

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The optimal placement problem studies how to optimally place orders in a limit order book to purchase/sell a fixed number of shares of a stock. Under a correlated random walk model with mean-reversion for the best ask/bid price, optimal placement strategies for both single-step and multi-step cases are derived in this thesis. In the single-step case, the optimal strategy involves only the market order, the best bid, and the second best bid. In the multi-step case, the optimal strategy is of a threshold type. Critical to the analysis is a generalized reflection principle for correlated random walks.

Furthermore, a simple linear price impact model is also presented, which gives insight into the price impact of executing large orders on placement decisions.

The availability of detailed limit order book information enables more accurate estimation of the order execution probability and price dynamics. Motivated by various optimization problems and models in algorithmic trading, a limiting behavior for order positions and related best bid/ask queues in a limit order book is studied in this thesis. In addition to the fluid and diffusion limits for the processes, fluctuations of the order position and related queues around their fluid limit are analyzed. As a corollary, explicit analytical expressions for various quantities of interests in a limit order book are derived.
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Acknowledgements

I would like to thank foremost my advisor, Professor Xin Guo, for her wise guidance, creative insights, kind understanding and strong support to my doctoral study, which has left me with an extremely rewarding experience at Berkeley. I have learned a lot from her professionally and personally via numerous discussions, which has extraordinarily improved my research capability, and will be helpful in the future. I am also grateful to Professor Ilan Adler, Professor James Pitman, and Professor Rhonda Righter for being my qualifying and dissertation committee and for their helpful feedback on my research, and their encouragement.

I want to thank Dan Bu, Kelly Choi, Long He, Te Ke, among all other friends here in Berkeley. The memorable time spent together with them on course studies, research discussions, and leisure activities will be valuable forever. I also want to thank my calibrators Adrien de Lerrard and Lingjiong Zhu for their support.

I would like to thank my mother, who spends most of her love on me and does not even expect any return. I would also like to thank my father, who passed away five years ago just before I came to the United States for my doctoral study. He lives in my heart forever and is with me whenever I need him. I thank all my relatives and friends in China, as their support helps me overcome all the difficulties in life.

Finally, I would like to thank Shiman Ding, my lovely girlfriend, for her continual and unconditional support. The joyful time spent with her will be an unparalleled treasure throughout my whole life.
Chapter 1

Introduction

1.1 Financial markets and algorithmic trading

1.1.1 History of financial markets

As early as the 12th century, the Courretiers de Change managed and regulated the debts of agricultural communities for banks in France. This could be viewed as the beginning of the financial markets. In 1602, the first stocks were issued by Dutch merchants to build the Dutch East India Company, which became the first joint-stock company, to protect the trade in the Indian Ocean. The company stock was traded on the Amsterdam Exchange, and various derivatives, including options and repos, emerged on the market as well. Moreover, short selling was allowed in practice in 1610 by the Dutch authorities. In the 1600s, more trade companies were built by British and French governments, raising funds from the stock market and sharing profit by paying dividends to shareholders. In 1792, the New York Stock Exchange (NYSE) was established by 24 stock brokers and soon became the most important exchange in the United States. In 1971, the National Association of Securities Dealers Automated Quotations (NASDAQ), as the first electronic market, was created. The NASDAQ trading volume increased quickly over the years as its automated trading system developed. Now, with approximately 3,200 companies from 37 countries and across all industry sectors listed, the NASDAQ has become the largest stock exchange in the United States in terms of market share and volume traded.
With the quick expansion of the NASDAQ, automatic and electronic order-driven trading platforms have largely replaced the traditional floor-based trading for virtually all financial markets.

### 1.1.2 Limit order book (LOB)

In an order-driven market, there are two types of buy/sell orders for market participants to post: market orders and limit orders. A *limit order* is an order to trade a certain amount of security (stocks, futures, etc.) at a specified price. The lowest price for which there is an outstanding limit sell order is called the *best ask price* and the highest limit buy price is called the *best bid price*. Limit orders are collected and posted in the LOB, which contains the quantities and the prices at different levels for all limit buy and sell orders. A *market order* is an order to buy/sell a certain amount of the equity at the best available price in the LOB. It is then matched with the best available price and a trade occurs immediately. The LOB is updated accordingly. A limit order stays in the LOB until it is executed against a market order or until it is canceled; cancellation is allowed at any time before getting executed. In essence, the closer a limit order is to the best bid/ask, the faster it may be executed. Most exchanges are based on the first-in-first-out (FIFO) policy for orders on the same price level, although some derivatives on some exchanges have the pro-rate microstructure, that is, an incoming market order is dispatched on all active limit orders at the best price, with each limit order contributing to execution in proportion to its volume.

Order books are available with different levels of details. For example, the so-called “level-1 order book” contains the best price level of the order book, while the “level-2 order book” provides the prices and quantities of the best five levels on both the ask and the bid sides. The following is an illustration of the top-5 level LOB:
The following table is a typical example showing the dynamics of the limit order book of the top 5 levels: a market sell order of size 1200, followed by a limit ask order of size 400 at the price 9.08, and then a cancellation of size 23 for limit ask order at the price 9.10.

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<td>1,235</td>
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Bid

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Bid

Table 1.1: A market sell order with size of 1200, and a limit ask order with size of 400 at 9.08 in sequence.
1.1.3 High frequency trading (HFT)

Algorithmic trading, also called automated trading, black-box trading, or algo trading, uses algorithms with pre-programmed execution instructions, which usually account for price, timing, volume, along with many other variables. HFT is considered as a primary form of algorithmic trading in finance (Lin 2014). More specifically, based on sophisticated mathematical models and algorithms, HFT moves in and out of positions in seconds or fractions of a second in the financial market and do not hold portfolios overnight. The executions are completed by computers automatically according to the execution rules that are pre-programmed. As of 2009, it is estimated that HFT accounts for at most 73% of all US equity trading volume (Iati 2009). Generally, HFT firms earn low margins with very high volumes of trading. There are very different opinions toward HFT in both academia and industry. Jones (2013) points out that HFT improves the market liquidity, reduces trading costs, and makes stock prices more efficient. However, concerns on the new risks brought by HFT rises as it was believed to be a critical factor in the May 6, 2010, Flash Crash. Moreover, there are also complaints against HFT by claiming that it takes profits from investors when index funds re-balance their portfolio.

HFT is based on modelling the microstructure of the market and attempts to take statistical arbitrage. In other words, LOB is the main source of data that high frequency traders are trying to understand and utilize. Unlike the traditional quantitative strategies based on price and volume, HFT takes into account all orders received in the LOB, including limit orders, market orders, and cancellations.

1.2 Optimal execution of large transactions

Large institutional investors, such as mutual funds, hedge funds, investment banks, etc., need to execute large portfolio transactions frequently. To them, the execution costs including bid/ask spreads, commission fees, opportunity costs of waiting, price impact from trading, and act as an important role in overall investment performance. Further discussion could be found in Loeb (1983) and Wagner (1993). How to measure and minimize the execution costs for a given portfolio transaction becomes critical to quantitative trading strategies. To purchase a large block of equity, if one places the large
order into the market directly, there will be a significant price impact which increases the execution cost. However, if one splits the large block into small packets and places them sequentially over time, the price impact is reduced while the transaction is exposed to more price uncertainty. Optimal execution models mainly focus on the price impact from large orders and the price uncertainty from the long execution horizon. However, they usually assume placing market orders, which means the execution is guaranteed. In fact, traders also face the trade-off between market orders and limit orders – the former one has higher cost with guaranteed execution while the latter one has lower cost with risk of non-execution.

1.2.1 TWAP and VWAP strategies

Time-weighted average price (TWAP) and Volume-weighted average price (VWAP) strategies are commonly used for executing large orders. The TWAP strategy is relatively simple. It allocates the total target volume uniformly over the given trading period. For example, suppose the agent is going to buy 240,000 shares of Bank of America from 10:00 AM to 2:00 PM, then the TWAP strategy first chooses how many orders to send, say one every 1 minute. Then the agent will send an order to buy 1,000 shares every minute to the exchange. A VWAP strategy is to allocate the total target volume according to the market trading volume for each period. For the previous example, the VWAP strategy first calculates the intra-day volume curve every minute from 10:00 AM to 2:00 PM to get \((V_1, \cdots, V_T), T = 240\). And then for the \(i^{th}\) minute, \(1 \leq i \leq T\), an order to buy \(240,000 \times \frac{V_i}{\sum_{i=1}^{T} V_i}\) shares will be sent to the exchange. Generally speaking, VWAP is less predictable than the TWAP and highly depends on the intra-day market trading volume curve. In practice, the VWAP is widely used as the benchmark for accessing large order execution quality (Madhavan (2002)). Mathematically, the market VWAP is defined as

\[
P^V_M = \frac{\int_0^T P(t) dV(t)}{V(T)},
\]

where \(V(t)\) is the market trading volume up to time \(t\). A strategy \(n = \{n(t), 0 \leq t \leq T\}\) is called a VWAP strategy if there exists a positive constant \(\gamma\), such that \(n(t) = \gamma V(t)\). Here, \(n(t)\) is the holding volume for the agent at
time $t$, and the VWAP of this strategy is defined as

$$P^V(n) = \frac{\int_0^T P(t)dn(t)}{n(T)}$$

(1.2)

It is easy to see that $P^V(n) = P^V_M$ when $n$ is a VWAP strategy. However, in practice, usually the number of shares to purchase, say $N$ shares, is given. Then a VWAP strategy is given by $n(t) = \frac{N}{V(T)} V(t)$. However, $V(T)$, the total market trading volume, is unknown at the beginning. Therefore, this strategy is not applicable in practice. The difference, $P^V(n) - P^V_M$, is called VWAP slippage. Minimizing the VWAP slippage is studied in the literature, for instance, [Konishi (2002), Frei and Westray (2013), and Guéant and Royer (2014)]. Moreover, in a recent paper [Kato (2014)] introduces a trading volume process into the Almgren–Chriss model, which is reviewed later, and shows the VWAP strategy is optimal for a risk-neutral trader.

1.3 Optimal placement problems and LOB dynamics modelling

This thesis is mainly devoted to solve two problems in the LOB.

1.3.1 Optimal placement problems

Optimal placement studies how to place an order in a LOB for optimizing an objective such as minimizing the cost. Given a number of shares to buy or sell, traders must decide between using market orders, limit orders, or both, decide on the number of orders to place at different price levels, and decide on the optimal sequence of order placement in a give time frame with multi-trades. Specifically, when using limit orders, traders do not need to pay the spread and most of the time even get a rebate$^1$. This rebate, however, comes with an execution risk as there is no guarantee of execution for limit orders. On the other hand, when using market orders, one has to pay both the spread between the limit and the market orders and the fee in

$^1$This rebate structure varies from exchange to exchange and leads to different optimization problems. For instance, in the Hong Kong stock exchange, successful executions of limit orders get a discount (i.e., a fixed percentage of the execution price) whereas in other places such as the London stock exchange, the discount may be a fixed amount.
exchange for a guaranteed immediate execution. Essentially, traders have to balance between paying the spread and fees when placing market orders vs. execution/inventory risks when placing limit orders.

Technically, the optimal placement problem can be stated as follows. Consider a setting where \( N \) shares are to be bought by time \( T > 0 \) (\( T \approx 1/5 \) minutes). One may split the \( N \) orders into \((S_{0,t}, S_{1,t}, \cdots)\), where \( S_{0,t} \geq 0 \) is the number of shares placed as the market order at time \( t = 0, 1, \ldots, T \), \( S_{1,t} \) is the number of shares placed at the best bid at time \( t \), \( S_{2,t} \) is the number of shares placed at the second best bid at time \( t \), and so on. If the limit orders are not executed by time \( T \), then one has to buy the non-executed orders at the market price at time \( T \). When one share of the limit order is executed, the market gives a rebate \( r > 0 \) and when a share of market order is submitted, a fee \( f > 0 \) is incurred. Although no intermediate selling is allowed at any time, one nevertheless can cancel any non-executed order and replace it with a new order at a later time. Now, given \( N \) and \( T \), the goal is to find the optimal strategy \((S_{0,t}, S_{1,t}, \cdots, S_{k,t})_{t=0,1,\ldots,T}\) to minimize the overall total expected cost.

1.3.2 LOB dynamics modelling

For the decisions on the optimal placement problem, how to estimate the probabilistic quantities in LOB is essential to the optimal strategies. Therefore, quantitative models for the dynamics of the LOB are needed to calculate quantities including the probability of a price increase, limit order execution, etc., given the current LOB details. Due to the high frequency nature of the LOB, heavy traffic limit is a useful tool for approximating the queueing system of the LOB as well as those quantities of interest. Generally, when people are interested in the evolution of the queueing system over time scales much larger than the interval between arrivals, the dynamics of the time scaled queue lengths could be described in terms of a simpler process, which is called the heavy traffic limit. There are many classical studies on this topic, for instance, Iglehart (1973a), Iglehart (1973b), Borovkov (1976), Harrison (1985), Billingsley (1968), and Whitt (2002). In mathematics, the heavy traffic limits are essentially based on the functional law of a large number (FLLN) and the functional central limit theorem (FCLT).

Technically, let \( Q^b(t) \) and \( Q^a(t) \) denote the best bid queue length and the best ask queue length at time \( t \), respectively. And \( Z(t) \) stands for the position of a particular order the agent placed in the best bid queue. We want
to derive the dynamics of \((Q^b(t), Q^a(t), Z(t))\) and compute the quantities of interest, for example, the probability of \(Z(t)\) hits zero level before \(Q^a(t)\) does so.

### 1.4 Outline of the thesis

Chapter 2 proposes a correlated random walk model for the best ask/bid price and solves the optimal placement problem under both single-step and multi-step settings. Chapter 3 reviews optimal execution models and market making problems, then proposes a simple optimal execution model with a linear price impact, where both market order and limit order are available. In Chapter 4, a queueing model is studied for the dynamics of a particular order and the best bid/ask. Fluid and diffusion limits are derived under different time scaling, and some applications are also presented. Reviews of some mathematical techniques used in my thesis and some proofs are hidden in the Appendix A.
Chapter 2

Optimal placement in an LOB

2.1 Review of the optimal placement models

To the best of my knowledge, there are few analytic results on the optimal placement problem. The most relevant work is from Hult and Kiessling (2010), who develops a high-dimensional Markov chain model for the limit order book and, therefore, the potential order placement strategies.

2.1.1 Hult and Kiessling’s model

Basically, their model is a Markov chain representation of a LOB. Let $\pi^1 < \pi^2 < \cdots < \pi^d$ denote all possible price levels in the LOB, and $X_t = (X_1^t, \cdots, X_d^t)$ represents the volume at time $t > 0$ of bid orders (negative value) and ask orders (positive value) at each price level. Assume that $X_j^t \in \mathbb{Z}$ for each $j = 1, \cdots, d$. Define $S \subset \mathbb{Z}^d$ as the state space of the Markov chain. For any $x \in S$, define $j_B = j_B(x) = \max(j, x_j < 0)$, which is the highest bid level, and $j_A = j_A(x) = \min(j, x_j > 0)$, which is the lowest ask level. Then define $\pi_B = \pi^{j_B}$ and $\pi_A = \pi^{j_A}$ as the best bid price and best ask price. Assume that $x^d > 0$ and $x^1 < 0$, $j_B < j_A$. By definition of $j_B$ and $j_A$, we have $x^j = 0$ for $j_B \leq j \leq j_A$. $j_A - j_B$ is the spread. Let $e^j = (0, 0, \cdots, 1, 0, \cdots, 0)$ denote a vector in $\mathbb{Z}^d$ with 1 in the $j^{th}$ position.

- A limit bid order of size $k$ at level $j$: $x \mapsto x - ke^j$, $j < j_A$,
- A limit ask order of size $k$ at level $j$: $x \mapsto x + ke^j$, $j > j_B$, 


• A market buy order of size $k$ at level $j$: $x \mapsto x - ke^j$, knock out ask order at $j_A$ if $k > x^{j_A}$,

• A market sell order of size $k$ at level $j$: $x \mapsto x + ke^j$, knock out bid order at $j_B$ if $k > x^{j_B}$,

• A cancellation of bid order of size $k$ at level $j$: $x \mapsto x + ke^j$, $j \leq j_B$ and $1 \leq k \leq |x^j|$,

• A cancellation of ask order of size $k$ at level $j$: $x \mapsto x - ke^j$, $j \geq j_A$ and $1 \leq k \leq x^j$.

Let $Q = (Q_{xy})$ denote the generator matrix of $X$, where $Q_{xy}$ is the transition intensity from state $(x^1, \ldots, x^d)$ to state $y = (y^1, \ldots, y^d)$. And let $P = (P_{xy})$ denote the transition matrix of the jump chain of $X$. First, we have

$$P_{xy} = \frac{Q_{xy}}{\sum_{y \neq x} Q_{xy}} \quad (2.1)$$

A Markov chain is completely determined from the initial state and $P$ or $Q$.

**Order execution probability** Now we consider the probability of an order being executed. Define $V_c = 0$ and

$$V_T = \begin{cases} 
1 & \text{if the position hits 0} \\
0 & \text{otherwise}
\end{cases} \quad (2.2)$$

To study the probability of the execution of a particular order, we need to track the position of that order, in addition to the state of the LOB. Suppose the agent put a limit buy order at level $J_0$. Moreover, we focus on the jump Markov chain and use $n \in \mathbb{N}$ as the index. Let $Y_n$ denote the number of orders at level $J_0$ that are in front of the agent’s order at time $n$. The initial condition gives $Y_0 = X_0^{J_0} - 1$ and $Y_n$ can only move up if $J_0$ becomes the best bid level and there is a market sell order, or if an order in front of the agent’s order is canceled. Note that $(X_n, Y_n)_{n \geq 0}$ is also a Markov chain with state space $\mathbb{S} \times \{0, -1, \cdots\}$ with transition matrix $\bar{P}$. We want to derive the probability that an order is executed before the best ask level is at least $J_1 > J_A(x_0)$. Therefore,

$$\partial D = \{(x, y) \in \mathbb{S}, y = 0 \quad \text{or} \quad x^j \leq 0 \quad \text{for all} \ J_0 < j < J_1\}, \quad (2.3)$$
\[ V_T(x, y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2.4) \]

\[ \Phi(s) = \mathbb{E} \left[ V_T(X_\tau, Y_\tau) \mathbb{1}_{\tau < \infty} | (X_0, Y_0) = s \right], \quad (2.5) \]

where \( \mathbb{1} \) is the indicator function.

**Optimal strategy for buying one unit**  Suppose the initial state of the LOB is \( X_0 \), and the agent wants to buy one unit. After \( n \) transitions of the LOB, \( (X_n, Y_n, j_n) \) defines a discrete Markov chain, where \( Y_n \) denotes the number of orders in front of the agent’s order, and \( j_n \) is the level where the agent’s order is posted. The set of possible actions depends on the current state \((x, y, j)\). In each state where \( y < 0 \), the agent has three options:

1. Do nothing and wait for a market transition.
2. Cancel the limit order and place a market order.
3. Cancel the existing limit order and place a new limit bid at level \( J_0 \leq j' < j_A(x) \).

The action set is \( \{-2, -1, 0, J_0, \ldots, j_A(x) - 1\} \), where \(-2\) represents \( y = 0 \) as the Markov chain is terminated; \(-1\) represents the agent cancels the limit order and use a market order to terminate the process; \(0\) represents the agent takes no actions; and \(j', J_0 \leq j' < j_A(x)\), represents the agent cancels the old limit order and submits a new limit order at level \( j' \). The expected buy price \( V(s, \alpha) \) from a state \( s = (x, y, j) \) with \( j_A(x) < J \) following a stationary policy \( \alpha = (\alpha, \alpha, \ldots) \) is

\[
V(s, \alpha) = \begin{cases} 
\sum_{s'} P_{ss'} V(s', \alpha) & \text{for } \alpha(s) = 0, \\
V(s', \alpha) & \text{for } \alpha(s) = j', J_0 \leq j' < j_A(x), \\
\pi^{j_A(x)} & \text{for } \alpha(s) = -1, \\
\pi^{j_B(x)} & \text{for } \alpha(s) = -2,
\end{cases} \quad (2.6)
\]

And it claims that an optimal buy strategy is the stationary policy \( \alpha_\infty \), with the expected buy price \( V_\infty \) satisfying,

\[
V_\infty(s) = \min \left( \sum_{s' \in S'} P_{ss'} V_\infty(s'), V_\infty(s_{J_0}), \ldots, V_\infty(s_{j_A(x)} - 1), \pi^{j_A(x)} \right), \quad (2.7)
\]
for \( j_A(x) < J_1, y < 0 \), and

\[
V(s, \alpha) = \begin{dcases}
\pi^j, & \text{for } j_A(x) = J_1, y < 0, \\
\pi^i, & \text{for } y = 0.
\end{dcases}
\] (2.8)

It would be very difficult to implement the Markov chain model directly in the LOB as the transition intensity matrix is huge. Therefore, further model specification is needed for applications. [Hult and Kiessling (2010)] provides a simple parameterization of the Markov chain for the LOB in order to allow for simple calibration. Assume the following:

- limit buy (sell) orders at a distance of \( i \) levels from the best ask (bid) level intensity \( \lambda^B_L(i) \) (\( \lambda^B_S(i) \)).
- Market buy (sell) orders arrive with intensity \( \lambda^B_M \) (\( \lambda^S_M \)).
- The size of limit and market orders follow discrete exponential distributions with parameters \( \alpha_L \) and \( \alpha_M \), respectively. That is, the distributions \( (p_k)_{k \geq 1} \) and \( (q_k)_{k \geq 1} \) of limit and market order sizes are given by

\[
p_k = (e^{\alpha_L} - 1)e^{-\alpha_L k}, \quad q_k = (e^{\alpha_M} - 1)e^{-\alpha_M k} \quad (2.9)
\]

- The size of cancellation orders is assumed to be 1. Each individual unit size buy (sell) order located at a distance of \( i \) levels from the best ask (bid) level is canceled with a rate \( \lambda^B_C(i) \) (\( \lambda^S_C(i) \)). At the cumulative level, the cancellations of buy (sell) orders at a distance of \( i \) levels from opposite best ask (bid) level arrive with a rate proportional to the volume at the level: \( \lambda^B_C(i)(x^{j_A-i} | (\lambda^S_C(i)(x^{j_B+i})) \).

\( P = (P_{xy}) \) is given as

- Limit order:
  - \( x \mapsto x + e^j k \) for \( j > j_B, k \geq 1 \) with rate \( p_k \lambda^S_L(j - j_B(x)) \)
  - \( x \mapsto x - e^j k \) for \( j < j_A, k \geq 1 \) with rate \( p_k \lambda^B_L(j_B(x) - j) \)

- Market order of size at least 2:
  - \( x \mapsto x + k e^j a \) for \( k \geq 2 \) with rate \( q_k \lambda^S_M \)
\(- x \mapsto x - k e^{jA} \) for \( k \geq 3 \) with rate \( q_k \lambda^R_M \)

- Cancellation:
  - \( x \mapsto x - e^j \) for \( j > j_A(x) \), with rate \( \lambda^S_C(j - j_B(x))x^j \)
  - \( x \mapsto x + e^j \) for \( j < j_B(x) \), with rate \( \lambda^B_C(j - j_A(x))x^j \)

- Market order of unit size and cancellation at the best ask/bid level:
  - \( x \mapsto x + e^{jB} \) with rate \( q_k \lambda^S_M + \lambda^B_C(j_A(x) - j_B(x))x^{jB} \)
  - \( x \mapsto x - e^{jA} \) with rate \( q_k \lambda^B_M + \lambda^S_C(j_A(x) - j_B(x))x^{jA} \)

Calibration of this model amounts to calculating the following parameters:

\[
(\lambda_L, \lambda_C, \lambda^B_M, \lambda^S_M, \alpha_L, \alpha_M).
\] (2.10)

With historical data, we can estimate those parameters by

\[
\lambda^B_L(i) = \frac{N^B_L(i)}{T},
\] (2.11)

where \( N^B_L(i) \) denote the number of limit bid orders arrived at the distance of \( i \) levels from the best ask, and \( T \) is the length of the data collection period. Similarly, we can estimate \( \lambda^S_L, \lambda^B_M, \) and \( \lambda^S_M \). Let \( b^*_i \) denote the number of bid orders at the distance \( i \) levels from the best ask at time \( t \), then

\[
b^*_i = \frac{1}{T} \int_0^T b^*_i dt.
\] (2.12)

and an estimation of the cancellation rate \( \lambda^B_C(i) \) is given by

\[
\lambda^B_C(i) = \frac{N^B_C(i)}{b^*_i T}.
\] (2.13)

And similarly for \( \lambda^S_C \). Let \( \bar{l} \) be the mean limit order size, then the maximum likelihood estimate for \( \alpha_L \) is

\[
\hat{\alpha}_L = \log \frac{\bar{l}}{\bar{l} - 1}.
\] (2.14)

And similarly for \( \alpha_M \).
2.1.2 Cont, Stoikov, and Talreja’s model

Similar to the calibration example in [Hult and Kiessling (2010), Cont et al. (2010)] models the LOB as a multiple queueing system. Assume that there are \( n \) price levels for the LOB, from 1 to \( n \). Let \( X(t) = \{X_1(t), \ldots, X_n(t)\}_{t \geq 0} \) denote the state of the LOB at time \( t \), where \( |X_p(t)| \) is the number of outstanding limit orders at price \( p \), \( 1 \leq p \leq n \). \( X_p(t) < 0 \) means there are \( -X_p(t) \) shares of limit bid orders at price \( p \), while \( X_p(t) > 0 \) means there are \( X_p(t) \) shares of limit ask orders at price \( p \). The best ask price \( p^A(t) = \inf\{p = 1, \ldots, n, X_p(t) > 0\} \land (n + 1) \) and the best bid price \( p^B(t) = \sup\{p = 1, \ldots, n, X_p(t) < 0\} \lor 0 \). Now define \( Q^B_i(t) \) as the queue length at the distance \( i \) below the best ask price,

\[
Q^B_i(t) = \begin{cases} 
X_{p^A(t) - i}(t) & 0 < i < p^A(t) \\
0 & p^A(t) \leq i < n,
\end{cases}
\tag{2.15}

Similarly, \( Q^A_i(t) \) is defined by

\[
Q^A_i(t) = \begin{cases} 
X_{p^B(t) + i}(t) & 0 < i < n - p^B(t) \\
0 & n - p^B(t) \leq i < n,
\end{cases}
\tag{2.16}

Assume the system evolves as follows:

- Limit orders arrive at a distance \( i \) from the opposite best price at independent, exponential times with rate \( \lambda(i) \),
- Market orders arrive at independent, exponential times with rate \( \mu \),
- Cancellations at a distance \( i \) from the opposite best price at a rate proportional to the corresponding queue length,
- The above event arrival processes are all independent,
- All orders and cancellations are of size 1 unit.
Let $x^{p+1} = x \pm (0, \ldots, 1, \ldots, 0)$, where 1 is in the $p^{th}$ component. Then the $\{X_t\}_{t \geq 0}$ is a continuous-time Markov chain with transition rates given by

$$
\begin{align*}
&x \rightarrow x^{p-1} \quad \text{with rate } \lambda(p^A(t) - p) \quad \text{for } p < p^A(t), \\
&x \rightarrow x^{p+1} \quad \text{with rate } \lambda(p - p^B(t)) \quad \text{for } p > p^B(t), \\
&x \rightarrow x^{p^B(t)+1} \quad \text{with rate } \mu, \\
&x \rightarrow x^{p^A(t)-1} \quad \text{with rate } \mu, \\
&x \rightarrow x^{p^A(t)} \quad \text{with rate } \theta(p^A(t) - p) \quad \text{for } p < p^A(t), \\
&x \rightarrow x^{p^B(t)} \quad \text{with rate } \theta(p - p^B(t)) \quad \text{for } p > p^B(t).
\end{align*}
$$

The first main result is

- They show that $\{X_t\}_{t \geq 0}$ is an ergodic Markov chain, which means it has a proper stationary distribution.

Moreover, focusing on the best bid/ask queue, both of them are birth-death processes. Therefore, when the best bid queue is depleted before the ask queue hits zero, then the mid-price moves down, and vice versa. Let $\hat{f}_Y(s) = \int e^{-st} f_Y(t) dt$ be the two-sided Laplace transform of $f_Y(t)$, where $f_Y(t)$ is the probability density function of random variable $Y$. Define

$$
\Phi_n^{\infty} = \frac{a_k}{b_k} = t_1 \circ t_2 \circ \cdots \circ t_n(0), \quad n \geq 1, \quad \tau_k(u) = \frac{a_k}{b_k + u}, \quad k \geq 1,
$$

where $\circ$ denotes the composition of functions. Let $\tau_m$ denote the first-passage time of this process to 0 with starting from $m$ and the spread is $S$. Then, define

$$
\hat{f}_{S_m}^s(s) = \hat{f}_{S_m}^s(s) = \left( -\frac{1}{\lambda} \right)^m \left( \prod_{k=1}^{m} \Phi_k \right) \frac{-\lambda(S)(\mu + k\theta(S))}{\lambda(S) + \mu + k\theta(S) + s},
$$

Then the second main result of this paper is

- Given that $X_{p^A(0)}(0) = a$, $X_{p^B(0)}(0) = b$, and $p^A(0) - p^B(0) = S$. Then the probability that the next price change is an increase is given by the
inverse Laplace transform of

\[ \hat{F}_{a,b}^S(s) = \frac{1}{s} (\hat{f}_{a}^S(\sum_{i=1}^{S-1} \lambda(i) + s) + \frac{\sum_{i=1}^{S-1} \lambda(i)}{\sum_{i=1}^{S-1} \lambda(i)} (1 - \hat{f}_{a}^S(\sum_{i=1}^{S-1} \lambda(i) + s))) \]

\[ \cdot (\hat{f}_{b}^S(\sum_{i=1}^{S-1} \lambda(i) + s) + \frac{\sum_{i=1}^{S-1} \lambda(i)}{\sum_{i=1}^{S-1} \lambda(i)} (1 - \hat{f}_{b}^S(\sum_{i=1}^{S-1} \lambda(i) + s))) \]

(2.17)

When \( S = 1 \), that is, the spread is 1 tick, then (2.17) reduces to

\[ \hat{F}_{a,b}^1(s) = \frac{1}{s} \hat{f}_{a}^1(s) \hat{f}_{b}^1(s). \]  

(2.18)

Now define \( \hat{g}_{m}^S = \prod_{i=1}^{m} \frac{\mu + \theta(S)(i-1)}{\mu + \theta(S)(i-1) + s} \). Then another important result given in this paper is

- Given that \( X_{p^A}(0) = a, p^A(0) - p^B(0) = S \), and there are \( b-1 \) orders at the best bid price in front of the agent’s order. Then the probability that the agent’s order gets executed before the price moves is given by the inverse Laplace transform of

\[ \hat{G}_{a,b}^S(s) = \frac{1}{s} \hat{g}_{b}^S(s) \left( \hat{f}_{a}^S(2 \sum_{i=1}^{S-1} \lambda(i) - s) \right. 

\[ \left. + \frac{2 \sum_{i=1}^{S-1} \lambda(i)}{2 \sum_{i=1}^{S-1} \lambda(i) - s} (1 - \hat{f}_{b}^S(2 \sum_{i=1}^{S-1} \lambda(i) - s)) \right) \]  

(2.19)

### 2.2 Summary of contributions

The optimal placement problem is analyzed with the additional constraint of no price impact of a large trade. With this constraint, it is without loss of generality to assume \( N = 1 \). Both single-step and multi-step trades are considered: the former means that one is only allowed to place the order at time 0, and the latter means that one is free to place orders and cancel and replace existing orders at any step between 0 and \( T \).

The analysis is based on a correlated random walk model with mean-reversion for the bid/ask price. Despite its apparent simplicity, this model is
strong enough to capture several key LOB characteristics such as the mean-reverting nature of algorithmic trading, the depth of the LOB (i.e., the probability that a best bid is executed within a certain time), and a proxy for the LOB imbalance, captured by the ‘initial’ probability of price movement.

Under this model, the optimal strategy for single-step trades is proved to involve placing orders only at three levels: the second best bid, the best bid, and the market order. This result significantly reduces the complexity and dimensionality for the optimal placement problem. It is also consistent with the empirical work in Cont and De Larrard (2013), which shows that most of the trading activities are concentrated at the top two levels of the LOB.

The optimal placement strategy for multi-step trades is shown to be a threshold type, with two thresholds that can be explicitly computed. As time goes by, the optimal strategy shifts from aggressive types to conservative types as illustrated in Figure 1. This particular behavior is intuitive given the Markovian structure and the mean-reverting nature of the underlying price model. It also provides some useful insight, albeit intuitive and may have been well used in practice: (i) as the LOB becomes more unbalanced, the optimal trading strategy shifts from the market order to the lower level of limit bid order (i.e., first the best bid, then to the second best bid); (ii) in the degenerate case of a simple symmetric random walk model, the optimal placement strategy is to always follow the best bid; and (iii) as the transaction cost between the market order and the limit order decreases, the optimal trading strategy moves toward the market order.

Technically speaking, the analysis with the correlated random walk model is harder than one would expect. The main difficulty is to establish the partial reflection principle and a certain monotonicity property for its running maximal process, which is critical for the dimension reduction: instead of comparing the expected cost at all levels, which is infeasible, we show that the critical ones are the top three levels for the one-step case. From here, a straightforward application of the Markov decision analysis via cumbersome calculations yields the threshold type optimal trading strategy for the multi-step case.

The explicit solution structure for the optimal placement problem under the simple correlated random walk model potentially paves the way for a better structural understanding of the more complex market-making problem. At the very least, these explicit solutions can be useful for comparing various numerical solutions under much more sophisticated models. We also hope that beyond analyzing the optimal placement problem, the technique
we develop in analyzing the correlated random walk may be of independent mathematical interest. The "reflection principle" for random walk or Brownian motion is one of the most well-known results in probability theory. This principle was initially introduced by Feller as a combinatorial trick for counting and comparing sample paths. The "mapping" technique exploited in our work differs from the scaling technique via its symmetry for the Brownian motion or Lévy process by Bayraktar and Nadtochiy (2013), though still in the spirit of Feller.

2.3 A correlated random walk model

The model. To analyze the optimal placement problem, a correlated random walk model is first proposed for the dynamics of the bid/ask price. In this model, we will assume that

1. the spread between the best bid price and the best ask price is always 1 tick;
2. the best ask price increases or decreases 1 tick at each time step \( t = 0, 1, \ldots, T \).

Let \( A_t \) be the best ask price at time \( t \). Then,

\[
A_t = \sum_{i=1}^{t} X_i, \quad A_0 = 0, \tag{2.20}
\]

where \( X_t \) \((1 \leq t \leq T)\) is a Markov chain on \( \{\pm 1\} \) with \( \mathbb{P}(X_1 = 1) = \bar{p} = 1 - \mathbb{P}(X_1 = -1) \) for some \( \bar{p} \in [0, 1] \), and for \( i = 1, \ldots, T - 1 \),

\[
\mathbb{P}(X_{i+1} = 1 \mid X_i = 1) = \mathbb{P}(X_{i+1} = -1 \mid X_i = -1) = p < \frac{1}{2}. \tag{2.21}
\]

Such a model is also called a correlated random walk model. (For earlier work on this type of model, see Goldstein (1951), Mohan (1955), Gillis (1955), and Renshaw and Henderson (1981).) The particular choice of \( p < \frac{1}{2} \) makes the price “mean revert”, a phenomenon often observed in high-frequency trading; see, for instance, Chen et al. (2013) for some statistical analysis on this phenomenon.
In order to analyze the optimal placement problem under this model, it is critical to characterize $Y_t$ ($0 \leq t \leq T$), where

$$Y_t = \min_{0 \leq s \leq t} A_s. \tag{2.22}$$

In the subsequent analysis, when there is little risk of confusion, we call a particular sample path $\omega$ instead of $\{X_t(\omega)\}_{1 \leq t \leq T}$. Consistent with this convention, $A_t(\omega)$ is the (best) ask price on a sample path $\omega$ at time $t$, $X_t(\omega)$ is the price change at the $t$th step of a particular sample path $\omega$, and $Y_T(\omega)$ is the lowest level that a particular sample path $\omega$ has ever hit by time $T$.

### 2.3.1 Probability distribution of the best ask price and running minimum price

It is easy to see from (2.21) that $\{(A_i, A_{i-1})\}_{i \geq 1}$ is a two-dimensional Markov chain. Compared to the simple random walk, the key to analyzing $A_t$ is to differentiate the sequences with different number of direction changes when they have the same number of “upward” edges (i.e., with the same value for $A_T$). We call it a direction change when a sequence of 1’s is followed by a $-1$ or when a sequence of $-1$’s is followed by a 1. For instance, both 1, 1, 1, $-1$, 1 and 1, 1, $-1$, 1, 1, 1, 1 have four 1’s (upward edges) and two $-1$’s (downward edges), yet the former one has two direction changes while the latter has four direction changes.

It is easy to see that if the numbers of 1’s and $-1$’s and the position of the direction changes are given, then the sequence is determined as long as the first edge is also given. For instance, if $T = 5$, $A_T = 1$, $X_1 = 1$, and the number of direction changes is 2, then we need to know the positions of the direction changes in order to identify the sequence from the two possible choices: 1, 1, $-1$, 1, 1 or 1, $-1$, 1, 1, 1.

A bit of thinking with some calculations yields

$$\mathbb{P}(A_T = k \text{ and number of direction change is } i) = \begin{cases} p^{T-i-1}(1-p)^i L_{T,k}, & \frac{T+k}{2} \in \mathbb{N} \text{ and } |k| \leq T, \\ 0, & \text{otherwise}, \end{cases} \tag{2.23}$$
where
\[ L_{T,k}^i = (1 - \bar{p}) \left( \frac{T + k}{2} - 1 \right) \left( \frac{T - k}{2} - 1 \right) + \bar{p} \left( \frac{T + k}{2} - 1 \right) \left( \frac{T - k}{2} - 1 \right) \]

From which, it is clear to see

**Proposition 2.1.**
\[
P(A_T = k) = \begin{cases} 
T - |k| 
\sum_{i=1}^{\frac{T-|k|}{2}} p^{T-i-1}(1 - p)^i L_{T,k}^i, & \text{if } \frac{T + k}{2} \in \mathbb{N} \text{ and } |k| \leq T, \\
0, & \text{otherwise.}
\end{cases}
\]

(2.25)

Using this, we obtain the following.

**Theorem 2.1 (Partial Reflection Principle).** If \( \frac{T - k}{2} \in \mathbb{N} \) and \( k > 0 \), then \( P(Y_T = -k) = P(A_T = -k) \).

Next, we show that the distribution of \( Y_t \) satisfies a monotone property, which seems intuitively clear although its proof is not that obvious. Note the “non-intuitive” part comes from \( P(Y_T = -k) \), which is different from \( P(Y_T \leq -k) \).

**Proposition 2.2.** \( P(Y_T = -k) \) is a decreasing function of \( k \) for \( k = 1, \ldots, T \).

This proposition turns out to be critical for our analysis of the optimal placement problem.
2.4 Optimal strategy for the placement problem

2.4.1 Single-step trade

We now consider the optimal placement with a single-step trade, where the investor needs to buy one share by time $T$. The investor decides where to place her buy order only at time $t = 0$ and could not change until $t = T$. If the order is placed as a limit order and it is not executed before time $t = T$, then she has to buy with a market order at time $t = T$.

In addition to our correlated random walk model, we need to specify the following:

- if $A_t \leq -k$ for some $t \leq T$, then a limit order at price $-k$ will be executed with probability 1;
- if $A_t > -k + 1$ for all $t \leq T$, then a limit order at price $-k$ will be executed with probability 0;
- If $A_t = -k + 1$ and $A_{t+1} = -k + 2$, then a limit order at price $-k$ has a chance $q$ to be executed between $t$ and $t + 1$.

Based on the above assumptions, if $Y_T(\omega) = -k + 1$ for a particular sample path $\omega$, then we have

$$q_k(\omega) = 1 - (1 - q)^{n_k(\omega)},$$

where $q_k(\omega)$ denotes the probability of a limit order placed at $-k$ being executed along that particular path $\omega$, and $n_k(\omega)$ is the number of times $A_t(\omega) = -k + 1$ strictly before $T$. Note that with $q = 0$, there is no chance for a limit order placed at $-k$ to be executed when $Y_T = -k + 1$; when $q = 1$, a limit order placed at $-k$ is guaranteed for execution when $Y_T = -k + 1$.

In addition, at time $T$, if there is a non-executed order at $A_T - 1$, this order will never be executed. Hence it suffices to count $n_k(\omega)$. Moreover, $q_k(\omega)$ increases as $n_k(\omega)$ increases, meaning that the longer a limit order stays at the best bid queue, the higher its chance of being executed.

Comparing expected costs at each level of LOB. Based on the above setup, solving the optimal placement problem amounts to comparing the
expected costs of placing one order at each level of the LOB. Clearly, the expected cost of an order placed at any price level depends on all the parameters $f, r, T, p, q, \bar{p}$. For simplicity, however, we will highlight only variables $k, q, T, \bar{p}$ to show the dependence of the cost on those variables. In particular, the expected cost of a limit order placed at the price $k$ ticks lower than the initial best ask price, given $q$ and the total number of price changes $T$ as well as the probability of the first price change being upward $\bar{p}$, is denoted by $C(k, q, T, \bar{p})$. Here $C(0, q, T, \bar{p})$ is the expected cost of a market order. Moreover, since all limit orders placed below $-T - 1$ will not be executed until $T$ and will have to be filled by market orders at time $T$, their expected costs are the same and will be denoted by $C(T + 1, q, T, \bar{p})$. That is, it suffices to consider the following minimization problem:

$$\min_{0 \leq k \leq T + 1} C(k, q, T, \bar{p}) \quad (= \min_{0 \leq k \leq \infty} C(k, q, T, \bar{p})).$$

It is easy to see

**Lemma 2.1.** For any $1 \leq k \leq T + 1$,

$$C(k, q, T, \bar{p}) = \sum_{\omega \in \{Y_T = -k+1\}} \mathbb{P}(\omega)q_k(\omega)(-k - r - A_T(\omega) - f) + C(k, 0, T, \bar{p}),$$

(2.27)

with $C(0, q, T, \bar{p}) = f$.

Since $A_T(\omega) \geq Y_T(\omega) = -k + 1$ for $\omega \in \{Y_T = -k + 1\}$, we have from the above lemma that $-k - r - f - A_T(\omega) < 0$. Note that $q_k(\omega)$ is an increasing function of $q$, we have the following:

**Lemma 2.2.** $C(k, q, T, \bar{p})$ is a decreasing function of $q$, i.e., if $0 \leq q_1 < q_2 \leq 1$, then

$$C(k, q_1, T, \bar{p}) > C(k, q_2, T, \bar{p}).$$

This result is quite intuitive: as the chance of execution increases for a limit order, its expected cost should decrease. And we also have the following result that leads to a partial order of the expected costs at different bid levels.

**Lemma 2.3.** For $1 \leq k \leq T - 2$,

$$\mathbb{P}(Y_T = -k)(k + \mathbb{E}[A_T|Y_T = -k])$$
$$\geq \mathbb{P}(Y_T = -k - 1)(k + 2 + \mathbb{E}[A_T|Y_T = -k - 1]).$$

(2.28)
Lemma 2.2 and Lemma 2.3 lead to the following.

**Proposition 2.3.** If \( q = 0 \), then for any given \( r, f, T, p, \) and \( \bar{p} \),

\[
C(1, 0, T, \bar{p}) < C(2, 0, T, \bar{p}) < \cdots < C(T + 1, 0, T, \bar{p}),
\]

i.e., if \( q = 0 \), then the best bid order is better than any other level of limit orders.

It is easy to see that the above proposition does not hold for a general \( q \). For example, \( C(1, q, T, \bar{p}) < C(2, q, T, \bar{p}) \) does not necessarily hold: if \( \bar{p} = 0 \) and \( q = 1 \), then the limit order placed at the level of \(-2\) will also be executed and is strictly better than placing an order at the level of \(-1\). However, when \( k \geq 2 \), due to the mean-reversion property of the price process, the order placed at a lower price level will have a lower chance of being executed, which leads to a partial order for the expected costs at different levels, as stated in the following.

**Proposition 2.4.** Given \( r, f, T, p, \) and \( \bar{p} \),

\[
C(2, q, T, \bar{p}) < C(3, q, T, \bar{p}) < \cdots < C(T + 1, q, T, \bar{p}),
\]

for general \( q \neq 0 \). In particular, an optimal placement strategy depends only on comparing the expected costs at three levels: \( C(0, q, T, \bar{p}) \) for the market order, \( C(1, q, T, \bar{p}) \) for the best bid, and \( C(2, q, T, \bar{p}) \) for the second best bid.

Comparison among \( C(0, q, T, \bar{p}) \), \( C(1, q, T, \bar{p}) \), and \( C(2, q, T, \bar{p}) \) can be more explicit with more information on \( \bar{p} \).

**Proposition 2.5.** For fixed values of \( r, f, p, q, \) \( C(1, q, T, \bar{p}) \) and \( C(2, q, T, \bar{p}) \) are both increasing linear functions of \( \bar{p} \).

Since both \( C(1, q, T, \bar{p}) \) and \( C(1, q, T, \bar{p}) \) are increasing linear functions of \( \bar{p} \), and \( C(0, q, T, \bar{p}) \) is a constant, the optimal placement strategy will be of a threshold type when focusing only on the parameter \( \bar{p} \). Depending on the intersections of \( C(0, q, T, \bar{p}) \), \( C(1, q, T, \bar{p}) \), \( C(2, q, T, \bar{p}) \), (some of which may lie beyond \([0, 1]\), the decision to use market order or limit order will depend
on at most two of the three intersections $\bar{p}_1^*, \bar{p}_2^*, \bar{p}_3^*$, which are given by

\[
\begin{align*}
\bar{p}_1^* &= \frac{r + f + 1}{(2 + r + C(2, q, T - 1, p))(1 - q)}, \\
\bar{p}_2^* &= \frac{1 + f - C(1, q, T - 1, 1 - p)}{2 + C(3, q, T - 1, p) - C(1, q, T - 1, 1 - p)}, \\
\bar{p}_3^* &= \frac{r + C(1, q, T - 1, 1 - p)}{(2 + r + C(2, q, T - 1, p))(1 - q) - (2 + C(3, q, T - 1, p) - C(1, q, T - 1, 1 - p))},
\end{align*}
\]

where $\bar{p}_1^*$ is the intersection of $C(1, q, T, \bar{p})$ and $C(0, q, T, \bar{p})$, $\bar{p}_2^*$ is the intersection of $C(2, q, T, \bar{p})$ and $C(0, q, T, \bar{p})$, and $\bar{p}_3^*$ is the intersection of $C(1, q, T, \bar{p})$ and $C(2, q, T, \bar{p})$. In particular, if all of them are outside $[0, 1]$, then it means we will stay with only 1 order for all $\bar{p} \in [0, 1]$. (See Figures 1 below.)

(a) Two thresholds

(b) One threshold

Figure 2.1: Expected costs against different $\bar{p}$. 
Figure 2.1: Expected costs against different $\bar{p}$.

**Proposition 2.6.** If $\bar{p} \geq 1 - p$, then $C(1, q, T, \bar{p}) < C(2, q, T, \bar{p})$, i.e., the optimal placement strategy involves only the market order and the best bid order.

### 2.4.2 Multi-step trade

Next we continue our analysis with a multi-step trade, where we assume that an investor needs to get one share of stock by time $T$ and she is allowed to place an order at any level at any time $t$ between 0 and $T$. Consistent with the single-step case, we make the following assumptions:

- a limit bid order placed at price of $A_t - 1$ will be executed with probability 1 if $A_{t+1} = A_t - 1$, and will be executed with probability $q$ if $A_{t+1} = A_t + 1$;
- if the order does not get executed by time $T$, she needs to cancel the previous order and use the market order at time $T$;
- after each price change at each time step $1, \ldots, T - 1$, she can do nothing (denote this action as $NO$) or cancel the non-executed order and replace it with a market order (denote this action as $MO$) or replace it with the best bid (denote this action as $BB$).

Note that due to these assumptions, among all limit orders, only the best bid will be used at any time $t$. The objective of the optimal placement problem is then to calculate the expected cost for using each action at each step and then determine the action with the lowest expected cost. This can be solved quite straightforwardly using Markov decision theory.
Optimal strategy for multi-step trading

A review of Markov decision process is provided in the appendix. Note that 
\{(A_t, X_t)\}_{1 \leq t \leq T} is a Markov chain. At each time \(t\), we can take actions from the set \(A = \{NO, BB, MO\}\). Since all the transactions are made in a short time, we do not discount the value function in the future. Moreover, let \(V_t((A_t, X_t), \alpha)\) be the expected cost for purchasing one share of stock by time \(T\) with policy \(\alpha\) at time \(t\). At time \(T\), we have

\[
V_T((A_T, X_T), \alpha) = A_T + f
\]
as we could only use market order at time \(T\). By symmetry, \(V_t((A_t, X_t), \alpha) = V_t((0, X_t), \alpha) + A_t\). Therefore, to simplify notation, we use \(V_t(X_t, \alpha)\) for \(V_t((0, X_t), \alpha)\).

Now denote the expected cost with adopting the optimal policy by

\[
\begin{align*}
V_t^*(x_t) &:= \min_{a \in A} \{V_t(x_t, a)\}, \\
\alpha_t^*(x_t) &= \arg \min_{a \in A} \{V_t(x_t, a)\}, \\
\alpha_T^*(x_T) &= MO, \\
V_T^*(x_T) &= f
\end{align*}
\]

(2.29)

Then, DP leads to the following backward recursion: for \(1 \leq t < T\),

\[
\begin{align*}
V_t(1, BB) &= p(1 - q)(V_{t+1}(1) + 1) + (1 - p + pq)(-1 - r), \\
V_t(1, NO) &= pV_{t+1}^*(1) + (1 - p)V_{t+1}^*(1) - 1 + 2p, \\
V_t(-1, BB) &= (1 - p - q + pq)(V_{t+1}^*(1) + 1) + (p + q - pq)(-1 - r), \\
V_t(-1, NO) &= pV_{t+1}^*(1) + (1 - p)V_{t+1}^*(1) + 1 - 2p, \\
V_t(1, MO) &= V_t(-1, MO) = f.
\end{align*}
\]

(2.30)

Let \(\alpha^t = (\alpha_t, \alpha_{t+1}, ..., \alpha_T)\) be a truncated policy starting at time \(t\). It is clear that at time \(t - 1\), \(\alpha^{t-1} = (\alpha^t, MO)\) gives \(V_{t-1}(x_{t-1}, \alpha^{t-1}) = V_t^*(x_t) = V_t^*(x_{t-1}).\) That is, \(V_{t-1}^*(1) \leq V_t^*(1) \) and \(V_{t-1}^*(-1) \leq V_t^*(-1).\) Therefore we have the monotonicity for \(V_t^*(1)\) and \(V_t^*(-1).\)

**Proposition 2.7.** Both \(V_t^*(1)\) and \(V_t^*(-1)\) are non-decreasing functions of \(t\).

To derive an analytic result for \(V_t^*(1)\) and \(V_t^*(-1)\), we first state the following lemma.

**Lemma 2.4.** For any \(t, 1 \leq t \leq T - 1\), the following inequalities always hold:

\[
\begin{align*}
f &\geq V_t^*(1) > -r - 2 + \frac{p}{(1 - p)(p + q - pq)}, \\
V_t(1, BB) &< V_t(1, MO), \\
V_t(-1, BB) &< V_t(-1, NO).
\end{align*}
\]

(2.31)
That is, if $X_t = 1$, then one should either wait or use the best bid order; if $X_t = -1$, then one should use either market order or the best bid.

Now, given the Markov property of $(A_t, A_{t-1})$ or equivalently that of $(A_t, X_t)$, a threshold-type optimal strategy is expected. Indeed, we can derive explicitly the optimal strategy and the corresponding value function.

**Theorem 2.2 (Optimal policy).** There exist two integers $t_1^*, t_2^*$ with $t_1^* < t_2^*$ such that

- for any $t < t_1^*$, $\alpha_t^*(-1) = BB$, $\alpha_t^*(1) = NO$;
- for any $t_1^* \leq t < t_2^*$, $\alpha_t^*(-1) = \alpha_t^*(1) = BB$;
- for any $t_2^* \leq t$, $\alpha_t^*(-1) = MO$, $\alpha_t^*(1) = BB$.

Here,

$$t_1^* = T - \left[ \frac{1}{\log(p - pq)} \cdot \log \left( \frac{q(1 - 2p)}{(1 - p + pq)(f + 2 + r) - 1}(1 - q)(p^2q + 1 + p^2 - 2p) \right) \right] - 1, \quad (2.32)$$

and

$$t_2^* = T - \left[ \frac{1}{\log(p - pq)} \cdot \log \left( \frac{(f + r + 1)(1 - p + pq) - (1 - p)(1 - q)}{(1 - p)(1 - q)((f + 2 + r)(1 - p + pq) - 1)} \right) \right]. \quad (2.33)$$

(See Figure 2.2.) Moreover, one has explicit expressions for $V_t^*(1)$ and $V_t^*(-1)$ ($t \leq T$) as follows:

$$V_t^*(1) =$$

$$\begin{cases} 
(p - pq)^{(T-t)} \left( f + 2 + r - \frac{1}{1 - p + pq} \right) - 2 - r + \frac{1}{1 - p + pq}, & t_1^* \leq t \leq T, \\
A \cdot \left( \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2} \right)^{t_1^* - t + 1} + B \cdot \left( \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2} \right)^{t_1^* - t + 1} + a_4, & t < t_1^*, 
\end{cases}$$

(2.34)

where $a_1 = p$, $a_2 = (1 - p)(1 - p - q + pq)$, $a_3 = 2p - 1 - (1 + r)(1 - p)(p + q - pq) + (1 - p)(1 - p - q + pq)$, $a_4 = \frac{a_3}{1 - a_1 - a_2}$, and coefficients $A$ and $B$ are given by the following linear equation system:

$$A + B = V_{t_1^* + 1}(1) - a_4,$$

$$A \cdot \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2} + B \cdot \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2} = V_{t_1^*}(1) - a_4.$$  

(2.35)
$$V_t^*(-1) = \begin{cases} f, & \text{for } t \geq t_2^*, \\ (p + (1-p)q)(-1 - r) + (1-p)(1 - q)(1 + V_{t+1}^*(1)), & \text{for } t < t_2^*, \end{cases}$$ (2.36)

Note that both $t_1^*$ and $t_2^*$ could be negative in these expressions. This is hardly surprising since what really matters here is the remaining time until $T$, i.e., $T - t$. At $T$, both $t_1^*$ and $t_2^*$ could then be negative.

![Diagram](image)

Figure 2.2: Illustration of the optimal trading strategy for $1 < t \leq T$

Given the expressions for $V_t^*(1)$ and $V_t^*(-1)$ for $t \leq T$, and the probability distribution of the first step $X_1$ without given $X_0$, the expected costs at time 0 as well as the optimal placement strategy can be derived in terms of $\bar{p}$.

**Corollary 2.1.**

$$\text{cost}^L = (1 - \bar{p} + \bar{p}q)(-1 - r) + (\bar{p} - \bar{p}q)(1 + V_1^*(1)) = \bar{p}(1 - q)(2 + r + V_1^*(1)) - 1 - r,$$ (2.37)
$$\text{cost}^N = (1 - \bar{p})(-1 + V_1^*(-1)) + \bar{p}(1 + V_1^*(1)) = \bar{p}(V_1^*(1) + 2 - V_1^*(-1)) - 1 + V_1^*(-1),$$ (2.38)
$$\text{cost}^M = f,$$ (2.39)

where $\text{cost}^L$, $\text{cost}^N$, $\text{cost}^M$ are the expected costs for purchasing 1 share if taking $BB$, $NO$, $MO$ at $t = 0$, respectively. The optimal strategy at time 0 is a threshold type, based on comparison among these three expressions.
Simple calculations show that cost $L$ and cost $N$ are linear functions with respect to $\bar{p}$ with positive first order coefficients. Since cost $M$ is a constant, the optimal placement strategy will be of a threshold type as we derived in the single step model, when focusing only on the parameter $\bar{p}$. Depending on the intersections of cost $L$, cost $N$, and cost $M$, (some of which may lie beyond $(0, 1)$,) the decision to use market order, the best bid order, or to wait, will depend on at most two of the three intersections $\bar{p}_1^*, \bar{p}_2^*, \bar{p}_3^*$, which are given by

$$\bar{p}_1^* = \frac{f + r + 1}{(1 - q)(2 + r + V_1^*(1))},$$
$$\bar{p}_2^* = \frac{f + 1 - V_1^*(-1)}{V_1^*(1) + 2 - V_1^*(-1)},$$
$$\bar{p}_3^* = \frac{V_1^*(-1) + r}{V_1^*(-1) + (1 - q)r - 2q - qV_1^*(1)},$$

where $\bar{p}_1^*$ is the intersection of cost $L$ and cost $M$, $\bar{p}_2^*$ is the intersection of cost $N$ and cost $M$, and $\bar{p}_3^*$ is the intersection of cost $N$ and cost $L$. In particular, if all of them are outside $[0, 1]$, then it means we will stay with only one order for all $\bar{p} \in [0, 1]$. Moreover, if $\bar{p} = 0$, then placing the best bid order is always better than market order and if $\bar{p} = 1$, then placing the best bid order is always better than to wait.

Note that if $T = 1$, then both the single-step model and multi-step model yield the same optimal trading strategy: use the best bid if $\bar{p} < \frac{r + f + 1}{(r + f + 2)(1 - q)}$ and use the market order otherwise.
Chapter 3

A model with price impact

In the last chapter, the optimal placement problem is studied with the assumption of no price impact from any market or limit orders. In this chapter, a martingale price model with an additive price impact is proposed. Moreover, the choices are restricted to the best bid orders and the market orders. Price impact is usually studied in the optimal execution models, as reviewed in the following section.

3.1 Review of optimal execution models

The basic task for the agent is to buy (sell) a large block of shares of stock over a given period. And the essentials of the optimal execution problem are how to model the price impact from the orders placed by the particular trader and what objective function to optimize. Intuitively, the price might be driven up fast if the trader places orders too fast, and one might lose opportunities and bear larger uncertainty if the trader places orders too slow.

3.1.1 Bertsimas and Lo’s model

Bertsimas and Lo (1998) propose an optimal execution model which minimizes the total expected trading cost when purchasing a large block of equity over a fixed time horizon. Price impact considered in their model leads to the optimal trading strategy which splits the volume as much as possible. Under a variety of circumstances, they showed that an optimal execution strategy could be derived from the dynamic programming principle.

The model is described as follows. Let \((0, 1, \cdots, T)\) be a sequence of time points where the trader places orders and \(N\) denote the number of shares the trader needs to purchase over time \([0, T]\). Assume that the price at time \(t\) is \(P_t\)
and the trader buys $S_t$ shares during the period $t$. Then the objective here is to minimize the expected total cost:

$$\min_{S_1, \ldots, S_T} \mathbb{E}\left[ \sum_{t=1}^{T} P_t S_t \right]$$

subject to $\sum_{t=0}^{T} S_t = N$ & dynamics of $\{P_t\}$

Note that if no intermediate selling is allowed, it is equivalent to add an additional constraint: $S_t \geq 0$, for $1 \leq t \leq T$.

The model for $\{P_t\}_{t \geq 1}$ is as follows. For $1 \leq t \leq T$,

$$P_t = P_{t-1} + \epsilon_t + \theta S_t$$

$$= P_0 + \sum_{i=1}^{t} \epsilon_i + \theta \sum_{i=1}^{t} S_t$$

where $\theta > 0$ represents the price impact from the particular trader, and $\{\epsilon_t\}_{1 \leq t \leq T}$ is an sequence of i.i.d. random variables with $\mathbb{E}[\epsilon_t|S_t, P_{t-1}]=0$. From the expression, they actually made the following assumptions:

- Without the particular trader’s order, the price dynamics itself is a martingale with independent price increments.
- The price impact from the particular trader is permanent.

Dynamic programming principle (DPP) is applied to solve this problem. Let $W_t = W_{t-1} - S_{t-1}$ denote the number of shares left to purchase at time $t$, with $W_1 = N$ and $W_{T+1} = 0$.

- $P_{t-1}, W_t$ are “State variables” at time $t$, containing all information to make decisions for the next period.
- $S_t$ is the decision variable at time $t$.

Suppose $\{S^*_1, \ldots, S^*_T\}$ is an optimal solution to (3.1), then $\{S^*_1, \ldots, S^*_T\}$ is still optimal for $\mathbb{E}_t[\sum_{k=t}^{T} P_k S_k]$. The Bellman equation could be formulated as follows. Let $V_t(P_{t-1}, W_t)$ denote the optimal expected cost at time $t$ with $W_t$ to purchase and previous price $P_{t-1}$, then for $t = T$,

$$V_T(P_{T-1}, W_T) = W_T(P_{T-1} + \theta W_T)$$

(3.3)
as we have no choice but \( S_T = W_T \). For a general \( t < T \),

\[
V_t(P_{t-1}, W_t) = \min_{S_t} \mathbb{E}[P_t S_t + V_{t+1}(P_{t+1}, W_{t+1})],
\]

(3.4)

where \( W_{t+1} = W_t - S_t \).

Now, observe the next-to-last period \( T - 1 \), the Bellman equation becomes

\[
V_{T-1}(P_{T-2}, W_{T-1}) = \min_{S_{T-1}} \mathbb{E}_{T-1}[P_{T-1} S_{T-1} + V_T(P_{T-1}, W_T)]
\]

\[
= \theta S_{T-1}^2 - \theta W_{T-1} S_{T-1} + \theta W_{T-1}^2 + P_{T-2} W_{T-1}.
\]

This is a convex and quadratic function w.r.t. \( S_{T-1} \). Take derivative w.r.t. \( S_{T-1} \) and set it to 0, we have

\[
S^*_{T-1} = \frac{W_{T-1}}{2}
\]

and

\[
V_{T-1}(P_{T-2}, W_{T-1}) = \frac{3}{4} \theta W_{T-1}^2 + P_{T-2} W_{T-1}.
\]

(3.5)

Continue this fashion, the optimal value function \( V_{T-k}(P_{T-k-1}, W_{T-k}) \) could be obtained recursively,

\[
S^*_{T-k} = \frac{W_{T-k}}{k+1}
\]

(3.6)

\[
V_{T-k}(P_{T-k-1}, W_{T-k}) = W_{T-k}(P_{T-k-1} + \frac{k+2}{2(k+1)} \theta W_{T-k}).
\]

(3.7)

Moreover, take \( k = T - 1 \), we have

\[
S^*_1 = \frac{W_1}{T} = \frac{N}{T}
\]

(3.8)

\[
V_1(P_0, W_1) = W_1(P_0 + \frac{T+1}{2T} \theta W_1) = P_0 N + \frac{\theta N}{2}(1 + \frac{1}{T})
\]

(3.9)

**Discussions** There are variants of Bertsimas and Lo (1998). One of them considers market condition/private information as follows:

\[
P_t = P_{t-1} + \theta S_t + \gamma X_t + \epsilon_t
\]

\[
X_t = \rho X_{t-1} + \eta_t \quad \rho \in (-1, 1)
\]

(3.10)

where \( \{\epsilon_t\}_{t \geq 1} \) and \( \{\eta_t\}_{t \geq 1} \) are two sequences of i.i.d. random variables with 0 mean and finite variance, and they are mutually independent as well. \( \{X_t\}_{t \geq 1} \) represents the market condition/private information. Similar calculation leads to the optimal solution:

\[
S^*_{T-k} = \delta_{W,k} W_{T-k} + \delta_{X,k} X_{T-k},
\]

(3.11)
where
\[ \delta_{W,k} = \frac{1}{1 + k}, \quad \delta_{X,k} = \frac{\rho b_{k-1}}{a_{k-1}}, \tag{3.12} \]
\[ a_k = \frac{\theta}{2} \left( 1 + \frac{1}{k + 1} \right) \quad (\rho = 0 \text{ implies } \delta_{X,k} = 0), \tag{3.13} \]
\[ b_k = \gamma + \frac{\theta \rho b_{k-1}}{2a_{k-1}} (> 0, \text{ if } \theta, \rho, \gamma > 0), \tag{3.14} \]

Essentially, the optimal trading strategy is “deterministic” (independent of \( P_t \)), with adjustment for market conditions.

Almgren and Chriss (2001) extend this model with considering the variance of the total trading cost as a component of the objective function. Moreover, they allow permanent price impact as well as temporary price impact for the trader’s order to the price dynamics of the underlying asset. There are further studies based this model, for instance, Almgren and Lorenz (2007) develop adaptive strategies, while Brunnermeier and Pedersen (2005), Carlin et al. (2007) introduce predatory trading strategies between two parties.

These models start with assumptions on the dynamics of the underlying asset price and the impact from the particular trader’s order and then formulate the objective functions. The LOB information does not appear in those studies. Obizhaeva and Wang (2013) propose an optimal execution model with the dynamics of supply/demand from the limit order book.

A lot of further generalizations inspired by Obizhaeva and Wang (2013) have been done later on. For instance, Alfonsi et al. (2008), Alfonsi et al. (2010), Alfonsi and Schied (2010), and Predoiu et al. (2011) develop models with more general settings on the shape functions for the density of the limit order book. Moreover, permanent price impact, as well as transient price impact are studied in the literature.

Alfonsi and Blanc (2014) use Hawkes processes for modelling market buy and sell orders issued by other traders, which enables them to solve the optimal execution problem explicitly. Moreover, a necessary and sufficient condition on the parameters of the Hawkes model is derived to rule out high-frequency arbitrages.

There are many studies on the optimal execution problem focusing on modelling the price impact and optimizing the total execution cost in different ways. Gatheral and Schied (2013) provide a detailed survey on price impact models with associated optimal execution strategies.
3.1.2 Optimal execution with market orders and limit orders

In the above studies, only market orders are used as the execution risk is not considered. Cartea and Jaimungal (2014) develop optimal execution strategies with both market orders and limit orders, which is similar to market making models with the micro-structure of the LOB. In this model, it is assumed that

- The mid-price of the underlying asset follows:
  \[ P_t = P_0 + \sigma (Z_t^+ - Z_t^-), \]  
  (3.15)
  where \( \sigma > 0 \) is the tick size and \( Z_t^\pm \) are mutually independent Poisson processes both with intensity \( \gamma \).

- At any time \( t \), there is always one unit of the limit order posted in the LOB at \( P_t - \delta_t \). \( \delta_t \) is a control variable that determines how deep the limit order is placed. Meanwhile, market orders will be executed at \( P_t + \Delta + f \), where \( \Delta \) is half of the spread and \( f \) denotes the transaction cost. Note that there is no price impact

- Between time \( t \) and \( t + dt \), the limit order is filled with probability \(-\kappa \delta_t\), where \( \kappa > 0 \).

- At the beginning, the agent has no positions and needs to have \( N \) shares by time \( T \).

- At time \( T \), if there are \( \bar{N}_T \) shares remaining to purchase, one has to use market orders to fulfil the position. The cost would be
  \[ \bar{N}_T (P_T + \Delta + f + c(\bar{N}_T)), \]  
  (3.16)
  where \( c(\bar{N}_T) \) represents the price impact from placing \( \bar{N}_T \) shares market order.

Let \( L_t \) denote the number of shares the agent bought via limit orders up to time \( t \), and similarly \( M_t \) is for the number of shares bought via market orders up to time \( t \). And let \( n_t = L_t + M_t \). With the above assumptions, the agent’s wealth process \( X_t \) satisfies
\[ dX_t = -(P_{t-} - \delta_{t-})dL_t - (P_{t-} + \Delta + f)dM_t, \]  
(3.17)

And the value function for the agent is
\[ H(t, x, P, n) = \sup_{a \in A} \mathbb{E}_{t,x,P,n}[X_T - \bar{N}_T (P_T + \Delta + f + c(\bar{N}_T)) - \phi \int_t^T (n_u - n_u)^2 du]. \]  
(3.18)
$\mathcal{A}$ is the set of strategies, $\phi > 0$, and $n$ denotes the target inventor schedule. The last term in the value function penalizes the deviation from the target schedule. Moreover, $\bar{N}_T = n - (n_T - n_t)$, and $E_{t,x,P,n}$ denotes the conditional expectation given that $X_{t-} = x$, $P_{t-} = P$, $n_{t-} = n$. As an example studied in this paper, the trading speed in the Almgren–Chriss model is used as the target inventory:

$$n_t = N(1 - \frac{\sinh(\kappa(T - t))}{\sinh(\kappa T)})$$

(3.19)

where $\kappa$ is determined by model parameters. The dynamic programming principle is used to claim that (3.18) is the unique viscosity solution to the quasi-variational inequality (QVI):

$$\max \left\{ H_t + \gamma(H(t, x, P - \sigma, n)) - 2H(t, x, P, n) + H(t, x, P + \sigma, n) \right. \left. \right.$$

$$\left. - \phi(n - n_u)^2 + \lambda \sup_\delta e^{-\kappa \delta} \left[ H(t, x - (P - \delta), P, n + 1) - H(t, x, P, n) \right] \right\} = 0,$$

(3.20)

for $n = 0, 1, \cdots, N - 1$ and the value function is subject to the terminal and boundary conditions

$$H(T, x, P, n) = x - (N - n)(P + \Delta + f + c(N - n)), \quad q = 0, 1, \cdots, N - 1,$$

$$H(t, x, P, N) = x - \phi \int_t^T (N - n_u)^2 du.$$  

Furthermore, with a fixed target inventory schedule like (3.19), numerical techniques are adopted for applications.

Compared to classical optimal execution model literature, this paper considers both limit orders and market orders and benefits from earning the spread via limit orders. However, this model has volume restriction and the price impact is not complete. Huijema (2014) considers linear permanent price impact from executed orders as well as outstanding limit orders, and allows multiple outstanding limit orders simultaneously in the LOB. Then they derive the Hamilton–Jacobi–Bellman equation by dynamic programming along with numerical techniques for practical applications. There are several more studies on the optimal use of market orders and limit orders in the execution problems have been discovered by different researchers.

### 3.1.3 Market making problems

The optimal placement problem and optimal execution problem are also related to the well-known market-making problem, one of the central problems in algorithmic
trading. Market-making strategies involve simultaneously placing limit and market orders to buy and sell. The goal of the market maker is to maximize the profit by playing with the spread between the bid and ask prices, while controlling the inventory risk and the execution risk. Stochastic control was applied to tackle the optimization problem in an order book in the early 1980s. Ho and Stoll (1981) study a single dealer with stochastic demand modelled by a Poisson process and return risk on stock modelled by diffusion processes and derived optimal bid and ask quotes around a “true” price for the asset. Based on this model, Avellaneda and Stoikov (2008) also take the microstructure of actual limit order books into account.

Avellaneda and Stoikov’s model

Let the underlying asset price evolve according to $dP_t = \sigma dW_t$, where $W_t$ is a standard one-dimensional Brownian motion and $\sigma$ is constant. The agent’s value function of simply holding $q$ shares of stock until the terminal time $T$ is

$$v(x, P, q, t) = \mathbb{E}_t[-\exp(-\gamma(x + qP_T))]$$

$$= -\exp(-\gamma x)\exp(-\gamma qP)\exp\left(\frac{\gamma^2 q^2 \sigma^2 (T-t)}{2}\right)$$

(3.21)

where $x$ is the initial wealth, $\gamma$ is the continuous discount rate. First, the agent’s reservation bid price $r_b$ could be derived by

$$v(x - r_b(P, q, t), P, q + 1, t) = v(x, s, q, t).$$

(3.22)

And this yields

$$r_b(P, q, t) = P + (-1 - 2q)\frac{\gamma \sigma^2 (T-t)}{2}$$

(3.23)

Similarly, the reservation ask price $r_a$ is given by

$$r_a(P, q, t) = P + (1 - 2q)\frac{\gamma \sigma^2 (T-t)}{2}$$

(3.24)

Now assume that the agent quotes the bid price $p_b$ and the ask price $p_a$, and is committed to respectively buy and sell one share of stock at these prices. Moreover, the market buy/sell orders against the agent’s ask/bid orders at the Poisson rate $\lambda^a(\delta^a)/\lambda^b(\delta^b)$, where $\delta^b = P - p^b$, $\delta^a = p^a - P$ are the distances between the quoted prices and the mid-price, and both $\lambda^a$ and $\lambda^b$ are decreasing functions. Let $N^a_t$ denote the amount of shares sold and $N^b_t$ denote the amount of shares bought. Then both of them are Poisson processes with intensities $\lambda^a(\delta^a)$ and $\lambda^b(\delta^b)$. The cash held by the agent is given by

$$dX_t = p^aDN^a_t - p^bDN^b_t.$$ 

(3.25)
The number of shares held at time $t$ is given by
\[ q_t = N_t^b - N_t^a. \] (3.26)

The objective of the agent is
\[ u(P, x, q, t) = \max_{\delta^a, \delta^b} \mathbb{E}_t \left[ -\exp(-\gamma(X_T + q_T S_T)) \right]. \] (3.27)

This type of optimization problem was first studied in [Ho and Stoll (1981)]. They applied dynamic programming principle to show that the function $u$ solves the Hamilton–Jacobi–Bellman equation
\begin{align*}
&u_t + \frac{1}{2}\sigma^2 u_{ss} + \max_{\delta^b} \lambda^b(\delta^b) [u(P, x - P + \delta^b, q + 1, t)] - u(P, x, q, t) \\
&\quad + \max_{\delta^a} \lambda^a(\delta^a) [u(P, x - P + \delta^a, q + 1, t)] - u(P, x, q, t) = 0 \quad (3.28) \\
&u(P, x, q, T) = -\exp(-\gamma(x + qP)).
\end{align*}

The remaining piece of this problem is to model the intensity functions $\lambda(\delta)$. The price impact model used here is from [Potters and Bouchaud (2003)]. Let $\delta P$ be the price change caused by a large market order $Q$, then
\[ \delta P \propto \log(Q). \] (3.29)

And a power law is used for the distribution of the size of market orders,
\[ f_Q(x) \propto x^{-1-\alpha}, \] (3.30)

where $x$ is the size of a large order. $\alpha$ varies in different literature; see [Gopikrishnan et al. (2000)], [Maslov and Mills (2001)], and [Gabaix et al. (2005)]. Then aggregating the above information, one could derive that
\[ \lambda(\delta) = A \exp(-k\delta), \] (3.31)

where $A$ and $k$ are parameters determined by the market. However, due to the computational difficulty in solving the HJB equation (3.28), an asymptotic expansion method is developed to approximate the optimal controls. In many other existing studies, numerical procedures to derive optimal strategies are also investigated; see, for example, [Bayraktar and Ludkovski (2014)], [Li and Horst (2013)], [Cartea et al. (2014)], [Cartea and Jaimungal (2013)], [Guilbaud and Pham (2013)], and [Guéant et al. (2012)].

Indeed, the market-making problem is the optimal

\[ \text{Indeed, the market-making problem is the optimal} \]
placement problem with the possibility of intermediate buying. In this regard, the optimal placement problem may be viewed as a basic question in algorithmic trading.

Given the complexity of the market-making problem and the difficulty to obtain any analytical results (so far), the natural question is: do we have a better luck with the simpler optimal placement problem? If so, can we obtain any useful insights from the results? This is the motivation for our work.

### 3.2 A price impact model with best bid orders and market orders

#### 3.2.1 Summary of contributions

Most of the existing literature on optimal execution focuses on how to optimally control the trading speed with using only market orders. The primary purpose here is to draw comparison between the optimal placement problem and the optimal execution problem. We will add the limit order option into Bertsimas and Lo’s model and analyze the structure of the optimal strategy. We will show that the optimal trading strategy never mixes the limit and market orders; consequently, the optimal strategy can be computed recursively. In a special and degenerate case of the model, our result reduces to the deterministic strategy of Bertsimas and Lo (1998) for the optimal execution problem. In a related work, Huitema (2014) studies the optimal execution problem with the price impact driven by imbalance of the LOB in a Brownian motion framework. Our model setting here is different and the main focus is to illustrate the connection between optimal execution and optimal placement. In addition, a linear price impact will be used just as in Cont et al. (2014).

#### 3.2.2 Dynamics of the price model and analysis.

Let $A_t$ be the best ask price at time $t$, and let $\{\epsilon_t\}_{t=1,2,\ldots,T}$ be a martingale with mean 0. Then the dynamics of $A_t$ is modeled as:

$$A_{t+1} = A_t + c_1 m_t + c_2 I_t l_t + \epsilon_{t+1},$$  \hspace{1cm} (3.32)

with $A_0 = 0$. Here,

$$I_t = \begin{cases} 
1 & \text{if the limit order placed at time } t \text{ was executed by time } t+1, \\
0 & \text{otherwise}, 
\end{cases}$$  \hspace{1cm} (3.33)
$c_1$ and $c_2$ are the price impact factor for market orders and limit orders, and $m_t$ and $l_t$ are the order sizes placed at time $t$ for market orders and limit orders, respectively.

Now, let $r$ be the rebate for an executed limit order and $f$ be the transaction fee for a market order. At each time, a limit order will be executed with probability one at the price of 1 tick lower than $A_t$. Independent of whether the limit order is executed or not, the transaction price for a market order and limit order at time $t$ will be $A_t + c_1 m_t + c_2 l_t + f + \epsilon_t$ and $A_t + c_1 m_t + c_2 l_t + \epsilon_t - 1 - r$, respectively. For ease of exposition and without much loss of generality, we assume that $c_1 = c_2 = c$. (In reality, it has been observed that $c_1 > c_2$ since the market order will affect the trading directly and immediately while a large limit order may take longer time. But it will be evident that this differentiation of $c_1$ and $c_2$ will not change much of the structural results.)

Clearly, because of the price impact, the reduction of a general $N$ to $N = 1$ is no longer valid. Now (3.32) is rewritten as

$$A_{t+1} = A_t + c(m_t + l_t) + \epsilon_{t+1}. \quad (3.34)$$

Moreover, we assume that the probability of the limit orders being executed is $p_l$. Let $C_t(n_t, m_t, l_t, A_t)$ be the expected cost at time $t$ with current price of $A_t$, $n_t$ (0 $\leq$ $n_t$ $\leq$ $N$) shares left to purchase, placing $l_t$ limit orders and $m_t$ market orders. Let $V_t(n_t, A_t)$ be the expected cost after optimization over $m_t$ and $l_t$. Then according to the dynamic programming principle, the optimal placement problem is to find

$$V_t(n_t, A_t) = \min_{0 \leq m_t, l_t \leq n_t} C_t(n_t, m_t, l_t, A_t), \quad 1 \leq t \leq T - 1, \quad (3.35)$$

$$V_T(n_T, A_T) = n_T(f + A_T + cn_T), \quad (3.36)$$

with

$$C_t(n_t, m_t, l_t, A_t) = \mathbb{E}\{m_t(f + A_t + c(m_t + l_t) + \epsilon_t) + p_l l_t(A_t - r - 1 + c(m_t + l_t) + \epsilon_t) + p_l V_{t+1}(n_t - m_t - l_t, A_t + c(m_t + l_t) + \epsilon_t) + (1 - p_l) V_{t+1}(n_t - m_t, A_t + cm_t + \epsilon_t)\}. \quad (3.37)$$

subject to (3.34).

By mathematical induction, it is easy to check the following.

**Lemma 3.1.** For $0 \leq t \leq T - 1$,$$
C_t(n_t, m_t, l_t, x) = C_t(n_t, m_t, l_t, 0) + xn_t,
$$
and

\[ V_t(n_t, x) = V_t(n_t, 0) + xn_t. \]

This implies that the influence of the current price is a linear shift of the total cost and will not affect the optimization over \( m_t \) and \( l_t \). Thus

\[
(M_t, L_t) = \arg \min_{0 \leq m_t, l_t} C_t(m_t, l_t, n_t, A_t),
\]

is independent of \( A_t \).

**Properties of the optimal strategy for period \( t \).** The above analysis suggests that \( M_t \) and \( L_t \) are functions of \( T - t \) and \( n_t \). To make the notation simpler, let \( a = \frac{r + f + 1}{c} \) be the normalized difference between market order and limit order and write \( C_t(n_t, m_t, l_t) \) and \( V_t(n_t) \) for \( C_t(n_t, m_t, l_t, r + 1) \) and \( V_t(n_t, r + 1) \), respectively. We see for \( 0 \leq t \leq T - 1 \),

\[
C_t(n_t, m_t, l_t) = \begin{align*}
&= m_t(m_t + l_t + a) + (1 - p_l)[V_{t+1}(n_t - m_t) + m_t(n_t - m_t)] \\
&\quad + p_l(l_t + l_t) + p_l[V_{t+1}(n_t - m_t - l_t) + (m_t + l_t)(n_t - m_t - l_t)] \\
&= am_t + n_tm_t + pl_t + (1 - pl)m_tl_t \\
&\quad + pl[V_{t+1}(n_t - m_t - l_t) + (1 - pl)V_{t+1}(n_t - m_t)]
\end{align*}
\]

**Theorem 3.1.** For any \( t \) and \( n_t \),

\[
M_t \cdot L_t = 0,
\]

i.e., the optimal strategy involves either market order or limit order at each step but never both. Moreover, \( V_t(n_t) \) is a continuous, piece-wise quadratic function of \( n_t \) with the second order coefficients within \((1/2, 1]\).

**Connection to Bertsimas and Lo (1998).** The study of the optimal execution problem is pioneered in Bertsimas and Lo (1998), where the stock price is modeled by a simple random walk with an additive impact of large sales. An intriguing consequence of this modeling approach is that the optimal strategy turns out to be more or less static or deterministic, i.e., evenly dividing the total number of shares \( N \) over the \( T \) selling period is optimal.

Obviously, \( pl = 0 \) will reduce our model with price impact to the model of Bertsimas and Lo (1998), with the optimal strategy the same as theirs. Indeed in the case of \( pl = 0 \) meaning the limit order will not be executed for sure, then the
market order always dominates the limit order. In (3.38), substituting $p_l$ with 0, we get

$$V_{T-1}(x, A_{T-1}) = \min_{0 \leq m_{T-1}, l_{T-1}} \left\{ m_{T-1}(A_{T-1} + c(m_{T-1} + l_{T-1}) + f) + (x - m_{T-1})(cx + A_{T-1} + f) \right\}.$$  

Since $(m_{T-1}, l_{T-1})$ is always dominated by $(m_{T-1}, 0)$ if both of them are feasible, we have $L_{T-1} = 0$, and

$$V_{T-1}(x, A_{T-1}) = \min_{0 \leq m_{T-1} \leq x} cm_{T-1}^2 - cxm_{T-1} + x(A_{T-1} + cx + f),$$

whose minimum is $3cx^2/4 + x(A_{T-1} + f)$ attained at $m_{T-1} = \frac{x}{2}$. Inductively, suppose $(M_{t+1} = x/(T - t), L_{t+1} = 0)$ is the optimal solution for the period $t, 0 \leq t \leq T - 1$ and $V_{t+1}(x, A_{t+1}) = cx^2(1 + 1/(T - t))/2 + x(A_{t+1} + f)$. Then at time $t$,

$$C_t(x, m_t, l_t, A_t) = m_t(A_t + c(m_t + l_t) + f) + c(x - m_t)^2(1 + 1/(T - t))/2 + (x - m_t)(A_t + c(m_t + l_t) + f)$$

$$= \frac{T - t + 1}{2(T - t)} c(m_t + l_t)^2 - \frac{1}{T - t}(cx - r - f - 1)(m_t + l_t)$$

$$+ \frac{T - t + 1}{2(T - t)} cx^2 + x(A_t - r - 1) - (r + f + 1)l_t.$$  

Obviously, $C_t$ increases as $l_t$ increases, therefore $L_t = 0$. Then,

$$C_t(x, m_t, 0, A_t) = m_t(A_t + cm_t + f) + c(x - m_t)^2(1 + 1/(T - t))/2 + (x - m_t)(A_t + cm_t + f)$$

$$= \frac{T - t + 1}{2(T - t)} cm_t^2 - \frac{1}{T - t} cxm_t + \frac{T - t + 1}{2(T - t)} cx^2 + x(A_t + f).$$

Now, simple calculations yield $M_t = \frac{x}{T - t + 1}$, and $E_t(x, A_t) = cx^2(1 + 1/(T - t + 1))/2 + x(A_t + f)$. Thus by induction, the optimal strategy is to always use market order, and if there are $T$ periods in total, then each time one places $1/T$ of the total volume.

Another extreme case of our model is when $p_l = 1$, which means the limit order will be executed for sure. Intuitively, since there is no immediate price impact to the execution price of the limit order and the limit order is getting executed with probability 1, it is then desirable to place all the orders at the same time to minimize the price impact. We can see this is indeed the case. For $t = T$, one will
choose to use the limit order for all the non-executed positions since it is cheaper and could be executed for sure. For \( t = T - 1 \),

\[
V_{T-1}(x, A_{T-1}) = \min_{0 \leq m_{T-1}, l_{T-1}} \left\{ m_{T-1}(A_{T-1} + c(m_{T-1} + l_{T-1}) + f) + (x - m_{T-1} - l_{T-1})(c(m_{T-1} + l_{T-1}) + A_{T-1} - r - 1) + l_{T-1}(A_{T-1} - r - 1) \right\}.
\]

Since \((m_{T-1}, l_{T-1})\) is dominated by \((0, l_{T-1} + m_{T-1})\) and since both are feasible, we have

\[
V_{T-1}(x, A_{T-1}) = \min_{0 \leq l_{T-1} \leq x} -cl_{T-1}^2 + cxl_{T-1} + (A_{T-1} - 1 - r)x
\]

Then \( V_{T-1}(x, A_{T-1}) = (A_{T-1} - 1 - r)x \) with \( L_{T-1} = 0 \) or \( L_{T-1} = x \). Recursively, it is easy to show that the optimal trading strategy for \( p_t \) is to place the total number of orders together at any of the period.

**An explicit result for the second last step**

Optimal placement problem with price impact is in general more complex despite the analytical structure that limit order and market order are never mixed in an optimal strategy. To get a sense of the complexity, we give some illustration for \( t = T - 1 \) with an explicit optimal strategy. Clearly,

\[
V(x, A_{T-1}) = \min_{0 \leq m_{T-1}, l_{T-1}} m_{T-1}(f + c(m_{T-1} + l_{T-1}) + A_{T-1}) + (1 - p_t)(x - m_{T-1})(f + cx + A_{T-1}) + p_t[l_{T-1}(-1 - r + A_{T-1}) + (x - m_{T-1} - l_{T-1})(f + cx + A_{T-1})].
\]

For any fixed \( x \) and \( A_{T-1} \), the RHS of (3.38) is a quadratic function of \( m_{t-1} \) and \( l_{t-1} \), whose Hessian is

\[
\begin{pmatrix}
2c & 1c \\
1c & 0
\end{pmatrix},
\]

whose determinant is \(-c^2 < 0\) for \( c > 0 \). Clearly this Hessian is neither positive definite nor negative definite for all \( m_{t-1} \) and \( l_{t-1} \). Thus the function will reach its minimum only on the boundary. Moreover,

- If \( p_t < \frac{1}{4} \), then there are two scenarios:
1. For \( 0 \leq x \leq \frac{4p_l(r+f+1)}{(1-4p_l)c} \), then \( M_{T-1} = 0, L_{T-1} = x \);

2. For \( \frac{4p_l(r+f+1)}{(1-4p_l)c} < x \), then \( M_{T-1} = \frac{1}{2}x, L_{T-1} = 0 \).

- If \( \frac{1}{4} \leq p_l \leq 1 \), then \( M_{T-1} = 0, L_{T-1} = x \).
Chapter 4

Dynamics of order positions in an LOB

4.1 Review of the heavy traffic limits and the LOB modelling

4.1.1 Heavy traffic limits

Generally speaking, heavy traffic limits are developed for simpler description and approximation of queueing systems. A simple $M/M/1$ queueing system could be described as follows. Requests arrive according to a Poisson process with intensity $\lambda$, while the service time is an exponential random variable with rate $\mu$. Let $C(t)$ denote the number of customers waiting in the system at time $t$. Then $\{C(t)\}_{t \geq 0}$ is a continuous time Markov chain with transition rate matrix

$$Q = \begin{pmatrix}
-\lambda & \lambda & & & \\
\mu & -(\lambda + \mu) & \lambda & & \\
& \mu & -(\lambda + \mu) & \lambda & \\
& & \mu & -(\lambda + \mu) & \lambda \\
& & & \cdots & \cdots & \cdots
\end{pmatrix} \tag{4.1}
$$

Let $\rho = \lambda/\mu$ and the scaled queue $C_\rho$ be defined by $C_\rho(t) = (1 - \rho)C(t)/(1 - \rho)^2$. Then for any $T > 0$, as $\rho \uparrow 1$,

$$C_\rho \Rightarrow B^R, \quad \text{in } (D[0,T], J_1) \tag{4.2}$$

where $B^R$ is a regulated Brownian motion defined by

$$B^R(t) = B(t) - \inf_{0 \leq s \leq t} B(s), \tag{4.3}$$
where $B(t)$ is a Brownian motion with drift term $-\mu$ and variance parameter $2\mu$. This result could be found in many standard references, for example, Ward and Glynn (2003). Harrison (1985) gives a detailed discussion on the regulated Brownian motion and a stochastic flow system based on Brownian motions. Specifically, let $I_t$ denote the cumulative input up to time $t$ and $O_t$ denote the cumulative potential output up to time $t$ with $I_0 = 0$ and $O_0 = 0$. That is, $O_t$ is the output if the server is busy all the time from 0 to $t$. In addition, the buffer capacity $b$ is introduced as the upper bound of the number of customers waiting in the system. Define $X_t := X_0 + I_t - O_t$ as the net flow process and $U_t, L_t, Z_t$ as follows:

- $L_t$ and $U_t$ are increasing and continuous with $L_0 = U_0 = 0$,
- $Z_t := X_t + L_t - U_t \in [0, b]$ for all $t \geq 0$,
- $L_t$ and $U_t$ increase only when $Z = 0$ and $Z = b$, respectively.

Here, $O_t - L_t$ represents the actual output as $L_t$, the virtual outflow during the idle time, is subtracted from $O_t$. Similarly, $I_t - U_t$ is the actual input as $U_t$, the input occurred when the buffer was full, is subtracted from $I_t$. $Z_t$ describes the dynamics of the system. When considering the limiting system, $X_t$ could be shown to be a Brownian motion with drift $\mu$ and instantaneous variance $\sigma$, and $Z_t$ is a two-sided regulated Brownian motion. Taking $b = \infty$ gives the one-sided regulated Brownian motion as we introduced by (4.3). Furthermore, one can formulate a stochastic control problem where $L_t$ and $U_t$ are used as the controllers and $Z_t$ is the controlled process.

There are a lot of studies on heavy traffic limits for more general queueing systems, with general arrival distributions, general service time distributions, and with the possibility of multiple servers. In addition to the $M/M/1$ queue, the customers in the line are assumed to be impatient and reneging is considered in Ward and Glynn (2003), Ward and Glynn (2005), Kang et al. (2010), among others. Each single individual leaves the queue according to an exponential random variable, that is, the reneging rate is proportional to the current queue length, denoted by $\gamma C(t)$. When $\mu = 0$, it is easy to see that this system is equivalent to the $M/M/\infty$ queue with a service rate $\gamma$. Under different parameter settings, “reflected” Ornstein–Uhlenbeck processes as the diffusion limit processes are derived.

### 4.1.2 Kruk’s model

Due to the complexity of the LOB, there are only few papers analyzing the dynamics of the LOB in a mathematically rigorous manner. Kruk (2003) is among the first to give a functional limit approximation of such a system with an appropriate
time scaling. A multi-price-level auction model, instead of the order flows and the LOB, was presented in this paper. However, the nature of this model is actually very similar to the LOB, except there is no cancellation being considered.

In the auction, suppose that there are $N$ price levels from $P_1$ to $P_N$, and $B_i(t)$ denotes the outstanding bid orders at price level $P_i$ at time $t$, while $A_i(t)$ denotes the outstanding bid orders at price level $P_1$ at time $t$. Obviously, $A_i(t)B_i(t) = 0$ for any $t$ as these bid and ask orders at the same price level are executed against each other. Moreover, to derive the functional limit, it is assumed that there is a sequence of systems indexed by $n$, that is, we have $\{(B_i^{(n)}(t), A_i^{(n)}(t))_{1 \leq i \leq N}\}_{n \geq 1}$.

The model can be described as follows. For each $n$,

- Buyers arrive according to a renewal process with inter-arrival times $\{u_k^{(n)}\}_{k \geq 1}$ with mean $1/\lambda^{(n)}$ and standard deviation $\alpha^{(n)}$, and $\{v_k^{(n)}\}_{k \geq 1}$. Sellers arrive also according to a renewal process with inter-arrival times $\{v_k^{(n)}\}_{k \geq 1}$ with mean $1/\mu^{(n)}$ and standard deviation $\beta^{(n)}$. And the two arrival processes are mutually independent.

- The $k^{th}$ buyer (seller) is willing to buy (sell) $l_k^{(n)} (m_k^{(n)})$ shares, where $\{l_k^{(n)}\}_{k \geq 1} (\{m_k^{(n)}\}_{k \geq 1})$ is a sequence of positive i.i.d. random variables with mean $l^{(n)} (m^{(n)})$ and standard deviation $\kappa^{(n)} (\nu^{(n)})$.

- The $k^{th}$ buyer (seller) is willing to buy (sell) at the price $b_k^{(n)} (a_k^{(n)})$, where $\{b_k^{(n)}\}_{k \geq 1} (\{a_k^{(n)}\}_{k \geq 1})$ is a sequence of i.i.d. random variables with

$$\mathbb{P}(b_k^{(n)} = i) = \mathbb{P}(a_k^{(n)} = i) = p_i, \quad i = 1, 2, \cdots, N, \quad k \geq 1. \quad (4.4)$$

The following assumptions are made.

- There exists positive constants $\bar{l}$, $\kappa$, $\nu$, $\lambda$, $\alpha$, $\beta$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, such that

$$\lim_{n \to \infty} l^{(n)} = \lim_{n \to \infty} m^{(n)} = \bar{l}, \quad \lim_{n \to \infty} \kappa^{(n)} = \kappa, \quad \lim_{n \to \infty} \nu^{(n)} = \nu,$$

$$\lim_{n \to \infty} \lambda^{(n)} = \lim_{n \to \infty} \mu^{(n)} = \bar{l}, \quad \lim_{n \to \infty} \alpha^{(n)} = \kappa, \quad \lim_{n \to \infty} \beta^{(n)} = \nu,$$

$$\lim_{n \to \infty} \sqrt{n}(\lambda - \lambda^{(n)}) = \gamma_1, \quad \lim_{n \to \infty} \sqrt{n}(\lambda - \lambda^{(n)}) = \gamma_2,$$

$$\lim_{n \to \infty} \sqrt{n}(\bar{l} - \bar{l}^{(n)}) = \gamma_3, \quad \lim_{n \to \infty} \sqrt{n}(\bar{l} - \bar{l}^{(n)}) = \gamma_4.$$

- The probability that there are customers of different types who come to the $n^{th}$ auction exactly at the same time converges to 0 as $n \to \infty$. 

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With a proper scaling, main results derived in this paper include:

- When $N = 2$, that is, when there are only two price levels, then with the scaling $\tilde{B}_k(n) = \frac{1}{\sqrt{n}} B_k(n) t$ and $\tilde{A}_k(n) = \frac{1}{\sqrt{n}} A_k(n) t$, a diffusion limit is derived as follows:

$$
(\tilde{A}_1^{(n)}, \tilde{B}_1^{(n)}, \tilde{B}_2^{(n)}, \tilde{A}_2^{(n)}) \Rightarrow (0, 0, Z).
$$

\hspace{1cm} (4.5)

where $Z$ is a semimartingale reflected Brownian motion.

- When $N \geq 3$, the following scaling is used. $\tilde{B}_k(n) = \frac{1}{n} B_k(n) t$ and $\tilde{A}_k(n) = \frac{1}{n} A_k(n) t$. Then

$$
(\tilde{A}^{(n)}, \tilde{B}^{(n)}) \Rightarrow (\bar{a} e, \bar{b} e),
$$

\hspace{1cm} (4.6)

where $\bar{a}, \bar{b}$ are two constants determined by the model parameters ($\lambda, \mu, \bar{l}, \bar{m}, (p_i)_{1 \leq i \leq N}$, etc.) and $e$ denote the identical mapping from $\mathbb{R}$ to $\mathbb{R}$.

### 4.1.3 Cont and de Larrard’s model

**Limit order book model: level-1**

[Cont and De Larrard (2013)](ref) propose a queueing model for approximating the LOB with the following assumptions:

- All orders are of sizes 1,

- Limit orders, market orders, and cancellations arrive as independent Poison processes with intensities $\lambda, \mu, \theta$ on both ask and bid queues,

- The best ask queue and the best bid queue are independent of each other,

- The bid-ask spread is always 1,

- When the best bid queue is depleted, both the best bid price and the best ask price will move down 1 tick. The new best bid queue and the new best ask queue will be sampled from a joint density $f(x, y)$, which is independent of the queue history,

- When ask queue is depleted, the bid and ask prices will both move up 1 tick. The new bid and ask size is sampled from a joint density $\tilde{f}(x, y)$ independent of the queue history.
This model can be viewed as a simplified version of Cont et al. (2010) as taking the spread $S = 1$ and a fixed total cancellation rate, instead of proportional to the queue length. This simplification allows analytic results for some key quantities. Actually, let $\{(q^b(t), q^a(t))\}_{t \geq 0}$ denote the number of shares on the best bid queue and best ask queue at time $t$, then when prior to hitting 0 on either axis, it is a planar random walk in the first quadrant. Moreover, it is easy to see,

- $(q^b(t), q^a(t)) \rightarrow (q^b(t) + 1, q^a(t))$ with rate $\frac{\lambda}{2(\lambda + \mu + \theta)}$,
- $(q^b(t), q^a(t)) \rightarrow (q^b(t) - 1, q^a(t))$ with rate $\frac{\mu + \theta}{2(\lambda + \mu + \theta)}$,
- $(q^b(t), q^a(t)) \rightarrow (q^b(t), q^a(t) + 1)$ with rate $\frac{\lambda}{2(\lambda + \mu + \theta)}$,
- $(q^b(t), q^a(t)) \rightarrow (q^b(t), q^a(t) - 1)$ with rate $\frac{\mu + \theta}{2(\lambda + \mu + \theta)}$.

Let $(q^b(0), q^a(0)) = (n, p) \in \mathbb{N}^2$ denote the number of orders on the bid side and ask side, respectively. Based upon the above level-1 model,

- The conditional distribution of the duration between price moves, given the state of the order book, is derived. (That is, the distribution for the time needed to deplete one of the queues.)
- If $\lambda = \mu + \theta$, then the probability $\phi(n, p)$ that the next price move is an increase, conditioned on having the $n$ orders on the bid side and $p$ orders on the ask side is
  \[
  \phi(n, p) = \frac{1}{\pi} \int_0^\pi \left( 2 - \cos(t) - \sqrt{(2 - \cos(t))^2 - 1} \right)^p \frac{\sin(nt) \cos(t/2)}{\sin(t/2)} dt,
  \]
- The autocorrelation and distribution of price changes. This result explains the negative lag-1 autocorrelation but not the other insignificant higher order autocorrelations.
- The volatility of the price.

**Heavy traffic limits for LOB**

Cont and De Larrard (2012) then relax the independence and the unit order size assumptions. Let $(V^a_i, V^b_i)$ be the change of level-1 ask and bid queue lengths after the $i^{th}$ event (market order, cancellation or limit order) happened. Let $(\tau^a_i, \tau^b_i)$ be the inter-event time. In the previous model, they have

- $P(V^a_i = 1) = \frac{\lambda}{\lambda + \mu + \theta}$, $P(V^a_i = -1) = \frac{\mu + \theta}{\lambda + \mu + \theta}$.
with similar equations for $V_i^b$. In that model, $\tau_i^a$ and $\tau_i^b$ are independent exponential with rate $\lambda + \mu + \theta$. However, in Cont and De Larrard (2012), $(V_i^a, V_i^b)$ and $(\tau_i^a, \tau_i^b)$ are assumed to be weakly dependent and covariance stationary. Let $(q^a(t), q^b(t))$ be the ask and bid size at time $t$. Under the assumption of a balanced limit order book in the sense that $E(V_i^a) = E(V_i^b) = 0$, the rescaled order book process converges weakly:
\[
\left( \frac{q^a(nt)}{\sqrt{n}}, \frac{q^b(nt)}{\sqrt{n}} \right)_{t \geq 0} \overset{d}{\to} \left( Q_t \right)_{t \geq 0},
\]
where $(Q_t)_{t \geq 0}$ is a planer Brownian motion with drift 0 and covariance matrix
\[
\left( \begin{array}{cc}
\lambda_a v_a^2 & \rho v_a v_b \sqrt{\lambda_a \lambda_b} \\
\rho v_a v_b \sqrt{\lambda_a \lambda_b} & \lambda_b v_b^2
\end{array} \right).
\]

with re-initialization distribution $F$ (resp., $\tilde{F}$) when it hits the x-axis (resp., y-axis). Both $F$ and $\tilde{F}$ are rescaled versions of $f$ and $\tilde{f}$ that in the previous section. Moreover, $1/\lambda_a = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \tau_i^a / n \right)$ is the average duration between events at the ask and $1/\lambda_b = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \tau_i^b / n \right)$ is the average duration between events at the bid. $v_a^2 = E\{(V_1^a)^2\} + 2 \sum_{i=2}^{\infty} \text{Cov}(V_1^a, V_i^a)$ is the variance of the event size at the ask, and $v_b^2 = E\{(V_1^b)^2\} + 2 \sum_{i=2}^{\infty} \text{Cov}(V_1^b, V_i^b)$ is the variance of the event size at the bid. Also,
\[
\rho = \frac{\text{Cov}(V_1^a, V_1^b) + 2 \sum_{i=2}^{\infty} \left[ \text{Cov}(V_1^a, V_i^b) + \text{Cov}(V_1^b, V_i^a) \right]}{v_a v_b}.
\]

According to Cont and De Larrard (2012), $\rho < 0$ for equity order books. They are able to compute the following quantities from the diffusion approximation:

- Duration until the next price change,
- Probability of an increase in the price,
- In particular, if $\lambda_a = \lambda_b$ and $v_a = v_b$, such probability depends only on the bid size $(x)$ and the ask size $(y)$ in the following manner:
\[
p_{up}(x, y) = \frac{1}{2} - \frac{\text{arctan}(\sqrt{\frac{1+\rho y-x}{1-\rho y+x}})}{2 \text{arctan}(\sqrt{\frac{1+\rho}{1-\rho}})}.
\]
4.1.4 Horst and Paulsen’s model

Horst and Paulsen (2015) propose a fluid limit model to address the best ask/bid prices, as well as the volume density function of the LOB. In their model, the state of the LOB is denoted by $S = (B, A, v_b(\cdot), v_a(\cdot))$, where $B(A)$ is the best bid/ask price, and $v_b(x)/v_a(x)$ denotes the volume density at price distance $x$ below/above the best bid/ask price. Moreover, $x$ can be negative, which represents a shadow book. Assuming that there is a sequence of systems indexed by $n, n \geq 1$, $S(n)_k$ denotes the $n$th system’s state after $k$ events occurred in that system. $x(n)_j = j \cdot \delta x(n)$ denotes the price level at which orders can be submitted, where $j \in \mathbb{Z}$ and $\delta x(n)$ is the tick size in the $n$th system. The volume density functions in the $n$th system after $k$ events are step functions as follows:

$$v(n)_{b,k}(x) = \sum_{j=0}^{\infty} v(n)_{k,j} B_j x(n)_j], v(n)_{a,k}(x) = \sum_{j=0}^{\infty} v(n)_{k,j} A_j x(n)_j],$$ (4.7)

There are eight types of events – four on the bid side and four on the ask side.

- $A = \{\text{market sell order}\}$
- $B = \{\text{limit bid placed in the spread}\}$
- $C = \{\text{cancellation on bid}\}$
- $D = \{\text{limit bid placed not in the spread}\}$
- $E = \{\text{market buy order}\}$
- $F = \{\text{limit ask placed in the spread}\}$
- $G = \{\text{cancellation on ask}\}$
- $H = \{\text{limit ask placed not in the spread}\}$

They can be classified into two categories according to different efforts caused in the LOB. Events which lead to price changes are called active orders. On the bid side, $A$ and $B$ are active orders. If the $k^{th}$ order in the $n^{th}$ system is of type $A$, then

$$B(n)_{k+1} = B(n)_k - \delta x(n), \quad A(n)_{k+1} = A(n)_k,$$

$$v(n)_{b,k+1}(\cdot) = v(n)_{b,k}(\cdot) + \delta x(n), \quad v(n)_{a,k+1}(\cdot) = v(n)_{a,k}(\cdot).$$

If the $k^{th}$ order in the $n^{th}$ system is of type $B$, then

$$B(n)_{k+1} = B(n)_k + \delta x(n), \quad A(n)_{k+1} = A(n)_k,$$

$$v(n)_{b,k+1}(\cdot) = v(n)_{b,k}(\cdot) - \delta x(n), \quad v(n)_{a,k+1}(\cdot) = v(n)_{a,k}(\cdot).$$

Both type $C$ and type $D$ orders are considered passive orders, which do not change the price. Let $\delta v^n$ denote the scaling parameter describing the impact of a single limit order (cancellation) on the state of the LOB. When the $k^{th}$ order is of type $C$, let $\omega_k$ be a random variable taking value in $(0, 1)$, representing the proportion
of cancellation caused by this order at the corresponding bid queue. When the $k^{th}$ order is of type $C$, let $\omega_k^D$ be a random variable taking value in $[0, M]$ for $M > 0$, representing the volume of the order. Moreover, $\pi_k^C, \pi_k^D$ are two random variables taking values in $[-M, M]$ representing the price level of the order. Then with a type $C$ order, the dynamics of the LOB follows:

$$v_{b,k+1}(\cdot) = v_{b,k}(\cdot) - \frac{\delta v_{(n)}(\cdot)}{\delta x_{(n)}} M_{k}^{(n),C} v_{b,k}(\cdot),$$

where

$$M_{k}^{(n),C} = \omega_k^C \sum_{j=-\infty}^{\infty} \mathbb{I}_{[x_j^{(n)}, x_{j+1}^{(n)}]}(x).$$

While with a type $D$ order, the dynamics of the LOB follows:

$$v_{b,k+1}(\cdot) = v_{b,k}(\cdot) + \frac{\delta v_{(n)}(\cdot)}{\delta x_{(n)}} M_{k}^{(n),D} v_{b,k}(\cdot),$$

where

$$M_{k}^{(n),D} = \omega_k^D \sum_{j=-\infty}^{\infty} \mathbb{I}_{[x_j^{(n)}, x_{j+1}^{(n)}]}(x).$$

In either case, the best bid/ask price and the volume density on the ask side remain unchanged.

Now let $\delta t^{(n)}$ denote the time scaling factor and $\tau_k^{(n)}$ denote the time the $k^{th}$ event occurs in the $n^{th}$ system. Then

$$\tau_{k+1}^{(n)} = \tau_k^{(n)} + \varphi_k(B_k^{(n)}, A_k^{(n)}) \delta t^{(n)},$$

where $\{\varphi_k\}_{k \geq 1}$ is a sequence of non-negative random variables. Furthermore, $\{\phi_k\}_{k \geq 1}$ is a sequence of random variables taking values in $\{A, \cdots, H\}$, representing the type of the $k^{th}$ order. Let $\delta p^{(n)}$ denote the scaling factor and $p_{k}^{(n)} = p^{(n),J}(B_k^{(n)}, A_k^{(n)}) = \mathbb{E}[\phi_k = I|S_k^{(n)}]$. Therefore, the dynamics of the process $\{S_k^{(n)} = (B_k^{(n)}, A_k^{(n)}, x_k^{(n)}, v_k^{(n)}), \tau_{k+1}^{(n)}\}_{k \geq 1}$ for each $n$ as a Markov process is defined. Now define

$$S^n(t) = S_k^{(n)} \text{ and } \tau(t) = \tau_k^{(n)} \text{ for } t \in [\tau_k^{(n)}, \tau_{k+1}^{(n)}]. \quad (4.8)$$

The following assumptions are made in Horst and Paulsen (2015):

- The initial volume density functions vanish out-side a compact interval $[-M, M]$ for some $M > 0$ and satisfies:

$$|v_{r,0}^{(n)}(\pm \delta x^{(n)}) - v_{r,0}^{(n)}(\cdot)|_{\infty} = O(\delta x^{(n)}),$$

$$\lim_{n \to \infty} \|v_{r,0}^{(n)} - v_{r,0}\|_{L^2} = 0,$$

$$\lim_{n \to \infty} (B_0^{(n)}, A_0^{(n)}) = (B, A),$$

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where \( \| \cdot \|_{L^2} \) denotes the \( L^2 \)-norm on \( \mathbb{R} \) with respect to Lebesgue measure and \( v_{r,0} \in L^2 (r \in \{b,a\}) \) is non-negative bounded and continuously differentiable.

- For \( I \in \{C, D, G, H\} \), the sequences \( \{\omega^I_k\}_{k \geq 1} \) and \( \{\pi^I_k\}_{k \geq 1} \) are independent sequences of i.i.d. random variables. Moreover, the random variables \( \pi^I_k \) have \( C^2 \)-densities \( f^I \) with compact support.

- There are bounded continuous functions with bounded gradients \( P^I : \mathbb{R}^2 \mapsto [0,1] \) such that
  \[
  p^{(n)}(\cdot,\cdot) = \delta p^{(n)}(\cdot,\cdot), \quad \text{for} \quad I = A, B, E, F,
  \]
  \[
  p^{(n)}(\cdot,\cdot) = (1 - \delta p^{(n)}) \cdot p^I(\cdot,\cdot), \quad \text{for} \quad I = C, D, G, H,
  \]
  \[
  p^A + p^B + p^E + p^F = 1,
  \]
  \[
  p^C + p^D + p^G + p^H = 1.
  \]

- The scaling constants \( \delta p^{(n)}, \delta x^{(n)}, \delta v^{(n)}, \) and \( \delta t^{(n)} \) are such that
  \[
  \lim_{n \to \infty} \frac{\delta x^{(n)} \delta p^{(n)}}{\delta v^{(n)}} = c_0, \quad \lim_{n \to \infty} \frac{\delta v^{(n)}}{\delta t^{(n)}} = c_1, \quad \text{and} \quad \frac{\delta p^{(n)}}{(\delta t^{(n)})^\alpha} = O(1),
  \]
  for some constant \( \alpha \in (1/2,1) \) and constants \( c_0, c_1 > 0 \).

The following is their main result (Theorem 1.13 in Horst and Paulsen (2015)).

**Theorem 4.1.** Let \( \{S^{(n)}_{(n)}\}_{n \geq 1} \) be the sequence of continuous time processes defined in (4.8) and suppose the above assumptions hold. Then, for all \( T > 0 \) there exists a deterministic process \( s \) such that

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \| S^{(n)}(t) - s(t) \| = 0 \quad \text{in probability.} \quad (4.9)
\]

The process \( s \) is of the form \( s(t) = (\gamma(t), v(\cdot, t))^T \), where \( \gamma(t) = (b(t), a(t))^T \) is the vector of the best bid and ask price at time \( t \in [0,T] \) and \( v(\cdot, t) = (v_b(\cdot, t), v_a(\cdot, t))^T \) denotes the vector of buy and sell volume densities at \( t \) relative to the best bid and ask price. Moreover, \( \gamma(t) \) and \( v(\cdot, t) \) could be specified as follows. Define

\[
A(\cdot) = \begin{pmatrix} p^A(\cdot) - p^B(\cdot) & 0 \\ 0 & p^E(\cdot) - p^F(\cdot) \end{pmatrix}, \quad
B(\cdot, x) = \begin{pmatrix} p^C(\cdot) f^C(x) & 0 \\ 0 & p^G(\cdot) f^G(x) \end{pmatrix}
\]
and \( c(\cdot, x) = (p^D(\cdot)f^D(x), p^H(\cdot)f^H(x))^T \), the function \( m(\cdot, \cdot) \) specifies the expected waiting time between two consecutive active order arrivals, the function \( \gamma \) is the unique solution to the 2-dimensional ODE system

\[
\begin{aligned}
\frac{d\gamma(t)}{dt} &= \frac{A(\gamma(t))}{m(\gamma(t))} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in [0, T] \\
\gamma(0) &= \begin{pmatrix} B_0 \\ A_0 \end{pmatrix}
\end{aligned}
\]  
(4.10)

and \((v_b, v_a)\) is the unique non-negative bounded classical solution of the PDE

\[
\begin{aligned}
v_t(t, x) &= \frac{1}{m(\gamma(t))} \left( A(\gamma(t))v_x(t, x) + B(\gamma(t), x)v(t, x) + c(\gamma(t), x) \right), \\
v(0, x) &= v_0(0, x), \quad x \in \mathbb{R}
\end{aligned}
\]  
(4.11)

Recently, Horst and Kreher (2015) derive some similar results for the one-sided limit order book with more general conditions.

### 4.2 Summary of contributions

Without loss of generality, let us focus on an order position in the best bid queue. Its dynamics will be affected by both the market orders and the cancellations, and its relative position in the queue will be affected by limit orders as well. Order positions in other queues will be simpler because of the absence of market orders. First, the fluid limits for the order positions and related (best) bid and ask queues are derived. This is in some way the first-order approximation for the processes. It is then showed that the rate of the order position approaching zero is proportional to the mean of order arrival intensities and the average size of the market orders, with appropriate modification by the cancellation orders on the queue; the (average) time it takes for the order position to be executed is also derived. The derivation is via two steps. The first fairly straightforward step is to establish the functional strong law of large numbers for the related bid/ask queues. The second step is trickier although intuitively clear. It requires more delicate analysis involving passing the convergence relation of stochastic processes in their corresponding càdlàg space with the Skorokhod topology to their integral equations.

Next, the second-order approximation is studied for order positions and related queues. The first step is to establish appropriate forms of the diffusion limit...
for the bid and ask queues. This multi-variate functional central limit theorem (FCLT) is established borrowing ideas from those for random fields. Under appropriate technical conditions, it is demonstrated that the queues are two-dimensional Brownian motions with mean and covariance structure explicitly given in terms of statistics of order sizes and order arrival intensities. This FCLT leads to analytical expression for the first hitting time of the queue depleting and the probability of price changes. These are useful quantities for LOBs and generalize existing results of Cont and De Larrard (2012). The second step is to combine the FCLTs and fluid limit results to show that fluctuations of the order positions are Gaussian processes with “mean-reversion”. The mean reverting level is essentially the fluid limit of order position relative to the queue length modified by the order book net flow defined as the limit order minus the market order and the cancellation. The speed of the mean reverting is proportional to the order arrival intensity and the rate of cancellations. As a corollary of the analysis, explicit expressions are obtained for the fluctuations of execution and hitting times. In addition, with the large deviation principle, the probability that the queues deviate from their fluid limits is also derived.

The analysis builds on techniques and results from classical probability theory, including the functional central limit theorems by Jacod and Shiryaev (1987), Glynn and Whitt (1988), and Bulinski and Shashkin (2007), the convergence of stochastic processes by Kurtz and Protter (1991), and the sample path large deviation principle by Dembo and Zeitouni (1998).

Practically speaking, studying order positions give more direct estimates for the “value” of order positions. This is useful for algorithmic trading. Indeed, based on the fluid limit, an explicit analytical comparison between the average time an order is executed and the average time of any related queues being depleted is derived. This is an important piece of information, especially when combined with the explicit form obtained in this chapter on the probability of a price increase. This latter quantity is a core quantity for the LOB, and has been studied in Avellaneda and Stoikov (2008) and Cont and De Larrard (2013) in a special case.

4.3 Fluid limits of order positions and related queues

First, let us introduce some notations for the analysis.

Notation. Without loss of generality, take the best bid and ask queues. Then there are six types of orders: best bid orders, market orders at the best bid, cancellation at the best bid, best ask, market orders at the best ask, and cancellation.
at the best ask. Consider order arrivals of any of these six types as point processes in the following way. Denote the order arrival process by $N = (N(t), t \geq 0)$ with the inter-arrival times $\{D_i\}_{i \geq 1}$. Here

$$N(t) = \max \left\{ m : \sum_{i=1}^{m} D_i \leq t \right\}. \quad (4.12)$$

Now, define a sequence of six-dimensional random vectors $\{\vec{V}_i = (V^j_i, 1 \leq j \leq 6)\}_{i \geq 1}$. For each $i$, the component $V^1_i$ represents the size of $i$-th order from the limit order at the best bid, $V^2_i$ the market order at the best bid, $V^3_i$ the cancellation at the best bid, $V^4_i$ the limit order at the best ask, $V^5_i$ the market order at the best ask, and $V^6_i$ the cancellation at the best ask. For ease of exposition, we assume that no simultaneous arrivals of different orders, i.e., each $\vec{V}_i$ always consists of one positive component and five zero’s. For instance, $\vec{V}_5 = (0, 0, 0, 4, 0, 0)$ means the fifth order is a best limit ask order of size 4. In this paper, we only consider càdlàg processes.

For ease of references in the main text, we also denote

- $D[0, T]$ the space of 1-dimensional càdlàg functions on $[0, T]$, while $D^K[0, T]$ the space of $K$-dimensional càdlàg functions on $[0, T]$. Consequently, the convergence in this space is, unless otherwise specified, in the sense of the weak convergence in $D^K[0, T]$ equipped with $J_1$ topology;

- $L_\infty[0, T]$ is the space of functions $f : [0, T] \rightarrow \mathbb{R}^d$, equipped with the topology of uniform convergence;

- $\mathcal{AC}_0[0, T]$ is the space of functions $f : [0, T] \rightarrow \mathbb{R}^d$, that is absolutely continuous and $f(0) = 0$;

- $\mathcal{AC}_0^+[0, T]$ is the space of non-decreasing functions $f : [0, T] \rightarrow \mathbb{R}^d$ that is absolutely continuous and $f(0) = 0$.

Similarly, we define $D[0, \infty)$, $D^K[0, \infty)$, $L_\infty[0, \infty)$, $\mathcal{AC}_0[0, \infty)$, $\mathcal{AC}_0^+[0, \infty)$ for $T = \infty$.

### 4.3.1 Fluid limits for order positions and related queue lengths

In order to study the fluid limit for order positions and the best ask/bid queues, we will need the following technical assumptions.
Assumption 4.1. \(\{D_i\}_{i \geq 1}\) is a stationary array of positive random variables with

\[
\frac{D_1 + D_2 + \ldots + D_i}{i} \rightarrow \frac{1}{\lambda}, \quad \text{in probability (4.13)}
\]
as \(i \rightarrow \infty\), where \(\lambda\) is a positive constant.

Assumption 4.2. \(\{\vec{V}_i\}_{i \geq 1}\) is a stationary array of square-integrable random vectors with

\[
\vec{V}_1 + \vec{V}_2 + \ldots + \vec{V}_i \rightarrow \vec{V}, \quad \text{in probability (4.14)}
\]
as \(i \rightarrow \infty\), where \(\vec{V} = (\vec{V}^j > 0, 1 \leq j \leq 6)\) is a constant vector.

Assumption 4.3. Cancelations are uniformly distributed on every queue.

We will see in Section 4.3.2 that this assumption on cancellation is not critical, except for affecting the exact form of the fluid limit for the order position.

Now, we define the scaled net order flow process \(\vec{C}_n\) as follows,

\[
\vec{C}_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} \vec{V}_i = \left( \frac{1}{n} \sum_{i=1}^{N(nt)} V^j_i, 1 \leq j \leq 6 \right) .
\]

(4.15)

Theorem 4.2. Given Assumptions 4.1 and 4.2, for any \(T > 0\),

\[\vec{C}_n \Rightarrow \lambda \vec{V} e, \quad \text{in} \ (D^0[0,T], J_1) \quad \text{as} \ n \rightarrow \infty.\]

(4.16)

Now define the scaled queue lengths with \(Q^b_n\) for the best bid queue and \(Q^a_n\) for the best ask queue, and the scaled order position \(Z_n\) by

\[
\begin{cases}
Q^b_n(t) = Q^b_n(0) + C^1_n(t) - C^2_n(t) - C^3_n(t), \\
Q^a_n(t) = Q^a_n(0) + C^4_n(t) - C^5_n(t) - C^6_n(t), \\
dZ_n(t) = -dC^2_n(t) - \frac{Z_n(t-)}{Q^b_n(t-)} dC^3_n(t).
\end{cases}
\]

(4.17)

The above equations are straightforward: bid/ask queue lengths increase with limit orders and decrease with market orders and cancellations according to their corresponding order flow processes; an order position will decrease and move towards zero with arrivals of cancellations and market orders; new limit orders arrivals will not change this particular order position; however, arrival of limit orders may change the speed of the order position approaching zero following Assumption 4.3, hence the factor of \(\frac{Z_n(t-)}{Q^b_n(t-)}\).
Strictly speaking, (4.17) only describes the dynamics of the triple \((Q^b_n, Q^a_n, Z_n)\) before any of them hits zero. Nevertheless, \(Z_n\) hitting zero means that the order placed has been executed, while \(Q^a_n\) hitting zero means that the best ask queues is depleted. Since our primary interest is in the order position, without little risk we may truncate the processes to avoid unnecessary technical difficulties on the boundary. That is, define
\[
\tau_n = \min\{\tau^z_n, \tau^a_n, \tau^b_n\},
\]
with
\[
\begin{align*}
\tau^b_n &= \inf\{t \geq 0 : Q^b_n(t) \leq 0\}, \\
\tau^a_n &= \inf\{t \geq 0 : Q^a_n(t) \leq 0\}, \\
\tau^z_n &= \inf\{t \geq 0 : Z_n(t) \leq 0\}.
\end{align*}
\]

Now, define the truncated processes
\[
\begin{align*}
\tilde{Q}^b_n(t) &= Q^b_n(t \wedge \tau_n), \\
\tilde{Q}^a_n(t) &= Q^a_n(t \wedge \tau_n), \\
\tilde{Z}_n(t) &= Z_n(t \wedge \tau_n).
\end{align*}
\]

Still, it is not immediately clear that these truncated processes would be well defined either: we do not know \textit{a priori} if the term \(-\frac{Z_n(t-)}{Q^b_n(t-)}\) is bounded when \(Q^b_n\) hits zero. This turns out not to be an issue.

**Lemma 4.1.** \(Z_n(t) \leq Q^b_n(t)\) for any time \(t \leq \min(\tau^z_n, \tau^a_n)\). That is, \(\tau^z_n \leq \tau^b_n\). In particular, (4.20) is well defined.

**Proof.** Note that \(\vec{C}^\uparrow_n\) is a positive jumping process. Therefore, when \(\delta C^1_n(t) > 0\), \(\delta Q^b_n(t) > 0\) while \(\delta Z_n(t) = 0\). And when \(\delta C^2_n(t) > 0\), \(\delta Q^b_n(t) = \delta Z_n(t)\). When \(\delta C^3_n(t) > 0\), \(\delta Q^b_n(t) = \frac{\delta Z_n(t)}{Q^b_n(t-)}\). Hence, when \(0 < Z_n(t-) \leq Q^b_n(t-)\), we have \(Z_n(t) \leq Q^b_n(t)\). 

This lemma, though simple, turns out to play an important role to ensure that fluid limits of order positions and related queues are well defined after rescaling. That is, we can extend the definition of \(\tilde{Q}^b_n, \tilde{Q}^a_n, \) and \(\tilde{Z}_n\) for any time \(t \geq 0\).

For simplicity, we will use for the rest of the paper with a bit of abuse of notations, \(Q^b_n, Q^a_n,\) and \(Z_n\) instead of \(\tilde{Q}^b_n, \tilde{Q}^a_n,\) and \(\tilde{Z}_n\), defined on \(t \geq 0\). The dynamics of the truncated processes could be described in the following matrix form.

\[
d \begin{pmatrix} Q^b_n(t) \\ Q^a_n(t) \\ Z_n(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -\frac{Z_n(t-)}{Q^b_n(t-)} & 0 & 0 & 0 \end{pmatrix} \mathbb{I}_{Q^b_n(t-), Q^a_n(t-), Z_n(t-)>0} \cdot d\vec{C}^\uparrow_n(t)
\]

(4.21)
The modified processes coincide with the original processes before hitting zero, which implies \( I_{t \leq \tau_n} = I_{Q_n^b(t-) > 0, Q_n^a(t-) > 0, Z_n(t-) > 0} \).

In order to establish the fluid limit for the joint process \((Q_n^b, Q_n^a, Z_n)\), we see that it is fairly standard to establish the the limit process for \((Q_n^b, Q_n^a)\) from the classical probability theory where various forms of functional strong law of large numbers exist. However, checking (4.21) for \(Z_n(t)\), we see that in order to pass from the fluid limit for \(Q_n^b(t)\) to that for \(Z_n(t)\), we effectively need to pass the convergence relation between some càdlàg processes \((X_n, Y_n)\) to \((X, Y)\) in the Skorohod topology to the convergence relation between \(\int X_n Y_n dX_n\) to \(\int X Y dY\). That is, given a sequence of stochastic process \(\{X_n\}_{n \geq 1}\) defined by a sequence of SDEs

\[
X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)dY_n(s),
\]

where \(\{U_n\}_{n \geq 1}, \{Y_n\}_{n \geq 1}\) are two sequences of stochastic processes and \(\{F_n\}_{n \geq 1}\) is a sequence of functionals, and assume that \(\{U_n, Y_n, F_n\} \Rightarrow \{U, Y, F\}\) as \(n \to \infty\) in some way, then would the sequence of the solutions to (4.22) converge to the solution to

\[
X(t) = U(t) + \int_0^t F(X, s-)dY(s) ?
\]

It turns out that such a convergence relation is delicate and can easily fail, as shown by the following simple example.

**Example 4.1.** Let \(\{X_i\}_{i \geq 1}\) be a sequence of identically distributed random variables taking values in \([-1,1]\) such that

\[
\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2},
\]

\[
\mathbb{P}(X_{i+1} = 1|X_i = 1) = \mathbb{P}(X_{i+1} = -1|X_i = 1) = \frac{1}{4} \quad \text{for} \quad i = 1 \geq 1.
\]

Define \(S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i\). Then it is easy to see that \(S_n(t)\) converges to \(\sqrt{3}B(t)\).

Now define a sequence of SDE’s \(dY_n(t) = Y_n(t)dS_n(t)\) with \(Y_n(0) = 1\). Clearly \(Y_n(t) = \prod_{i=1}^{\lfloor nt \rfloor} (1 + \frac{X_i}{\sqrt{n}})\) and \(Y_n(t)\) converges to \(\exp\{\sqrt{3}B(t) - \frac{t}{2}\}\), as \(n \to \infty\).

However, the solution to \(dY(t) = Y(t)d(\sqrt{3}B(t))\) with \(Y(0) = 1\) is given by \(Y(t) = \exp\{\sqrt{3}B(t) - \frac{t^2}{2}\}\).

Nevertheless, under proper conditions as specified in Assumptions 4.1, 4.2, 4.3 one can establish the desired convergence relation. Such assumptions prove to be sufficient, in light of Theorem A.3 in Appendix A by Kurtz and Protter (1991).
Theorem 4.3. Given Assumptions 4.1, 4.2, and 4.3. If there exist constants $q^b$, $q^a$, and $z$ such that

\[(Q_n^b(0), Q_n^a(0), Z_n(0)) \Rightarrow (q^b, q^a, z),\]

(4.23)

then for any $T > 0$,

\[(Q_n^b, Q_n^a, Z_n) \Rightarrow (Q^b, Q^a, Z) \quad \text{in} \quad (\mathcal{D}^3[0,T], J_1),\]

where $(Q^b, Q^a, Z)$ is given by

\[Q^b(t) = q^b - \lambda v^b(t \wedge \tau), \quad (4.24)\]
\[Q^a(t) = q^a - \lambda v^a(t \wedge \tau), \quad (4.25)\]

and for $t < \tau$,

\[\frac{dZ(t)}{dt} = -\lambda \left( \bar{V}^2 + \bar{V}^3 \frac{Z(t-)}{Q^b(t-)} \right), \quad Z(0) = z. \quad (4.26)\]

Here $\tau = \min\{\tau^a, \tau^b, \tau^z\}$ with

\[\tau^a = \frac{q^a}{\lambda v^a}, \quad \tau^b = \frac{q^b}{\lambda v^b}, \quad (4.27)\]

and

\[
\tau^z = \begin{cases} 
(\frac{(1 + c)z}{a} + b)^{c/(c+1)} & c \notin \{-1,0\}, \\
(1 - e^{-\frac{z}{ab}}) & c = -1, \\
\frac{z}{ab} + 1 & c = 0.
\end{cases} \quad (4.28)
\]

Moreover, if $v^b > 0$, $v^a > 0$, and $q^a/v^a > q^b/v^b$. Then $\tau^z_n \rightarrow \tau^z$ a.s. as $n \rightarrow \infty$.

Here

\[a = \lambda \bar{V}^2, \quad b = \frac{q^b}{(\lambda \bar{V}^3)}, \quad c = -\frac{v^b}{\bar{V}^3}, \quad (4.29)\]
\[v^b = -\bar{V}^1 + \bar{V}^2 + \bar{V}^3, \quad v^a = -\bar{V}^4 + \bar{V}^5 + \bar{V}^6. \quad (4.30)\]

The following figure is an illustration of the fluid limits of $(Q_n^b(t), Q_n^a(t), Z(t))$, with $Q_n^b(0) = Q_n^a(0) = Z(0) = 100$, $\lambda = 1$, $\bar{V}^1 = \bar{V}^4 = 1$, $\bar{V}^2 = 0.6$, $\bar{V}^3 = 0.8$, $\bar{V}^5 = 0.7$, $\bar{V}^6 = 0.8$.

For the following graph, $\bar{V}^3 = 1.3$ and $\bar{V}^2$ varies from 1.3 to 3.3.

For the following graph, $\bar{V}^2 = 1.3$ and $\bar{V}^3$ varies from 1.3 to 3.3.
Figure 4.1: Illustration of the fluid limit \((Q^b(t), Q^a(t), Z(t))\)

Figure 4.2: Illustration of the ratio \(Z(t)/Q^b(t)\) with different \(\bar{V}^2\)
4.3.2 Discussions

General assumptions for cancellation.

In the above section, we have derived the fluid limit for the order positions under the simple assumption that cancellation is uniform on the queue. This assumption can be easily relaxed and the analysis and the form of fluid limit can be modified accordingly. For instance, one may assume (more realistically) that the closer the order to the queue head, the less likely they are being cancelled. As result, we may replace the term $\frac{Z_n(t)}{Q_n(t)}$ in (4.17) with $\Upsilon \left( \frac{Z_n(t)}{Q_n(t)} \right)$ where $\Upsilon$ is a Lipschitz continuous increasing function from $[0, 1]$ to $[0, 1]$ with $\Upsilon(0) = 0$ and $\Upsilon(1) = 1$. That is, the dynamics of the scaled processes are described as

$$
\frac{d}{dt} \begin{pmatrix} Q_n^h(t) \\ Q_n^a(t) \\ Z_n(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -\Upsilon \left( \frac{Z_n(t)}{Q_n(t)} \right) & 0 & 0 & 0 \end{pmatrix} \cdot I_{Q_n(t)>0} \cdot d\vec{C}_n(t) \tag{4.31}
$$

Figure 4.3: Illustration of the ratio $Z(t)/Q^b(t)$ with different $\bar{V}^3$
Then the limit processes would follow

\[
\begin{pmatrix}
Q_b(t) \\
Q_a(t) \\
Z(t)
\end{pmatrix} =
\begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \\
0 & -1 & -1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{Z(t{-})}{Q^b(t{-})} \\
\frac{Q^a(t{-})}{Q^b(t{-})} \\
\end{pmatrix}
\cdot \mathbb{I}_{Q^a(t{-}), Q^b(t{-}), Z(t{-}) > 0} \cdot dC(t)
\]

(4.32)

**Theorem 4.4.** Given Assumptions 4.1 and 4.2, and also the scaled processes \((Q^b_n, Q^a_n, Z_n)\) defined by (4.31). If there exist constants \(q^b, q^a, \) and \(z\) such that

\[
(Q^b_n(0), Q^a_n(0), Z_n(0)) \Rightarrow (q^b, q^a, z),
\]

(4.33)

then for any \(T > 0,\)

\[
(Q^b_n, Q^a_n, Z_n) \Rightarrow (Q^b, Q^a, Z) \quad \text{in} \quad (D^3[0, T], J_1),
\]

where \((Q^b, Q^a, Z)\) is defined by (4.32) and

\[
(Q^b(0), Q^a(0), Z(0)) = (q^b, q^a, z).
\]

(4.34)

Linear dependence between the order arrival and the trading volume.

One may also replace Assumption 4.1 by the assumption that order arrival rate is linearly correlated with trading volumes. The fluid limit can be analyzed in a similar way with few modifications.

**Assumption 4.4.** \(\{N(nt)\}_{t \geq 0}\) is a simple point process with an intensity \(n\lambda + \alpha nQ^a_n(t-) + \beta nQ^b_n(t-)\) at time \(t\), where \(\alpha, \beta\) are positive constants.

**Assumption 4.5.** For any \(1 \leq j \leq 6, \{V^j_i\}_i \geq 1\) is a sequence of stationary, ergodic and uniformly bounded sequence. Moreover, for any \(i \geq 2,\)

\[
\mathbb{E}[\bar{V}_i | G_{i-1}] = \bar{V}.
\]

(4.35)

**Theorem 4.5.** Given Assumptions 4.2, 4.3, 4.4, and 4.5, then Theorem 4.3 holds except that the limit process will be replaced by

\[
Q^b(t) = -\frac{\alpha q^a v^b - \alpha q^b v^a + \lambda v^b}{v^a \alpha + v^b \beta} + \frac{v^b (\beta q^b + \alpha q^a + \lambda)}{\beta v^b + \alpha v^a} e^{-(v^b \beta + v^a \alpha) t \wedge \tau},
\]

(4.36)

\[
Q^a(t) = -\frac{\beta q^a v^a - \beta q^a v^b + \lambda v^a}{v^a \alpha + v^b \beta} + \frac{v^a (\beta q^b + \alpha q^a + \lambda)}{\beta v^b + \alpha v^a} e^{-(v^b \beta + v^a \alpha) t \wedge \tau},
\]

(4.37)
\[ Z(t) = z e^{-\int_{0}^{t} \tilde{V}_3 \left[ \frac{\lambda}{Q^b(s)} + \beta + \frac{\alpha Q^a(s)}{Q^a(s)} \right] ds} \]

\[ - \int_{0}^{t} \tilde{V}_2 \left[ \lambda + \beta Q^b(s) + \alpha Q^a(s) \right] e^{-\int_{s}^{t} \tilde{V}_3 \left[ \frac{\lambda}{Q^b(u)} + \beta + \frac{\alpha Q^a(u)}{Q^a(u)} \right] du} ds. \]

**Corollary 4.1.** Given Assumptions 4.2, 4.3, 4.4, and 4.5. Assume further that \( v^b \beta + v^a \alpha > 0 \) and \(-\frac{\lambda v^b}{\alpha} < \frac{q^a}{v^b} - \frac{q^b}{v^a} < \frac{\lambda v^b}{\alpha}\). Then \( Q^b(t) \) and \( Q^a(t) \) will hit zero at some finite times \( \tau^b \) and \( \tau^a \) respectively. Moreover,

\[ \tau^b = -\frac{1}{v^b \beta + v^a \alpha} \log \left( \frac{v^b \lambda + q^a v^b \alpha - q^b v^a \alpha}{v^b \beta q^b + v^b \alpha q^a + \lambda v^b} \right), \]

\[ \tau^a = -\frac{1}{v^b \beta + v^a \alpha} \log \left( \frac{-q^a v^b \beta + q^b v^a \beta + \lambda v^a}{\beta q^b v^a + \alpha q^a v^a + \lambda v^a} \right), \]

and \( \tau^z \) is determined via the equation

\[ z = \int_{0}^{\tau^z} \tilde{V}_2 \left( \lambda + \beta Q^b(s) + \alpha Q^a(s) \right) e^{-\int_{s}^{\tau^z} \tilde{V}_3 \left[ \frac{\lambda}{Q^b(u)} + \beta + \frac{\alpha Q^a(u)}{Q^a(u)} \right] du} ds. \]
process. Here, to establish appropriate forms of FCLTs for $\{\vec{V}_i\}_{i \geq 1}$, we used the result in Burton et al. (1986). Readers can find more details in the framework of Bulinski and Shashkin (Bulinski and Shashkin (2007), Chapter 5, Theorem 1.5).

**Assumption 4.6.** $\{N(t)\}$ is independent of $\{\vec{V}_i\}_{i \geq 1}$.

**Assumption 4.7.** $\{N(i, i + 1)\}_{i \in \mathbb{Z}}$ is a stationary and ergodic sequence, with $\lambda := \mathbb{E}[N(0, 1)] < \infty$, and

$$\sum_{n=1}^{\infty} \|\mathbb{E}[N(0, 1)] - \lambda F_{-n}^{-\infty}\|_2 < \infty,$$

(4.42)

where $\|Y\|_2 = (\mathbb{E}[Y^2])^{1/2}$ and $F_{-n}^{-\infty} := \sigma(N(i, i + 1), i \leq -n)$.

**Assumption 4.8.** Let $n \in \mathbb{N}$ and $\mathcal{M}(n)$ denote the class of real-valued bounded coordinate-wise non-decreasing Borel functions on $\mathbb{R}^n$. Let $|I|$ denote the cardinality of $I$ when $I$ is a set, and $||\cdot||$ denote the $L^\infty$-norm. Let $\{\vec{V}_i\}_{i \geq 1}$ be a stationary sequence of $\mathbb{R}^6$ valued random vectors and for any finite set $I \subset \mathbb{N}$, $J \subset \mathbb{N}$, and any $f, g \in \mathcal{M}(6|I|)$, one has

$$\text{Cov}(f(\vec{V}_I), g(\vec{V}_J)) \geq 0.$$  

Moreover, for $1 \leq j \leq 6,$

$$v_j^2 = \text{Var}(V_1^j) + 2 \sum_{i=2}^{\infty} \text{Cov}(V_1^j, V_i^j) < \infty.$$  

(4.43)

Note that an i.i.d. sequence $\{\vec{V}_i\}_{i \geq 1}$ clearly satisfies the above assumption if $\vec{V}_1$ is square-integrable. It is not difficult to see that Assumption 4.7 implies Assumption 4.1, and Assumption 4.8 implies Assumption 4.2.

With these assumptions, we can define the centered and scaled net order flow $\vec{\Psi}_n = (\vec{\Psi}_n(t), t \geq 0)$ by

$$\vec{\Psi}_n(t) = \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{N(nt)} \vec{V}_i - \lambda \vec{V} nt \right) = \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{N(nt)} V_i^j - \lambda \vec{V}^j nt, 1 \leq j \leq 6 \right).$$  

(4.44)

Here,

$$\vec{V} = (\vec{V}^j, 1 \leq j \leq 6) = (\mathbb{E}[V_j^j], 1 \leq j \leq 6)$$  

(4.45)

is the mean vector of order sizes.
Next, define $R^b_n$ and $R^a_n$, the time rescaled queue length for the best bid and best ask respectively, by

$$
dR^b_n(t) = d(\Psi^1_n(t) + \lambda \vec{V}^1 t) - d(\Psi^2_n(t) + \lambda \vec{V}^2 t) - d(\Psi^3_n(t) + \lambda \vec{V}^3 t),
$$

$$
dR^b_n(t) = d(\Psi^4_n(t) + \lambda \vec{V}^4 t) - d(\Psi^5_n(t) + \lambda \vec{V}^5 t) - d(\Psi^6_n(t) + \lambda \vec{V}^6 t).
$$

The definition of the above equations is intuitive just as their fluid limit counterparts. The only modification here is that the drift terms is added back to the dynamics of the queue lengths because $\vec{\Psi}$ has been re-centered. The equations can also be written in a more compact matrix form,

$$
d\begin{pmatrix} R^b_n(t) \\
R^a_n(t) \end{pmatrix} = A \cdot d\begin{pmatrix} \vec{\Psi}_n(t) + \lambda \vec{V} t \end{pmatrix}, \quad \text{(4.46)}
$$

with the linear transformation matrix

$$
A = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}. \quad \text{(4.47)}
$$

However, (4.46) may not be well defined, unless $R^b_n(t) > 0$ and $R^a_n(t) > 0$. As in the fluid limit analysis, one may truncate the process at the time when one of the queues vanishes. That is, define

$$
i^a_n = \inf\{t : R^a_n(t) \leq 0\}, \quad i^b_n = \inf\{t : R^b_n(t) \leq 0\}, \quad i_n = \inf\{i^a_n, i^b_n\}, \quad \text{(4.48)}
$$

and define the truncated process $(R^b_n, R^a_n)$ by

$$
d\begin{pmatrix} R^b_n(t) \\
R^a_n(t) \end{pmatrix} = A\mathbb{I}_{t \leq i_n} \cdot d\begin{pmatrix} \vec{\Psi}_n(t) + \lambda \vec{V} t \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} R^b_n(0) \\
R^a_n(0) \end{pmatrix} = \begin{pmatrix} R^b_n(0) \\
R^a_n(0) \end{pmatrix}, \quad \text{(4.49)}
$$

Now, we will show

**Theorem 4.6.** Given Assumptions 4.6, 4.7, and 4.8 for any $T > 0$,

$$
\vec{\Psi}_n \Rightarrow \vec{\Psi} \overset{d}{=} \Sigma \vec{W} \circ \lambda e - \vec{V} v_d \vec{W}_1 \circ \lambda e \quad \text{in} \quad (D^6[0, T], J_1). \quad \text{(4.50)}
$$

Here $\vec{W}_1$ is a standard scalar Brownian motion, $v_d$ is given by Eqn. (A.71), $\vec{W}$ is a standard six-dimensional Brownian motion independent of $\vec{W}_1$, $\circ$ denotes the composition of functions, and $\Sigma$ is given by $\Sigma \Sigma^T = (a_{jk})$ with

$$
a_{jk} = \begin{cases} v_j^2 & \text{for } j = k, \\ \rho_{j,k} v_j v_k & \text{for } j \neq k, \end{cases} \quad \text{(4.51)}
$$
\[ v_j^2 = \text{Var}(V_j^1) + 2 \sum_{i=2}^{\infty} \text{Cov}(V_j^1, V_i^1), \]
\[ \rho_{j,k} = \frac{1}{v_j v_k} \left( \text{Cov}(V_j^1, V_1^1) + \sum_{i=2}^{\infty} \left( \text{Cov}(V_j^1, V_i^1) + \text{Cov}(V_i^1, V_j^1) \right) \right). \]

That is, \( \vec{\Psi} = (\Psi^j, 1 \leq j \leq 6) \) is a six-dimensional Brownian motion with zero drift and variance-covariance matrix \( (\lambda \Sigma^T \Sigma + \lambda v_2^2 \vec{V} \cdot \vec{V}^T) \).

- If \( (R^n_b(0), R^n_a(0)) \Rightarrow (q^b, q^a) \), then for any \( T > 0 \),
  \[ \left( \begin{array}{c} R^n_b \\ R^n_a \end{array} \right) \Rightarrow \left( \begin{array}{c} R^b \\ R^a \end{array} \right) \text{ in } (D^2[0, T], J_1). \] (4.53)

Here, the diffusion limit process \( (R^b, R^a)^T \) up to the first hitting time of the boundary is a two-dimensional Brownian motion with drift \( \vec{\mu} \) and the variance-covariance matrix as
\[ \vec{\mu} := (\mu_1, \mu_2)^T = \lambda A \cdot \vec{V} \quad \text{and} \quad \sigma \sigma^T := A \cdot (\lambda \Sigma^T \Sigma + \lambda v_2^2 \vec{V} \cdot \vec{V}^T) \cdot A^T. \] (4.54)

### 4.4.2 Applications to LOB

**Examples**

Having established the fluid limit and the fluctuations of the queue lengths and order positions, we will give some examples of the order arrival process \( N(t) \) that satisfy the assumptions in our analysis.

**Example 4.2 (Poisson Process).** Let \( N(t) \) be a Poisson process with intensity \( \lambda \). Clearly assumptions 4.1 and 4.7 are satisfied.

**Example 4.3 (Hawkes Process).** Let \( N(t) \) as a Hawkes process Brémaud and Massoulié (1996), a simple point process with intensity
\[ \lambda(t) := \lambda \left( \int_{-\infty}^{t} h(t-s)N(ds) \right), \] (4.55)
at time \( t \), where we assume that \( \lambda(\cdot) : R_{\geq 0} \rightarrow R^+ \) is an increasing function, \( \alpha \)-Lipschitz, where \( \alpha \| h \|_{L^1} < 1 \) and \( h(\cdot) : R_{\geq 0} \rightarrow R^+ \) is a decreasing function and
\[ \int_0^\infty h(t)dt < \infty. \] Under these assumptions, there exists a stationary and ergodic Hawkes process satisfying the dynamics \((4.55)\) (see e.g. Brémaud and Massoulié (1996)). By the Ergodic theorem,

\[ \frac{N(t)}{t} \to \lambda := E[N(0,1)], \tag{4.56} \]

a.s. as \( t \to \infty \). Therefore, the Assumption \((4.1)\) is satisfied. It was proved in Zhu (2013), that \( \{N(i, i + 1]\}_{i \in \mathbb{Z}} \) satisfies the Assumption \((4.7)\) and hence \( \frac{N_n - \lambda n}{\sqrt{n}} \Rightarrow v_d W_1(\cdot) \), on \((D[0,T], J_1)\) as \( n \to \infty \).

In the special case \( \lambda(z) = \nu + z \), \((4.55)\) becomes

\[ \lambda(t) = \nu + \int_{-\infty}^{t} h(t - s)N(ds), \tag{4.57} \]

which is the original self-exciting point process proposed by Hawkes (1971), where \( \nu > 0 \) and \( \|h\|_{L^1} < 1 \). In this case,

\[ \lambda = \frac{\nu}{1 - \|h\|_{L^1}}, \quad v^2_d = \frac{\nu}{(1 - \|h\|_{L^1})^2}. \tag{4.58} \]

**Example 4.4** (Cox Process with Shot Noise Intensity). Let \( N(t) \) be a Cox process with shot noise intensity (see e.g. Asmussen and Albrecher (2010)). That is, \( N(t) \) is a simple point process with intensity at time \( t \) given by

\[ \lambda(t) = \nu + \int_{-\infty}^{t} g(t - s)\bar{N}(ds), \tag{4.59} \]

where \( \bar{N} \) is a Poisson process with intensity \( \rho \), \( g(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^+ \) is decreasing, \( \|g\|_{L^1} < \infty \), and \( \int_0^\infty tg(t)dt < \infty \). \( N(t) \) is stationary and ergodic and

\[ \frac{N(t)}{t} \to \lambda := \nu + \rho \|g\|_{L^1}, \tag{4.60} \]

a.s. as \( t \to \infty \). Therefore, Assumption \((4.1)\) is satisfied. Moreover one can check that condition \((4.42)\) in Assumption \((4.7)\) is satisfied. Indeed, by stationarity,

\[ \|E[N(0,1) - \lambda(F_{-\infty}^{-\infty})]\|_2 = \|E[N(n - 1, n - \lambda(F_{-\infty}^{-\infty})]\|_2. \tag{4.61} \]

We have

\[ E[N(n - 1, n - \lambda(F_{-\infty}^{-\infty}) = E \left[ \int_{n-1}^{n} \lambda(t)dt - \lambda(F_{-\infty}^{-\infty}) \right], \tag{4.62} \]

where

\[ \lambda(t) = \nu + \int_{-\infty}^{0} g(t - s)\bar{N}(ds) + \int_{0}^{t} g(t - s)\bar{N}(ds), \tag{4.63} \]
therefore,
\[
\mathbb{E}[N(n - 1, n) - \lambda |F_0^-] = \int_{n-1}^{n} \int_{-\infty}^{0} g(t - s) \bar{N}(ds)dt + \rho \int_{n-1}^{n} \int_{0}^{t} g(t - s) dsdt - \rho \|g\|_{L^1}. \tag{4.64}
\]

By Minkowski’s inequality,
\[
\|\mathbb{E}[N(n - 1, n) - \lambda |F_0^-]\|_2 \leq \left\| \int_{n-1}^{n} \int_{-\infty}^{0} g(t - s) \bar{N}(ds) dt \right\|_2 + \left\| \rho \int_{n-1}^{n} \int_{0}^{t} g(t - s) ds dt - \rho \|g\|_{L^1} \right\|_2. \tag{4.65}
\]

Note that
\[
\left\| \rho \int_{n-1}^{n} \int_{0}^{t} g(t - s) ds dt - \rho \|g\|_{L^1} \right\|_2 = \rho \int_{n-1}^{n} \int_{t}^{\infty} g(s) ds dt, \tag{4.66}
\]
therefore,
\[
\sum_{n=1}^{\infty} \left\| \rho \int_{n-1}^{n} \int_{0}^{t} g(t - s) ds dt - \rho \|g\|_{L^1} \right\|_2 = \int_{0}^{\infty} \int_{t}^{\infty} g(s) ds dt = \int_{0}^{\infty} t g(t) dt. \tag{4.67}
\]

Furthermore,
\[
\sum_{n=1}^{\infty} \left\| \int_{n-1}^{n} \int_{-\infty}^{0} g(t - s) \bar{N}(ds) dt \right\|_2 \leq \sum_{n=1}^{\infty} \left\| \int_{-\infty}^{0} g(n - 1 - s) \bar{N}(ds) \right\|_2 \tag{4.68}
\]
\[
= \sum_{n=1}^{\infty} \sqrt{\int_{-\infty}^{0} g^2(n - 1 - s) ds} + \rho^2 \left( \int_{-\infty}^{0} g(n - 1 - s) ds \right)^2 \leq \sum_{n=1}^{\infty} \sqrt{\int_{-\infty}^{0} g^2(n - 1 - s) ds} + \sum_{n=1}^{\infty} \rho \int_{-\infty}^{0} g(n - 1 - s) ds \leq \sqrt{\rho} \sum_{n=1}^{\infty} \sqrt{g(n - 1)} \sqrt{\int_{-\infty}^{0} g(n - 1 - s) ds} + \rho \int_{0}^{\infty} t g(t) dt \leq \sqrt{\frac{\rho}{4}} \left[ \sum_{n=1}^{\infty} g(n - 1) + \sum_{n=1}^{\infty} \int_{-\infty}^{0} g(n - 1 - s) ds \right] + \rho \int_{0}^{\infty} t g(t) dt \leq \sqrt{\frac{\rho}{4}} \left[ g(0) + \|g\|_{L^1} + \int_{0}^{\infty} t g(t) dt \right] + \rho \int_{0}^{\infty} t g(t) dt < \infty.
\]

Hence Assumption 4.7 is satisfied. $\frac{N_n - \lambda n}{\sqrt{n}} \Rightarrow v_d W_1(\cdot)$ on $(D[0, T], J_1)$ as $n \to \infty$, where
\[
v_d^2 = \nu + \rho \|g\|_{L^1} + \rho \|g^2\|_{L^1}. \tag{4.69}
\]
Probability of price increase and hitting times.

Given the diffusion limit to the queue lengths for the best bid and ask, we can also compute the distribution of the first hitting time $\tau$ and the probability of price increase/decrease. Our results generalize those in Cont and De Larrard (2013) which correspond to the special case of zero drift.

Given Theorem 4.6, let us first parameterize $\sigma$ by

$$
\sigma = \left( \begin{array}{cc} \sigma_1 \sqrt{1 - \rho^2} & \sigma_1 \rho \\ 0 & \sigma_2 \end{array} \right).
$$

Next, denote $I_\nu$ the Bessel function of the first kind of order $\nu$ and $\nu_n := n\pi/\alpha$, and define

$$
\alpha := \begin{cases} 
\frac{\pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right)}{2} & \rho > 0, \\
\frac{\pi}{2} & \rho = 0, \\
\tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \rho < 0,
\end{cases}
$$

(4.70)

$$
r_0 := \sqrt{\frac{(q^b/\sigma_1)^2 + (q^a/\sigma_2)^2 - 2\rho(q^b/\sigma_1)(q^a/\sigma_2)}{1 - \rho^2}},
$$

(4.71)

$$
\theta_0 := \begin{cases} 
\frac{\pi + \tan^{-1}\left(q^a/\sigma_2\sqrt{1-\rho^2}/q^b/\sigma_1 - \rho q^a/\sigma_2\right)}{\pi/2} & q^b/\sigma_1 < \rho q^a/\sigma_2, \\
\frac{\pi}{2} & q^b/\sigma_1 = \rho q^a/\sigma_2, \\
\tan^{-1}\left(-\frac{q^a/\sigma_2\sqrt{1-\rho^2}/q^b/\sigma_1 - \rho q^a/\sigma_2}{\pi/2}\right) & q^b/\sigma_1 > \rho q^a/\sigma_2.
\end{cases}
$$

(4.72)

Then according to Zhou (2001), we have

**Corollary 4.2.** Given Theorem 4.6 and the initial state $(q^b, q^a)$, the distribution of the first hitting time $\tau$

$$\mathbb{P}_{\mu}(\tau > t) = \frac{2}{\alpha t} e^{l_1 q^b + l_2 q^a + l_3 t} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) e^{-\frac{\pi^2}{\alpha t}} \int_0^\alpha \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta,
$$

(4.73)

where

$$
g_n(\theta) := \int_0^\infty r e^{-\frac{r^2}{\alpha t}} e^{l_1 r \sin(\theta - \alpha) - l_2 r \cos(\theta - \alpha)} I_{\frac{r^2}{\alpha t}} \left(\frac{rr_0}{t}\right) dr,
$$

\[
\begin{align*}
l_1 &:= -\frac{\mu_1 \sigma_2 + \rho \mu_2 \sigma_1}{(1 - \rho^2)\sigma_1^2 \sigma_2}, & l_2 &:= \frac{\rho \mu_1 \sigma_2 - \mu_2 \sigma_1}{(1 - \rho^2)\sigma_2^2 \sigma_1}, \\
l_3 &:= \frac{l_1^2 \sigma_1^2}{2} + \rho l_1 l_2 \sigma_1 \sigma_2 + \frac{l_2^2 \sigma_2^2}{2} + l_1 \mu_1 + l_2 \mu_2, \\
l_4 &:= l_1 \sigma_1 + \rho l_2 \sigma_2, & l_5 &:= l_2 \sigma_2 \sqrt{1 - \rho^2}.
\end{align*}
\]
Note that when $\mu > 0$, it is possible to have $P_{\mu}(\iota = \infty) > 0$. This means the measure $P_{\mu}$ here might be a sub-probability measure, depending on the value of $\mu$. In that case, $P_{\mu}(\iota > t)$ actually includes $P_{\mu}(\iota = \infty)$.

Moreover, based on the results in Iyengar (1985) and Metzler (2010),

Corollary 4.3. Given Theorem 4.6 and the initial state $(q^b, q^a)$, the probability of price decrease given

$$P_{\mu}(\iota^b < \iota^a) = \int_0^\infty \int_0^\infty \exp(\mu_1(r \cos \alpha - q^b/\sigma_1) + \mu_2(r \sin \alpha - q^a/\sigma_2) - |\mu|^2 t/2)g(t, r)drdt,$$

(4.74)

where

$$g(t, r) = \frac{\pi}{\alpha^2} e^{-(r^2 + r_0^2)/4t} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi (\pi - \theta_0)}{\alpha} \right) I_n \left( \frac{rr_0}{t} \right).$$

(4.75)

Similarly, when $\mu > 0$, with positive probability, we might have $\tau^b = \infty$ and $\tau^a = \infty$. Therefore $P_{\mu}(\iota^b < \iota^a)$ we compute here implicitly refer to $P_{\mu}(\iota^b < \iota^a$ and $\tau^b < \infty$ in that case.

Note that both expressions for $\iota$ and the probability of price decrease are semi-analytic. However, in the special case of $\mu = 0$, i.e., when $V_1 = V_2 + V_3$ and $V_4 = V_5 + V_6$, they become analytic.

Corollary 4.4. Given Theorem 4.6 and the initial state $(q^b, q^a)$. If $\mu = 0$, then

$$P(\iota > t) = \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{n: \text{odd}} \frac{1}{n} \sin \frac{n\pi \theta_0}{\alpha} \left[ I_{n-1}(l_0^2/4t) + I_{n+1}(l_0^2/4t) \right].$$

(4.76)

Corollary 4.5. Given Theorem 4.6 and the initial state $(q^b, q^a)$. If $\mu = 0$, the probability that the price decreases is $\frac{\theta_0}{\alpha}$.

Proof.

$$P(\iota^b < \iota^a) = \int_0^\infty \frac{(r/r_0)^{\pi/\alpha} - 1}{\sin^2(\pi \theta_0/\alpha) + [(r/r_0)^{\pi/\alpha} + \cos(\pi \theta_0/\alpha)]^2} \frac{dr}{\alpha r_0}$$

$$= \int_0^\infty \frac{\sin(\pi \theta_0/\alpha)}{\sin^2(\pi \theta_0/\alpha) + [(r/r_0)^{\pi/\alpha} + \cos(\pi \theta_0/\alpha)]^2} \frac{d(r/r_0)^{\pi/\alpha}}{\pi}$$

$$= \int_0^\infty \frac{\sin(\pi \theta_0/\alpha)}{\sin^2(\pi \theta_0/\alpha) + [x + \cos(\pi \theta_0/\alpha)]^2} \frac{dx}{\pi} = \frac{\theta_0}{\alpha}.$$

$\square$
4.4.3 Remarks and discussions

Remark 4.1. Assumption 4.6 in Theorem 4.6 may be relaxed to allow dependence between the arrival process $N$ and the order size sequence $\{V_i\}_{i \geq 1}$ as long as $(\Phi^D_n, \Phi^V_n)$ is guaranteed to converge jointly.

Remark 4.2. Theorem 4.6 is different from diffusion limits in Cont and De Larrard (2012). The difference comes from Assumption 3.2 in Cont and De Larrard (2012), which assumes that the mean of the aggregated queue length is relatively small compared to its variance. This assumption does not in general hold in our framework where individual order type is considered instead of aggregations of market, limit and cancellation orders. Without this assumption, we need different scaling method and proof. Nevertheless, if we have to impose Assumption 3.2 as in Cont and De Larrard (2012), then our result will be reduced to theirs because the second term in Equation (4.50) would then vanish.

From the above remarks, it is clear that there are more than one possible alternative sets of assumptions under which appropriate forms of diffusion limits may be derived. For instance, one may impose a weaker condition than Assumption 4.7 for $\{D_i\}_{i \geq 1}$.

Assumption 4.9. For any time $t$,

$$\lim_{n \to \infty} \frac{N(nt)}{n} = \lambda t, \text{ a.s.} \quad (4.77)$$

Moreover, there exists $K > 0$, such that $\mathbb{E}[N(t)] \leq Kt$, for any $t$.

This assumption holds, for example, if the point process $N(t)$ is stationary and ergodic with finite mean. To compensate for the weakened assumption 4.9, one may need a stronger condition on $\{V_i\}_{i \geq 1}$, for instance, Assumption 4.5.

Note that under this alternative set of assumptions, the resulting limit process will in fact be simpler than Theorem 4.6. This is because Assumption 4.5 implies that $V_i^j$ is actually uncorrelated to $V_i^j$ for any $i \neq i'$ and $1 \leq j \leq 6$. Hence the covariance of $V_i^j$ and $V_i^j$, $i \geq 2$ in the limit process may vanish. We illustrate this in details as follows.

Take Assumptions 4.5 and 4.9, define a modified version of the scaled net order flow process $\Psi^*_n$ by

$$\Psi^*_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{N(nt)} (\vec{V}_i - \bar{V}), \quad (4.78)$$

while the scaled processes $R^a_n(t)$, $R^b_n(t)$ still follows (4.46), the first hitting time the same as in (4.48), and the corresponding limit processes in (A.76), and (A.75). Then we have
Theorem 4.7. Given Assumptions 4.5, 4.6, and 4.9, then for any $T > 0$,

- $\Psi_n^* \Rightarrow \Psi^*$ where $\Psi^* = (\sigma_j W_j, 1 \leq j \leq 6)$, where $(W_j, 1 \leq j \leq 6)$ is a standard six-dimensional Brownian motion and $\sigma_j^2 = \lambda \text{Var}(V^j_1)$.

- $(R^b_n, R^a_n) \Rightarrow (R^b, R^a)$ in $(D^2[0, T], J_1)$.

4.4.4 Fluctuation analysis

Based on the diffusion and fluid limit analysis for the order position and related queues, it is possible to consider fluctuations of queues, order positions and execution times around their corresponding fluid limits.

Fluctuations of queues and order positions.

First, we consider fluctuation of $(Q^b_n, Q^a_n, Z_n)$ around its fluid limit $(Q^b, Q^a, Z)$.

Theorem 4.8. Given Assumptions 4.3, 4.6, 4.7, and 4.8

$$\sqrt{n} \begin{pmatrix} Q^b_n - Q^b \\ Q^a_n - Q^a \\ Z_n - Z \end{pmatrix} \Rightarrow \begin{pmatrix} \Psi^1 - \Psi^2 - \Psi^3 \\ \Psi^4 - \Psi^5 - \Psi^6 \\ Y \end{pmatrix}, \quad \text{in } (D^3[0, \tau], J_1) \quad (4.79)$$

as $n \to \infty$. Here $(Q^b_n, Q^a_n, Z_n), (Q^b, Q^a, Z)$ are given in (4.17) and Theorem 4.3, $(\Psi^j, 1 \leq j \leq 6)$ is given in (4.50), and $Y$ satisfies

$$dY(t) = \left( \frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{Q^b(t)} - Y(t) \right) \lambda V^3 dt - d\Psi^2(t) - \frac{Z(t)}{Q^b(t)} d\Psi^3(t), \quad (4.80)$$

with $Y(0) = 0$.

Fluctuations of execution and hitting times.

In addition, we can study the fluctuations of the execution time $\tau^z_n$.

Proposition 4.1. Given Assumptions 4.3, 4.6, 4.7, and 4.8, for any $x$ (say, $x < 0$),

$$\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau^z_n - \tau^z) \geq x) = \mathbb{P}(Y(\tau^z) > ax). \quad (4.81)$$
Proof. For any $x < 0$,

\begin{align}
\mathbb{P}(\sqrt{n}(\tau_n^z - \tau^z) \geq x) &= \mathbb{P}\left( Z_n\left( \tau^z + \frac{x}{\sqrt{n}} \right) > 0 \right) \\
&= \mathbb{P}\left( \sqrt{n}\left( Z_n\left( \tau^z + \frac{x}{\sqrt{n}} \right) - Z\left( \tau^z + \frac{x}{\sqrt{n}} \right) \right) > -\sqrt{n}Z\left( \tau^z + \frac{x}{\sqrt{n}} \right) \right).
\end{align}

Note that

\[ \lim_{n \to \infty} \sqrt{n}Z\left( \tau^z + \frac{x}{\sqrt{n}} \right) = xZ'\left( \tau^z \right), \]

and for any $t > 0$, $c \neq -1$, equations (A.58) and (4.28) lead to

\[ Z'(\tau^z) = -\frac{ac}{1+c} - \left( z + \frac{ab}{1+c} \right) b^\frac{1}{c+1} \left( \frac{(1+c)z}{a} + b \right)^{\frac{c+1}{c+1}} = -a. \]

Similarly, when $c = -1$, we have $Z(t) = [a \log(b-t) + \frac{z}{b} - a \log b](b-t)$. Thus $Z'(t) = -a - \left[ a \log(b-t) + \frac{z}{b} - a \log b \right]$, and $Z'(\tau^z) = -a - \left[ a \log(be^{-\frac{\tau^z}{b}}) + \frac{z}{b} - a \log b \right] = -a$. Finally, recall that $\sqrt{n}(Z_n(t) - Z(t)) \to Y(t)$ on $(D[0, \tau^z), J_1)$ as $n \to \infty$, hence the desired result. \hfill \Box

In fact, the above results can be more explicit because $Y(t)$ is a Gaussian process with zero mean and variance $\sigma^2_Y$, the latter of which can be computed explicitly, albeit in a messy form as follows.

**Proposition 4.2.** $Y(t)$ defined in Eqn (4.80) is a Gaussian process for $t < \tau^z$,
with mean 0 and variance $\sigma_Y^2(t)$. In particular, when $c < 0$ and $c \neq -1$,

\[
\sigma_Y^2(t) := \frac{(b + ct)^{\frac{3}{2}} + 1 - b t^{\frac{3}{2}} + 6}{(2 + c)(b + ct)^{\frac{3}{2}}} \sum_{j=1}^{6} \left[ \lambda \left( \Sigma_{2j} - \frac{\Sigma_{3j}a}{(1 + c)\lambda V^3} \right)^2 + \frac{\lambda^2}{6} \left( \frac{c}{1 + c} \right)^2 \bar{V}^2 \right] \\
+ \frac{b^2}{\lambda^2 V^3} \left( \frac{b + ct}{2} \right)^{\frac{1}{2}} \left( z + \frac{ab}{1 + c} \right) \\
\cdot \sum_{j=1}^{6} \left[ \lambda \left( \Sigma_{2j} - \frac{\Sigma_{3j}a}{(1 + c)\lambda V^3} \right) \Sigma_{3j} + \frac{\lambda^2}{6} \bar{V}^2 \right] \\
+ \frac{t}{(b + ct)^{\frac{3}{2} + 1} \lambda^2 (V^3)^2} \sum_{j=1}^{6} \left[ \lambda (\Sigma_{3j})^2 + \frac{\lambda^2}{6} (\bar{V})^2 \right] \left( z + \frac{ab}{1 + c} \right)^2 \\
- \frac{2a}{(b + ct)^{\frac{3}{2} + 1} (1 + c)\lambda V^3} \cdot \left[ \lambda \left( \frac{(b + ct)^{\frac{3}{2}} + 1 - b t^{\frac{3}{2}}}{2 + c} \right) \\
+ \left[ (\beta - \gamma)c \right] \left( b + ct \right) \left( b^{\frac{1}{2}} - b \right) + \frac{b}{b + ct} \left( (b + ct)^{\frac{1}{2} - 1} - b^{\frac{1}{2} - 1} \right) \right] \\
+ \frac{2}{(b + ct)^{\frac{3}{2}} + z^{\frac{1}{2}} + \frac{ab}{1 + c}} \frac{b^{\frac{1}{2}}}{\lambda V^3} \cdot \left[ \frac{\lambda}{\alpha} \left( (b + ct)^{\frac{1}{2}} - b^{\frac{1}{2}} \right) + \frac{t}{b(b + ct)} \right] \\
+ \frac{\hat{\gamma}}{c} \left[ \frac{\log b}{b} - \frac{\log(b + ct)}{b + ct} \right] \\
+ \frac{\hat{\delta}}{1 - c} \left[ (b + ct)^{\frac{1}{2} - 1} - b^{\frac{1}{2} - 1} \right] + \frac{\hat{\eta}}{2c} \left[ b^{\frac{1}{2} - 1} - (b + ct)^{\frac{1}{2} - 1} \right].
\]

Here

\[
\hat{\alpha} = \frac{\alpha}{c + 1}, \quad \hat{\beta} = -\frac{b^{\frac{1}{2} + 1}}{1 + c} - \gamma b^{\frac{1}{2}} + \frac{\delta}{bc} \frac{\beta}{c}, \quad \hat{\gamma} = \frac{\beta}{c}, \quad \hat{\delta} = \gamma, \quad \hat{\eta} = \frac{\delta}{c},
\]

with

\[
\begin{align*}
\alpha &:= -(\psi_{12} - \psi_{22} - \psi_{32}) + (\psi_{13} - \psi_{23} - \psi_{33}) \frac{a}{(1 + c)\lambda V^3} - \frac{a \varphi}{c(1 + c)\lambda V^3}, \\
\beta &:= -(\psi_{13} - \psi_{23} - \psi_{33}) \left( z + \frac{ab}{1 + c} \right) \frac{b^{\frac{1}{2}}}{\lambda V^3} + \left( z + \frac{ab}{1 + c} \right) \frac{\varphi b^{\frac{1}{2}}}{c\lambda V^3}, \\
\gamma &:= \frac{ab \varphi}{c(1 + c)\lambda V^3}, \quad \delta := -\varphi \left( z + \frac{ab}{1 + c} \right) \frac{b^{\frac{1}{2} + 1}}{c\lambda V^3}, \\
\varphi &:= \psi_{11} + \psi_{22} + \psi_{33} - \psi_{12} - \psi_{13} - \psi_{21} - \psi_{31} + \psi_{23} + \psi_{32}.
\end{align*}
\]
Remark 4.3. Proposition 4.2 only gives the formula for the variance of $Y(t)$ for the case $c \neq -1, c < 0$. The variance $\sigma^2_Y(t)$ for the case $c = -1$ can be taken as a continuum limit as $c \to -1$.

Corollary 4.6. Given Assumptions 4.3, 4.6, 4.7, and 4.8, then for any $x$ (say $x > 0$),

$$
\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau^{\pm}_n - \tau^{\mp}) \geq x) = 1 - \Phi \left( \frac{ax}{\sigma_Y(\tau^{\mp})} \right),
$$

(4.87)

where $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$ is the cumulative probability distribution function of a standard Gaussian random variable.

Proposition 4.3. Given Assumptions 4.6, 4.7, and 4.8, with $v^b, v^a > 0$. Then for any $x$ (say $x < 0$),

$$
\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau^b_n - \tau^b) \geq x) = 1 - \Phi \left( \sqrt{\frac{q^b \lambda v^b}{\psi_{11} + \psi_{22} + \psi_{33} - 2\psi_{12} - 2\psi_{13} + 2\psi_{23}}} x \right),
$$

(4.88)

where $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$ is the cumulative probability distribution function of normal random variable with mean zero and variance one, and

$$
\psi_{ij} := \sum_{k=1}^{6} \Sigma_{ijk} \Sigma_{jk} \lambda + V^i V^j v^2 \lambda^3, \quad 1 \leq i, j \leq 6.
$$

(4.89)

Moreover,

$$
\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau^a_n - \tau^a) \geq x) = 1 - \Phi \left( \sqrt{\frac{q^a \lambda v^a}{\psi_{44} + \psi_{55} + \psi_{66} - 2\psi_{45} - 2\psi_{46} + 2\psi_{56}}} x \right).
$$

(4.90)

Proof. Similar to the proof of the fluctuation of the execution time $\tau^\pm_n$, we can show that, for any $x < 0$,

$$
\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau^\pm_n - \tau^\pm) \geq x) = \mathbb{P}((\Psi^1 - \Psi^2 - \Psi^3)(\tau^b) > -(Q^b)'(\tau^b)x),
$$

(4.91)

From the expression of $Q^b, \tau^b$ in Eqns. (4.24), (4.27), and (4.50), it is clear that $(Q^b)'(\tau^b) = -q^b$ and the mean of $(\Psi^1 - \Psi^2 - \Psi^3)(t)$ is zero and the variance is

$$
(\psi_{11} + \psi_{22} + \psi_{33} - 2\psi_{12} - 2\psi_{13} + 2\psi_{23})t.
$$

(4.92)
Therefore,
\[
\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau_n^b - \tau^b) \geq x) = 1 - \Phi \left( \sqrt{\frac{q^b \lambda u^b}{\psi_{11} + \psi_{22} + \psi_{33} - 2\psi_{12} - 2\psi_{13} + 2\psi_{23}}} x \right).
\]
(4.93)

Similarly, we can show that (4.90) holds.

**Large deviations**

In addition to the fluctuation analysis in the previous section, one can further study the probability of the rare events that the scaled process \((Q_n^b(t), Q_n^a(t))\) deviates away from its fluid limit. Informally, we are interested in the probability \(\mathbb{P}(\mathcal{T}(Q_n^b(t), Q_n^a(t)) > \varepsilon)\) as \(n \to \infty\), where \((f^b(t), f^a(t))\) is a given pair of functions that can be different from the fluid limit \((Q^b(t), Q^a(t))\).

Recall that a sequence \((P_n)_{n \in \mathbb{N}}\) of probability measures on a topological space \(\mathbb{X}\) satisfies the large deviation principle with rate function \(\mathcal{I} : \mathbb{X} \to \mathbb{R}\) if \(\mathcal{I}\) is non-negative, lower semicontinuous and for any measurable set \(A\), we have
\[
- \inf_{x \in A} \mathcal{I}(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P_n(A) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(A) \leq - \inf_{x \in \overline{A}} \mathcal{I}(x).
\]
The rate function is said to be good if the level set \(\{x : \mathcal{I}(x) \leq \alpha\}\) is compact for any \(\alpha \geq 0\). Here, \(A^o\) is the interior of \(A\) and \(\overline{A}\) is its closure. Finally, the contraction principle in large deviation says that if \(P_n\) satisfies a large deviation principle on \(X\) with rate function \(\mathcal{I}(x)\) and \(F : X \to Y\) is a continuous map, then the probability measures \(Q_n := P_n F^{-1}\) satisfies a large deviation principle on \(Y\) with rate function \(I(y) = \inf_{x : F(x) = y} \mathcal{I}(x)\). Interested readers are referred to the standard references by [Dembo and Zajic (1995)] and [Varadhan (1984)] for the general theory of large deviations and its applications.

**Assumption 4.10.** Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of stationary \(\mathbb{R}^d\)-valued random vectors with the \(\sigma\)-algebra \(\mathcal{F}_m^\ell\) defined as \(\sigma(X_i, m \leq i \leq \ell)\). For every \(C < \infty\), there is a non-decreasing sequence \(\ell(n) \in \mathbb{N}\) with \(\sum_{n=1}^{\infty} \frac{\ell(n)}{n(n+1)} < \infty\) such that
\[
\sup \left\{ \mathbb{P}(A) \mathbb{P}(B) - e^{\ell(n)} \mathbb{P}(A \cap B) : A \in \mathcal{F}_0^{k_1}, B \in \mathcal{F}_{k_1+\ell(n)}^{k_1+k_2+\ell(n)} , k_1, k_2 \in \mathbb{N} \right\} \leq e^{-Cn},
\]
\[
\sup \left\{ \mathbb{P}(A \cap B) - e^{\ell(n)} \mathbb{P}(A) \mathbb{P}(B) : A \in \mathcal{F}_0^{k_1}, B \in \mathcal{F}_{k_1+\ell(n)}^{k_1+k_2+\ell(n)}, k_1, k_2 \in \mathbb{N} \right\} \leq e^{-Cn}.
\]

Assumption 4.10 holds under the hypermixing condition of Section 6.4. in [Dembo and Zajic (1995)] under the \(\psi\)-mixing condition (1.10) and (1.12) of [Bryc (1992)], and under the hyperexponential \(\alpha\)-mixing rate for stationary processes of Proposition 2 in [Bryc and Dembo (1996)]. It is clear that Assumption 4.10 holds if \(X_i\) are \(m\)-dependent.

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Assumption 4.11. For all $0 \leq \gamma, R < \infty$,

$$g_R(\gamma) := \sup_{k,m \in \mathbb{N}, k \in [0,Rm]} \frac{1}{m} \log \mathbb{E} \left[ e^{\gamma \| \sum_{i=k+1}^{k+m} X_i \|} \right] < \infty,$$

and $A := \sup_{\gamma} \limsup_{R \to \infty} R^{-1} g_R(\gamma) < \infty$.

Assumption 4.11 is trivially satisfied if $X_i$ are bounded. If $X_i$ are i.i.d. random variables, which is a standard assumption for Mogulskii’s theorem that will be used in this section, then Assumption 4.11 reduces to the assumption that the logarithmic moment generating function of $X_i$ is finite.

Under Assumption 4.10 and Assumption 4.11, Dembo and Zajic (1995) proved a sample path large deviation principle for $\mathbb{P}(\frac{1}{n} \sum_{i=1}^{n} X_i \in \cdot)$ (see Theorem A.5 in Appendix B). From this, we can show the following,

**Lemma 4.2.** Assume that both $(\overrightarrow{V}_i)_{i \in \mathbb{N}}$ and $(N_i - N_{i-1})_{i \in \mathbb{N}}$ satisfy Assumption 4.10 and Assumption 4.11. Then, for any $T > 0$, $\mathbb{P}(C_n(t) \in \cdot)$ satisfies a large deviation principle on $L_\infty[0,T]$ with the good rate function

$$\mathcal{I}(f) = \inf_{\substack{h \in AC_0[0,T], g \in AC_0[0,\infty) \\text{ s.t. } g(h(t))=f(t),0 \leq t \leq T}} [I_V(g) + I_N(h)], \quad (4.94)$$

with the convention that $\inf_{\emptyset} = \infty$ and

$$I_V(g) = \int_0^\infty \Lambda_V(g'(x))dx, \quad (4.95)$$

if $g \in AC_0^+[0,\infty)$ and $I_V(g) = \infty$ otherwise, where

$$\Lambda_V(x) := \sup_{\theta \in \mathbb{R}^6} \{ \theta \cdot x - \Gamma_V(\theta) \}, \quad \Gamma_V(\theta) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\sum_{i=1}^{n} \theta \cdot \overrightarrow{V}_i} \right], \quad (4.96)$$

and

$$I_N(h) = \int_0^T \Lambda_N(h'(x))dx, \quad (4.97)$$

if $h \in AC_0^+[0,T]$ and $I_N(h) = \infty$ otherwise, where

$$\Lambda_N(x) := \sup_{\theta \in \mathbb{R}^6} \{ \theta \cdot x - \Gamma_N(\theta) \}, \quad \Gamma_N(\theta) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\theta N_n} \right]. \quad (4.98)$$

Moreover, by the contraction principle,
Theorem 4.9. Under the same assumptions as in Lemma 4.2, \( P((Q^b_n(t), Q^a_n(t)) \in \cdot) \) satisfies a large deviation principle on \( L^\infty[0, \infty) \) with the rate function

\[
I(f^b, f^a) = \inf_{\phi \in G^f} \mathcal{I}(\phi),
\]

where \( \mathcal{I}(\cdot) \) is defined in Lemma 4.2, \( G^f \) is the set consists of absolutely continuous functions \( \phi(t) \) starting at 0 that satisfy

\[
d(f^b(t), f^a(t))^T = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} d\phi(t),
\]

with the initial condition \((f^b(0), f^a(0)) = (q^b, q^a)\). Otherwise \( I(f) = \infty \).

Let us now consider a special case:

Corollary 4.7. Assume that \( N(t) \) is a standard Poisson process with intensity \( \lambda \) independent of the i.i.d. random vectors \( \vec{V}_1 \) in \( \mathbb{R}^6 \) such that \( \mathbb{E}[e^{\theta \cdot \vec{V}_1}] < \infty \) for any \( \theta \in \mathbb{R}^6 \). Then, the rate function \( I(f) \) in (4.94) in Lemma 4.2 has an alternative expression

\[
\mathcal{I}(f) = \int_0^\infty \Lambda(f'(t)) dt,
\]

for any \( f \in AC_0[0, \infty) \), the space of absolutely continuous functions starting at 0 and \( I(\phi) = +\infty \) otherwise, where

\[
\Lambda(x) := \sup_{\theta \in \mathbb{R}^6} \left\{ \theta \cdot x - \lambda(\mathbb{E}[e^{\theta \cdot \vec{V}_1}] - 1) \right\}.
\]

Tails of the hitting time. Since

\[
Q^b(t) = q^b - \lambda v^b t \wedge \tau,
Q^a(t) = q^a - \lambda v^a t \wedge \tau,
\]

and the first hitting time \( \tau_n = \tau^b_n \wedge \tau^a_n \) of \((Q^b_n(t), Q^a_n(t))\) coincides with the first hitting time of \((Q^b_n(t), Q^b_n(t))\), from the fluid limit, we have

\[
\tau^b_n \to \tau^b := \frac{q^b}{\lambda v^b},
\tau^a_n \to \tau^a := \frac{q^a}{\lambda v^a},
\]

and \( \tau_n \to \tau := \tau^b \wedge \tau^a \). Here \( v^a, v^b \) are from (4.30).

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Using the large deviations result, we can study the tail probabilities of the hitting time \( \tau_n \) as \( n \) goes to \( \infty \). Note that for any \( t > \tau \),

\[
\mathbb{P}(\tau_n \geq t) = \mathbb{P}(Q^b_n(s) > 0, Q^a_n(s) > 0, 0 \leq s < t)
\]

\[
= \mathbb{P}(Q^b_n(s) > 0, Q^a_n(s) > 0, 0 \leq s < t).
\]

And for any \( t < \tau \),

\[
\mathbb{P}(\tau_n \leq t) = \mathbb{P}(Q^b_n(s) \leq 0 \text{ or } Q^a_n(s) \leq 0, \text{ for some } 0 \leq s \leq t)
\]

\[
= \mathbb{P}(Q^b_n(s) \leq 0 \text{ or } Q^a_n(s) \leq 0, \text{ for some } 0 \leq s \leq t).
\]

From the large deviation principle for \( \mathbb{P}(Q^b_n(\cdot) \in \cdot, Q^a_n(\cdot) \in \cdot) \), i.e. Theorem 4.9, we have, for any \( t > \tau \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\tau_n \geq t) = - \inf_{f^b(s) \geq 0, f^a(s) \geq 0, \text{ for any } 0 \leq s \leq t} I(f^b, f^a) = - \inf_{f^b(s) \geq 0, f^a(s) \geq 0, \phi \in \mathcal{G}_f} \mathcal{I}(\phi).
\]

Similarly, for any \( t < \tau \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\tau_n \leq t) = - \inf_{f^b(s) \leq 0 \text{ for some } 0 \leq s \leq t \text{ or } f^a(s) \leq 0 \text{ for some } 0 \leq s \leq t} I(f^b, f^a) = - \inf_{f^b(s) \leq 0 \text{ for some } 0 \leq s \leq t \text{ or } f^a(s) \leq 0 \text{ for some } 0 \leq s \leq t, \phi \in \mathcal{G}_f} \mathcal{I}(\phi).
\]

Recall that \( \mathcal{G}_f \) consists of the functions \( \phi = (\phi^j(t), 1 \leq j \leq 6) \in \mathcal{AC}_0[0, \infty) \) and

\[
f^b(t) = q^b + \phi^1(t) - \phi^2(t) - \phi^3(t),
\]

\[
f^a(t) = q^a + \phi^4(t) - \phi^5(t) - \phi^6(t).
\]

Therefore, we have the following,

**Corollary 4.8.** For any \( t > \tau \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\tau_n \geq t) = - \inf_{q^b + \phi^1(s) - \phi^2(s) - \phi^3(s) \geq 0, q^a + \phi^4(s) - \phi^5(s) - \phi^6(s) \geq 0, \text{ for any } 0 \leq s \leq t, \phi \in \mathcal{AC}_0[0, \infty)} \mathcal{I}(\phi). \tag{4.103}
\]

Similarly, for any \( t < \tau \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\tau_n \leq t) = - \inf_{q^b + \phi^1(s) - \phi^2(s) - \phi^3(s) \leq 0 \text{ for some } 0 \leq s \leq t \text{ or } q^a + \phi^4(s) - \phi^5(s) - \phi^6(s) \leq 0 \text{ for some } 0 \leq s \leq t, \phi \in \mathcal{AC}_0[0, \infty)} \mathcal{I}(\phi). \tag{4.104}
\]

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Bibliography


Appendix A

A.1 Technical Reviews

A.1.1 Markov chain and Markov decision process review of continuous time Markov chain (CTMC)

A stochastic process \( (X_t)_{t \geq 0} \) is called a continuous time Markov chain if it takes values in a countable set and the time spent in each state has an exponential distribution. Denote the set of states by \( S = \{1, \cdots, i, \cdots\} \). That is,

\[
P(X(t + h) = j | X(t) = i) = \delta_{ij} + q_{ij} h + o(h)
\]

with \( q_{ii} = -\sum_{j \neq i} q_{ij} < 0 \) \( (A.1) \)

\[
q_{ij} \geq 0
\]

\( Q = (q_{ij})_{i,j} \) is called the generator matrix of CTMC. E.g., A CTMC on \( S = 1, 2 \), the generator matrix for such a process could be written as

\[
Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.
\]

(A.2)

For \( i = 1, 2 \),

\[
P(X(t + h) \neq i | x(t) = i) = \alpha h + o(h) \quad (A.3)
\]

\[
P(X(t + h) = i | x(t) = i) = 1 - \alpha h + o(h) \sim \exp(-\alpha h), \quad (A.4)
\]

where \( \exp(-\alpha h) \) represents the exponential distribution with parameter \( \alpha h \). Define \( p_{ij}(t) = P(X_t = j | X_0 = i) \), then \( P(t) = (p_{ij}(t)) \) solves \( P'(t) = P(t)Q \), which gives \( P(t) = \exp(tQ) \).
Proposition A.1. For any $t, s > 0$, 
\[
P(t + s) = e^{(t+s)Q} = e^tQe^sQ = P(s)P(t) \tag{A.5}
\]

This corresponds to the Kolmogorov-Chapman equation in the discrete time Markov chain. For the example above, 
\[
P(t) = e^tQ = \left( \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} e^{-(\alpha+\beta)t} \right) \tag{A.6}
\]

Let $t \to \infty$, we have 
\[
P(t) = e^tQ = \left( \frac{\beta}{\alpha + \beta} \right) \tag{A.7}
\]

A review of Markov decision process

Suppose $X = (X_n)_{n \geq 1}$ is a MC on a countable state space $\mathbb{S}$ with transition matrix $P$. Consider a subset $D \subset \mathbb{S}$, we define 
\[
\partial D = \mathbb{S} \setminus D
\]
as the boundary of $D$. Let $\tau = \inf\{n, X_n \not\in D\} = \inf\{n, X_n \in \partial D\}$. Suppose a running cost function $V_c = (V_c(s))_{s \in D} \geq 0$ and $V_T = (V_T(s))_{s \in \partial D} \geq 0$. The expected cost starting from state $s$ and ending up at $\partial D$ is given by 
\[
\Phi(s) = \mathbb{E} \left[ \sum_{n=1}^{\tau-1} V_c(X_n) + V_T(X_\tau)1_{\tau < \infty} | X_0 = s \right] \tag{A.8}
\]

Then $\Phi$ satisfies a linear system of equations
\[
\begin{cases}
\Phi = P\Phi + V_c & \text{in } D \\
\Phi = V_T & \text{in } \partial D
\end{cases} \tag{A.9}
\]

$\mathcal{A} = \mathcal{C} \cup \mathcal{T}$ is a finite set of actions, where $\mathcal{C}$ is the set of continuous actions and $\mathcal{T}$ is the set of terminal actions. Define $A: \mathbb{S} \mapsto 2^\mathcal{A}$ as the function associating a non-empty set of actions $A(s)$ to each state $s \in \mathbb{S}$. Here $2^\mathcal{A}$ is the power set consisting of all subsets of $\mathcal{A}$. Naturally, we have $C(s) = A(s) \cap \mathcal{C}$ and $T(s) = A(s) \cap \mathcal{T}$.

For every $s \in \mathbb{S}$, and $a \in C(s)$, the transition probability from $s$ to $s'$ when selecting action $a$ is denoted as $P_{ss'}(a)$.

- $V_c(s, a)$ is the cost of continuation when at $s$, with action $a \in C(s)$.
- $V_T(s, a)$ is the cost of termination when at $s$, with action $a \in T(s)$.
• $V_c(s,a), V_T(s,a) \geq 0$ and bounded.

• $\alpha = (\alpha_0, \alpha_1, \cdots)$ is a sequence of actions, where $\alpha_n : S^{n+1} \to A$ such that $\alpha_n(s_0, \cdots, s_n) \in A(s_n)$ for each $n \geq 0$ and $(s_0, \cdots, s_n) \in S^{n+1}$.

• $V(s,\alpha)$ is the expected cost starting at $X_0 = s$, following a policy $\alpha$ until termination.

A policy $\alpha^*$ is called optimal if for any policy $\alpha$,

$$V(s,\alpha^*) \leq V(s,\alpha). \quad (A.10)$$

The optimal expected cost $V^*(s)$ is defined by

$$V^*(s) = \inf_{\alpha} V(s,\alpha). \quad (A.11)$$

**The Bellman equation** The value function of an optimization problem is associated with the Bellman equation of the following form

$$W(s) = \min( \min_{\alpha \in C(s)} V_c(s,\alpha) + \sum_{s' \in S} P_{ss'}(\alpha) W(s'), \min_{\alpha \in T(s)} V_T(s,\alpha)) \quad (A.12)$$

which comes from

$$V_0(s) = \min_{\alpha \in T(s)} V_T(s,\alpha) \quad (A.13)$$

$$V_{n+1}(s) = \min( \min_{\alpha \in C(s)} V_c(s,\alpha) + \sum_{s' \in S} P_{ss'}(\alpha) V_n(s'), \min_{\alpha \in T(s)} V_T(s,\alpha)) \quad (A.14)$$

**A.1.2 Some basic reviews on stochastic process limits**

Let $X$ be a mapping from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a measurable space $(E, \mathcal{E})$. We call $X$ a random element if $X$ is $(\mathcal{F}, \mathcal{E})$–measurable, i.e., $\forall B \in \mathcal{E}$, it follows $X^{-1}B \in \mathcal{F}$. Moreover, $P = \mathbb{P}X^{-1}$ is the probability measure on $(E, \mathcal{E})$ introduced by $X$. Let $\{X_n\}_{n \geq 1}$ be a sequence of random elements with values in $E$. We say $\{X_n\}_{n \geq 1}$ converges to $X$ in distribution ($X_n \Rightarrow X$) if $P_n \Rightarrow P$, where $P_n \Rightarrow P$ if and only if

$$P_n f \rightarrow P f$$

for any bounded, continuous real function $f$ on $E$.

We call a function on $[0,1]$ is càdlàg if it is right-continuous and has left-hand limits. Let $D[0,1]$ denote the set of all càdlàg functions from $[0,1]$ to $\mathbb{R}$. Let $A$ be
the class of strictly increasing, continuous mappings of $[0, 1]$ onto itself. For $\lambda \in \Lambda$ one defines
\begin{equation}
||\lambda|| = \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|.
\end{equation}
(A.15)

Then the standard $J_1$ metric on $D[0, 1]$ is
\begin{equation}
d_{J_1}(x_1, x_2) \equiv \inf_{\lambda \in \Lambda} \{ ||x_1 \circ \lambda - x_2|| \vee ||\lambda - \epsilon|| \},
\end{equation}
where $a \vee b = \max\{a, b\}$ and $\circ$ denotes the composition of functions. Let $B$ be the Borel set on $D[0, 1]$ with $J_1$, then a stochastic process $X_t, 0 \leq t \leq 1$ with càdlàg sample paths is a $D[0, 1]$ valued random element. Therefore we can define the weak convergence for a sequence of càdlàg stochastic processes on $[0, 1]$ in the space $D[0, 1]$ with metrics $J_1$.

Further more, the space $D([0, 1], \mathbb{R})$ could be generalized in two ways. First the range of the functions could be generalized from $\mathbb{R}$ to $\mathbb{R}^k$ and second the domain of the functions could be generalized from $[0, 1]$ to $[0, \infty)$. More detailed discussion could be found in [Whitt (2002)].

**Donsker’s Theorem**

This is a basic theorem in stochastic process limits, and could be treated as a generalization of classical central limit theorem.

**Theorem A.1.** Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables with $E[X_1] = m$ and $\text{Var}(X_1) = \sigma^2$, define the $n$th normalized partial-sum process $S_n$ as
\begin{equation}
S_n(t) \equiv n^{-1/2} \left( \sum_{i=1}^{[nt]} X_i - m nt \right)
\end{equation}
(A.16)

Then, as $n \to \infty$,
\begin{equation}
S \Rightarrow \sigma B \quad \text{in} \quad (D[0, \infty), J_1),
\end{equation}
(A.17)

where $B$ is a standard Brownian motion.

It is easy to extend Donsker’s Theorem to the multidimensional case via Cramér-Wold device. For applications, one might need to relax the i.i.d. conditions as well. One widely used relaxation technique is so called uniform mixing condition. We give some basic concepts here and refer to [Bradley (1985), Billingsley (1968), Jacod and Shiryaev (1987), and Whitt (2002)] for more details.
Let \( \{X_n\}_{-\infty < n < \infty} \) be a two-sided stationary sequence with
\[
\mathbb{E}[X_n] = m < \infty, \quad \mathbb{E}[X_n^2] < \infty.
\] (A.18)
Moreover, let \( \mathcal{F}_n = \sigma(X_k, k \leq n) \) denote the \( \sigma \)-field generated by \( \{X_k, k \leq n\} \), \( \mathcal{G}_n = \sigma(X_k, k \geq n) \) denote the \( \sigma \)-field generated by \( \{X_k, k \geq n\} \). Define
\[
\rho_n \equiv \sup\{|\mathbb{E}[XY]| : X \in \mathcal{F}_k, \mathbb{E}X = 0, \mathbb{E}X^2 \leq 1, Y \in \mathcal{G}_{n+k}, \mathbb{E}Y = 0, \mathbb{E}Y^2 \leq 1\}.
\] (A.19)
We say that \( \{X_n\}_{-\infty < n < \infty} \) satisfies the uniform mixing condition if
\[
\sum_{n=1}^{\infty} \rho_n < \infty.
\] (A.20)
Note that it is always possible to extend the one-sided stationary sequence to a two-sided stationary sequence, for example, see Breiman (1968).

**Theorem A.2.** If \( \{X_n\}_{-\infty < n < \infty} \) is a two-sided stationary sequence satisfying the uniform mixing condition and
\[
\mathbb{E}[X_n] = m < \infty, \quad \mathbb{E}[X_n^2] < \infty.
\] (A.21)
Then \( \sigma^2 = \text{Var}X_n + 2 \sum_{k=1}^{\infty} \text{Cov}(X_1, X_{1+k}) < \infty \) and (A.17) holds.

### A.1.3 Convergence of stochastic processes by Kurtz and Protter

Kurtz and Protter (1991) provides a powerful tool to show convergence of stochastic processes defined by SDE’s with certain conditions. Define \( h_\delta(r) : [0, \infty) \rightarrow [0, \infty) \) by \( h_\delta(r) = (1 - \delta/r)^+ \). Define \( J_\delta : D_{\mathbb{R}^m}[0, \infty) \rightarrow D_{\mathbb{R}^m}[0, \infty) \) by
\[
J_\delta(x)(t) = \sum_{s \leq t} h_\delta(|x(s) - x(s-)|)(x(s) - x(s-))
\]
Let \( Y_n \) be a sequence of stochastic processes adapted to \( \mathcal{F}_t \). Define \( Y_n^\delta = Y_n - J_\delta(Y_n) \). Let \( Y_n^\delta = M_n^\delta + A_n^\delta \) be a decomposition of \( Y_n^\delta \) into an \( \mathcal{F}_t \)-local martingale and a process with finite variation.

**Condition A.1.** For each \( \alpha > 0 \), there exist stopping times \( \tau_n^\alpha \) such that \( P\{\tau_n^\alpha \leq 1\} \leq 1/\alpha \) and \( \sup_n \mathbb{E}[\|M_n^\delta\|_{t \leq \tau_n^\alpha} + T(A_n^\delta)_{t \leq \tau_n^\alpha}] < \infty \), where \( [M_n^\delta]_{t \leq \tau_n^\alpha} \) denotes the total quadratic variation of \( M_n^\delta \) up to time \( \tau_n^\alpha \), and \( T(A_n^\delta)_{t \leq \tau_n^\alpha} \) denotes the total variation of \( A_n^\delta \) up to time \( \tau_n^\alpha \).
Let $T_1[0, \infty)$ denote the collection of non-decreasing mappings $\lambda$ of $[0, \infty)$ to $[0, \infty)$ such that $\lambda(h + t) - \lambda(t) \leq h$ for all $t, h \geq 0$. Let $M_{km}$ be the space of real-valued $k \times m$ matrices, and $D_{\text{càdlàg}}[0, \infty)$ be the space of càdlàg functions from $[0, \infty)$ to $M_{km}$. Assume that there exist mappings $G_n, G : D^k[0, \infty) \times T_1[0, \infty) \to D_{\text{càdlàg}}[0, \infty)$ such that $F_n \circ \lambda = G_n(x \circ \lambda, \lambda)$ and $F(x) \circ \lambda = G(x \circ \lambda, \lambda)$ for $(x, \lambda) \in D^k[0, \infty) \times T_1[0, \infty)$.

Condition A.2. i. For each compact subset $\mathcal{H} \subset D^k[0, \infty)$ and $t > 0$, it follows that $\sup_{(x, \lambda) \in \mathcal{H}} \sup_{s \leq t} |G_n(x, \lambda, s) - G(x, \lambda, s)| \to 0$.

ii. For $\{(x_n, \lambda^n)\} \in D^k[0, \infty) \times T_1[0, \infty)$, if $\sup_{s \leq t} |x_n(s) - x(s)| \to 0$ and $\sup_{s \leq t} |\lambda^n(s) - \lambda(s)| \to 0$ for each $t > 0$, then it asserts that $\sup_{s \leq t} |G(x_n, \lambda^n, s) - G(x, \lambda, s)| \to 0$.

Theorem A.3. Suppose that $(U_n, X_n, Y_n)$ satisfies

$$X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)dY_n(s)$$

$(U_n, Y_n) \Rightarrow (U, Y)$ in the Skorokhod topology and that $\{Y_n\}$ satisfies Condition A.1 for some $0 < \delta \leq \infty$. Assume that $\{F_n\}$ and $F$ have representations in terms of $\{G_n\}$ and $G$ satisfying Condition A.2. If there exists a global solution $X$ of

$$dX(t) = U(t) + \int_0^t F(X, s-)dY(s),$$

and the local uniqueness holds, then $(U_n, X_n, Y_n) \Rightarrow (U, X, Y)$.

A.1.4 Some large deviations results

According to Theorem 5.1.2. in Dembo and Zeitouni (1998), we have

Theorem A.4 (Mogulskii’s Theorem). Assume $(X_i)_{i \geq 1}$ are i.i.d. random vectors in $\mathbb{R}^d$. If $\Gamma(\theta) := \log \mathbb{E}[e^{\theta \cdot X_1}] < \infty$ for any $\theta \in \mathbb{R}^d$ and let

$$\Lambda(x) := \sup_{\theta \in \mathbb{R}^d} \{\theta \cdot x - \Gamma(\theta)\},$$

then $P(\frac{1}{n} \sum_{i=1}^{[nt]} X_i \in \cdot)$ follows a large deviation principle on $L^\infty[0, \infty)$ with the rate function

$$\mathcal{I}(\phi) = \int_0^T \Lambda(\phi(t))dt,$$
for any \( \phi \in \mathcal{AC}_0[0, \infty) \), the space of absolutely continuous functions starting at 0 and \( \mathcal{I}(\phi) = +\infty \) otherwise.

According to Theorem 2 in [Dembo and Zajic (1995)], we have

**Theorem A.5.** Let \( (X_i)_{i \in \mathbb{N}} \) be a sequence of stationary \( \mathbb{R}^d \)-valued random vectors satisfying Assumption 4.10 and Assumption 4.11. Then, the empirical mean process \( S_n(t) := \frac{1}{n} \sum_{i=1}^{[nt]} X_i, 0 \leq t \leq T \), satisfies a large deviations principle on \( D[0, T] \) equipped with the topology of uniform convergence with the convex good rate function

\[
I(\phi) := \int_0^T \Lambda(\phi'(t))dt,
\]

(A.24)

for any \( \phi \in \mathcal{AC}_0[0, \infty) \), the space of absolutely continuous functions starting at 0 and \( \mathcal{I}(\phi) = +\infty \) otherwise, where

\[
\Lambda(x) := \sup_{\theta \in \mathbb{R}^d} \{ \theta \cdot x - \Gamma(\theta) \},
\]

(A.25)

with \( \Gamma(\theta) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[e^{\sum_{i=1}^{n} \theta \cdot X_i}] \).

**Remark A.1.** Note that the original Theorem 2 in [Dembo and Zajic (1995)] applies to Banach space valued \( (X_i)_{i \in \mathbb{N}} \). For the purpose in our paper, we only need to consider \( \mathbb{R}^d \) valued \( (X_i)_{i \in \mathbb{N}} \).

### A.2 Technical proofs

#### A.2.1 Proof of Theorem [2.1]

**Proof.** Clearly

\[
P(Y_T \leq -k) = P(Y_T \leq -k, A_T \leq -k) + P(Y_T \leq -k, A_T > -k)
\]

\[
= P(A_T \leq -k) + P(Y_T \leq -k, A_T > -k),
\]

and

\[
P(Y_T = -k) = P(Y_T \leq -k) - P(Y_T \leq -k - 1)
\]

\[
= P(A_T = -k) + P(Y_T \leq -k, A_T \geq -k + 1)
\]

\[
- P(Y_T \leq -k - 1, A_T \geq -k).
\]

Since \( \frac{T-k}{2} \in \mathbb{N}, \{A_T \geq -k + 1\} = \{A_T \geq -k + 2\} \), it suffices to show that

\[
P(Y_T \leq -k, A_T \geq -k + 2) = P(Y_T \leq -k - 1, A_T \geq -k).
\]
Consider \( \forall \omega \in \{ Y_T \leq -k, A_T \geq -k + 2 \} \) and let \( \tau(\omega) = \sup \{ t : A_t(\omega) = Y_T(\omega) \} \). We have \( \tau(\omega) \leq T - 2 \) from \( A_T(\omega) \geq -k + 2 \) and \( Y_T(\omega) \leq -k \). Now, define a mapping \( h : \{ Y_T \leq -k, A_T \geq -k + 2 \} \rightarrow \{ Y_T \leq -k - 1, A_T \geq -k \} \) as follows

\[
X_t(h(\omega)) = \begin{cases} 
X_t(\omega), & t \neq \tau(\omega) + 1, \\
-1, & t = \tau(\omega) + 1.
\end{cases}
\]

We will show that \( h \) is well-defined and is in fact a bijection.

![An Example of the Mapping h](image)

**Figure A.1:** Illustration of the mapping \( h \)

First, we show that \( h \) is well-defined, i.e., \( h(\omega) \in \{ Y_T \leq -k - 1, A_T \geq -k \} \). By the definition of \( \tau(\omega) \), together with \( A_T(\omega) - Y_T(\omega) \geq 2 \), we see that \( X_{\tau(\omega)}(\omega) = -1 \) and \( X_{\tau(\omega)+1}(\omega) = X_{\tau(\omega)+2}(\omega) = 1 \). Since \( h(\omega) \) only changes the movement at the step \( \tau(\omega) + 1 \) from 1 to \(-1\), with \( A_{\tau(\omega)+1}(h(\omega)) = -Y_T(\omega) - 1 \) and \( A_T(h(\omega)) = A_T(\omega) - 2 \geq -k \), we have \( h(\omega) \in \{ Y_T \leq -k - 1, A_T \geq -k \} \). Moreover, it is easy to see that \( h \) does not alter the number of direction changes from \( \omega \) to \( h(\omega) \). Also note that \( \tau(\omega) \geq 1 \) for all \( k > 0 \), that means the first price movement will not be changed by \( h \). Then we have \( X_1(\omega) = X_1(h(\omega)) \) and

\[
P(\omega) = P(h(\omega)). \tag{A.26}
\]

Second, we show that \( h \) is an injection. Suppose \( \omega_1, \omega_2 \in \{ Y_T \leq -k, A_T \geq -k + 2 \} \) with \( h(\omega_1) = h(\omega_2) \). If \( \tau(\omega_1) = \tau(\omega_2) \), then obviously \( \omega_1 = \omega_2 \). If \( \tau(\omega_1) \neq \tau(\omega_2) \), wlog., we assume that \( \tau(\omega_1) < \tau(\omega_2) \). Then for any \( t \leq \tau(\omega_1) \) and

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Proof. For $t > \tau(\omega_2)$, we have $X_t(\omega_1) = X_t(\omega_2)$. For $\tau(\omega_1)+1$, we have $X_{\tau(\omega_1)+1}(\omega_1) = -1$ and $X_{\tau(\omega_1)+1}(\omega_2) = 1$. Thus $A_{\tau(\omega_1)+1}(\omega_2) = A_{\tau(\omega_1)}(\omega_1) - 1 = Y_T(\omega_1) - 1$. Therefore,

$$Y_T(\omega_2) \leq A_{\tau(\omega_1)+1}(\omega_2) \leq Y_T(\omega_1) - 1.$$ 

Note that $A_{\tau(\omega_2)+1}(\omega_1) = A_{\tau(\omega_2)+1}(\omega_2) = Y_T(\omega_2) + 1$, we have $A_{\tau(\omega_2)+1}(\omega_1) \leq Y_T(\omega_1)$, i.e., $h$ is an injection.

Next, we show that $h$ is a surjection, i.e., for any $\omega \in \{Y_T \leq -k-1, A_T \geq -k\}$, there exists $\omega \in \{Y_T \leq -k, A_T \geq -k+2\}$ such that $h(\omega) = \omega$. Let $\overline{\tau}(\omega) = \inf\{t : A_t(\omega) = Y_T(\omega)\}$, and define $\omega$ by

$$X_t(\omega) = \begin{cases} X_t(\omega), & t \neq \overline{\tau}(\omega), \\ 1, & t = \overline{\tau}(\omega). \end{cases}$$

By the definition of $\overline{\tau}(\omega)$, $A_\tau(\omega) \geq Y_T(\omega) + 1$ for any $t < \overline{\tau}(\omega)$, and $A_t(\omega) \geq A_\tau(\omega) + 2 \geq Y_T(\omega) + 2$ for any $t \geq \overline{\tau}(\omega)$, Thus $\overline{\tau}(\omega) - 1$ is the last time that $A_t(\omega)$ reaches its lowest position. Then by the definition of $h$, $h(\omega) = \omega$. Hence $h$ is a surjection.

Note that since $h$ is a bijection from $\{Y_T \leq -k, A_T \geq -k+2\}$ to $\{Y_T \leq -k-1, A_T \geq -k\}$, using \eqref{A.26}, we obtain

$$\mathbb{P}(Y_T \leq -k, A_T \geq -k+2) = \mathbb{P}(Y_T \leq -k-1, A_T \geq -k).$$

$\square$

### A.2.2 Proof of Proposition 2.2

Proof. For $\forall \omega \in \{Y_T = -k-1\}$, let $\tau(\omega) = \inf\{t : A_t(\omega) = Y_T(\omega)\}$. Define a mapping $g$ on $\{Y_T = -k-1\}$ as follows

$$X_t(g(\omega)) = \begin{cases} X_t(\omega), & t \neq \tau(\omega), \\ 1, & t = \tau(\omega). \end{cases}$$

In other words, $g$ changes $X_{\tau}(\omega)(\omega)$ from a “downward” edge to an “upward” edge. By the definition of $\tau(\omega)$, it is clear that $Y_T(g(\omega)) = -k$. Thus $g(\omega) \in \{Y_T = -k\}$. Now we show that $g$ in an injection, i.e., $\forall \omega_1$ and $\omega_2 \in \{Y_T = -k-1\}$, if $g(\omega_1) = g(\omega_2)$, then $\omega_1 = \omega_2$. If $\tau(\omega_1) = \tau(\omega_2)$, then it is easy to see that $X_t(\omega_1) = X_t(\omega_2)$ for $1 \leq t \leq T$, which is equivalent to $\omega_1 = \omega_2$. If $\tau(\omega_1) \neq \tau(\omega_2)$, wlog., we assume that $\tau(\omega_1) < \tau(\omega_2)$. Then for any $t < \tau(\omega_1)$, $X_t(\omega_1) = X_t(\omega_2)$; $A_t(\omega_1) = A_t(\omega_2) - 2$ for $\tau(\omega_1) \leq t < \tau(\omega_2)$; and $X_t(\omega_2) = -1$ and $X_t(\omega_1) = 1$ for
Figure A.2: Illustration of the mapping $g$

t = \tau(\omega_2). Thus $A_{\tau(\omega_2)}(\omega_2) = A_{\tau(\omega_1)}(\omega_2) \geq Y_T(\omega_1) + 1 = -k$, contradictory to the definition of $\omega_2$. Hence $\tau(\omega_1) = \tau(\omega_2)$ and $\omega_1 = \omega_2$. Thus $g$ is an injection. Note that since the mapping $g$ does not reduce the number of direction changes, we also have $P(g(\omega)) \geq P(\omega)$. Thus $P\{Y_T = -k - 1\} \leq P\{Y_T = -k\}$. \qed
A.2.3 Proof of Lemma 2.1

\[ C(k, q, T, \bar{p}) = \sum_{\omega \in \{ Y_T = -k+1 \}} \mathbb{P}(\omega)[q_k(\omega)(-k - r) + (1 - q_k(\omega))(A_T(\omega) + f)] \]

\[ + \mathbb{P}(Y_T \leq -k)(-k - r) + \mathbb{P}(Y_T > -k + 1)(\mathbb{E}[A_T | Y_T > -k + 1] + f) \]

\[ = \sum_{\omega \in \{ Y_T = -k+1 \}} \mathbb{P}(\omega)q_k(\omega)(-k - r - A_T(\omega) - f) \]

\[ + \sum_{\omega \in \{ Y_T = -k+1 \}} \mathbb{P}(\omega)(A_T(\omega) + f) \]

\[ + \mathbb{P}(Y_T \leq -k)(-k - r) + \mathbb{P}(Y_T > -k + 1)(\mathbb{E}[A_T | Y_T > -k + 1] + f) \]

\[ = \sum_{\omega \in \{ Y_T = -k+1 \}} \mathbb{P}(\omega)q_k(\omega)(-k - r - A_T(\omega) - f) \]

\[ + \sum_{\omega \in \{ Y_T = -k+1 \}} \mathbb{P}(\omega)(A_T(\omega) + f) \]

\[ + \mathbb{P}(Y_T \leq -k)(-k - r) + \mathbb{P}(Y_T > -k + 1)(\mathbb{E}[A_T | Y_T > -k + 1] + f) \]

\[ = \sum_{\omega \in \{ Y_T = -k+1 \}} \mathbb{P}(\omega)q_k(\omega)(-k - r - A_T(\omega) - f) + C(k, 0, T, \bar{p}). \]

A.2.4 Proof of Lemma 2.3

Define the same mapping \( g \) from \( \{ Y_T = -k - 1 \} \) to \( \{ Y_T = -k \} \) as in the proof of Proposition 2.2. Note that \( g \) is an injection and \( \mathbb{P}(\omega) \leq \mathbb{P}(g(\omega)) \) for any \( \omega \in \{ Y_T = -k - 1 \} \), because \( g(\omega) \) has at least as many direction changes as \( \omega \) has. Moreover, because \( g \) changes one “downward” edge to one “upward” edge, we have \( A_T(g(\omega)) = A_T(\omega) + 2 \). Thus

\[ \mathbb{P}(Y_T = -k - 1)(k + 2 + \mathbb{E}[A_T | Y_T = -k - 1]) \]

\[ = \sum_{\omega \in \{ Y_T = -k-1 \}} \mathbb{P}(\omega)(k + 2 + A_T(\omega)) \]

\[ \leq \sum_{g(\omega) \in g(\{Y_T = -k-1\})} \mathbb{P}(g(\omega))(k + A_T(g(\omega))) \]

\[ \leq \sum_{g(\omega) \in \{Y_T = -k\}} \mathbb{P}(g(\omega))(k + A_T(g(\omega))). \]

The second to the last inequality holds because \( g \) is an injection and \( k + 2 + A_T(\omega) \geq 0 \), and the last inequality follows from \( A_T(g(\omega)) + k \geq 0 \) for \( \forall g(\omega) \in \{ Y_T = -k \} \).
A.2.5 Proof of Proposition 2.3

Proof. We first show the last inequality in the sequence, i.e., $C(T, 0, T, \bar{p}) \leq C(T + 1, 0, T, \bar{p})$. Indeed,

$$C(T + 1, 0, T, \bar{p}) - C(T, 0, T, \bar{p})$$

$$= \mathbb{E}[A_T] + f + \mathbb{P}(Y_T = -T)(T + r) - \mathbb{P}(Y_T \geq -T + 1)(\mathbb{E}[A_T \mid Y_T \geq -T + 1] + f)$$

$$= \mathbb{P}(Y_T = -T)(r + f + T + \mathbb{E}[A_T \mid Y_T = -T])$$

$$= (f + r) \mathbb{P}(Y_T = -T) = 0.$$

Now let $b_k = C(k + 1, 0, T, \bar{p}) - C(k, 0, T, \bar{p})$ for $k = 1, 2, \ldots, T - 1$, then

$$b_k = \mathbb{P}(Y_T \leq -k - 1)(-k - 1 - r) + \mathbb{P}(Y_T \geq -k)(\mathbb{E}[A_T \mid Y_T \geq -k] + f)$$

$$+ \mathbb{P}(Y_T \leq -k)(-k - r) + \mathbb{P}(Y_T \geq -k + 1)(\mathbb{E}[A_T \mid Y_T \geq -k + 1] + f)$$

$$= -\mathbb{P}(Y_T \leq -k - 1) + \mathbb{P}(Y_T = -k)(r + f + k + \mathbb{E}[A_T \mid Y_T = -k]).$$

Since $r, f \geq 0$, it suffices to check this for the extreme case of $r = f = 0$, i.e., to show

$$b_k = -\mathbb{P}(Y_T \leq -k - 1) + k \mathbb{P}(Y_T = -k) + \mathbb{P}(Y_T = -k) \mathbb{E}[A_T \mid Y_T = -k] > 0.$$

First, by Proposition 2.2, we have

$$\tilde{b}_{T-1} = -\mathbb{P}(Y_T = -T) + (T - 1) \mathbb{P}(Y_T = -T + 1)$$

$$+ \mathbb{P}(Y_T = -T + 1) \mathbb{E}[A_T \mid Y_T = -T + 1]$$

$$= -\mathbb{P}(Y_T = -T) + \mathbb{P}(Y_T = -T + 1) > 0;$$

and for $k = 1, 2, \ldots, T - 2$, by Lemma 2.3, we have

$$\tilde{b}_k - \tilde{b}_{k+1} = -\mathbb{P}(Y_T = -k - 1) + k \mathbb{P}(Y_T = -k) + \mathbb{E}[A_T Y_T = -k] \mathbb{P}(Y_T = -k)$$

$$- (k + 1) \mathbb{P}(Y_T = -k - 1) + \mathbb{E}[A_T Y_T = -k - 1] \mathbb{P}(Y_T = -k - 1)$$

$$= \mathbb{P}(Y_T = -k)(k + \mathbb{E}[A_T Y_T = -k])$$

$$- \mathbb{P}(Y_T = -k - 1)(k + 2 + \mathbb{E}[A_T Y_T = -k - 1])$$

$$\geq 0.$$

Thus recursively $\tilde{b}_k > 0$ for $k = 1, 2, \ldots, T - 1$. \qed
A.2.6 Proof of Proposition 2.4

Proof. For ease of exposition, define the first term on the right hand side of (2.27) as

\[ D_k^q = \sum_{\omega \in \{Y_T = -k + 1\}} \mathbb{P}(\omega) q_k(\omega)(-k - r - A_T(\omega) - f). \]

Since \(C(k, 0, T, \bar{p})\) is increasing in \(k\) for \(1 \leq k \leq T + 1\), it suffices to show that \(D_k^q\) is also increasing in \(k\) for \(2 \leq k \leq T + 1\). The key idea here is to define a mapping \(F: \{Y_T = -k\} \rightarrow \{Y_T = -k + 1\}\) such that \(F\) is an injection and keeps the number of direction changes and the number of hitting the lowest level. Then we can argue that \(D_k^q < D_{k+1}^q\).

To construct \(F\), we first define \(N(\omega) = \sum_{t=1}^{T} \mathbb{1}_{A_t(\omega) = Y_T(\omega)}\), and \(\tau_n(\omega) = \inf\{i : \sum_{t=1}^{i} \mathbb{1}_{A_t(\omega) = Y_T(\omega)} = n\}\) for \(n \leq N(\omega)\). Then \(F\) will be defined in two steps. First, construct \(F_1: \{Y_T = -k\} \rightarrow \{Y_T = -k + 1\}\) so that

\[
X_t(F_1(\omega)) = \begin{cases} 
X_t(\omega), & t < \tau_1(\omega), \\
X_{t+1}(\omega), & \tau_1(\omega) \leq t < T, \\
0, & t = T.
\end{cases}
\]

Second, construct \(F_2: \{Y_{T-1} = -k + 1\} \times \mathbb{N} \rightarrow \{Y_T = -k + 1\}\) so that

\[
X_t(F_2(\omega, n)) = \begin{cases} 
X_t(\omega), & t \leq \tau_n(\omega), \\
1, & t = \tau_n(\omega) + 1, \\
X_{t-1}(\omega), & t \geq \tau_n(\omega) + 2.
\end{cases}
\]

Now \(F\) is defined as

\[ F(\omega) = F_2(F_1(\omega), N(\omega)). \]

In other words, \(F\) first deletes the “downward” edge when \(\omega\) first hits \(Y_T(\omega)\) and then adds an “upward” edge after the modified \(\omega\) hits \(Y_T(\omega) + 1\) for \(N(\omega)\) times. It is easy to see that \(Y_T(F(\omega)) = Y_T(\omega) + 1\). And for each \(t\) where \(A_t(\omega) = Y_T(\omega)\), we have \(A_{t-1}(F_1(\omega)) = Y_T(\omega) + 1\). Thus \(N(F_1(\omega)) \geq N(\omega)\) and \(F_2\) is well-defined on \((F_1(\omega), N(\omega))\), and

\[ q_{k+1}(\omega) = q_k(F(\omega)). \]

Moreover, when we remove the “downward” edge in mapping \(F_1\), since \(k \geq 2\) (hence \(\tau_1(\omega) \geq 2\)), we change neither the first step nor the number of direction changes. When adding the “upward” edge in mapping \(F_2\), the number of direction changes...
Figure A.3: Illustration of the mapping $F$

changes of $F_2(F_1(\omega))$ is the same as the number of direction changes of $\omega$ and hence

$$P(\omega) = P(F(\omega)).$$

Furthermore, since $F$ replaces a “downward” edge by an “upward” edge for $\omega$, $A_T(f(\omega)) = A_T(\omega) + 2$.

Next, we show that $F$ is an injection. Let $\omega_1, \omega_2 \in \{Y_T = -k\}$ and $F(\omega_1) = F(\omega_2)$. First, since $N(\omega) = N(F(\omega))$, we have $N(\omega_1) = N(\omega_2)$. Note that after operating $F_2$ at $\tau_{N(\omega)}$, $A_t(F(\omega)) > Y_T(f(\omega))$ for all $t > \tau_{N(\omega)}$. In order to have $F(\omega_1) = F(\omega_2)$, we must have $\tau_{N(\omega_1)}(\omega_1) = \tau_{N(\omega_2)}(\omega_2)$. Therefore, by the definition of $F_2$, $F(\omega_1) = F(\omega_2)$ implies $F_1(\omega_1) = F_1(\omega_2)$. Now if $\tau_1(\omega_1) = \tau_1(\omega_2)$, then by the definition of $F_1$, we have $X_t(\omega_1) = X_t(\omega_2)$ for all $t \neq \tau_1(\omega_1)$. Since $\omega_1$ and $\omega_2$ first hit the lowest level $-k$ at $\tau_1(\omega_1)$, we have $X_{\tau_1(\omega_1)}(\omega_1) = X_{\tau_1(\omega_1)}(\omega_2) = -1$, implying $\omega_1 = \omega_2$. If $\tau_1(\omega_1) \neq \tau_1(\omega_2)$, wlog., we assume that $\tau_1(\omega_1) < \tau_1(\omega_2)$. Then $X_t(\omega_1) = X_t(\omega_2)$ for all $t < \tau_1(\omega_1)$ and $t > \tau_1(\omega_2)$, and $A_t(\omega_1) = A_t(\omega_2)$ for $t \geq \tau_1(\omega_2)$ and $A_t(\omega_1) = A_t(\omega_2) - 1$ for all $\tau_1(\omega_1) \leq t < \tau_1(\omega_2)$. Now $A_{\tau_1(\omega_1)}(\omega_1) = Y_T(\omega_1)$ suggests that $N(\omega_1) \geq N(\omega_2) + 1$, which is a contradiction to the assumption $F(\omega_1) = F(\omega_2)$. Therefore, $F$ is an injection.
Finally, calculating the difference of the sequence $D_k^q$

\[ D_k^q - D_{k+1}^q = \sum_{\omega \in \{Y_T = -k\}} \mathbb{P}(\omega)q_{k+1}(\omega)(k + 1 + r + A_T(\omega) + f) \]

\[ - \sum_{\omega \in \{Y_T = -k+1\}} \mathbb{P}(\omega)q_k(\omega)(k + r + A_T(\omega) + f) \]

\[ \leq \sum_{\omega \in \{Y_T = -k\}} \mathbb{P}(\omega)q_{k+1}(\omega)(k + 1 + r + A_T(\omega) + f) \]

\[ - \sum_{\omega \in \{Y_T = -k\}} \mathbb{P}(F(\omega))q_k(F(\omega))(k + r + A_T(F(\omega)) + f) \]

\[ = \sum_{\omega \in \{Y_T = -k\}} \left( \mathbb{P}(\omega)q_{k+1}(\omega)(k + 1 + r + A_T(\omega) + f) \right) \]

\[ - \mathbb{P}(\omega)q_{k+1}(\omega)(k + r + A_T(F(\omega)) + f) \]

\[ = \sum_{\omega \in \{Y_T = -k\}} \mathbb{P}(\omega)q_{k+1}(\omega)((k + 1 + r + A_T(\omega) + f) \]

\[ - (k + r + A_T(\omega) + 2 + f) \]

\[ = - \sum_{\omega \in \{Y_T = -k\}} \mathbb{P}(\omega)q_{k+1}(\omega) \leq 0. \]

Thus $D_k^q$ is increasing as $k$ increases for $k \geq 2$. \hfill \Box

A.2.7 Proof of Proposition 2.5

Proof. Notice that $C(2, q, T - 1, p) \geq -2 - r$, and $C(1, q, T, \bar{p}) = (1 - \bar{p} + \bar{p}q)(-1 - r) + \bar{p}(1 - q)(1 + C(2, q, T - 1, p)) = \bar{p}(2 + r + C(2, q, T - 1, p))(1 - q) - 1 - r$, where $C(2, q, T - 1, p)$ is independent of $\bar{p}$ and $1 + C(2, q, -1, p) \geq -1 - r$. Thus $C(1, q, T, \bar{p})$ is an increasing linear function of $\bar{p}$. Similarly, $C(2, q, T, \bar{p}) = (1 - \bar{p})(-1 + C(1, q, T - 1, 1 - p)) + \bar{p}(1 + C(3, q, T - 1, p)) = \bar{p}(2 + C(3, q, T - 1, p) - C(1, q, T - 1, 1 - p) - 1) + C(1, q, T - 1, 1 - p)$. Thus $C(2, q, T, \bar{p})$ is a linear function of $\bar{p}$. Moreover, $C(3, q, T - 1, p) - C(1, q, T - 1, 1 - p) > -2$ since for each sample path, the difference between the limit orders placed at $-3$ and $-1$ will not be more than 2. Therefore $C(2, q, T, \bar{p})$ is also increasing in $\bar{p}$. \hfill \Box

A.2.8 Proof of Proposition 2.6

Proof. Note that the proof of the monotonicity of $D_k^q$ with respect to $k \geq 2$ in Proposition 2.4 does not work for $k = 1$. This is because the first step of \omega
may be an “upward” edge and may be changed into a “downward” edge by the mapping $F$. That will decrease the probability of this sample path. However, in the case of $\bar{p} \geq 1 - p$, even if the first step of $\omega$ is changed from “downward” to “upward”, the probability of the sample path will not decrease and the inequality 

$$P(f(\omega)) = P(\omega)\frac{pp}{(1-p)(1-p)} \geq P(\omega)$$

as well as the other properties of $F$ still hold. That is, the same method will show that

$$C(1, q, T, \bar{p}) < C(2, q, T, \bar{p}),$$

for $\bar{p} \geq 1 - p$.

\[\square\]

### A.2.9 Proof of Lemma 2.4

**Proof.** We will show by mathematical induction.

First, for $t = T - 1$, simple algebra shows that all $V_{T-1}(1, BB), V_{T-1}(1, MO) = f), V_{T-1}(1, NO) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$, clearly so is

$$V_{T-1}^*(1) = \min\{V_{T-1}(1, BB), V_{T-1}(1, MO), V_{T-1}(1, NO)\}.$$

Moreover,

$$V_{T-1}(1, BB) = (pq + 1 - p)(-1 - r) + p(1 - q)(1 + f)$$

$$= -1 - 2pq + 2p - (pq + 1 - p)r + p(1 - q)f < f = V_{T-1}(1, MO),$$

$$V_{T-1}(-1, NO) = p(-1 + f) + (1 - p)(1 + f)$$

$$> (p+q-pq)(-1 - r) + (1 - p - q + pq)(1 + f) = V_{T-1}(-1, BB).$$

Hence the desired result holds for $t = T - 1$.

Now suppose this lemma holds for $t = T - 1$. Then it is easy to verify $V_{t-1}(1, Act) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$ for $Act = MO, BB$. Meanwhile,

$$V_{t-1}(1, NO) = (1 - p)(-1 + \min\{V_{t}(1, BB), V_{t}(-1, NO), V_{t}(-1, MO)\})$$

$$+ p(1 + V_{t}^*(1))$$

$$= (1 - p)(-1 + \min\{V_{t}(1, BB), V_{t}(-1, MO)\}) + p(1 + V_{t}^*(1))$$

$$> -2 - r + \frac{p}{(1-p)(p+q-pq)},$$

because $V_{t}(1, NO) > V_{t}(-1, BB)$, and $(1 - p)(-1 + V_{t}(-1, Act)) + p(1 + V_{t}^*(1)) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$ for $Act = BB, MO$.

Therefore, $V_{t-1}^*(1) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$, with $V_{t-1}^*(1) \leq f$ from the fact $V_{t-1}(1, MO) = f$.

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Hence the lemma holds for $t - 1$, and for any $1 \geq t \geq T - 1$ by induction.  

**A.2.10 Proof of Theorem 2.2**

*Proof.* By Lemma 2.7 and Lemma 2.4 to show the existence of $t^*_1$, it suffices to consider

$$V_t(1, BB) - V_t(1, NO) = (pq + 1 - p)(-1 - r) + p(1 - q)(1 + V_{t+1}^*(1))$$

$$- (1 - p)(-1 + V_{t+1}^*(1)) + p(1 + V_{t+1}^*(1))$$

$$= -pq(2 + r + V_{t+1}^*(1)) - (1 - p)(r + V_{t+1}^*(1)).$$

This is a non-increasing function with respect to $t$. Therefore if $V_t(1, BB) \leq V_t(1, NO)$ for some $t^*_1$, then the inequality holds for all $t > t^*_1$. $V_{T-1}(1, BB) - V_{T-1}(1, NO) = -pq(2 + r + f) - (1 - p)(r + f) < 0$ implies $t^*_1 \leq T - 1$ if $t^*_1$ exists. Similarly, $V_t(-1, MO) - V_t(-1, BB)$ is also a non-increasing function with respect to $t$. Therefore if $V_t(-1, MO) \leq V_t(-1, BB)$ for some $t^*_2$, then the inequality holds for all $t > t^*_2$. Notice that when $V_{t+1}(-1, MO) \leq V_{t+1}(-1, BB)$, it follows that $V_t(1, BB) - V_t(1, NO) \leq 0$ as $V_{t+1}^*(-1) = f \geq 0$. Therefore $t^*_1 < t^*_2$ if both of them exist. And the existence will be clear when the expressions of $t^*_1$ and $t^*_2$ are given.

In fact, $t^*_1$ and $t^*_2$ can be computed explicitly as follows. For $t \geq t^*_1$ the sequence $V_t^*(1)$ can be computed recursively from the following equation

$$V_t^*(1) = p(1 - q)(V_{t+1}^*(1) + 1) + (1 - p + pq)(-1 - r) \quad \text{for} \quad t^*_1 \leq t \leq T - 1$$

$$V_t^*(1) = p(1 - q)(V_{t+1}^*(1) + 1) + (1 - p + pq)(-1 - r) \quad \text{for} \quad t^*_1 \leq t \leq T - 1$$
Moreover, the fact that \( t = \frac{p - 2pq - 1 - r(1 - p + pq)}{1 - p + pq} \)
for \( t_1^* \leq t \leq T \).

Moreover, the fact that \( V_{t_1^*}(-1, BB) \leq V_{t_1^*}(-1, MO) \) from the showed claim that
\( V_{t+1}(-1, MO) \leq V_{t+1}(-1, BB) \) implies \( V_t(1, BB) - V_t(1, NO) \leq 0 \). Therefore
\( V_{t_1^*}(-1) = V_{t_1^*}(-1, BB) \) and
\[
V_{t_1^*-1}(1, BB) - V_{t_1^*-1}(1, NO) = pq(2 + r + V_{t_1^*}(1)) - (1 - p)(r + V_{t_1^*}(-1))
\]
\[
= pq \left( 2 + r + (p - pq)T^{-t_1^*} \left( f + 2 + r - \frac{1}{1 - p + pq} \right) - 2 - r + \frac{1}{1 - p + pq} \right)
- (1 - p)(r + (p + (1 - p)q)(-1 - r) + (1 - p)(1 - q)(1 + V_{t_1^*+1}(1)))
\]
\[
= \frac{q(1 - 2p)}{1 - p + pq} - (f + 2 + r - \frac{1}{1 - p + pq})(p - pq)^{T-t_1^*-1}(1 - q)(p^2 q + 1 + p^2 - 2p) \geq 0.
\]

Note that from the expression of the second last line, it is clear that \( V_{t_1^*-1}(1, BB) - V_{t_1^*-1}(1, NO) \) is positive when \( t_1^* \) is negative infinity, and negative when \( t_1^* = T - 1 \). Thus the existence of \( t_1^* \) is guaranteed. Moreover, we derive the formula for \( t_1^* \) as shown in (2.32). Simple calculation from the above expression confirms \( t_1^* \leq T - 1 \).

Now, for \( t \leq t_1^* \) we have
\[
V_{t-1}(1) = (1 - p)(-1 + V_{t}(1)) + p(1 + V_{t}(1))
= pV_{t}(1) + 2p - 1
+ (1 - p)[(1 - p - q + pq)(1 + V_{t+1}(1)) + (-1 - r)(p + q - pq)]
\]
\[
= a_1V_{t}(1) + a_2V_{t+1}(1) + a_3
\]
where \( a_1 = p, a_2 = (1 - p)(1 - p - q + pq), a_3 = 2p - 1 - (1 + r)(1 - p)(p + q - pq) + (1 - p)(1 - p - q + pq). \) And the boundary condition for the iteration is
\[
V_{t_1^*}(1) = (p - pq)^{T-t_1^*} \left( f + 2 + r - \frac{1}{1 - p + pq} \right) - 2 - r + \frac{1}{1 - p + pq},
\]
\[
V_{t_1^*+1}(1) = (p - pq)^{T-t_1^*-1} \left( f + 2 + r - \frac{1}{1 - p + pq} \right) - 2 - r + \frac{1}{1 - p + pq}.
\]
Rewriting the above relation as

\[ V_{t-1}^\ast(1) - a_4 = a_1(V_t^\ast(1) - a_4) + a_2(V_{t+1}^\ast(1) - a_4), \]

with \( a_4 = \frac{a_3}{1-a_1-a_2} \) leads to the solution for \( V_t^\ast(1) \) and \( t_2^\ast \) (as well as the existence of \( t_2^\ast \)). One can verify from the expression of \( t_2^\ast \) that it is less than or equal to \( T \). Moreover, we can verify that \( t_1^\ast < t_2^\ast \) from their expression, which is consistent with our previous claim.

Finally, from the iterative expression of \( V_t^\ast(-1) \), we can compute it by (2.36).

A.2.11 Proof of Proposition 4.2

Proof. By multiplying Eqn.(4.80) by the integrating factor \( e^{\int_0^t \lambda^\ast V^3(s)ds} \) and integrating from 0 to \( t \), and finally dividing the integrating factor, we get

\[
Y(t) = -\int_0^t e^{-\int_0^u \lambda^\ast V^3(s)ds} d\Psi^2(s) - \int_0^t e^{-\int_0^u \lambda^\ast V^3(s)ds} \frac{Z(s)}{Q^b(s)} d\Psi^3(s) + \int_0^t e^{-\int_0^u \lambda^\ast V^3(s)ds} \frac{Z(s)(\Psi^1(s) - \Psi^2(s) - \Psi^3(s))}{(Q^b(s))^2} \lambda V^3 ds,
\]

which implies that \( Y(t) \) is a Gaussian process since \( \Psi^\ast \) is a Gaussian process. Since \( \Psi^\ast \) is centered, i.e., with mean zero, it is easy to see that \( Y(t) \) is also centered. Next, let us determine the variance of \( Y(t) \). By Itô’s formula, we have

\[
d(Y(t)^2) = 2Y(t)dY(t) + d(Y)_t \quad \text{(A.28)}
\]

\[
d(Y)_t = -2Y(t) \frac{Y(t)}{Q^b(t)} \lambda V^3 dt - 2Y(t)d\Psi^2(t) - 2Y(t) \frac{Z(t)}{Q^b(t)} d\Psi^3(t) + 2Y(t) \frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{(Q^b(t))^2} \lambda V^3 dt.
\]

From Eqn. (4.80),

\[
d(Y)_t = d(\Psi^2)_t + \frac{Z(t)^2}{Q^b(t)^2} d(\Psi^3)_t + \frac{2Z(t)}{Q^b(t)} d(\Psi^2, \Psi^3),
\]

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\]
Plugging (A.29) into (A.28), and taking expectations on the both hand sides of the equation, we get

\[
d\mathbb{E}[Y(t)^2] = d\langle \Psi^2 \rangle_t + \frac{Z(t)^2}{Q^b(t)^2} d\langle \Psi^3 \rangle_t + \frac{2Z(t)}{Q^b(t)} d\langle \Psi^2, \Psi^3 \rangle_t \\
- 2\mathbb{E}[Y(t)^2] \frac{1}{Q^b(t)} \lambda V^3 dt \\
+ 2 \frac{Z(t) (\mathbb{E}[Y(t)\Psi^1(t)] - \mathbb{E}[Y(t)\Psi^2(t)] - \mathbb{E}[Y(t)\Psi^3(t)])}{Q^b(t)^2} \lambda V^3 dt
\]  

(A.30)

By using the integrating factor \( e^{\int_0^t \frac{2\lambda V^3}{Q^b(s)} ds} \), we conclude that

\[
\mathbb{E}[Y(t)^2] = \int_0^t e^{-\int_s^t \frac{2\lambda V^3}{Q^b(u)} du} Z(s) \frac{2\lambda V^3}{Q^b(s)^2} \mathbb{E}[Y(s)\Psi^1(s)] - \mathbb{E}[Y(s)\Psi^2(s)] - \mathbb{E}[Y(s)\Psi^3(s))] ds \\
+ \int_0^t e^{-\int_s^t \frac{2\lambda V^3}{Q^b(u)} du} \frac{Z(s)^2}{Q^b(s)^2} d\langle \Psi^3 \rangle_s \\
+ \int_0^t e^{-\int_s^t \frac{2\lambda V^3}{Q^b(u)} du} \frac{2Z(s)}{Q^b(s)} d\langle \Psi^2, \Psi^3 \rangle_s
\]  

(A.31)

Let us recall that

\[
\vec{\Psi} = \Sigma \vec{W} \circ \lambda e - \vec{V} v_d \lambda \vec{W}_1 \circ \lambda e.
\]  

(A.33)

We also recall that \((\psi_{ij})_{1 \leq i, j \leq 6}\) is a symmetric matrix defined as

\[
\psi_{ij} := \sum_{k=1}^6 \Sigma_{ik} \Sigma_{jk} \lambda + \vec{V}^i \vec{V}^j v_3^2 \lambda^3, \quad 1 \leq i, j \leq 6.
\]  

(A.34)

Therefore, we have

\[
\langle \Psi^2 \rangle_t = \psi_{22} t, \quad \langle \Psi^3 \rangle_t = \psi_{33} t, \quad \langle \Psi^2, \Psi^3 \rangle_t = \psi_{23} t.
\]  

(A.35)

For any \(i, j\) and \(t > s\),

\[
\mathbb{E}[\Psi^i(t)\Psi^j(s)] = \sum_{k=1}^6 \Sigma_{ik} \Sigma_{jk} \lambda s + \vec{V}^i \vec{V}^j v_3^2 \lambda^3 s = \psi_{ij} s.
\]  

(A.36)

For any \(i = 1, 2, 3\), from (A.27), we can compute \(\mathbb{E}[Y(t)\Psi^i(t)]\) as

\[
\mathbb{E}[Y(t)\Psi^i(t)]
\]  

(A.37)
Next, combining equations (A.35), (A.36), (A.37), (A.58), and (4.28), after some computation, we see

\[
\int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \, d\langle \Psi^2 \rangle_s + \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \, \frac{Z(s)^2}{Q^3(s)} \, d\langle \Psi^3 \rangle_s \\
= \lambda \sum_{j=1}^6 \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \left( \sum_{j=1}^6 \frac{Z(s)}{Q^3(s)} \Sigma_{3j} \right)^2 \, ds \\
+ \lambda^3 v_d^2 \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \left( V^2 + \frac{Z(s)}{Q^3(s)} \overline{\Psi^3} \right)^2 \, ds \\
= \frac{(b + ct)^{\frac{2}{\epsilon} + 1} - b^\frac{1}{\epsilon} + 1}{(2 + c)(b + ct)^{\frac{2}{\epsilon}}} \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \left( \frac{\Sigma_{3j} - \frac{\Sigma_{3j} a}{(1 + c)\lambda V^3}}{(1 + c severity) \lambda V^3} \right) \, ds \\
\Sigma_{3j} + \frac{\lambda^3 v_d^2}{6} \left( \frac{c}{1 + c} \overline{V} \overline{\Psi^3} \right)^2 \\
\cdot \sum_{j=1}^6 \left[ \lambda \left( \Sigma_{3j} - \frac{\Sigma_{3j} a}{(1 + c)\lambda V^3} \right) \right] \left( \Sigma_{3j} + \frac{\lambda^3 v_d^2}{6} \left( \frac{c}{1 + c} \overline{V} \overline{\Psi^3} \right)^2 \right) \\
+ \frac{t}{(b + ct)^{\frac{2}{\epsilon} + 1}} \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \left( \lambda \Sigma_{4j}^2 + \frac{\lambda^3 v_d^2}{6} \left( \frac{c}{1 + c} \overline{V} \overline{\Psi^3} \right)^2 \right) \left( z + \frac{ab}{1 + c} \right)^2, \quad (A.38)
\]

and

\[
E[Y(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))] \\
= -(\psi_{12} - \psi_{22} - \psi_{32}) \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \, ds \\
- (\psi_{13} - \psi_{23} - \psi_{33}) \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \, \frac{Z(s)}{Q^3(s)} \, ds \\
+ (\psi_{11} + \psi_{22} + \psi_{33} - \psi_{12} - \psi_{13} - \psi_{21} - \psi_{31} + \psi_{23} + \psi_{32}) \\
\cdot \int_0^t e^{-\int_s^t \frac{2\lambda v^3}{Q^3(u)} \, du} \, \frac{Z(s)b}{(Q^3(s))^2} \, ds \\
= \alpha (b + ct) + \beta (b + ct)^{-\frac{1}{\epsilon}} + \gamma \frac{\log(b + ct)}{(b + ct)^{\frac{1}{\epsilon}}} + \delta + \eta (b + ct)^{-\frac{1}{\epsilon} - 1}, \quad (A.39)
\]

where \(\alpha, \beta, \gamma, \delta\) are defined in (4.86) and \(\alpha, \beta, \gamma, \delta, \eta\) are defined in (4.85). There-
fore,

\[
\int_0^t e^{-\int_0^t \frac{2Z(s)}{Q(s)} ds} \frac{2Z(s)}{(Q(s))^2} \lambda V^3 \mathbb{E}[Y(s)(\Psi^1(s) - \Psi^2(s) - \Psi^3(s))] ds
\]

\[
= \frac{2}{(b + ct)^2} \int_0^t (b + cs)^{\frac{3}{2} - 1} \left[ -\frac{a}{(1 + c)\lambda V^3} + \left( z + \frac{ab}{1 + c} \right) \frac{b^2}{\lambda V^3} (b + cs)^{\frac{1}{2} - 1} \right]
\cdot \left[ \hat{\alpha}(b + cs) + \hat{\beta}(b + cs)^{\frac{1}{2}} + \hat{\gamma} \frac{\log(b + cs)}{(b + cs)^{\frac{1}{2}}} + \hat{\delta} \right] ds
\]

(A.40)

Hence, we get the desired result by substituting (A.38) and (A.40) into (A.31). \( \square \)

### A.2.12 Proof of Lemma 3.1

**Proof.** First, this lemma is valid for \( t = T - 1 \) since

\[
C_{T-1}(n_{T-1}, m_{T-1}, l_{T-1}, A_{T-1})
= \mathbb{E}\{ m_{T-1}(f + A_{T-1} + c(m_{T-1} + l_{T-1}) + \epsilon_{T-1})
+ p_l l_{T-1}(A_{T-1} - r - 1 + c(m_{T-1} + l_{T-1}) + \epsilon_{T-1})
+ p_l(n_{T-1} - m_{T-1} - l_{T-1})
\cdot [A_{T-1} + c(m_{T-1} + l_{T-1}) + c(n_{T-1} - m_{T-1} - l_{T-1}) + \epsilon_{T-1} + \epsilon_T]
+ (1 - p_l)(n_{T-1} - m_{T-1} + c(m_{T-1} - n_{T-1} - m_{T-1}) + \epsilon_{T-1} + \epsilon_T)\}
= m_{T-1}(f + A_{T-1} + c(m_{T-1} + l_{T-1}))
+ p_l l_{T-1}(A_{T-1} - r - 1 + c(m_{T-1} + l_{T-1}))
+ p_l(n_{T-1} - m_{T-1} - l_{T-1})
\cdot [A_{T-1} + c(m_{T-1} + l_{T-1}) + c(n_{T-1} - m_{T-1} - l_{T-1}) + \epsilon_{T-1} + \epsilon_T]
= m_{T-1} A_{T-1} + m_{T-1}(f + c(m_{T-1} + l_{T-1})) + p_l l_{T-1}(-r - 1 + c(m_{T-1} + l_{T-1}))
+ p_l(n_{T-1} - m_{T-1} - l_{T-1})(c n_{T-1}) + (1 - p_l)(n_{T-1} - m_{T-1})(c n_{T-1})
= m_{T-1} A_{T-1} + C_{T-1}(n_{T-1}, m_{T-1}, l_{T-1}, 0).
\]

And since the coefficient of \( A_{T-1} \) is \( n_{T-1} \), we have

\[
V_{T-1}(n_{T-1}, A_{T-1}) = \min_{0 \leq m_{T-1}, l_{T-1}} \{ C_{T-1}(n_{T-1}, m_{T-1}, l_{T-1}, A_{T-1}) \}
= m_{T-1} A_{T-1} + \min_{0 \leq m_{T-1}, l_{T-1}} \{ C_{T-1}(n_{T-1}, m_{T-1}, l_{T-1}, 0) \}
= n_{T-1} A_{T-1} + V_{T-1}(n_{T-1}, 0).
\]
Thus the hypothesis holds for \( t = T - 1 \). Assume it is true for some \( t + 1 < T \) with \( t \geq 0 \). Then, for \( t \),

\[
C_t(n_t, m_t, l_t, A_t) = \mathbb{E}\{m_t(f + A_t + c(m_t + l_t) + \epsilon_t) + p_l l_t(A_t - r - 1 + c(m_t + l_t) + \epsilon_t) + p_l V_{t+1}(n_t - m_t - l_t, A_t + c(m_t + l_t) + \epsilon_t)\} + \mathbb{E}\{p_l(V_{t+1}(n_t - m_t - l_t, A_t + c(m_t + l_t) + \epsilon_t))\}
\]

Thus the hypothesis holds for \( k = t \) if it holds for \( k = t + 1 \). By induction, the proof is complete.

\[\square\]

**A.2.13 Proof of Theorem 3.1**

**Proof.** The proof is by induction. First, for \( t = T \), \( M_T = n_T \) and \( L_T = 0 \). Thus \( V_T(n_T) = n_T^2 + an_T \).

Second, suppose the theorem holds for \( t + 1 \). Let \( 0 = A_1 < A_2 < \ldots < A_k = 1 \), for \( i = 1, 2, \ldots, k - 1 \),

\[
V_{t+1}(x) = z_{i,1}x^2 + z_{i,2}x + z_{i,3} \quad \text{when} \quad x \in [A_i, A_{i+1}).
\]

Suppose \( n_t - m_t - l_t \in [A_i, A_{i+1}) \) and \( n_t - m_t \in [A_j, A_{j+1}) \) Hessian of \( C_t \) is

\[
\mathbb{H}(C_t) = \begin{pmatrix}
2p_l z_{i,1} + 2(1 - p_l)z_{j,1} & 1 - p_l + 2z_{i,1}p_l \\
1 - p_l + 2z_{i,1}p_l & 2p_l z_{i,1}
\end{pmatrix}.
\]

And the determinant of this matrix is

\[
4p_l^2 z_{i,1}^2 + 4p_l(1 - p_l)z_{i,1}z_{j,1} - 1 - p_l^2 - 4z_{i,1}^2p_l^2 + 2p_l + 4p_l^2 z_{i,1} - 4p_l z_{i,1} = -(1 - p_l)^2 - 4p_l(1 - p_l)z_{i,1}(1 - z_{j,1}) < 0.
\]

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Therefore the Hessian of $C_t$ is neither semi-positive nor semi-negative. Thus the possible minimum value will only be reached on the boundary. Note that $V_{t+1}$ is a piece-wise quadratic function, we also need to consider the pairs of $(m_t, l_t)$ making $n_t - m_t - l_t = A_i$ or $A_{i+1}$, or $n_t - m_t = A_j$ or $A_{j+1}$. When $n_t - m_t - l_t = A_i$, then $C''(l_t) = -2(1 - p_t) + 2(1 - p_t)z_{j,1} \leq 0$, implying that $L_t = 0$, $M_t = n_t - A_i$ or $L_t = n_t - A_i, M_t = 0$. Similarly, we can show that in the case of $L_t = A$ or $M_t = A$, $C_t$ will be a quadratic function with a negative second order coefficient. Thus $M_t \cdot L_t = 0$ for $t$.

Next, we will calculate $V_t$. In the case of $l_t = 0$, $C_t = \alpha m_t + n_t m_t + V_{t+1}(n_t - m_t)$, which is a piece-wise quadratic function of $m_t$. On each piece, if not on the boundary of that piece, the optimal solution $M_t$ is a linear function of $n_t$ with coefficient $a_1 = 1 - 1/(2z_1)$, which is in the range of $(0, 1/2]$. Then the second order coefficient of $V_t$ is $z_1(1 - a_1)^2 + a_1^2 < 1$. And if $M_t$ is on the boundary of that piece, then $M_t$ is still a linear function of $n_t$ with the first order coefficient of $1$. Then the second order coefficient of $V_t$ is $z_1(1 - a_1)^2 + a_1^2 = 1$. Similarly for the case of $m_t = 0$, $C_t = \beta n_t l_t + l_t V_{t+1}(n_t - l_t) + (1 - p_t) V_{t+1}(n_t)$. Therefore $L_t$ is a linear function of $n_t$ with the first order coefficient no greater than 1, denoted by $b_1$. Then the second order coefficient of $V_t$ is $p_t z_1(1 - b_1)^2 + p_t b_1^2 + (1 - p_t) z_1 \leq 1$. By induction, the statement holds for all $t \leq T$.

\[ A.2.14 \quad \text{Proof of Theorem 4.2} \]

Proof. First, we define the scaled processes $S^D_n$ and $\overline{S}^V_n$ by

\[ S^D_n(t) = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} D_i, \quad (A.41) \]

\[ \overline{S}^V_n(t) = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} V_i = \left( \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} V_i, 1 \leq j \leq 6 \right). \quad (A.42) \]

Then by Assumption 4.1 and according to Glynn and Whitt (Glynn and Whitt (1988), Theorem 5), the strong Law of Large Numbers (SLLN) also follows. That is,

\[ \lim_{i \to \infty} \frac{D_1 + D_2 + \ldots + D_i}{i} = \frac{1}{\lambda}, \quad \text{a.s.} \quad (A.43) \]

Then by the equivalence of SLLN and FSLLN (Glynn and Whitt (1988), Theorem 4), it is clear that for any $T > 0$,

\[ S^D_n = \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} D_i \Rightarrow \frac{\mathbf{e}}{\lambda}, \quad \text{a.s. in } (D[0, T], J_1) \text{ as } n \to \infty. \quad (A.44) \]

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Moreover, since $\overrightarrow{V}_1$ is square-integrable, it follows that $\mathbb{E}[V_j^2] < \infty$ for $1 \leq j \leq 6$. Note that $\{V_j^i\}_{i \geq 1}$ is stationary, applying Birkhoff’s Ergodic Theorem (Breiman (1968), Theorem 6.28) leads to

$$\frac{1}{n} \sum_{i=1}^{n} V_j^i \rightarrow \mathbb{E}[V_j^i | \mathcal{I}]$$

a.s. as $n \rightarrow \infty$, (A.45)

where $\mathcal{I}$ is the invariant $\sigma$-algebra of $\{V_j^i\}_{i \geq 1}$. Given the WLLN for $\{V_j^i\}_{i \geq 1}$, it follows that

$$\mathbb{E}[V_j^i | \mathcal{I}] = \bar{V}_j ,$$

(A.46)

and

$$\frac{1}{n} \sum_{i=1}^{n} V_j^i \rightarrow \bar{V}_j ,$$
a.s. as $n \rightarrow \infty$. (A.47)

Therefore, again by Theorem 4 in Glynn and Whitt (1988),

$$\overrightarrow{S}_n^{V,j} = \frac{1}{n} \sum_{i=1}^{n} V_j^i \Rightarrow \bar{V}_j e, \quad \text{a.s. in } (D[0,T], J_1) \text{ as } n \rightarrow \infty$$

(A.48)

Since the limit processes for $\{S_n^P\}_{n \geq 1}$ and $\{S_n^{V,j}\}_{n \geq 1}$, $1 \leq j \leq 6$, are deterministic, according to Theorem 11.4.5 in Whitt (2002),

$$(\overrightarrow{S}_n^V, S_n^P) \Rightarrow (\overrightarrow{V} e, \frac{e}{\lambda}), \quad \text{a.s. in } (D^7[0,T], J_1) \text{ as } n \rightarrow \infty.$$ (A.49)

Finally, from Theorem 9.3.4 in Whitt (2002),

$$\overrightarrow{C}_n \Rightarrow \lambda \overrightarrow{V} e, \quad \text{in } (D^6[0,T], J_1) \text{ as } n \rightarrow \infty.$$ (A.50)

A.2.15 Proof of Theorem 4.3

Proof. First, note that (4.24), (4.25), (4.26) are the solutions to the following SDE’s

$$d \begin{pmatrix} Q^b(t) \\ Q^a(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & Z(t^-) & 0 & 0 \end{pmatrix} 1_{Q^a(t^-), Q^b(t^-), Z(t^-) > 0} \overrightarrow{V} dt$$

(A.51)

$$(Q^b(0), Q^a(0), Z(0)) = (q^b, q^a, z).$$
Hence it suffices to show the convergence to (A.51). Now, set $Y_n = \overrightarrow{C_n}$, $X_n = (Q_n^b, Q_n^a, Z_n)$, and
\[
F_n(x, s-) = F(x, s-) = \begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & -1 \\
-1 & 0 & \frac{x^3(s-)}{x^1(s-)} & 0 & 0 & 0
\end{pmatrix} \mathbb{1}_{x(s-) > 0}.
\]

To decompose $Y_n$, define the filtrations $F_{nt} := \sigma(\{N(s)\}_{0\leq s\leq nt}, \{\overrightarrow{V_i}\}_{1\leq i\leq N(nt)})$ and $G_i := \sigma(\{\overrightarrow{V_k}\}_{1\leq k\leq i})$, take $\delta = \infty$,

\[
M_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} \overrightarrow{V_i} - \mathbb{E}[\overrightarrow{V_i}|G_{i-1}],
\]

(A.52)

and

\[
A_n(t) = Y_n(t) - M_n(t).
\]

(A.53)

We will show that $M_n$ is a martingale with respect to $F_{nt}$ and $\{Y_n\}_{n \geq 1}$ satisfies Condition A.1 in Theorem A.3.

For $\forall s, 0 \leq s < t$, it is easy to see that $F^n_s \cap (N(ns) < i) \subseteq F^n_{\frac{1}{n} \sum_{k=1}^i D_k} \cap (N(ns) < i)$. Assumption 4.6 implies that $\mathbb{E}[\overrightarrow{V_i}|F^n_{\frac{1}{n} \sum_{k=1}^i D_k-} \cap (N(ns) < i)] = \mathbb{E}[\overrightarrow{V_i}|G_{i-1}].$ Thus

\[
\mathbb{E}\left[ \mathbb{E}\left[ \overrightarrow{V_i}|G_{i-1} \right] \left| F^n_{s} \cap (N(ns) < i) \right. \right] = \mathbb{E}\left[ \mathbb{E}\left[ \overrightarrow{V_i}|F^n_{\frac{1}{n} \sum_{k=1}^i D_k-} \right] \left| F^n_{s} \cap (N(ns) < i) \right. \right]
\]

(A.54)

Meanwhile, $F^n_{\frac{1}{n} \sum_{k=1}^i D_k-} \cap (N(ns) \geq i) \subseteq F^n_s \cap (N(ns) \geq i)$. Thus

\[
\mathbb{E}\left[ \mathbb{E}\left[ \overrightarrow{V_i}|G_{i-1} \right] \left| F^n_{s} \cap (N(ns) \geq i) \right. \right] = \mathbb{E}\left[ \mathbb{E}\left[ \overrightarrow{V_i}|F^n_{\frac{1}{n} \sum_{k=1}^i D_k-} \right] \left| F^n_{s} \cap (N(ns) \geq i) \right. \right]
\]

(A.55)
Moreover, \( \mathbb{E} \left[ \widehat{V}_i | \mathcal{F}_s^n \cap (N(ns) < i) \right] = \widehat{V}_i \) since \( \widehat{V}_i \) is measurable with respect to \( \mathcal{F}_s^n \cap (N(ns) < i) \). Therefore,

\[
\mathbb{E} \left[ M_n(t) | \mathcal{F}_s^n \right] = \mathbb{E} \left[ \sum_{i=1}^{N(ns)} \frac{\widehat{V}_i - \mathbb{E}[\widehat{V}_i | \mathcal{G}_{i-1}]}{n} \right]_{| \mathcal{F}_s^n}
\]

\[
= \frac{1}{n} \sum_{i=1}^{N(ns)} \left( \mathbb{E} \left[ \widehat{V}_i | \mathcal{F}_s^n \cap (N(ns) \geq i) \right] - \mathbb{E} \left[ \mathbb{E}[\widehat{V}_i | \mathcal{G}_{i-1}] | \mathcal{F}_s^n \cap (N(ns) \geq i) \right] \right) + \frac{1}{n} \mathbb{E} \left[ \sum_{i=N(ns)+1}^{N(nt)} \widehat{V}_i - \mathbb{E}[\widehat{V}_i | \mathcal{G}_{i-1}] \right]_{| \mathcal{F}_s^n \cap (N(ns) < i)}
\]

\[
= \frac{1}{n} \sum_{i=1}^{N(ns)} \left( \widehat{V}_i - \mathbb{E}[\widehat{V}_i | \mathcal{G}_{i-1} \cap (N(ns) \geq i)] \right) + \frac{1}{n} \lambda n(t-s) \left( \mathbb{E}[\widehat{V}_i | \mathcal{F}_s^n \cap (N(ns) < i)] - \mathbb{E}[\mathbb{E}[\widehat{V}_i | \mathcal{G}_{i-1}] | \mathcal{F}_s^n \cap (N(ns) < i)] \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{N(ns)} \left( \widehat{V}_i - \mathbb{E}[\widehat{V}_i | \mathcal{G}_{i-1} \cap (N(ns) \geq i)] \right) = M_t(s).
\]

And \( \mathbb{E}[M_n(t)] < \infty \) follows directly from Assumption 4.2. Hence it follows that \( M_n(t) \) is a martingale. The quadratic variance of \( M_n(t) \) is as follows:

\[
\mathbb{E} \left[ [M_n]_t \right] = \frac{nt^2}{n^2} \sum_{j=1}^{6} \mathbb{E} \left[ \lambda \left( V_{ij}^2 - \mathbb{E}[V_{ij}^2 | \mathcal{G}_{i-1}] \right)^2 \right]
\]

\[
= \frac{t}{n} \sum_{j=1}^{6} \lambda \mathbb{E} \left[ \left( V_{ij}^2 \right)^2 - 2V_{ij}^2 \mathbb{E}[V_{ij}^2 | \mathcal{G}_{i-1}] + \left( \mathbb{E}[V_{ij}^2 | \mathcal{G}_{i-1}] \right)^2 \right]
\]

\[
= \frac{t}{n} \sum_{j=1}^{6} \lambda \left( \mathbb{E} \left( V_{ij}^2 \right)^2 - \mathbb{E} \left( \mathbb{E}[V_{ij}^2 | \mathcal{G}_{i-1}] \right)^2 \right) \leq \frac{t}{n} \sum_{j=1}^{6} \lambda \mathbb{E} \left( V_{ij}^2 \right)^2,
\]

because

\[
\mathbb{E} \left[ V_{ij}^2 | \mathcal{G}_{i-1} \right] = \mathbb{E} \left[ \mathbb{E}[V_{ij}^2 | \mathcal{G}_{i-1}] \right] = \mathbb{E} \left( \mathbb{E}[V_{ij}^2 | \mathcal{G}_{i-1}] \right)^2.
\]

Thus \( \mathbb{E} \left[ [M_n]_t \right] \) is bounded uniformly in \( n \) since \( \widehat{V}_i \) is square-integrable. Let \( [T(A_n)]_t \) denote the total variation of \( A_n \) up to time \( t \). Then \( \mathbb{E} \left[ [T(A_n)]_t \right] \) is also
uniformly bounded in $n$, as

$$E \left[ |T(A_n)| t \right] = t \sum_{j=1}^{6} \lambda E \left[ |V_j| G_{i-1} \right] \leq t \sum_{j=1}^{6} \lambda E \left[ |V_j| |G_{i-1}| \right]$$

(A.56)

$$= t \sum_{j=1}^{6} \lambda E |V_j| < \infty.$$ (A.57)

where the inequality in (A.56) uses the Jensen’s inequality for conditional expectations and (A.57) follows from the square-integrability assumption. Thus, $Y_n$ satisfies Condition A.1 with $\tau_n^\alpha = \alpha + 1$.

Now taking $G_n(x \circ e, e) = F_n(x) = F(x)$, it is easy to see that Condition A.2 is satisfied according to Kurtz and Protter (1991).

It remains to check the existence of a global solution and the strong local uniqueness for the limit equations (A.51), which can be in fact be solved explicitly hence these conditions naturally satisfied. Clearly, (4.24), (4.25) are solutions to $Q^b(t)$, $Q^a(t)$ before hitting 0. Moreover, $Z(t)$ satisfies (4.26).

Now $Q^a(t) = 0$ when $t = \tau^a$ as given in (4.27). $\tau^a > 0$ if $\bar{V}^4 - \bar{V}^5 - \bar{V}^6 < 0$; otherwise $Q^a(t)$ never hits zero in which case define $\tau^a = \infty$. The case for $\tau^b$ is similar.

The equation for $Z(t)$ when $Z(t-) > 0$ is a first order linear ODE with the solution

$$Z(t) = \begin{cases} 
- \frac{a}{1+c} \left( b + c(t \wedge \tau) \right) + \left( z + \frac{ab}{1+c} \right) \left[ \frac{b}{b+c(t \wedge \tau)} \right]^{1/c} & c \notin \{-1,0\}, \\
[a \log(b - (t \wedge \tau)) + z/b - a \log b] (b - (t \wedge \tau)) & c = -1, \\
(z + ab) e^{-t/b} - ab & c = 0.
\end{cases}$$

(A.58)

From the solution, we can solve $\tau^z$ explicitly as given in (4.28). Note that the expression of $Z(t)$ may not be monotonic and there might be multiple roots when $c \neq 0$. Nevertheless, it is easy to check that the solution given in (4.28) is the smallest positive root. For instance, when $c \notin \{-1,0\}$, there are two roots $-b/c$ and $\left( \frac{(1+c)x + b}{a} \right)^{c/(c+1)} b^{1/(c+1)} c^{-1} - b/c$ and when $c = -1$, there are two roots $b$ and $b(1-e^{-\frac{z}{ab}})$. More computations confirm that indeed the smallest positive roots are $\tau^z = \left( \frac{(1+c)x + b}{a} \right)^{c/(c+1)} b^{1/(c+1)} c^{-1} - b/c$ for $c \notin \{-1,0\}$ and $\tau^z = b(1-e^{-\frac{z}{ab}})$ for $c = -1$. Moreover, $\tau^z < \tau^b$ from the calculation. Therefore $\tau = \min\{\tau^a, \tau^z\}$ is well defined and finite. 

\[\square\]
A.2.16 Proof of Theorem 4.4

Proof. First, let us extend the definition of $\Upsilon$ from $[0,1]$ to $\mathbb{R}$ by

$$\Upsilon(x) = x\mathbb{1}_{0 \leq x \leq 1} + \mathbb{1}_{1 < x}.$$  \hspace{1cm} (A.59)

Then $\Upsilon$ is (still) Lipschitz continuous and increasing on $\mathbb{R}$. That is, there exists $K > 0$, such that for any $z_1, z_2 \in \mathbb{R}$, $|\Upsilon(z_1) - \Upsilon(z_2)| \leq K|z_1 - z_2|$. Next, define $\tau = \min\{\tau^b, \tau^a, \tau^z\}$ with $\tau^b = \inf\{t : Q^b(t) \leq 0\}$, $\tau^a = \inf\{t : Q^a(t) \leq 0\}$, and $\tau^z = \inf\{t : Z(t) \leq 0\}$. Similar to the argument for Lemma 4.1, $\Upsilon \in [0,1]$ and $z, q^b > 0$ imply that $Z_n(t) \leq Q^b_n(t)$ and $Z(t) \leq Q^b(t)$ for any time before hitting zero. Thus $\tau^z \leq \tau^b$. Now the remaining part of the proof is similar to that of Theorem 4.3 except for the global existence and local uniqueness of the solution to (4.32) and (4.34). It is easy to see that while $Q^b(t)$ and $Q^a(t)$ remain unchanged compared to those in Theorem 4.3,

$$\frac{dZ(t)}{dt} = -\lambda \left( V^2 + V^3 \Upsilon \left( \frac{Z(t)}{Q^b(t)} \right) \right) \mathbb{1}_{t \leq \tau}. \hspace{1cm} (A.60)$$

Let the right hand side of (A.60) be denoted by $\vartheta(Z,t)$, and define $\vartheta(Z, q^b/(\lambda v^b)) = 1$. Let $\{T_i\}_{i \geq 1}$ be an increasing positive sequence with $\lim_{i \to \infty} T_i = \tau$. Then for any $z_1, z_2 \geq 0$ and $0 \leq t \leq T_i$,

$$|\vartheta(z_1, t) - \vartheta(z_2, t)| = \lambda V^3 \left| \Upsilon \left( \frac{z_1}{q^b - \lambda v^b t} \right) - \Upsilon \left( \frac{z_2}{q^b - \lambda v^b t} \right) \right| \leq \lambda V^3 K \left| \frac{z_1}{q^b - \lambda v^b t} - \frac{z_2}{q^b - \lambda v^b t} \right| \leq \frac{\lambda V^3 K}{q^b - \lambda v^b T_i} |z_1 - z_2|.$$

Therefore $\vartheta(Z,t)$ is Lipschitz continuous in $Z$ and continuous in $t$ for any $t < T_i$ and $Z > 0$. By the Picard’s existence theorem, there exists a unique solution to (A.60) with the initial condition $Z(0) = z$ on $[0, T_i]$. Now letting $i \to \infty$, the unique solution exists on $[0, \tau]$. Moreover, by the boundedness of $\vartheta(Z, \tau)$ and the continuity of $Z(t)$ at $\tau$, the unique solution also exists at $t = \tau$. For $t > \tau$, $\vartheta(Z,0) = 0$ and $Z(t) = Z(\tau)$. Hence there exists a unique solution $Z(t)$ for $t \geq 0$. Note that $\tau^a = \infty$ (resp. $\tau^b = \infty$) when $v^a < 0$ (resp. $v^b < 0$). However, since the right hand side of (A.60) is less than or equal to $-\lambda V^2$, it follows that $Z(t)$ is decreasing in $t$ and hits 0 within finite time. Therefore $\tau$ is well defined. \qed

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**A.2.17 Proof of Theorem 4.5**

*Proof.* Let us recall that before $t \leq \tau$, with Assumption 4.4,

\[
d \begin{pmatrix} Q_b^n(t) \\ Q_a^n(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -Z_n(t-) & 0 & 0 & 0 \end{pmatrix} 1_{Q_n^b(t-)>0,Z_n(t-)>0} \cdot d \vec{C}_n(t),
\]

(A.61)

where

\[
\vec{C}_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} \vec{V}_i = M_n(t) + \int_0^t (\lambda + \beta Q_n^b(s-) + \alpha Q_n^a(s-)) ds \vec{V}.
\]

(A.62)

Here

\[
\vec{M}_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} [\vec{V}_i - \vec{V}] + \frac{1}{n} \vec{V} \left[ N(nt) - n \int_0^t (\lambda + \beta Q_n^b(s-) + \alpha Q_n^a(s-)) ds \right]
\]

(A.63)

is a martingale. Similar to the arguments before, we can show that $(Q_n^b, Q_n^a, Z_n) \Rightarrow (Q^b, Q^a, Z)$, where $(Q^b, Q^a, Z)$ satisfies the ODE:

\[
d \begin{pmatrix} Q_b(t) \\ Q_a(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -Z_n(t-) & 0 & 0 & 0 \end{pmatrix} 1_{Q_n(t-)>0,Z_n(t-)>0} \cdot (\lambda + \beta Q_b(t-) + \alpha Q_a(t-)) \vec{V} dt,
\]

with the initial condition $(Q_b(0), Q_a(0), Z(0)) = (q_b, q_a, z)$. The equations for $Q_b(t)$ and $Q_a(t)$ can be written down more explicitly as

\[
dQ_b(t) = (\lambda + \beta Q_b(t-) + \alpha Q_a(t-))(\check{V}_1 - \check{V}_2 - \check{V}_3) dt, \quad (A.64)
\]

\[
dQ_a(t) = (\lambda + \beta Q_b(t-) + \alpha Q_a(t-))(\check{V}_4 - \check{V}_5 - \check{V}_6) dt, \quad (A.65)
\]

which can be further simplified as

\[
d \begin{pmatrix} Q_b(t) \\ Q_a(t) \end{pmatrix} = \begin{pmatrix} -v^b \beta & -v^b \alpha \\ -v^a \beta & -v^a \alpha \end{pmatrix} \begin{pmatrix} Q_b(t) \\ Q_a(t) \end{pmatrix} + \begin{pmatrix} \lambda v_b \\ \lambda v_a \end{pmatrix}.
\]

(A.66)

Hence, for $t \leq \tau$, we get

\[
\begin{pmatrix} Q_b(t) \\ Q_a(t) \end{pmatrix} = c_1 \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} + c_2 e^{-(v^b + v^a) t} \begin{pmatrix} v^b \\ v^a \end{pmatrix} - \begin{pmatrix} \lambda \beta \\ 0 \end{pmatrix}, \quad (A.67)
\]

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where $c_1, c_2$ are constants that can be determined from the initial condition,
\[ c_1 = -q^a v^b - \frac{\lambda v^a}{v^a + v^b}, \quad c_2 = \frac{\beta q^b + \alpha v^a + \lambda}{\beta v^b + \alpha v^a}. \] (A.68)
Hence Eqns (4.36) and (4.37).

Finally, $Z(t)$ satisfies the first order ODE
\[ dZ(t) + Z(t) V_3 \left[ \frac{\lambda Q^b(t) + \beta + \alpha Q^a(t)}{Q^b(t)} \right] dt = -V_2 [\lambda + \beta Q^b(t) + \alpha Q^a(t)] dt \] (A.69)
with the solution given by (4.38).

### A.2.18 Proof of Theorem 4.6

**Proof.** First, define $N_n$ by
\[ N_n(t) = \frac{N(nt) - n\lambda t}{\sqrt{n}}. \]

Now recall the FCLT from Page 197 Billingsley (1968). For a stationary, ergodic and mean zero sequence $(X_n)_{n \in \mathbb{Z}}$, that satisfies $\sum_{n \geq 1} \|\mathbb{E}[X_0 | F_{-\infty}^-]\|_2 < \infty$, then
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nT]} X_i \Rightarrow W_1(\cdot) \] on $(D[0, T], \mathcal{F}, \mathbb{P})$ with
\[ v_2^2 = \mathbb{E}[(N(0, 1) - \lambda)^2] + 2 \sum_{j=1}^{\infty} \mathbb{E}[(N(0, 1) - \lambda)(N(j, j + 1) - \lambda)] < \infty. \] (A.71)

Next, for any $\epsilon > 0$ and $n$ sufficiently large,
\[ P \left( \sup_{0 \leq s \leq T} \left| \frac{N_{[ns]} - \lambda [ns]}{\sqrt{n}} - \frac{N_{ns} - \lambda ns}{\sqrt{n}} \right| > \epsilon \right) \leq \frac{1}{n^2} \max_{0 \leq k \leq \lfloor nT \rfloor, k \in \mathbb{Z}} N(k, k + 1) > \epsilon \sqrt{n} - \lambda \]
\[ \leq (\lfloor nT \rfloor + 1) P(N[0, 1] > \epsilon \sqrt{n} - \lambda) \]
\[ \leq \frac{\lfloor nT \rfloor + 1}{(\epsilon \sqrt{n} - \lambda)^2} \int_{N[0, 1] > \epsilon \sqrt{n} - \lambda} N[0, 1]^2 d\mathbb{P} \to 0, \]
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as $n \to \infty$. Hence, $N_n \Rightarrow v_dW_1$ on $(D[0,T],J_1)$ as $n \to \infty$.

Moreover, thanks to [Burton et al. (1986)], Assumption 4.8 implies

$$\overrightarrow{\Phi}_n^V \Rightarrow \Sigma \overrightarrow{\bar{W}} \quad \text{in} \quad (D^6[0,T],J_1),$$

(A.73)

where $\overrightarrow{\bar{W}}$ is a standard six-dimensional Brownian motion and $\Sigma$ is a $6 \times 6$ matrix representing the covariance scale of the limit process. Furthermore, the expression of $\Sigma$ by (4.51) and (4.52) can be explicitly computed following Burton et al. (1986).

Now, by Assumption 4.6, the joint convergence is guaranteed by Theorem 11.4.4. in Whitt (2002), i.e.,

$$(N_n, \overrightarrow{\Phi}_n^V) \Rightarrow (v_dW_1, \Sigma \overrightarrow{\bar{W}}) \quad \text{in} \quad (D^7[0,T],J_1).$$

(A.74)

Moreover, by Corollary 13.3.2. in Whitt (2002), we see

$$\overrightarrow{\Psi}_n \Rightarrow \overrightarrow{\Psi} = \Sigma \overrightarrow{\bar{W}}$$

in $(D^6[0,T],J_1)$.

To establish the second part of the theorem, it is clear that the limiting process would satisfy

$$d \begin{pmatrix} R^b(t) \\ R^a(t) \end{pmatrix} = A d \left( \overrightarrow{\Psi}(t) + \lambda \overrightarrow{\bar{V}} t \right),$$

(A.75)

$$(R^b(0), R^a(0)) = (q^b, q^a),$$

with

$$\iota^b = \inf \{ t : R^b(t) \leq 0 \}, \quad \iota^a = \inf \{ t : R^a(t) \leq 0 \}, \quad \iota = \min \{ \iota^a, \iota^b \}. \quad \text{(A.76)}$$

We now show that

$$(R^b_n, R^a_n) \Rightarrow (R^b, R^a) \quad \text{in} \quad (D^2[0,T],J_1).$$

(A.77)

According to the Cramér-Wold device, it is equivalent to showing that for any $(\alpha, \beta) \in \mathbb{R}^2$,

$$\alpha R^b_n + \beta R^a_n \Rightarrow \alpha R^b + \beta R^a \quad \text{in} \quad (D^2[0,T],J_1).$$

(A.78)

Since $\overrightarrow{\Psi}_n \Rightarrow \overrightarrow{\Psi}$ in $(D^2[0,T],J_1)$, by the Cramér-Wold device again,

$$(\alpha, \beta) \cdot A \cdot \overrightarrow{\Psi}_n \Rightarrow (\alpha, \beta) \cdot A \cdot \overrightarrow{\Psi} \quad \text{in} \quad (D^2[0,T],J_1).$$

(A.79)

By definition, it is easy to see that

$$\alpha R^b_n(t) + \beta R^a_n(t) = (\alpha, \beta) \cdot A \left( \overrightarrow{\Psi}_n(t \wedge \iota_n) + \overrightarrow{\bar{V}} (t \wedge \iota_n) \right) + \alpha q^b + \beta q^a.$$
Since the truncation function is continuous, by continuous-mapping theorem, it asserts that (A.78) holds and the desired convergence follows.

Moreover, because $\vec{V}_e$ is deterministic and $\alpha q^b + \beta q^a$ is a constant, we have the convergence in (A.78), as well as the convergence in (A.77). Note that $\tau_n, n \geq 1$ and $\tau$ are first passage times, by Theorem 13.6.5 in Whitt (2002),

$$(\tau_n, R_n^b(\tau_n-), R_n^a(\tau_n-)) \Rightarrow (\tau, R^b(\tau-), R^a(\tau-)). \quad (A.81)$$

A.2.19 Proof of Theorem 4.7

Proof. Under Assumption 4.5, it is clear that

$$\Psi^*_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{N(nt)} (\vec{V}_i - \mathbb{E}[\vec{V}_i \mid G_{i-1}])$$

(A.82)

is a martingale. Now define for $j = 1, 2, \ldots, 6,$

$$M^j_{nt} := \sum_{i=1}^{N(nt)} \left( V^j_i - \mathbb{E}[V^j_i \mid G_{i-1}] \right) = \sum_{i=1}^{N(nt)} (V^j_i - \bar{V}^j). \quad (A.83)$$

First, the jump size of $M^j_{nt}$ is uniformly bounded since $N(nt)$ is a simple point process and by Assumption 4.5, $V^j$'s are uniformly bounded. Next, the quadratic variation of $M^j_{nt}$ is given by

$$[M^j]_{nt} = \sum_{i=1}^{N(nt)} (V^j_i - \bar{V}^j)^2. \quad (A.84)$$

By Assumptions 4.5 and 4.9 and the ergodic theorem, as $t \to \infty,$

$$\frac{[M^j]_t}{t} \to \lambda \text{Var}[V^j], \text{a.s..} \quad (A.85)$$

Moreover, since $M^j$ and $M^k$ have no common jumps for $j \neq k,$

$$[M^j, M^k]_t \equiv 0. \quad (A.86)$$

Therefore, applying the FCLT for martingales of Theorem VIII-3.11 of Jacod and Shiryaev (1987), for any $T > 0,$ we have

$$\Psi^*_n \Rightarrow \Psi^*, \quad \text{in } (D^6[0,T], J_1), \quad (A.87)$$

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To see the second part of the claim, first note that by Assumption 4.9,

\[
\frac{1}{n} \sum_{i=1}^{N(nt)} \vec{V} \to \lambda \cdot \vec{V}, \quad \text{in } (D[0,T], J_1)
\] (A.88)

a.s. as \( n \to \infty \). The remaining of the proof is to check the conditions for Theorem A.3 as in the proof of Theorem 4.3. The quadratic variance of 

\( M_{nt} := (M_{nt})_{1 \leq j \leq 6} \)

is given by

\[
\mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} M_{nt} \right)^2 \right] = \frac{1}{n} \sum_{1 \leq j \leq 6} \mathbb{E}[N(nt)] \mathbb{E} \left[ \left( V_j^i - \mathbb{E}[V_j^i|\mathcal{F}_{T_i^j}] \right)^2 \right]
\]

\[
\leq K t \sum_{1 \leq j \leq 6} \mathbb{E} \left[ \left( V_j^i \right)^2 \right],
\] (A.89)

which is uniformly bounded in \( n \). The total variation of

\( A_n := \frac{1}{n} \sum_{i=1}^{N(nt)} \vec{V} \)

satisfies

\[
\mathbb{E}[|T(A_n)_t|] \leq \sum_{1 \leq j \leq 6} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{N(nt)} |\vec{V}^i| \right) \right] \leq \sum_{1 \leq j \leq 6} K t \mathbb{E}[||\vec{V}^j||],
\] (A.90)

which is uniformly bounded in \( n \). Hence the desired result. \( \square \)

### A.2.20 Proof of Theorem 4.8

*Proof.* Given Assumptions 4.3, 4.6, 4.7, and 4.8 we have from Theorem 4.6

\[
\vec{\Psi}_n = 1/\sqrt{n} \left( \sum_{i=1}^{N(nt)} \vec{V}^i - \lambda n \vec{V} \right) \Rightarrow \vec{\Psi}, \quad \text{in } (D^6[0,\tau], J_1)
\] (A.91)

Hence, we have the following convergence in \((D[0,\tau], J_1)\),

\[
\sqrt{n}(Q^b_n - Q^b) \Rightarrow \Psi_1 - \Psi_2 - \Psi_3,
\]

\[
\sqrt{n}(Q^a_n - Q^a) \Rightarrow \Psi_4 - \Psi_5 - \Psi_6.
\] (A.92)

Recall the dynamics of \( Z_n(t) \) in (4.21) and \( Z(t) \) in Theorem 4.3 we see

\[
d(Z_n(t) - Z(t)) = -d(C^2_n(t) - C^2(t)) - \frac{Z_n(t)}{Q^b_n(t)} dC^3_n(t) + \frac{Z(t)}{Q^b(t)} dC^3(t)
\]

\[
= -d(C^2_n(t) - C^2(t)) - \frac{Z_n(t)}{Q^b_n(t)} d(C^3_n(t) - C^3(t))
\]

\[
+ \left[ \frac{Z(t)}{Q^b(t)} - \frac{Z_n(t)}{Q^b_n(t)} \right] dC^3(t).
\] (A.93)
We can rewrite it as
\[
\frac{Z_n(t) - Z(t)}{Q^b(t -)} dC^3(t) = dX_n(t),
\]
\[
X_n(t) = -(C^2_n(t) - C^2(t)) - \int_0^t \frac{Z_n(s^-) dC^3_n(s)}{Q^b_n(s^-)}
\]
\[
+ \int_0^t \frac{Z_n(s^-)(Q^b_n(s^-) - Q^b(s^-))}{Q^b(s^-)Q^b_n(s^-)} dC^3(s).
\]
Now,
\[
\sqrt{n}X_n \Rightarrow -\Psi^2 - \int_0^\cdot \frac{Z(s^-)}{Q^b(s^-)} d\Psi^3(s)
\]
\[
+ \int_0^\cdot \frac{Z(s^-)(\Psi^1(s^-) - \Psi^2(s^-) - \Psi^3(s^-))}{(Q^b(s^-))^2} \lambda V^3 ds \quad (A.94)
\]
As the limit processes \( \Psi \) and \( Q^b, Q^a \) are continuous, this could be changed into
\[
\sqrt{n}X_n \Rightarrow -\Psi^2 - \int_0^\cdot \frac{Z(s)}{Q^b(s)} d\Psi^3(s) + \int_0^\cdot \frac{Z(s)(\Psi^1(s) - \Psi^2(s) - \Psi^3(s))}{(Q^b(s))^2} \lambda V^3 ds \quad (A.95)
\]
Hence,
\[
\sqrt{n}(Z_n - Z) \Rightarrow Y, \quad (A.96)
\]
where \( Y \) satisfies (4.80).

### A.2.21 Proof of Lemma 4.2

**Proof.** Under Assumption [4.10] and Assumption [4.11] by Theorem [A.5] in Appendix B, \( P(\frac{1}{n} \sum_{i=1}^{|n|} V_i \in \cdot) \) satisfies a large deviation principle on \( L^\infty[0, M] \) with the good rate function
\[
I_V(f) = \int_0^M \Lambda_V(f'(x)) dx,
\]
if \( f \in AC^+_0[0, M] \) and \( I_V(f) = \infty \) otherwise, where
\[
\Lambda_V(x) := \sup_{\theta \in \mathbb{R}^6} \{ \theta \cdot x - \Gamma_V(\theta) \}, \quad \Gamma_V(\theta) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\sum_{i=1}^n \theta \cdot V_i} \right],
\]
and \( P(\frac{1}{n} N_n \in \cdot) \) satisfies a large deviation principle on \( L^\infty[0, T] \) with the good rate function
\[
I_N(f) = \int_0^T \Lambda_N(f'(x)) dx,
\]
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if \( f \in \mathcal{AC}_0^+[0,T] \) and \( I_N(f) = \infty \) otherwise, where

\[
\Lambda_N(x) := \sup_{\theta \in \mathbb{R}^d} \{ \theta \cdot x - \Gamma_N(\theta) \}, \quad \Gamma_N(\theta) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\theta N_n} \right].
\]

Since \( (\hat{V}_i)_{i \in \mathbb{N}} \) and \( N_t \) are independent, \( \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \hat{V}_i \in \cdot \right) \) satisfies a large deviation principle on \( L_\infty[0,M] \times L_\infty[0,T] \) with the good rate function \( \Gamma_V(\cdot) + \Gamma_N(\cdot) \).

We claim that the following superexponential estimate holds,

\[
\limsup_{M \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( N_n \geq nM \right) = -\infty. \tag{A.97}
\]

Indeed, for any \( \gamma > 0 \), by Chebychev's inequality,

\[
\mathbb{P} \left( N_n \geq nM \right) \leq e^{-\gamma n \mathbb{E} [ e^{\gamma N_n} ]}. \tag{A.98}
\]

From Assumption 4.11, \( \sup_{\gamma > 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} [ e^{\gamma N_n} ] < \infty \). Hence, by letting \( \gamma \to \infty \) in (A.98), we have (A.97).

For any closed set \( C \in L_\infty[0,T] \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{N_n} \hat{V}_i \in C \right) \tag{A.99}
\]

\[
= \limsup_{M \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{N_n} \hat{V}_i \in C, \frac{1}{n} N_n T \leq M \right) \tag{A.100}
\]

\[
= - \inf_{M \in \mathbb{N}} \inf_{f_i \in C} \left[ I_V(g) + I_N(h) \right] \tag{A.101}
\]

\[
= - \inf_{f \in C} \inf_{h \in \mathcal{AC}_0^+[0,T], g \in \mathcal{AC}_0[0,\infty)} \left[ I_V(g) + I_N(h) \right], \tag{A.102}
\]

where (A.100) follows from (A.97) and (A.101) follows from the contraction principle. The contraction principle applies here since for \( h(t) = \frac{1}{n} N_n t \) and \( g(t) = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \hat{V}_i \) we have \( \frac{1}{n} \sum_{i=1}^{N_n} \hat{V}_i = g(h(t)) \) and moreover, the map \( (g,h) \mapsto g \circ h \) is continuous since for any two functions \( F_n, G_n \to F, G \) in uniform topology and are...
absolutely continuous, sup$_t |F_n(G_n(t)) - F(G(t))| \leq$ sup$_t |F_n(G_n(t)) - F(G_n(t))| + sup$_t |F(G_n(t)) - F(G(t))| \rightarrow 0 \text{ as } n \rightarrow \infty.

For any open set $G \in L_{\infty}[0, T],$

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P \left( \frac{1}{n} \sum_{i=1}^{N_n} \hat{V}_i \in G \right)$$

$$\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P \left( \frac{1}{n} \sum_{i=1}^{N_n} \hat{V}_i \in G, \frac{1}{n} N_n \leq M \right)$$

$$= - \inf_{h \in AC_{[0,T]}^+, g \in AC_{[0,M]}^0} [I_V(g) + I_N(h)].$$

Since it holds for any $M \in \mathbb{N},$ the lower bound is proved.

A.2.22 Proof of Theorem 4.9

Proof. Since $\mathbb{P}(\mathcal{C}_n(t) \in \cdot)$ satisfies a large deviation principle on $L^\infty[0, \infty)$ with the rate function $I(\phi),$ $\mathbb{P}((Q^b_n(t), Q^a_n(t)) \in \cdot)$ satisfies a large deviation principle on $L^\infty[0, \infty)$ with the rate function

$$I(f) := I(f^b, f^a) = \inf_{\phi \in \mathcal{G}_f} I(\phi),$$

where $\mathcal{G}_f$ is the set consists of absolutely continuous functions $\phi(t) = (\phi^j(t), 1 \leq j \leq 6)$ starting at 0 that satisfy

$$d \begin{pmatrix} f^b(t) \\ f^a(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} d\phi(t),$$

with the initial condition $(f^b(0), f^a(0)) = (q^b, q^a).$ It is clear that

$$f^b(t) = q^b + \phi^1(t) - \phi^2(t) - \phi^3(t),$$

$$f^a(t) = q^a + \phi^4(t) - \phi^5(t) - \phi^6(t),$$

and the mapping $\phi \mapsto (f^b, f^a)$ is continuous, since it is easy to check that if

$$\phi_n(t) := (\phi^1_n(t), \ldots, \phi^6_n(t)) \rightarrow \phi(t) = (\phi^1(t), \ldots, \phi^6(t))$$

in the $L^\infty$ norm, then $(f^b_n(t), f^a_n(t)) \rightarrow (f^b(t), f^a(t))$ in the $L^\infty$ norm. Since the mapping $\phi \mapsto (f^b, f^a)$ is continuous, the large deviation principle follows from the contraction principle. \qed
A.2.23 Proof of Corollary 4.7

Proof. By Lemma 4.2

\[ I_V(g) + I_N(h) = \int_0^T \Lambda_V(g'(t)) dt + \int_\infty^0 \Lambda_N(h'(t)) dt, \]

where

\[ \Lambda_V(x) = \sup_{\theta \in \mathbb{R}_+} \left\{ \theta \cdot x - \log \mathbb{E}[e^{\theta \cdot V_1}] \right\}, \]

and

\[ \Lambda_N(x) = x \log \left( \frac{x}{\lambda} \right) - x + \lambda. \]

Since \( f(t) = g(h(t)) \), we have \( f'(t) = g'(h(t))h'(t) \) and

\[ \int_0^\infty \Lambda_V(g'(t)) dt = \int_0^T \Lambda_V(g'(h(t))h'(t)) dt = \int_0^T \Lambda_V \left( \frac{f'(t)}{h'(t)} \right) h'(t) dt. \]

Therefore,

\[ \inf_{h \in AC^+_c[0,T], g \in AC_0[0,\infty), g(h(t)) = f(t), 0 \leq t \leq T} [I_V(g) + I_N(h)] \]

\[ = \inf_{h \in AC^+_c[0,T], g \in AC_0[0,\infty)} \int_0^T \left[ \Lambda_V \left( \frac{f'(t)}{h'(t)} \right) h'(t) + h'(t) \log \left( \frac{h'(t)}{\lambda} \right) - h'(t) + \lambda \right] dt. \]

Now,

\[ \inf_y \left\{ \Lambda_V \left( \frac{x}{y} \right) y + y \log \left( \frac{y}{\lambda} \right) - y + \lambda \right\} \]

\[ = \inf_y \sup_{\theta} \left\{ \theta \cdot x - y \log \mathbb{E}[e^{\theta \cdot V_1}] + y \log \left( \frac{y}{\lambda} \right) - y + \lambda \right\} \]

\[ = \sup_{\theta} \inf_y \left\{ \theta \cdot x - y \log \mathbb{E}[e^{\theta \cdot V_1}] + y \log \left( \frac{y}{\lambda} \right) - y + \lambda \right\} \]

\[ = \sup_{\theta} \left\{ \theta \cdot x - \lambda \mathbb{E}[e^{\theta \cdot V_1}] - 1 \right\}. \]

Therefore, (4.94) reduces to (4.101).