Representing Sato-Levine Invariants by Whitney Tower Intersections

by

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Abstract

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In this thesis we explore connections between the (mod 2 reduction of the) first nonvanishing Milnor invariants of links in the 3-sphere and the spin-bordism groups over certain appropriately defined nilpotent groups. We focus our attention on the generalized Sato-Levine invariants of Conant, Schneiderman, and Teichner [6], and using their lens of twisted Whitney towers. Though we use very different tools, our results extend results of Igusa and Orr relating Milnor invariants to the homology of nilpotent groups [13].
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This dissertation is dedicated to all the people who came before me and who will come after me and who—by word and by deed—illustrate the undying power of curiosity, empathy, and persistence.

May those who are blind see
May those who see feel
May those who feel act
May those who act teach the others.
Chapter 1

Introduction and Summary

A (classical) link is an ordered collection of smoothly embedded circles in $S^3$. Links naturally arise in many contexts—e.g. it is a classic result of Lickorish [18] that every closed, connected orientable 3-manifold can be realized as the integral surgery on a link in $S^3$. Closer to our immediate interest, links arise when studying singular points of immersed surfaces in 4-manifolds: look at a 4-ball neighborhood of a singularity and on the 3-sphere boundary of that 4-ball you’ll see a link. A link is slice if its components bound disjoint properly embedded disks in the $D^4$ which is bounded by the $S^3$. If the link coming from a singular point as above is a slice link then the singularities can be removed.

The Whitney move—removing paired intersections between $n$-dimensional submanifolds of $M^{2n}$ by using framed embedded Whitney disks—plays a central role of in Whitney’s strong embedding theorem, surgery theory, h-cobordism and the classification of higher dimensional manifolds. However, because two-dimensional disks don’t generically embed in a 4-manifold, the Whitney trick fails in dimension 4. Thus, understanding how to remove these intersections and singularities among disks—understanding how to “slice” links—is a key goal in the classification of 4-manifolds. [5]

A long-standing and powerful tool in studying link concordance has been the Milnor invariants, which capture information about the nilpotent quotients of the fundamental group of the link complement. These invariants are obstructions to slicing a link. However, from Milnor’s original definiton, these invariants are also very difficult to compute.

Any link in the 3-sphere bounds a collection of immersed disks in the 4-ball, and any algebraically cancelling pair of intersections yields an immersed Whitney disk. Iteratively removing intersections of progressively high-order Whitney disks leads to the Whitney towers of Conant, Schneiderman, and Teichner. These towers—accompanied by intersection trees—geometrically embody the information presented in the first nonvanishing Milnor invariants. Conant, Schneiderman, and Teichner construct a Whitney tower filtration of links that makes it relatively easy to compute Milnor invariants, and among other things, define higher order Sato-Levine invariants as certain mod 2 reductions of Milnor invariants, use it to generalize the notion of $k$-sliceness.

Our main result is:
Theorem 7.3. $\beta$ induces a homomorphism

$$h : W_{2k-2}^o \rightarrow \ker (\Omega_3^{spin}(B_F^{F_k}) \rightarrow \Omega_3(B_F^{F_k}))$$

where $\beta$ is determined by a basing for the link (cf. Section 2.9), and $W_{2k-2}$ is in the associated graded group of the twisted Whitney tower filtration of Conant, Schneiderman, and Teichner. Our construction is summarized in Diagram 1.1 below.

The image of the homomorphism lies in the kernel of the forgetful map $\Omega_3^{spin}(B_F^{F_k}) \rightarrow \Omega_3(B_F^{F_k})$. (Theorem 5.9) The kernel of this homomorphism contains $W_{2k-2}$, the corresponding term in the framed Whitney tower filtration. (Theorem 7.4) For links with a nontrivial generalized Sato-Levine invariant—which measures the obstruction to framing a twisted Whitney tower—the image of this map lies in a quotient of $H_2(B_F^{F_k};\mathbb{Z}_2) = \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2$, via a correspondence taking twisted trees to the class indicated by applying the bracket map. Moreover (Theorem 8.1), the image corresponds to the secondary edge invariant ($\text{sec}$) of Teichner [30].

In the diagram, $\xi$ comes from the correspondence between Whitney concordance classes and intersection trees (cf. Lemma 2.3). The singular bordism groups $\Omega_3^{spin}(B_F^{F_k})$ and $\Omega_3(B_F^{F_k})$ are explained in Section 2.8. $p$ is the natural projection coming from the spectral sequence for $\Omega_3^{spin}(B_F^{F_k})$ (cf. Section 7).

In Section 2 we review classic theorems of Stallings and Dwyer which connect the lower central series to low-dimensional topology (2.1), Whitney disks and the special challenge they present in 4-dimensional setting (2.2), Milnor invariants background (2.5), key aspects of Conant, Schneiderman, Teichner’s results on Whitney tower concordance of classical links and their geometric characterizations of Milnor’s link invariants (2.3), gropes (2.4), and the generalized Sato-Levine invariants to which our results apply (2.6).

In our main constructions, we consider as given an $m$-component link in $L \subset S^3 = \partial B^4$ with the only requirement being that $L$ lies deep enough in the Whitney tower filtration (that is, that $L$ bounds a twisted Whitney tower of order $2k - 2$). It follows that the longitudes are sufficiently deep in the lower central series of the link exterior, and so we get isomorphisms between nilpotent quotients of $\pi_1(M_L)$ and those of the free group on $m$ generators, $F(m)$ (Up to the $(2k - 1)$st quotient, in fact). These isomorphisms allow us to define maps from $M_L$ to the appropriate classifying spaces. (Specifically, $M_L \rightarrow B_{F_k}^F$, cf. Section 2.9)

Then in Section 3 we build a four-manifold, $N_W$, that will turn out to geometrically embody the intersection invariant of the link $L$. We do surgery on the twisted Whitney
CHAPTER 1. INTRODUCTION AND SUMMARY

tower which the link bounds—specifically, on the top-level (unframed) Whitney circle. This turns the pair \((D^4, S^3)\) into \((S^2 \times S^2 \# D^4, S^3)\). It also creates auxilliary slice disks for the link components and thus allows us to complete the Whitney move, removing the slice disks. We then see the four-manifold we’ve constructed has as boundary \(M_L\), the zero surgery of the link.

After proving a general result on Alexander duality (Section 4) and then specializing it to our situation, in Section 5 we see how the homology and fundamental groups of \(N_w\) allow us—via Stallings’s Theorem, Dwyer’s refinement, and Teichner and Freedman’s related results—to extend the map \(\partial N_w = M_L \to BF_k^E \) to \(N_w \to BF_k^E \). We also see how the second homology of \(N_w\) is generated by a grope of order \(k\), \(\Gamma\), along with the grope’s meridinal sphere. The order \(k\) grope embodies the intersection invariant data of the original link, and is Poincaré dual to \(w_2(N_w, M_L)\).

Then, in Section 6 we show how the geometric meaning of the transgression in the Leray-Serre spectral sequence for the relevant fibration allows us to connect the grope in the second homology of \(N_w\) to the corresponding commutator (mod the image of \(d^2\)) in the second homology of \(BF_k^E\). From our construction we also see this is the same commutator as that captured by the generalized Sato-Levine invariant of the link.
Chapter 2

Background and Definitions

Throughout this discussion, by “manifold” we will mean smooth manifold unless we qualify it otherwise.

A link of $m$ components (aka an $m$-component link), $L$, is an ordered collection of smoothly embedded circles in $S^3$. A framed link is one whose normal bundle $\nu(L)$ is trivialized. Two links $L$ and $L'$ are concordant if there is smooth, oriented, properly embedded, $m$-component submanifold $V$ of $S^3 \times [0, 1]$, diffeomorphic to $L \times [0, 1]$ and such that $V \cap S^3 \times \{0\} = L$ and $V \cap S^3 \times \{1\} = L'$. [3]

Given an open tubular neighborhood of $L$, $\hat{\nu}(L)$, the link exterior, $E_L$, is $S^3 \setminus \hat{\nu}(L)$.

Given a connected codimension 2 submanifold $A^n \hookrightarrow B^{n+2}$, with trivialized normal bundle, $\nu(A) \cong A \times D^2$, a meridian of $A$ is (the isotopy class of) the $S^1$ boundary of a normal disk. Regardless of codimension, a normal disk may also be called a meridinal disk.

The $i$th longitude of a link, $\lambda_i$, is a push-off of a component $L_i$ that is nullhomologous in $S^3 \setminus L_i$.

Given a basepoint in $S^3$ and a choice (up to homotopy) of paths within $E_L$ from the basepoint to each boundary component of $E_L$, along with the corresponding meridians. This defines a map $\delta : \bigvee^m S^1 \to E_L$. Passing to fundamental groups we get a basing, $\delta : F(m) \to \pi_1(E_L)$.

2.1 Lower Central Series of Groups

We denote the free group on $m$ generators, $x_1, x_2, \ldots, x_m$, by $F(m)$, or simply by $F$ when $m$ is understood. For any group $G$, let $G_1 = G$ and $G_{k+1} = [G, G_k]$ denote the lower central series for $G$.

For convenience, we recall two key theorems connecting the lower central series to homology and thus to the rest of topology: Stallings’ Theorem and Dwyer’s Theorem, which extends Stallings’ Theorem.

Theorem 2.1. (Stallings [28]) If $g : \sigma \to \pi$ is a group homomorphism inducing an isomor-
A Hopf link in $S^3$. b) The link exterior, along with a basing.

Figure 2.1: a) A Hopf link in $S^3$. b) The link exterior, along with a basing.

If $\delta$ is an epimorphism on $H_1$ and an epimorphism on $H_2$, then the induced maps $g_i : \sigma \sigma_i \rightarrow \pi \pi_i$ are isomorphisms for all $1 \leq i < \omega$.

Given a group $G$, and the canonical map on classifying spaces induced by the quotient $G \rightarrow G/G_{k-1}$, define $\phi_k(G) := \ker(H_2(G) \rightarrow H_2(G/G_{k-1}))$. There is the Dwyer filtration $\phi_\omega(G) \subset \cdots \phi_{k+1}(G) \subset \phi_k(G) \subset \cdots \phi_3(G) \subset \phi_2(G) = H_2(G)$. Dwyer extended Stallings’ result to show

**Theorem 2.2.** (Dwyer [9]) If $f : \sigma \rightarrow \pi$ is a group homomorphism inducing an isomorphism on $H_1$, then, for $2 \leq k < \omega$ the following are equivalent:

1. $f$ induces a surjection $H_2(\sigma)/\phi_k(\sigma) \rightarrow H_2(\pi)/\phi_k(\pi)$

2. $f$ induces an isomorphism $\sigma/\sigma_k \cong \pi/\pi_k$

3. $f$ induces an isomorphism $H_2(\sigma)/\phi_k(\sigma) \cong H_2(\pi)/\phi_k(\pi)$ and an injection $H_2(\sigma)/\phi_{k+1}(\sigma) \rightarrow H_2(\pi)/\phi_{k+1}(\pi)$

Given any space $X$ with fundamental group $\pi_1(X)$, both Stallings’ and Dwyer’s results easily extend to such a space by attaching cells of dimension $\geq 3$ to form $K(\pi_1(X), 1)$:
recall, $H_1(X) = H_1(\pi_1(X))$ and there is the well-known exact sequence $\pi_2(X) \to H_2(X) \to H_2(\pi_1(X))$ induced by the Hurewicz map.

In this light, Freedman and Teichner [11] define Dwyer’s subspace, $\phi_k(X)$, to be the kernel of

$$H_2(X) \to H_2(\pi_1(X)) \to H_2(\pi_1(X)_k)$$

where the first map is the Hurewicz map, and they define Dwyer’s filtration for $H_2(X)$

$$\pi_2(X) \subset \phi_\omega(X) \subset \cdots \phi_{k+1}(X) \subset \phi_k(X) \subset \cdots \phi_3(X) \subset \phi_2(X) = H_2(X)$$

2.2 Whitney disks, the Whitney move, and framing

In a simply-connected oriented 4-manifold, $X$, a pair of (transverse) intersection points between oriented connected surfaces $A$ and $B$ is called a cancelling pair if they have opposite signs, where the signs are determined via the standard sign convention comparing the orientations of the surfaces at an intersection point with the orientation of the ambient manifold. If $A$ and $B$ are also simply connected, for example, disks, such a cancelling pair, $p, q$ in $A \cap B$, determine an embedded Whitney circle comprised of curves in $\alpha \subset A$ and $\beta \subset B$ joining $p$ and $q$. Since $X$ is simply-connected, the Whitney circle bounds an immersed Whitney disk, $W$. (Henceforward, we assume all intersections are transverse.) If $W$ is embedded with interior disjoint from the paired surfaces $A$ and $B$, then we can do the Whitney move (aka the Whitney trick), removing the cancelling pair of intersections. In dimensions higher than four, the Whitney disk can be framed, and, by general position, the Whitney disk is generically embedded and disjoint, which allows the Whitney trick to work. The Whitney move is the key ingredient in Whitney’s strong embedding theorem, as well as the $s$-cobordism theorem and the surgery exact sequence. [7]

In dimension 4, it is no longer the case that the Whitney disk is generically embedded and disjoint. Moreover, even if $W$ is embedded and disjoint, a further framing obstruction appears in dimension 4.

Although the normal bundle of an embedded Whitney disk, $W$, is necessarily trivial since $D^2$ is contractible, there is a special Whitney framing that is compatible with doing the Whitney move. This Whitney framing allows the move to proceed without introducing new intersections. To do the Whitney move, the normal bundle of $W$ must contain a 1-dimensional subbundle—the Whitney section—that splits the tangent and normal bundles, respectively, of the surfaces paired by the disk, so that the surfaces remain transverse and no new intersection points appear between the surfaces during the Whitney move.

A canonical, nonvanishing Whitney section—$\nu(W)|_{\partial W}$—always exists over the boundary of the Whitney disk. It is given given by pushing $\partial W$ tangentially along one sheet and normally along the other. The problem is to extend it over the whole disk. In dimensions greater than 4, this problem is taken care of by orientability of the subbundle, which in turn is taken care of by the cancelling intersection points. However, in dimension 4 this is not
sufficient. This has to do with the fact that, whereas for $n > 4$, $\pi_1(G_1(n - 2)) = \mathbb{Z}_2$, for $n = 4$, $\pi_1(G_1(n - 2)) = \mathbb{Z}$, where $G_k(m)$ is the Grassmanian of $k$-planes in $\mathbb{R}^m$. Note that $k = 1$ here corresponds to the dimension of subbundle of the normal bundle of $W$ that is trivialized over the boundary of $W$. (Equivalently, for $n > 4$, $\pi_1(O(n - 2)) = \mathbb{Z}_2$, but for $n = 4$, $\pi_1(O(n - 2)) = \mathbb{Z}$. [27, section 1.7], [12, section 4.1]

Thus, in dimension 4, the Whitney disk comes with a relative Euler number $\omega(W) \in \mathbb{Z}$, which is the obstruction to extending the Whitney section across the disk. [10, section 1.3]. When $\omega(W) = 0$, we say the disk is framed. Otherwise, it is twisted, or $k$-twisted if we want to specify the number of rotations.

![Figure 2.2](image_url)

Figure 2.2: a) A Whitney disk $W$ pairing local surface sheets $A$ and $B$, with Whitney section. b) Doing the Whitney move to remove intersections.

### 2.3 Whitney towers

In dimension 4, the Whitney disk is not generically embedded. Moreover, generic intersections between a Whitney disk and surfaces can obstruct a Whitney move. However, it can be made disjointly embedded (and framed) at the cost of creating intersections with the surfaces paired by $W$. [7] See Figure 2.3. This leads to Conant, Schneideman, and Teichner’s construction of Whitney towers, built up by pairing as many intersections as possible with iterated Whitney disks.

As in [26] we recall the definitions below:

An order 0 surface $S$ in a 4-manifold $M$ is a properly immersed surface—i.e. $\partial A$ is embedded in $\partial M$, and the interior of $A$ is generically immersed in $M \setminus \partial M$. A Whitney tower of order 0 in $M$ is a collection of order 0 surfaces.

Given a (transverse) intersection point $p$ between surfaces of order $m$ and $n$, we say the order of $p$ is $(m + n)$.

The order of a Whitney disk is $(n + 1)$ if it pairs intersection points of order $n$.

For $n \geq 0$, a Whitney tower of order $(n + 1)$ is a Whitney tower $W$ of order $n$ together with Whitney disks pairing all order $n$ intersection points.
In the definition of Whitney tower given above, all surfaces and Whitney disks are required to be framed.

There is also the notion of a twisted Whitney tower. An order 0 twisted Whitney tower is a collection of properly immersed surfaces that are not necessarily framed.

For $n > 0$, an order $(2n - 1)$ twisted Whitney tower is just a (framed) Whitney tower of order $(2n - 1)$ as above.

A twisted Whitney tower of order $2n$, for $n > 0$, is a Whitney tower all of whose surfaces of order $< n$ are framed, except the disks of order $n$ are allowed to be twisted.

A framed link $L \subset S^3 = \partial(B^4)$ bounds an order $n$ twisted Whitney tower $W \subset B^4$ if the order 0 surfaces of $W$ are bounded by $L$. Henceforward, all links will be assumed to be framed, and inside $S^3$ unless otherwise stated.

In all Whitney towers, twisted or framed, the Whitney disks must have disjointly embedded boundaries and generically immersed interiors.

For $n \geq 1$ two links $L_0$ and $L_1$ are (twisted) Whitney tower concordant of order $n$ if for each $i$ their $i$th components, $L_{0,i} \subset S^3 \times 0$ and $-L_{1,i} \subset S^3 \times 1$ cobound an immersed annulus $A_i \subset S^3 \times I$ which supports a (twisted) Whitney tower of order $n$.

Let $W^\circ_n = W^\circ_n(m)$ be the set of framed links of $m$ components in $S^3$ bounding twisted Whitney towers of order $n$ in the 4-ball.

There is the twisted Whitney tower filtration:

$$\cdots \subset W^\circ_3 \subset W^\circ_2 \subset W^\circ_1 \subset W^\circ_0 = \mathbb{L}$$

Let $W^\circ_n = W^\circ_n(m)$ be the associated graded Whitney concordance class of links, given by the quotient of $W^\circ_n$ modulo order $(n + 1)$ twisted Whitney tower concordance. It is shown in [7, Lemma 3.4] that, although in general the band sum is not a well-defined operation on concordance classes of links, it is a well-defined operation in $W^\circ_n$. 
Intersection Trees

A tree is a connected graph without loops. A unitrivalent tree is a tree all of whose “internal” vertices are trivalent, i.e. connect three edges. A rooted tree is a tree with a preferred univalent (“external”) vertex called the root. The order of a tree is the number of trivalent vertices. All trees will be considered oriented, with orientation given by cyclic orderings of adjacent edges around each trivalent vertex.

\[
\begin{diagram}
\node{I_1} \ar[dr] \ar[ddr] & & \node{I_2} \ar[ll] & & & & \node{J_1} \ar[ll] & & \node{J_2} \ar[ll] & & \\
&&&&&&&&&\end{diagram}
\]

Figure 2.4: Two trees \( I = (I_1, I_2) \) and \( J = (J_1, J_2) \).

In what follows we will use the bijective correspondence between oriented rooted trees with non-root univalent vertices labeled by elements from the index set \( \{1, 2, \ldots, m\} \) and formal non-associative bracketing of elements from the index set.

Given two rooted trees \( I \) and \( J \), the rooted product, \( (I, J) \) is the rooted tree gotten by identifying the roots of \( I \) and \( J \) to a single vertex \( v \) and sprouting a new rooted edge from \( v \).

The inner product, \( \langle I, J \rangle \) is the unrooted tree gotten from identifying the roots into a single nonvertex point. [7]. See Figure 2.5

\[
\begin{diagram}
\node{I_2} \ar[ddr] & & \node{J_1} \ar[ll] & & & & \node{I_2} \ar[ll] & & \node{J_1} \ar[ll] & & \\

\node{I_1} \ar[dr] & & &&&\node{J_1} \ar[ll] & & \node{J_2} \ar[ll] & & \\

\end{diagram}
\]

Figure 2.5: a) The rooted product \( (I, J) \). b) The inner product \( \langle I, J \rangle \).

As in [6] we define \( \mathcal{T} = \mathcal{T}(m) \) to be the abelian group freely generated by oriented, unitrivalent trees, with vertex labels in \( \{1, \ldots, m\} \) corresponding to link components, modulo the antisymmetry (AS) and Jacobi (IHX) relations. (cf. Figure 2.6.) A \( \infty \)-tree is a rooted tree with root labelled \( \infty \).

\( \mathcal{T} \) is graded according to the order of the trees, \( \mathcal{T} = \bigoplus_n \mathcal{T}_n \), where \( \mathcal{T}_n = \mathcal{T}_n(m) \) is the free abelian group on order \( n \) trees, modulo the AS and IHX relations.
Figure 2.6: Local pictures of the antisymmetry (AS) and Jacobi (IHX) relations in $T$. The univalent vertices extend to fixed subtrees in each equation.

\[ \{ (i, J), J \} = 0 \]

$T_{2k-1}^\infty$ is the quotient of $T_{2k-1}$ by the boundary twist relations:

\[ \{ (i, J), J \} = 0 \]

These relations correspond to the geometric fact that an intersection point of order $2n - 1$ of the form $p = \{ (i, J), J \}$ can be removed by a boundary twist on $W_{(i, J)}$ which changes the framing by $\pm 1$. [10, Section 1.4] This merely creates a twisted disk of order $n$, which is allowed in an order 2n twisted Whitney tower and thus represents no obstruction to raising the order from $2n - 1$ to $2n$.

$T_{2k}^\infty$ is the free abelian group on order $2k$ trees and order $k$ trees, modulo the same relations as above (only applied to order 2k trees) and certain additional relations: symmetry ($-J)^\infty = J^\infty$, twisted IHX, and interior twist, $2 \cdot J^\infty = \{ J, J \}$. Note that $T_{2k}$ is a subgroup of $T_{2k}^\infty$.

Conant, Schneiderman, and Teichner define summation maps $\eta_n : T_n^\infty \to L_1 \otimes L_{n+1}$. The image of $\eta_n$ turns out to be $D_n$ (cf. Section 2.3), with $\eta_n$ an isomorphism for $n \equiv 0, 1, 3 \mod 4$. For the last case, $n = 4k - 2$, they show $\ker(\eta_{4k-2}) \cong \mathbb{Z}_2 \otimes L_k$, generated by symmetric $\propto$-trees of the form $(J, J)^\infty$. [7]

From Towers to Trees

To each order 0 surface $\Sigma_i$ we associate the order 0 rooted tree with just 1 edge with one vertex labelled $i$. The tree is indicated simply by $i$. To each intersection point $p$ between order 0 surfaces $\Sigma_i$ and $\Sigma_j$ we associate the order 0 tree $t_p := \{ i, j \}$. To a Whitney disk $W_{(i, j)}$ pairing order 0 surfaces $\Sigma_i$ and $\Sigma_j$, we associate the rooted tree $(i, j)$. Recursively, for a Whitney disk $W_{(I, J)}$ pairing surfaces $W_I$ and $W_J$, where these surfaces may be order 0 or higher and $I$, $J$ denote the corresponding rooted subtrees, we associate the rooted tree $(I, J)$. (If $I = i$ then $W_I$ means $\Sigma_i$.) Similarly, to any intersection point $p$ between surfaces $W_I$ and $W_J$ with corresponding trees $I$ and $J$, we associate the tree $t_p := \{ I, J \}$. 
The trees may be thought of as embedded in the Whitney tower as in Figure 2.7.

Figure 2.7: a) The rooted tree \((A,B)\) associated to the Whitney disk \(W_{(A,B)}\). b) The unrooted tree \(\langle (A,B), C \rangle\) associated to the intersection point \(p \in W_{(A,B)} \cap W_C\).

For a twisted Whitney tower, \(W\), the order \(n\) intersection invariant \(\tau_n^\circ(W)\) is the sum of signed trees associated to all unpaired intersections \(p\) in \(W\), along with \(\circ\)-trees associated to all twisted Whitney disks of \(W\). It is given by

\[
\tau_n^\circ(W) := \sum_p \epsilon_p \cdot t_p + \sum J \omega(W_J) \cdot J^\circ
\]

where \(\epsilon_p\) is the sign of \(p\) and \(\omega(W_J)\) is the twisting of \(W_J\).

**Some Facts on Whitney Tower Filtrations**

It is shown in [7, Theorem 1.8] that \(L\) bounds a twisted Whitney tower, \(W\), of order \(n\) with \(\tau_n^\circ(W) = 0\) if and only if \(L\) bounds a twisted Whitney tower of order \((n + 1)\).

By [7, Remark 3.1] links \(L_0\) and \(L_1\) in \(W_n^\circ\) represent the same element in \(W_n^\circ\) if and only if there exists order \(n\) twisted Whitney towers \(W_0\) and \(W_1\) such that \(\tau_n^\circ(W_0) = \tau_n^\circ(W_1) \in \mathcal{T}_n^\circ\).

**Bracketing Map**

Suppose we are given a link \(L \in W_{2k-2}^\circ\). Then it bounds a twisted Whitney tower \(W\) of order \(2k - 2\), with corresponding intersection invariant \(\tau_L\).

We define the reduced bracketing map, \(b : \mathcal{T}^\circ \to \mathbb{Z}_2 \otimes L\), on \(\circ\)-trees via the usual correspondence (cf. 2.3) between rooted oriented labeled trees and nonassociative bracketings. We call it “reduced” because, since antisymmetry but not self-annihilation relations hold in \(\mathcal{T}^\circ\), the target for the usual bracketing would be \(\mathbb{Z}_2 \otimes L'\), but, as we see in Section 2.5, exact sequence 2.5, this then maps to \(\mathbb{Z}_2 \otimes L\) after modding out symmetric trees. (Also cf. [7, Definition 5.2]) Thus, this map descends in the quotient to \(b : \mathcal{T}_{2k-2}^\circ/\mathcal{T}_{2k-2} \to \mathbb{Z}_2 \otimes L_k(m) = \mathbb{Z}_2 \otimes F_k\), where \(L_k\) is the degree \(k\) portion of the free \(\mathbb{Z}\)-Lie algebra \(L(m) = \oplus L_n\) on \(m\) generators. By treating the Lie bracket as the usual commutator, \(\mathbb{Z}_2 \otimes L_k\) is isomorphic to \(\frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2\) [19].
CHAPTER 2. BACKGROUND AND DEFINITIONS

Figure 2.8: a) A link in $S^3$, with twisted Whitney disk. b) The corresponding Whitney tower in $D^4$. c) The intersection tree.

**Interior Twist**

The operation of *interior twisting*—cf. [10, Section 1.3]—allows the framing to be changed by $\pm 2$ only at the cost of introducing a self-intersection in $W$. There are several ways to see why this works as it does. First, we can recall that for a generic immersion of oriented manifolds, $\Sigma^2 \emb X^4$, we have

$$\Sigma \cdot \Sigma = \omega(\nu \Sigma) + 2(\text{signed number of self intersections})$$

where $\omega(\nu \Sigma)$ is the (integer) Euler number of $\nu \Sigma$.

Second, by examining Figure 2.9 we can see that there are two loops along which the normal bundle undergoes a full rotation, in one case from the twisting of the original strip at time $t=0$, in the second case, from the shifting from past to future along the paths indicated. [10, Section 1.7] This leads to the *interior twist* relation of Conant, Schneiderman, and Teichner: Given a rooted tree, $J$, and letting $k \cdot J^\omega$ represent the corresponding $k$–twisted tree, we have $2 \cdot J^\omega = \langle J, J \rangle$. Because it introduces only intersections of higher order than in the original Whitney tower, it will sometimes be useful to consider only $\omega(W) \mod 2$.

**From Trees to Links and Back**

There are surjective realization maps, $R^\omega_n : T^\omega_n \to W^\omega_n$. [7, Section 3.2] The maps are defined using a modification of Cochran’s “Bing-doubling along a tree” algorithm to turn trees into links. [4] Given any link $L \in W^\omega_n$ with corresponding class $[L] \in W^\omega_n$, there exists a tree $\tau_L \in T^\omega_n$ such that $[R^\omega_n(\tau_L)] = [L]$. See Figure 2.10 for an illustration of how this works.

For $n \equiv 0, 1, 3 \pmod 4$, $R^\omega_n$ is an isomorphism. For $n \equiv 2 \pmod 4$, the kernel of $R^\omega_n$ is generated by symmetric $\approx$-trees of the form $(J, J)^\omega$ of the appropriate order, i.e. where the order of the rooted tree $J$ is $\frac{n+2}{4}$. 
Figure 2.9: A view of the interior twist operation. (Based on the diagram appearing in [10, Section 1.7].) Two paths along which the normal bundle of the disk gets a $2\pi$ twist. Path A—along the boundary of the twisted piece—is fully in the present. Path B is in the present, goes into the future as it crosses portion of the fill-in piece, returns to the present, then goes into the past as it crosses part of the interior of the twisted piece.

\[ W_2^{2k-2} \xrightarrow{\xi} T_2^{2k-2} \quad \xrightarrow{q} \quad \frac{T_2^{2k-2}}{T_2^{2k-2}} \]

(2.1)

**Lemma 2.3.** The map $\xi : W_2^{2k-2} \to \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2$ (defined by diagram 2.1 above) is well-defined and surjective.

**Proof.** $L \in W_2^{2k-2}$ determines, via $R_{2k-2}^{-1}$, a corresponding intersection invariant $\tau_L$. From the discussion of the realization map above, we see that this is unambiguous for $2k - 2 \equiv 0 \pmod{4}$, and given up to symmetric rooted trees $(J, J)^\omega$ for $2k - 2 \equiv 2 \pmod{4}$. By the self-annihilation relation (cf. Section 2.3), the symmetric trees are trivial under the bracketing map, $b$, so this map factors through the quotient $\frac{T_2^{2k-2}}{T_2^{2k-2}}$ as illustrated in diagram 2.1 and explained in Section 2.3. (In the geometric construction below we will also see that the symmetric tree ambiguity is of no consequence.)
2.4 Gropes

A grope of class 1 is a circle. A class 2 grope is a compact oriented surface, $\Sigma$, with a single boundary component. Recursively, for $n \geq 2$, a class $n$ grope is formed by attaching to each pair of dual circles in a symplectic basis for $\Sigma$ a pair of gropes whose classes add up to $n$. [6, Section 1.6] “Attaching a class 1 grope” is taken to mean not attaching anything at all. The surfaces are called stages.

A closed $n$-grope—also known as a sphere-like grope—is the 2-complex you get when you replace a 2-cell in $S^2$ by a class $n$ grope. [16] See Figure 2.11.

Generalizing the easy observation that the boundary circle $\gamma$ of a compact oriented surface
with standard symplectic basis \( \{ \alpha_i, \beta_i \} \) is the product of commutators, \( \prod[\alpha_i, \beta_i] \), Freedman and Teichner show in [11] that

**Lemma 2.4.** (Freedman and Teichner) For a space \( X \), a loop \( \gamma \) lies in \( \pi_1(X)_k \) if and only if \( \gamma \) bounds some \( k \)-grope \( \Gamma \subset X \). Moreover, the class of \( \Gamma \) is the maximal \( k \) such that \( \gamma \in \pi_1(X)_k \).

Recall the *Dwyer Filtration* of the second homology of a space (cf. Section 2.1), where *Dwyer’s subspace*, \( \phi_k(X) \), was defined to be the kernel of \( H_2(X) \to H_2(\pi_1(X)) \to H_2(\pi_1(X)_k) \).

Freedman and Teichner also show

**Lemma 2.5.** (Freedman and Teichner [11]) Dwyer’s subspace \( \phi_k(X) \) of \( H_2(X) \) is precisely the subset of homology classes represented by maps of closed \( k \)-gropes into \( X \).

### 2.5 Milnor Invariants

By a result of Milnor [21], given a basing \( \delta : F \to \pi_1(E_L) \) we have the presentation for the nilpotent quotients

\[
\frac{\pi_1(E_L)}{\pi_1(E_L)_j} \cong \langle y_1, y_2, \ldots, y_m | [y_i, \lambda_i], F_j \rangle
\]

where \( y_i = \delta(x_i) \) is the image of the \( i \)th meridian and \( \lambda_i \) is the image of the \( i \)th longitude in \( \pi_1(E_L) \).

![Figure 2.11: A class 5 closed grope.](image-url)
It follows that if the longitudes lie in $\pi_1(E_L)_{n+1}$ then the basing induces isomorphisms on the nilpotent quotients

$$\frac{F}{F_i} \to \frac{\pi_1(E_L)}{\pi_1(E_L)_i}, \quad \text{for } i \leq n + 2$$ (2.2)

By the 5-Lemma, this implies

$$\frac{F_{n+1}}{F_{n+2}} \cong \frac{\pi_1(E_L)_{n+1}}{\pi_1(E_L)_{n+2}}$$ (2.3)

The group $\frac{F_{n+1}}{F_{n+2}}$ is abelian—$[F_n, F_n] \subset F_{2n} \subset F_{n+2}$—and is generated by $n$-fold simple commutators of the form $[x_{i_1}, [x_{i_2}, \ldots, [x_{i_{n-1}}, x_{i_n}]] \ldots]$. [19]

The classic Milnor invariants of a link $L$, $\overline{\mu}_L$, are defined via the Magnus expansion. First recall that the Magnus expansion, $g : F(m) \to A(Z, m)$, embeds $F(m)$ into $A(Z, m)$, the associative $Z$-algebra of formal power series in $m$ noncommuting variables $\kappa_1, \ldots, \kappa_m$, by mapping the generator $x_i$ to $1 + \kappa_i$ and $x_i^{-1}$ to $1 - \kappa_i + \kappa_i^2 - \kappa_i^3 + \ldots$. So given $z \in F(m)$, $g(z)$ will have the form $1 + \sum \mu(i_1, \ldots, i_s)\kappa_{i_1} \cdots \kappa_{i_s}$. Magnus shows that $z \in F_k$ if and only if all coefficients $\mu(i_1, \ldots, i_s)$ vanish for $s < k$. [19, section 5.5]

The classic Milnor invariants are then the coefficients in the Magnus expansion representing the link longitudes, once the longitudes have been written in terms of the meridian generators $y_i = \delta(x_i)$ of $E_L$. The length of a Milnor invariant is the degree plus 1 of the term where it appears. [21]. Milnor showed that the first nonvanishing such coefficients are well-defined—i.e. independent of basing—and are isotopy invariants for the link. Later, Stallings showed them to be concordance invariants. [28]

Let $L = L(m)$ denote the free Lie algebra over $Z$ on $m$ generators, $X_1, X_2, \ldots, X_m$, with product given by the usual Lie bracket. This is $\mathbb{N}$-graded, $L = \bigoplus_n L_n$, where the degree $n$ part, $L_{n+1}$, is the additive abelian group of length $n + 1$ brackets, modulo Jacobi identities and self-annihilation relations $[X, X] = 0$. It is a classic result (e.g. cf. [19]) that

$$L_{n+1} \cong \frac{F_{n+1}}{F_{n+2}}$$ (2.4)

with the generators of the Lie algebra mapping to meridians and Lie brackets mapping to group commutators. [6]

Let $L' = L'(m)$ denote the quasi-Lie algebra on $m$ generators defined by Levine. [17] The difference between $L'$ and $L$ is that the self-annihilation relation $[X, X] = 0$ no longer holds in $L'$, being replaced by the weaker antisymmetry (AS) relation: $[X, Y] = -[Y, X]$. $L'$ has the same grading as $L$. Levine shows that the natural projection induces $L'_{2k-1} \cong L_{2k-1}$, while for even orders (i.e. odd degrees) we have the split exact sequence

$$\mathbb{Z}_2 \otimes L_k \to L'_{2k} \to L_{2k}$$ (2.5)

where the first map sends $X$ to $[X, X]$. [7, Section 5]
Let $\mu_n^i(L)$ denote the image of the $i$th longitude in $L_{n+1}$ under the isomorphism in (2.3) above. As in [6] we define the order $n$ Milnor invariant by

$$\mu_n(L) := \sum_i X_i \otimes \mu_n^i(L) \in L_1 \otimes L_{n+1}$$

The order $n$ Milnor invariant, $\mu_n(L)$, is the first non-vanishing “total” Milnor invariant, corresponding to all length $n + 2$ classical Milnor invariants. It should be noted that Magnus’ work on the Magnus expansion shows a link $L$ has vanishing Milnor invariants of length $\leq n + 1$ if and only if the longitudes lie in $\pi_1(E(L)_{n+1})$. [22]

By results of Cochran, Schneiderman, and Teichner [7], for $L \in W_n$ the Milnor invariants of length $\leq n + 1$ vanish (and thus the longitudes lie in $\pi_1(E(L)_{n+1})$ and the length $n + 2$ invariants can be computed from the intersection tree $\tau^\omega_n(W)$ of any twisted Whitney tower $W$ of order $n$ bounded by $L$. Thus, this computes the order $n$ total Milnor invariant as well. It is a fact that $\mu_n(L)$ lies in $D_n$, the kernel of the bracketing map $L_1 \otimes L_{n+1} \rightarrow L_{n+2}$. Indeed, by [6, Theorem 7] there is a surjection $\mu_n : W_n^\omega \rightarrow D_n$ that is an isomorphism for $n \equiv 0, 1, 3 \pmod{4}$.

Results of Igusa & Orr

Let $F$ be the free group on $m$ generators. Igusa and Orr showed that [13]:

$$H_3(B \frac{F}{F_k}; \mathbb{Z}) = \bigoplus_{i=k}^{2k-2} (mN_i - N_{i+1})\mathbb{Z}$$

where $N_i$ is the rank of $\frac{F}{F_{i+1}} \cong L_i$, and thus the number of basic commutators of length $i$. For a link of $m$-components with vanishing Milnor invariants of length $k$, Orr showed that the expression in parentheses above, $mN_i - N_{i+1}$, is the number of independent Milnor invariants of length $i + 1$. [22]

For such a link, Igusa and Orr used calculations of the homology of nilpotent groups $\frac{F}{F_k}$, along with calculations of elements of $\pi_3(B \frac{F}{F_k})$, determined by maps from $S^3$ constructed using a basing on the link with target a certain simply-connected quotient space of $B \frac{F}{F_k}$, to show that $H_3(B \frac{F}{F_k}; \mathbb{Z})$ essentially captures the Milnor invariant information in the range given (i.e. length $k + 1$ to $2k - 1$).

Also note that the expression in parentheses above, $mN_i - N_{i+1}$, is the rank of $D_{i-1}$, the kernel of the bracketing map $L_1 \otimes L_i \rightarrow L_{i+1}$. [6]

2.6 Generalized Sato-Levine and Arf Invariants

The original Sato-Levine invariant [24] was defined for all 2-component links (not necessarily spherical) in $S^{m+2}$ that are codimension 2 and bound Seifert surfaces in the complement of one...
CHAPTER 2. BACKGROUND AND DEFINITIONS

another. In the classical case it is defined for any 2-componenet link with zero linking number: Given components $L_1$ and $L_2$ bounding Seifert surfaces $\Sigma_1$ and $\Sigma_2$ which meet transversally, consider the 1-manifold intersection $\Sigma_1 \cap \Sigma_2$, and compute it's framing with respect to either Seifert surface. By the Pontrjagin-Thom construction, the framing corresponds to an element of of $\pi_3(S^2)$. Sato showed this to be a concordance invariant and defined a coarser equivalence, $\beta$-equivalence, that determined an abelian group structure under connect sum and mapped isomorphically onto $\pi_3(S^2) \cong \mathbb{Z}$, with generator the Whitehead link. [24, Theorem 4.1]

The Sato-Levine invariant was shown by Cochran [2] to be identical to the first nonvanishing length 4 Milnor invariant, $\bar{\mu}(1122)$. Generalizing Sato’s construction—using derivatives of links to investigate higher-order intersections of “derived” Seifert surfaces—Cochran defines a series of concordance invariants—$\beta_i$, with $\beta_1$ being the Sato-Levine invariant—which are much easier to calculate than the classic Milnor invariants. Cochran shows that they represent well-defined “integral lifts” of otherwise under-determined higher-order Milnor invariants. [4]

Moving in a different direction, Conant, Schneiderman, and Teichner generalize the Sato-Levine invariant to $m$-component links, and show that it represents the obstruction to framing a twisted Whitney tower of even order.

Specifically, denoting the cokernel of $W_{2n} \rightarrow W_{2n}^\ast$ by $K_{2n-1}^\mu$, Conant, Schneiderman, and Teichner define the Generalized Sato-Levine Invariant to be the map $SL_{2n-1} : K_{2n-1}^\mu \rightarrow \mathbb{Z}_2 \otimes L_{n+1}$, defined via the commutative diagram below (2.6). They show $SL_{2n-1}$ is an isomorphism when $n$ is even. When $n$ is odd, say $n = 2j - 1$, the kernel of $K_{2n-1}^\mu = K_{4j-3}^\mu \rightarrow \mathbb{Z}_2 \otimes L_{2j}$ is shown to be isomorphic to $K_{4j-2}^\ast$, the kernel of the Milnor invariant map $\mu_{4j-2} : W_{4j-2}^\ast \rightarrow D_{4j-2}$. This kernel is the source of their higher-order Arf invariants and is generated by symmetric $\ast$-trees. [7, Section 5.2]

\[
\begin{array}{c}
\begin{array}{ccc}
W_{2n} & \rightarrow & W_{2n}^\ast \\
\cong & \downarrow & \mu_{2n} \\
\mu_{2n} & \downarrow & \mu_{2n} \\
\downarrow & \downarrow & \downarrow \\
D_{2n} & \rightarrow & D_{2n}^{sL_{2n}} \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{ccc}
K_{2n-1}^\mu & \rightarrow & K_{2n-1}^\mu \\
\downarrow & \downarrow & \downarrow \\
\mu_{2n} & \rightarrow & \mu_{2n} \\
\downarrow & \downarrow & \downarrow \\
L_{n+1} & \rightarrow & L_{n+1} \\
\end{array}
\end{array}
\]

(2.6)

The map $sL_{2n}$ is defined by Conant, Schneiderman, and Teichner by applying the snake-lemma in the diagram below (2.7), which relates the quasi-Lie algebra bracketing map to the usual Lie algebra bracketing map. In the diagram, the $\mathbb{Z}_2 \otimes L_{n+1}$ is the kernel of the projection $L_{2n+2} \rightarrow L_{2n+2}$, made up of two torsion elements of the form $[X, X]$, and the map $sq$ is the squaring map, taking elements $1 \otimes X$ to $[X, X]$, and fitting into the exact sequence on the right side of the diagram, as proved by Levine [17].
Conant, Schneiderman, and Teichner show that in all odd orders there are links with $\mu_{2n-1} = 0$ and which thus bound \textit{twisted} Whitney towers of order $2n - 1$, but which don’t bound \textit{framed} Whitney towers of order $2n - 1$. So the cokernel of $W_{2n} \hookrightarrow W_{2n}^\omega$ is nonempty, and $\text{SL}_{2n-1}$ represents a \textit{framing obstruction} for even-order twisted Whitney towers.

$\text{SL}_1$ corresponds to the classical (mod 2) Sato-Levine invariants of the 2-component sub-links.

The simplest example of this phenomenon is the Whitehead link. Recall that this link is the generator of the classic Sato-Levine invariant. It’s longitudes are 3-fold commutators of the fundamental group generators, and thus, the Magnus coefficients of order 2 are zero, and thus, the classic Milnor invariants of length 3 are zero, and thus, $\mu_1 = 0$. So, by [7, Theorem 1.8] the link is in $W_2$. (cf. Figure 2.8.) By viewing the link alternately as an internal band sum on the Borromean rings, we see it also bounds a framed tower of order 1, and thus also lies in $W_1$. However, the first order intersection invariant $\tau_1 = \langle 1, (2, 2) \rangle \neq 0$, and so the Whitehead link does not bound a framed tower of order 2. See Figure 2.12.

Figure 2.12: An internal band sum on the Borromean rings yields the Whitehead link. Moving into the 4-ball we see the link bounds embedded disks $D_1$ and $D_2$, paired by a framed embedded Whitney tower, $W_{(1,2)}$. The corresponding order 1 framed tree, $\langle 1, (2, 2) \rangle$, is shown in the last pane.
The classical Arf invariants of the link components live in $\mathbb{Z}_2^m$ and give an isomorphism \( \ker(\mu_2 : W_2^m \to D_2) \xrightarrow{\sim} \mathbb{Z}_2^m \). [6, Lemma 9]

2.7 Spin Structures and 3-manifolds

Let \( E \) be an oriented vector bundle over a manifold \( M^n \). Recall that a spin structure on \( E \) is a fiber homotopy class of lifts, \( \tilde{f} : M \to BSpin(n) \),

\[
\begin{array}{c}
\text{BSpin}(n) \\
\downarrow p \\
M \xrightarrow{f} BSO(n)
\end{array}
\]

where \( f \) is the classifying map of the bundle, \( Spin(n) \) is the simply connected double cover of \( SO(n) \), and \( p \) is the induced map on classifying spaces.

Alternatively, if the fiber dimension of \( E \) is greater than 2, a spin structure on \( E \) is trivialization of \( E \) over the 1-skeleton of \( M \) that extends over the 2-skeleton of \( M \), considered up to homotopy.

A vector bundle that can be given a spin structure is called spinnable. A (necessarily smooth) manifold \( M \) is spinnable (or spin) if its tangent bundle can be given a spin structure. A vector bundle \( E \to M \) is spinnable if and only if the second Stiefel-Whitney class, \( w_2(E) \in H^2(M; \mathbb{Z}_2) \), is zero. The spin structures on a spin manifold \( M \) are classified by \( H^1(M; \mathbb{Z}_2) \).

One nice geometric way to think about the free and transitive action of \( H^1(M; \mathbb{Z}_2) \) on the spin structures of \( E \to M \) is by using dual homology classes and counting rotations as one moves along the 1-skeleton. See [27].

We recall a few facts on 3-manifolds and spin structures. Every orientable 3-manifold is spin. Even more, every orientable 3-manifold is parallelizable. (The only remaining obstruction to trivialization over the 3-skeleton would be a cohomology class with coefficients in \( \pi_2(SO(3)) \), which is zero.)

Any connected orientable 3-manifold bounds a 2-handlebody—a 4-manifold with only 0 and 2 handles. Moreover, any spin 3-manifold spin bounds—has matching spin structures with—such a 2-handlebody. [15, Theorem VII.3]

A nice geometric way of thinking about spin structures on 3-manifold boundaries of 4-manifolds comes from links. Recall the classic result of Lickorish that every closed, connected orientable 3-manifold can be realized as the integral surgery on a link in \( S^3 \). [18] (In fact, because any spin \( M^3 \) spin bounds, the framing coefficients in the integral surgery given by Lickorish can be assumed to be even. [12, Section 5.7])

Let \( L = \{L_1, \ldots, L_m\} \) be a framed link in \( S^3 \). Again let \( M_L \) denote the zero-surgery of \( L \). A sublink \( L' \) of \( L \) is characteristic if, for each \( i \), \( \ell k(L_i, L_i) \equiv \ell k(L', L_i) \mod 2 \). By interpreting linking numbers in terms of the intersection form of a 2-handlebody bounded
by $M_L$, and thus connecting characteristic sublinks to characteristic surfaces, a bijection is established between the set of characteristic sublinks of $L$ and the spin structures on $M_L$. [14]

Given $M_L$ bounding a 2-handlebody $N$, the spin structure corresponding to a given sublink $L'$ is characterized by the fact that it extends to a 2-handle $h$ attached to a knot $K$ if and only if the framing on $h$ is the same as $\ell k(L', K) \pmod 2$. [12] But a 2-handlebody has a spin structure if and only if the framings are even, and then this structure is unique since $H^1$ is zero. Thus, given a link $L \subset S^3$ with the zero-surgery $M_L$, there is a canonical spin structure, $s_0$, corresponding to the empty sublink: this is the unique spin structure that extends to the 2-handlebody built by attaching 2-handles to the zero-framed components of $L \subset S^3 = \partial D^4$.

### 2.8 Bordism and Spin Bordism

Let $X$ be a topological space. The $n$-dimensional oriented bordism group over $X$, $\Omega^SO_n(X)$, is the set of equivalence classes of pairs $(M, f)$ where $M$ is a smooth, closed, oriented $n$-manifold and $f : M \to X$ is continuous, via the following equivalence: Two pairs $(M_0, f_0)$ and $(M_1, f_1)$ belong to the same class—and thus are said to be (oriented) bordant over $X$—if there exists a pair $(W, F)$ where $W$ is a compact, oriented $(n + 1)$-manifold, $F : W \to X$ is continuous, $\partial W = M_0 \sqcup -M_1$, and $F|_{M_1} = f_1$.

Bordism groups over a point $\Omega^SO_n(\cdot)$ are denoted simply $\Omega^SO_n$.

Since (via the Pontrjagin-Thom construction) $\Omega^SO_n(X) = H_n(X; \text{MSO})$ and is an additive homology theory, the Atiyah-Hirzebruch spectral sequence exists, with $H_p(X; \Omega^SO) \Rightarrow \Omega^SO_{p+q}(X)$. Examining the spectral sequence, and using the facts that $\Omega^SO_0 \cong \mathbb{Z}$ and $\Omega^SO_1, \Omega^SO_2$, and $\Omega^SO_3$ are all trivial, it is easily shown that $\Omega^SO_3(X) \cong H_3(X; \mathbb{Z})$, with the isomorphism given by the Hurewicz map: $[(M, f)] \mapsto f_*([M])$. [8, Section 9.3]

Similarly, the $n$-dimensional spin bordism group over $X$, $\Omega^Spin_n(X)$, is the set of equivalence classes of triples $(M, f, s)$ where $M$ is a closed, spin $n$-manifold, with a given spin structure $s$, and $f : M \to X$ is continuous, via the following equivalence: Two triples $(M_0, f_0, s_0)$ and $(M_1, f_1, s_1)$ belong to the same class—and thus are said to be spin bordant over $X$—if there exists a triple $(W, F, S)$ where $W$ is a compact, spin $(n + 1)$-manifold, with spin structure $S$, $F : W \to X$ is continuous, $\partial(W, S) = (M_0, s_0) \sqcup -(M_1, s_1)$, and $F|_{M_1} = f_1$.

Since $\Omega^Spin_n(X) = H_n(X; \text{MSPIN})$ and is an additive homology theory, the Atiyah-Hirzebruch spectral sequence gives $H_p(X; \Omega^Spin_q) \Rightarrow \Omega^Spin_{p+q}(X)$.

By ignoring spin structure there is “forgetful” map, $\Omega^Spin_n(X) \to \Omega^SO_n(X)$.

### 2.9 From Links to 3-manifolds

As usual assume we are given a link $L \in W^\infty_{2k-2}$, which therefore bounds a twisted Whitney tower $W$ of order $2k - 2$. Also, recall $E_L \subset S^3 = \partial D^4$ denotes the link exterior while $M_L$ denotes the 3-manifold resulting from zero-surgery on $L$. Again, let $\delta : F \to \pi_1(E_L)$ be a
basing map for \( m \) link meridians generating \( \pi_1(E_L) \). By [6, Theorem 5] the length \( 2k - 1 \) Milnor invariants vanish and the link longitudes lie in \( \pi_1(E_L)_{2k-1} \).

Not only only we have the isomorphisms already mentioned in Section 2.5 (2.2), but, by [11, Lemma 2.4] we also get isomorphisms

\[
\frac{\pi_1(M_L)}{\pi_1(M_L)_i} \to \frac{F}{F_i}
\]

(2.9)

for \( i \leq 2k - 1 \).

Passing to classifying spaces and letting \( i = k \), we get a map \( M_L \to B \frac{\pi_1(M_L)}{\pi_1(M_L)_k} \to B \frac{F}{F_k} \).

\[
\begin{array}{ccc}
M_L & \to & B\pi_1(M_L) \\
& \downarrow{\beta} & \downarrow{}
\end{array}
\]

\[
B\pi_1(M_L) \to B \frac{\pi_1(M_L)}{\pi_1(M_L)_k} \to B \frac{F}{F_k}
\]

(2.10)

Thus, endowing \( M_L \) with the canonical spin structure \( s_0 \) as described in Section 2.7, we have \([M_L, \beta, s_0] \in \Omega^{\text{spin}}_3(B \frac{F}{F_k})\).

In what follows, we will see that \( \beta \) defines a homomorphism \( h : \Omega^{\omega}_{2k-2} \to \Omega^{\text{spin}}_3(B \frac{F}{F_k}) \), whose image lies in the kernel of the forgetful map, \( \Omega^{\text{spin}}_3(B \frac{F}{F_k}) \to \Omega_3(B \frac{F}{F_k}) \). (cf. section 7)
Chapter 3

Construction of $N_{W}$

Suppose we are given a link $L \in W_{2k-2}$ which bounds a twisted Whitney tower $W$ with corresponding intersection invariant $\tau_L$. In this and the following sections we will show that there exists a 4-manifold $N$ with $\partial N = M_L$ and with $\beta : M_L \to B_{F_k} F_k$ (cf. section 2.9) extending to $N$. Moreover, there exists a closed grope $\Gamma \subset N$ of class $k$, such that $\langle \Gamma \rangle = \text{PD}(w_2(N, M))$, and $\beta_*[\Gamma] \in H_2(B_{F_k} F_k; \mathbb{Z}_2) = \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2$ is given by $\beta(\tau_L)$. What we will show is summarized in Diagram 3.1.

\[
\begin{array}{c}
H^2(N, \partial N, \mathbb{Z}_2) \ni w_2(N, M) \\
\text{PD} \cong \\
\langle [\Gamma] \rangle \\
5.5 \\
H_2(N; \mathbb{Z}_2) \xrightarrow{\beta_*} H_2(B_{F_k} F_k; \mathbb{Z}_2) \\
5.5 \\
H_2(B_{\pi_1(N)} F_k; \mathbb{Z}_2) \xrightarrow{\tau} \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2 \\
\gamma := \partial(\Gamma \setminus \bar{D}^2) \in \frac{\pi_1(N)}{\pi_1(N)_{k+1}} \otimes \mathbb{Z}_2 \cong \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2 \\
3.1 \\
\beta(\tau_L)
\end{array}
\]

By assumption we have a twisted Whitney tower $\mathcal{W}$ of order $2k-2$ properly embedded in $D^4$, with $\partial\mathcal{W} = L$. Assume for now that $\mathcal{W}$ is nontrivially twisted, i.e. $L \in W_{2k-2} \setminus W_{2k-2}$. We will deal with the easier framed case in Theorem 7.4. Let $\mathcal{W}^0$ stand for $\mathcal{W}$ minus the twisted disk.

Let $\mathcal{W}$ be $j$-twisted, with $j \neq 0$. If $j$ is even then, by doing interior twists, and using the interior twist relation of 2.3, we can reduce to the framed case. So we may assume $j$ is odd.

Let $A$ denote the twisted top Whitney disk, with $\alpha := \partial A$ the corresponding Whitney circle. Do surgery on a tangential push-off of $\alpha$—call it $\alpha'$—with framing specified by the $j$-twisting of $\nu(\alpha)$. In other words, replace $\alpha' \times D^3$ with $D^2 \times S^2$ with gluing map determined
by the embedding and framing of $\alpha'$ in $D^4$. We now have an auxiliary 2-disk—the core of the 2-handle we’ve attached—with which we can perform the Whitney move. Moreover, this core disk, together with $A$, forms a sphere $S^2_\alpha$, whose normal bundle $\nu(S^2_\alpha)$ has Euler class $j$ corresponding to the relative Euler number for $A$. Note that $\nu(S^2_\alpha)$ lies in the complement of $W^0$. Let the boundary of the surgered out $D^3$ that was normal to $\alpha'$ be denoted by $S^2_\beta$.

Figure 3.1: A half-dimensional picture of how the surgery introduces the auxilliary disk and allows the Whitney move to proceed. In the first pane we see a linked pair of circles—a black and a white $S^0$—in the boundary sphere of a disk, $D^2$. The circles bound disks—$D^1$—that intersect in the interior of $D^2 I$. In the second pane we surger an $S^0$—you can even imagine this bounding a kind of Whitney disk—to glue on a 1-handle. In the last pane, after sliding part of the white $S^0$ over the auxilliary disk in 1-handle, the spheres are unlinked.

Figure 3.2: Surfaces paired by a twisted Whitney disk. Surgery along the Whitney circle using the $j$-twisting introduces an auxilliary disk to do the Whitney move.

**Lemma 3.1.** After performing $j$-surgery on $\alpha' \subset D^4$, the new manifold is $D_\alpha := D^4 \# S^2_\alpha \times S^2_\beta$

**Proof.** The following argument is taken in large part from [12, section 5.2]. We have $D^4 \cong D^4 \# S^4$. Let $\alpha_0 \subset D^4 \# S^4$ be an embedded circle inside $S^4$ and away from the area where the connect sum occurs. To be specific, let $\alpha_0$ be $\partial D^2 \times 0 \subset \partial(D^2 \times D^3) = S^4$. Since everything’s simply connected, $\alpha'$ and $\alpha_0$ are homotopic. By general position, a generic
homotopy, $I \times S^1 \hookrightarrow M^4$ will be an immersion with isolated double points. The double points correspond to distinct $I$ coordinates, and hence, the homotopy is actually an isotopy. Therefore, we can assume $\alpha = \alpha_0$, and the surgery on $\alpha$ produces $D^4 \# S$ where $S$ is obtained by surgering $S^4$ on $\alpha_0$. But on $S^4 = \partial D^5$ this is the same as attaching a 5-dimensional 2-handle $h$ along $\alpha_0$, and so $S = \partial(D^5 \cup h)$. Now $(D^5 \cup h)$ is a $D^3$ bundle over $S^2$, and these are of two flavors, according to whether the framing is even or odd. (Recall that $\pi_1(O(3)) = \mathbb{Z}_2$). Therefore, since $j$ is odd, the new manifold is $D_\alpha := D^4 \# S^2_\alpha \times S^2_\beta$. Note that the intersection form is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the boundary of $D_\alpha$ is still $S^3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{kirby.png}
\caption{Kirby diagram for $D_\alpha$}
\end{figure}

Now we remove the slice disk neighborhoods from $D_\alpha$ and call the result $N_W$. Since these zero-order disks are framed for towers of order greater than zero (cf. Section 2.3), and since $\partial D_\alpha = S^3$, the boundary of $N_W$ is the zero-surgery of our link $L$, $S^0(L)$, which we denote by $M_L$.

The meridian of $S^2_\alpha$ is the same as the meridian of $A$, and this bounds the Clifford “torus” formed by the top surfaces paired by the Whitney disk. This is summarized in Diagram 3.2 below.

\begin{equation}
\begin{array}{c}
W^0 \subset W \\
\downarrow \text{surgery} \\
W^0 \subset D_\alpha \\
\downarrow \text{slice disks} \\
N_W \subset M_L
\end{array}
\end{equation}

Recall: The region near a transverse intersection point between two surfaces in a four-manifold can be modeled by the origin in $\mathbb{R}^2 \times \mathbb{R}^2$. The Clifford or linking torus is $S^1 \times S^1$ in this model. Also note that the Clifford torus lies in the complement of the surfaces and intersects the Whitney disk once.

As we see in the Figure 3.4, which illustrates $W_{(I,J)}$, the Clifford torus corresponding to the intersection between the surfaces $\Sigma_I$ and $\Sigma_J$ hits $W_{(I,J)}$ in a single point. In Figure 3.5 we make a puncture in the Clifford torus and join it by a tube to $\mu_{(I,J)}$. (As the top surfaces may themselves be Whitney disks, of course, the “torus” is in general a grope.)
Figure 3.4: Clifford torus linking an intersection point, dual to the Whitney disk.

Figure 3.5: Where commutators come from. A Whitney disk, $W$, pairs surfaces $I$ and $J$. The Clifford torus intersects $W$ in a point. The meridians of $W$ and $I$ are indicated. The curve $m_J \subset I$ is a parallel displacement of the meridian of $J$.

By induction we show that this meridian is a very special commutator in the meridians of the slice disks on the link:

**Lemma 3.2.** Upon doing the Whitney move, the meridian $\mu_A$ of the twisted Whitney disk becomes, in $\pi_1(N_w)$, the commutator in the meridians of the slice disks on the link, given precisely by treating the $\ang$ of $\tau_L$ as a root, converting to a formal bracketing as in [6] and viewing the brackets as commutators in the usual way, $[a,b] := aba^{-1}b^{-1}$.

**Proof.** First, for the base case, suppose the order of our twisted Whitney disk is 1, i.e. $A = W_{i,j}$ where $i$ and $j$ come from zero-order surfaces $\Sigma_i$ and $\Sigma_j$. Since the dual torus has symplectic basis $\mu_i, \mu_j$, we see that $\mu_A = [\mu_i, \mu_j]$. This is precisely what we get from applying the bracketing operation to the corresponding rooted tree for $A, (i,j)$. 
CHAPTER 3. CONSTRUCTION OF $N_w$  

Next, suppose the order of our disk is $k$, i.e. $A = W(I,J)$ where $|I| + |J| + 1 = k$. Suppose we know that for any surface $\Sigma$ of order $< k$, $\mu(\Sigma)$ is the bracketing of the subtree with root at $\Sigma$. By the inductive hypothesis, since $|I| < k$, $\mu_I$ is the commutator indicated by the corresponding tree. And the same goes for $\mu_J$. But by the same reasoning as before, $\mu_A = [\mu_I, \mu_J]$. Therefore, $\mu_A$ is the commutator given by the corresponding tree.

Also, by a result of Freedman and Teichner [11, Lemma 2.1], since $\mu_A \in \pi_1(N_w)_k$, it bounds a grope of class $k$ (and none higher). Our construction exhibits such a grope. To see this grope, we use the following recipe, starting with the meridian of the top (surgery) Whitney disk: When we’re at the meridian of a zero-order surface, do nothing. When we’re at the meridian of a Whitney disk, $W(I,J)$, tube this meridian via (the normal circle bundle over) an arc in the disk to the Clifford torus, $T(I,J)$—formed from the meridians of the surfaces $\Sigma_I$ and $\Sigma_J$—which intersects the Whitney disk. Attach this torus, $T(I,J)$, to what has already been built. $T(I,J)$ has symplectic basis $\mu_I$ and $\mu_J$. For each of these meridians do the same thing, either attaching another torus—if the meridian belongs to another Whitney disk, or attaching nothing—if the meridian corresponds to a zero-order surface. And so on. See Figure 3.6 below.

Since $S^2_\alpha$ and $S^2_\beta$ intersect in a single point, we see the meridian $\mu_A$ is a circle in the sphere, $S^2_\beta$. Thus, at the start of our process of attaching tori to meridians as above, the first torus attaches to the $S^2_\beta$.

One way to see how $\mu_A$ goes from being the nullhomotopic normal circle for $S^2_\alpha$ in $\pi_1(D_\omega)$ to being a nontrivial commutator in $\pi_1(N_w)$ is to recall that the Whitney move is the inverse of the finger move. [1] A finger move (Figure 3.7) between surfaces $\Sigma_A$ and $\Sigma_B$ with meridians $\mu_A$ and $\mu_B$ respectively, kills the commutator $[\mu_A, \mu_B]$ in the complement of the surface since the fundamental group of the linking torus is abelian. Thus, doing the Whitney move on these surfaces “resurrects” this commutator in the fundamental group of the complement—i.e., we remove it as a relation in $\pi_1$ of the complement of our Whitney tower. In effect, we “hollow out” the Clifford torus in doing the Whitney move. Now if one of the meridians in the Clifford torus belongs to a Whitney disk that hasn’t been “Whitney moved” yet, the meridian is still nullhomotopic. So we proceed with the sequence of Whitney moves: as we go down the Whitney tower, at each stage as we do the Whitney move we remove another relator which the meridian of that Whitney disk represents in the fundamental group of the complement of the Whitney tower. We thus inductively “hollow out” our grope.

For future reference, let us note that we have shown that

Lemma 3.3. Upon removal of the slice disks neighborhoods, $S^2_\beta \subset D_\omega$ becomes $\Gamma \subset N_w$, a closed grope of class $k$.

Remark 1 (On multiple trees in the forest). Although we started by assuming we had a single top-level Whitney disk, $A$, if our Whitney tower has multiple top-level disks—i.e. an intersection forest—the construction still applies. First, as in [6, Section 2.9] , arrange by finger moves that Whitney towers are semi-split, meaning in particular that no Whitney disk
Figure 3.6: An intersection tree for a twisted Whitney tower of order $2k - 2 = 8$, along with the grope—in this case, class $k = 5$—that arises from our construction. The meridians of the Whitney disks of the original tower are indicated, with $\mu_\infty$ the meridian of the topmost Whitney disk.

appears in more than one tree in the intersection forest for $L$. (We use “semi” here because for our purposes we do not need the extra step—as in [6]—of reducing framings on twisted disks to $\pm 1$.) Then we simply apply the surgery multiple times—for each top-level twisted Whitney disk—get auxiliary disks, do the Whitney move, and get a separate grope for each
Figure 3.7: A finger move between two local sheets with meridians $\alpha$ and $\beta$. In the third pane we see the paired intersection points, and in the last pane we see a closeup of the Clifford torus generated by $\alpha$ and $\beta$.

_subtree in the forest. The intersection forms sum appropriately._
Chapter 4

On Alexander duality

We will recall a general result on Alexander duality for properly immersed subcomplexes of $D^n$, then we will modify the argument and specialize to the case of the Whitney disks in $D_\omega$, so we can obtain information about $N_W$.

In what follows, unless otherwise indicated, all homology will be assumed to be with $\mathbb{Z}$ coefficients.

**Theorem 4.1.** Let $\Sigma$ be a properly immersed subcomplex of $D^n$, where $n \geq 2$. Then $H^k(\Sigma, \partial \Sigma) \cong \tilde{H}_{n-k-1}(D^n \setminus \Sigma)$.

*Proof.* Consider the $\Sigma$ properly immersed in the “northern” hemisphere $D_N$ of $S^n$. Let $\nu(\Sigma)$ denote a tubular neighborhood of $\Sigma$ in $D_N$. (See Figure 4.1.) Then the complement in $S^n$ of $D_N \setminus \nu(\Sigma)$ is $D_S \cup \bar{\nu}(\Sigma)$, where the union is along the boundaries of $\Sigma$ and $D_S$, with disjoint union in the event that $\Sigma$ has no boundary.

$$H^k(\Sigma, \partial \Sigma) \cong H^k(D_S \cup \bar{\nu}(\Sigma), D_S) \quad \text{(by homotopy)}$$

$$\cong H^k(S^n \setminus (D_N \setminus \nu(\Sigma)), S^n \setminus D_N)$$

$$\cong H_{n-k}(D_N, D_N \setminus \nu(\Sigma)) \quad \text{(by Lefschetz duality)}$$

$$\cong \tilde{H}_{n-k-1}(D_N \setminus \nu(\Sigma)) \quad \text{(from the sequence for the pair)}$$

Notice that the only step where the homology of $D_N$ matters is the last one. Thus, by applying the reasoning in the first three steps to the properly embedded $m$ order-zero Whitney disks in $D_\omega$, we have the following

**Corollary 4.2.** $H_2(D_\omega, N_W) \cong H^2(\bigsqcup_{i=1}^m D_i^2 / \partial D_i^2) \cong \mathbb{Z}^m$, generated by the dual meridional disks of the $D_i^2$.

**Corollary 4.3.** $H_3(D_\omega, N_W) \cong H^1(\bigsqcup_{i=1}^m D_i^2 / \partial D_i^2) = 0$. 
Figure 4.1: $\Sigma$ properly immersed in the northern hemisphere of $S^n$
Chapter 5

Facts about $N_W$

**Lemma 5.1.** $H_1(N_w)$ is freely generated by the meridional circles dual to the (now removed) $m$ properly embedded slice disks, i.e. the meridians of the link components in $M_L$.

*Proof.* Consider the following portion of the exact sequence in homology for the pair $(D_\infty, N_w)$:

$$H_2(D_\infty) \longrightarrow H_2(D_\infty, N_w) \xrightarrow{\partial} H_1(N_w) \longrightarrow H_1(D_\infty)$$

The last map is the zero map as $H_1(D_\infty) = 0$. The first map is also the zero map: since the generators of $H_2(D_\infty)$ have zero algebraic intersection with the slice disks, they can be represented by cycles completely inside $N_w$. (Just add tubes joining cancelling pairs of intersection points.)

Thus we see that $\partial : H_2(D_\infty, N_w) \to H_1(N_w)$ is an isomorphism, and the generators of $H_1(N_w)$ are the boundaries of the meridional disks for the slice disks. \hfill $\square$

**Lemma 5.2.** $H_2(N_w) \cong \mathbb{Z}^2$, with generators mapping to $S_\alpha^2$ and $S_\beta^2$ under the inclusion $N_w \hookrightarrow D_\infty$.

*Proof.* Consider the following portion of the exact sequence in homology for the pair $(D_\infty, N_w)$:

$$H_3(D_\infty, N_w) \xrightarrow{\partial} H_2(N_w) \xrightarrow{i} H_2(D_\infty) \xrightarrow{p} H_2(D_\infty, N_w) \xrightarrow{\partial} H_1(N_w)$$

By corollary 4.3 the first map is zero. Thus, $i$ is injective. By the proof of Lemma 5.1 the last map is an isomorphism. Thus $p$ is zero and $i$—induced by inclusion—is in fact an isomorphism. \hfill $\square$

**Theorem 5.3.** $H_2(N_w, M_L)$ is generated by $[S_\alpha^2]$ and $[\Gamma]$, where $[S_\alpha^2]^2 = j$ with $j$ odd, $[\Gamma]^2 = 0$, $[S_\alpha^2 \cdot \Gamma] = 1$, and $\Gamma$ is a closed grope of class $k$.

*Proof.* See the observations on the intersection form of $N_w$ and $[\Gamma]$ in the construction of $N_w$ in section 3. (In particular, Lemma 3.3.) \hfill $\square$

**Lemma 5.4.** $H_2(N_w) \cong H_2(N_w, M_L) \cong \mathbb{Z}^2$
CHAPTER 5. FACTS ABOUT $N_w$

Proof. Consider the long exact sequence for the pair $(N_w, M_L)$.

$$
\begin{align*}
H_3(N_w, M_L) &\xrightarrow{\partial_3} H_2(M_L) \xrightarrow{i_*} H_2(N_w) \xrightarrow{j_*} H_2(N_w, M_L) \xrightarrow{\partial_2} \\
&\quad \quad H_1(M_L) \xrightarrow{\cong} H_1(N_w)
\end{align*}
$$

The last map is an isomorphism by Lemma 5.1. The first map, $\partial_3$, is an isomorphism for the same reason: Poincaré duality yields the dual map, $i^*: H^1(N_w) \to H^1(M_L)$, which is also an isomorphism by Lemma 5.1. Thus, both $i_*$ and $\partial_2$ are zero maps, and $j_*$ is an isomorphism.

Lemma 5.5. $\partial(\Gamma \setminus \hat{D}^2)$ is given by the natural transgression in the 5-term exact sequence for the fibration $B(\pi_1(N)_k) \to B(\pi_1(N)) \to B(\pi_1(N)/\pi_1(N))$.

Proof. Since $\Gamma$ is a closed $k$-grope, $\gamma := \partial(\Gamma \setminus \hat{D}^2)$ represents a class $[\gamma] \in \pi_1(N)_k$, well-defined in $\pi_1(N)/\pi_1(N)$, thus, $(\Gamma \setminus \hat{D}^2, \gamma)$ defines a class in $H_2(B(\pi_1(N)), B(\pi_1(N)))$. By Corollary 6.7 to Theorem 6.1, $[\gamma]$ maps to $\tau([\Gamma])$.

Lemma 5.6. The inclusion $M_L \hookrightarrow N_w$ induces an isomorphism

$$
\frac{\pi_1(N_w)}{\pi_1(N_w)_k} \cong \frac{\pi_1(M_L)}{\pi_1(M_L)_k}
$$

Proof. $N_w = (D_0 \setminus \bigsqcup_{i=1}^m \nu(D_i))$, where $D_0$ is the result of surgery along the boundary of the twisted Whitney disk and the $D_i$ are the slice disks bounded by the link components as described in Section 3.

$H_1(M_L)$ is freely generated by the meridians of the link components, since $M_L = S^0(L)$ and all linking numbers are zero [12]. Similarly, by Lemma 5.1, $H_1(N_w)$ is generated by the meridians to the $D_i$. The inclusion $M_L \hookrightarrow N_w$ thus induces an isomorphism on $H_1$ since the meridians of the link go to the meridians of the disks.

By Theorem 5.3 and Lemma 5.4, $H_2(N_w)$ is generated by gropes of class $k$ or higher. Therefore, by a result of Freedman and Teichner [11], $H_2(N_w)$ is the same as $\phi_k(N)$, the $k$th term in its Dwyer filtration. Therefore, $M_L \hookrightarrow N_w$ trivially induces a surjection

$$
\frac{H_2(M_L)}{\phi_k(M_L)} \twoheadrightarrow \frac{H_2(N_w)}{\phi_k(N_w)}
$$

The result then follows from Dwyer’s Theorem (2.2).

Lemma 5.7. The basing map induces an isomorphism

$$
\frac{F_k}{F_{k+1}} \cong \frac{\pi_1(N)_k}{\pi_1(N)_{k+1}}
$$
Proof. Consider the exact sequences for $\pi_1(N)_k \to \pi_1(N) \to \pi_1(N)_k$ and $\pi_1(M)_k \to \pi_1(M) \to \pi_1(M)_k$. The inclusion map induces the following commutative diagram involving the corresponding 5-term exact sequences, where the isomorphisms labelled $A$, $B$, and $C$ come from Lemma 5.6:

\[
\begin{array}{ccc}
H_2(\pi_1(M)) & \xrightarrow{q} & H_2\left(\frac{\pi_1(M)}{\pi_1(M)_k}\right) \\
\downarrow A & & \downarrow f \\
H_2(\pi_1(N)) & \xrightarrow{q'} & H_2\left(\frac{\pi_1(N)}{\pi_1(N)_k}\right)
\end{array}
\]

We will show that the middle map $f$ is an isomorphism.

First, $f$ is injective. For suppose $x \in \frac{\pi_1(M)_k}{\pi_1(M)}$ goes to zero under $f$. It has a preimage $y \in H_2\left(\frac{\pi_1(M)}{\pi_1(M)_k}\right)$. This corresponds to $y' \in \ker \tau'$. By the reasoning of Lemma 5.5, $\partial(y' \setminus \hat{D}^2) \in \pi_1(N)_{k+1}$. Thus, $y'$ is a $(k+1)$-grope. But by [11], the $(k+1)$-gropes are the kernel of $q'$. Therefore, $y' = 0$, $y = 0$, and $x = 0$.

And now a standard diagram chase—using only the fact that $A$ and $B$ are surjective while $C$ is injective—shows that $f$ is also surjective. Let $y' \in \frac{\pi_1(N)_k}{\pi_1(N)}$, since $B$ is surjective, there exists $z \in H_1(M_L)$ such that $B(z) = j'(y)$. By commutativity and the fact that $C$ is injective, $z \in \ker h$. So there is a $w \in \frac{\pi_1(M)_k}{\pi_1(M)}$ such that $j'(f(w)) = j'(y)$ and hence $f(w)^{-1}y' =: v' \in \ker j'$. Moreover, there exists $u' \in H_2\left(\frac{\pi_1(N)}{\pi_1(N)_k}\right)$ with $\tau'(u') = v'$ and thus, by the surjectivity of $A$, there is $u \in H_2\left(\frac{\pi_1(M)}{\pi_1(M)_k}\right)$ such that $f(\tau(u)) = v'$. Therefore, $y' = f(w)v' = f(w)f(\tau(u)) = f(w\tau(u))$.

Now compose with the isomorphisms in Equation 2.9 of Section 2.9.

\]

Corollary 5.8. The basing map $\beta : M_L \to B_{F_k}^E$ extends to a map $\bar{\beta} : N_W \to B_{F_k}^E$.

Proof. Define the extension of $\beta$ to $N_W$ by precomposing the induced map $\pi_1(N_W)_k \to \pi_1(N)$ with the natural quotient.

\[
[M_L, \beta, s_0] \in \ker(\Omega^1_{\text{spin}}(B_{F_k}^E) \to \Omega_{\beta}(B_{F_k}^E))
\]

Theorem 5.9. $[M_L, \beta, s_0] \in \ker(\Omega^1_{\text{spin}}(B_{F_k}^E) \to \Omega_{\beta}(B_{F_k}^E))$

Proof. From the Lemmas above and Corollary 5.8, we see that $[M_L, \beta] = 0 \in \Omega_{\beta}(B_{F_k}^E)$

Lemma 5.10. The isomorphism above sends $\partial(\Gamma \setminus \hat{D}^2)$ to the element given by $b(\tau_L)$ under the bracket map.
Proof. As we saw in the construction of $N_w$, $\partial(\Gamma \setminus \bar{D}^2)$ is the meridian of the twisted Whitney disk, $A$. But by Lemma 3.2, the meridian, $\mu_A$, is simply the commutator of the top-level surfaces that $A$ is attached to. This is the same as the element of $\frac{F_k}{F_{k+1}}$ given by $b(\tau_L)$ under the bracket map.

Lemma 5.11. $[\Gamma]$ is Poincaré dual to $w_2(N_w, M_L)$

Proof. By the earlier result on the structure of $H_2(N_w)$, the Poincaré dual of $w_2(N_w, M_L)$ is represented by a mod 2 homology class $[\Sigma] = a[\Gamma] + b[S^2_\alpha]$. By the Wu formula, $\langle w_2(N_w, M_L), [\Gamma] \rangle = [\Gamma]^2 = 0$ and $\langle w_2(N_w, M_L), [S^2_\alpha] \rangle = [S^2_\alpha]^2 = 1$. But $\langle w_2(N_w, M_L), x \rangle = [\Sigma] \cdot x = a[\Gamma] \cdot x + b[S^2_\alpha] \cdot x$. Thus, we see $a = 1$ and $b = 0$. \qed
Chapter 6

Some facts about fibrations

Recall that, given a fibration $F \hookrightarrow E \overset{p}{\rightarrow} B$, with path-connected fiber and base, one corollary of the Leray-Serre spectral sequence for the fibration is the 5-term exact sequence for the fibration:

$$
\begin{align*}
H_2(E) & \xrightarrow{p_*} H_2(B) \xrightarrow{\tau} H_1(F)_{\pi_1(B)} \rightarrow H_1(E) \xrightarrow{p_*} H_1(B)
\end{align*}
$$

where $\tau$ is the transgression (cf. Definition 6.2 below) and the third map $H_1(F)_{\pi_1(B)} \rightarrow H_1(E)$ precomposed with the surjection $H_1(F) \rightarrow H_1(F)_{\pi_1(B)}$ gives the induced map from inclusion $F \hookrightarrow E$.

**Theorem 6.1.** Given a fibration $F \hookrightarrow E \rightarrow B$ with $B$ and $F$ path connected, there is the following commutative diagram

$$
\begin{array}{ccccccc}
H_2(E) & \rightarrow & H_2(E,F) & \rightarrow & H_1(F) & \rightarrow & H_1(E) \\
\| & & \| & & \| & & \\
H_2(E) & \rightarrow & H_2(B) & \rightarrow & H_1(F)_{\pi_1(B)} & \rightarrow & H_1(E)
\end{array}
$$

where the top exact sequence comes from the homology sequence for the pair $(E,F)$ and the bottom exact sequence comes from the 5-term sequence for the fibration.

In the process of proving this theorem we will need to show the equivalence of two notions of the transgression. Key elements of the argument can be found in [20].

First, note the following:
Given a fibration $F \hookrightarrow E \xrightarrow{p} B$ we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\pi_q(F) & \xrightarrow{i_*} & \pi_q(E) & \xrightarrow{p_*} & \pi_q(B) & \xrightarrow{\partial} & \pi_{q-1}(F) \\
\downarrow{h} & & \downarrow{h} & & \downarrow{h} & & \downarrow{h} \\
H_q(F) & \xrightarrow{i_*} & H_q(E) & \xrightarrow{j_*} & H_q(E,F) & \xrightarrow{\partial} & H_{q-1}(F) \\
\downarrow{p_*} & & \downarrow{\tilde{p}_*} & & \downarrow{\tau} & & \downarrow{\tau} \\
H_q(\ast) & \xrightarrow{j_*} & H_q(B) & \xrightarrow{\tilde{p}_*} & H_q(B,\ast) & \xrightarrow{\partial} & H_{q-1}(\ast)
\end{array}
$$

(6.1)

where the first row is the homotopy exact sequence for the fibration, the other rows are the homology sequences for the pairs, $h$ is the Hurewicz map, $\tilde{p}_*$ is the induced fibration on pairs. We have the following:

**Definition 6.2 (Transgression).** The transgression, $\tau$, is the homomorphism,

$$
\tau : j_*^{-1}(\text{im} \tilde{p}_*) \to H_{q-1}(F)/\partial(\text{ker} \tilde{p}_*)
$$

given by $\tau(z) = \partial r + \partial(\text{ker} \tilde{p}_*)$ where $z \in j_*^{-1}(\text{im} \tilde{p}_*)$, and $\tilde{p}_*^{-1}(j_*(z)) = r + \text{ker}(\tilde{p}_*)$. (The various maps are as defined in Diagram 6.1 above.)

*(Preview of the proof of Theorem 6.1)* A convenient reformulation of the definition of the transgression is given by the following diagram:

$$
\begin{array}{ccccccc}
\ker \tilde{p}_* & \xrightarrow{\partial} & H_q(E,F) & \xrightarrow{\tilde{p}_*} & \text{im} \tilde{p}_* & \xrightarrow{\tau} & \text{im} \tilde{p}_* \\
\downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\tau} & & \downarrow{\tau} \\
\partial(\text{ker} \tilde{p}_*) & \xrightarrow{\partial} & H_{q-1}(F) & \xrightarrow{\tilde{p}_*} & H_{q-1}(F)/\partial(\text{ker} \tilde{p}_*)
\end{array}
$$

(6.2)

We will show that the two geometrically defined terms on the right correspond to terms in the spectral sequence for the fibration. We will also show that a similar diagram with these spectral sequence terms commutes. Then we will identify a certain differential appearing in the new diagram with the transgression. Then, translating back into the geometric setting, the proof will be complete.

Given the fibration $F \hookrightarrow E \xrightarrow{p} B$, let $E_{s,*}$ denote the corresponding Leray-Serre spectral sequence. We then have the following:

**Lemma 6.3.** $E_{n,0}^n \cong j_*^{-1}(\text{im} \tilde{p}_*) \subseteq H_n(B)$

**Lemma 6.4.** $E_{0,n-1}^n \cong H_{n-1}(F)/\partial(\text{ker} \tilde{p}_*)$
Lemma 6.5. \( d^n : E^n_{n,0} \rightarrow E^n_{0,n-1} \) is the transgression of the fibration.

where, again, the various maps are as defined in Diagram 6.1 above.

Proof of Lemma 6.3. Consider the map of fibrations:

\[
\begin{array}{ccc}
E & \xrightarrow{i} & (E, F) \\
p & & \downarrow \bar{p} \\
B & \xrightarrow{j} & (B, *)
\end{array}
\]

Let \( E_{*,*}^n \) and \( \tilde{E}_{*,*}^n \) denote the corresponding Leray-Serre spectral sequences. By naturality the inclusions \( j \) induce \( j_* : E_{p,q}^r \rightarrow \tilde{E}_{p,q}^r \) for all \( r, p, q \). Their \( E^2 \) pages are the same save for the first column, since \( H_{p>0}(B; H_*(F)) \cong H_{p>0}((B, *); H_*(F)) \). When \( p = 0 \) we get \( \tilde{E}_0^2 \cong H_0((B, *); H_*(F)) = 0 \) since \( B \) is path connected. This implies \( j_* : E_{p,q}^3 \rightarrow \tilde{E}_{p,q}^3 \) is injective for \( p > 0 \) and an isomorphism for \( p \geq 3 \). We then get by induction that \( \tilde{E}_{n,0} \cong \tilde{E}_{n,0} \). Since the \( p = 0 \) column for the \( \tilde{E}_n \) page is still zero, the differentials originating in the \( p = n \) column are zero, we get \( \tilde{E}_{n,0}^n = \tilde{E}_{n,0}^\infty \).

Now for any fibration \( G \hookrightarrow X \xrightarrow{\pi} B \) with connected fiber, consider the commutative diagram below, taken from applying naturality to the induced map of fibrations between \( G \hookrightarrow X \xrightarrow{\pi} B \) and \( * \hookrightarrow B \xrightarrow{id} B \).

\[
\begin{array}{ccc}
H_n(X) & \xrightarrow{\pi_*} & H_n(B) \\
\downarrow & & \downarrow \cong \\
E_{n,0}^\infty(X) & \longrightarrow & E_{n,0}^\infty(B)
\end{array}
\]

(6.3)

The map at the bottom is the composition \( E_{n,0}^\infty(X) \hookrightarrow E_{n,0}^2(X) \rightarrow E_{n,0}^2(B) = E_{n,0}^\infty(B) \). The second map is an isomorphism since the fiber is assumed to be path connected. Thus, the diagram gives a factorization of \( \pi_* \) as a surjection followed by an injection. This must be equivalent to the canonical factorization \( H_n(X) \twoheadrightarrow \text{im} \pi_* \twoheadrightarrow H_n(B) \). Thus \( E_{n,0}^\infty(X) \) is identified with \( \text{im} \pi_* \).

Applying this to our relative fibration above, we have \( \tilde{E}_{n,0}^n = \tilde{E}_{n,0}^\infty \) and hence, the isomorphism \( j_* : E_{n,0}^n \cong \tilde{E}_{n,0}^n \) gives \( E_{n,0}^n \cong j_*^{-1}(\text{im} \tilde{p}_*) \). \( \square \)
Next, from naturality of the spectral sequence we get the following diagram:

\[
\begin{array}{cccc}
\ker \tilde{p}_* & \xrightarrow{\partial} & \partial(\ker \tilde{p}_*) \\
H_n(E) & \xrightarrow{j_*} & H_n(E, F) & \xrightarrow{\partial} & H_{n-1}(F) & \xrightarrow{i_*} & H_{n-1}(E) \\
\downarrow{p_*} & & \downarrow{\tilde{p}_*} & & \downarrow{(#)} & & \downarrow{q} \\
E_{n,0}^\infty & \xrightarrow{d^n} & E_{n,0}^m & \xrightarrow{d^n} & E_{0,n-1}^n & \xrightarrow{d^n} & E_{0,n-1}^\infty
\end{array}
\]

(6.4)

The two longer rows are exact, as is the first column, as we have just shown. We will show that the middle square commutes and that the second column is exact. This will identify \(d^n\) with the transgression. The map \(q\) in the second column is the natural quotient map. To complete the proof of Lemmas 6.4 and 6.5 we need the following:

**Lemma 6.6.** The middle square in the diagram above commutes.

**Proof of Lemma 6.6.** To prove the middle square commutes we invoke the definitions of the spectral sequence terms appearing:

\[
E_{n,0}^m = Z_{n,0}^n/B_{n,0}^{n-1} + Z_{n-1,1}^{n-1}, \quad E_{0,n-1}^m = Z_{0,n-1}^n/B_{0,n-1}^{n-1}
\]

where

\[
\begin{align*}
Z_{n,0}^n &= \{x \in F_nC_n(E) \mid \partial(x) \in F_0C_{n-1}(E) = C_{n-1}(F)\} \quad (6.5) \\
B_{n,0}^{n-1} &= \{x \in F_nC_n(E) \mid \exists y \in F_{2n-1}C_{n+1}(E) \text{ s.t. } \partial y = x\} = B_n(E) \quad (6.6) \\
Z_{n-1,1}^{n-1} &= \{x \in F_nC_{n-1}(E) \mid \partial(x) \in F_0C_{n-1}(E) = C_{n-1}(F)\} \quad (6.7) \\
Z_{0,n-1}^n &= \{x \in F_0C_{n-1}(E) \mid \partial(x) \in F_0C_{n-2}(E)\} = F_0C_{n-1}(E) \quad (6.8) \\
B_{0,n-1}^{n-1} &= \{x \in F_0C_{n-1}(E) \mid \exists y \in F_nC_{n}(E) \text{ s.t. } \partial y = x\} \quad (6.9) \\
F_sC_t(E) &= \text{im}(C_t(p^{-1}(B(s))) \to C_t(E)) \quad (6.10)
\end{align*}
\]

and \(C_s\) denotes relevant chain complexes for the spaces involved.

Let \(x \in H_n(E, F) = H_n(C_n(E)/F_0C_n(E))\). We have \(\tilde{p}_*(x) = x + B_n(E) + Z_{n,0}^{n-1}\). Moreover, from the definitions we see that the differential \(d^n : E_{n,0}^n \to E_{0,n-1}^n\) can be written as \(d^n(x + B_n(E) + Z_{n,0}^{n-1}) = \partial(x) + B_{0,n-1}^{n-1}\). Since \(\partial(x) \in F_0C_{n-1}(E)\) this is well-defined. Also, we have \(\partial(x) \in H_{n-1}(F)\) given by \(\partial(x) + B_{n-1}(F)\). So the map \(q : H_{n-1}(F) \to E_{0,n-1}^n\) takes \(x + B_{n-1}(F)\) to \(x + B_{0,n-1}^{n-1}\). Since \(B_{n-1} = \{\partial(y) \mid y \in F_0C_n(E)\} \subset \{\partial(y) \mid y \in F_{n-1}C_n(E)\}\) \(= B_{0,n-1}^{n-1}\), we see that \(q\) is also well-defined.

Finally, chasing around the square in both directions we see that \(x + F_0C_{n}(E)\) with \(\partial(x) \in F_0C_{n-1}(E)\) is taken to \(\partial(x) + B_{0,n-1}^{n-1}\) both by \(d^n \circ \tilde{p}_*\) and by \(q \circ \partial\). \(\square\)
CHAPTER 6. SOME FACTS ABOUT FIBRATIONS

Proof of Lemma 6.4. Now we can complete the proof of Lemma 6.4. Let \( u \in \ker(q) \). By commutativity of the rightmost square, \( u \in \im(\partial) \). So \( \exists t \in H_n(E, F) \) s.t. \( \partial t = u \). If \( t \in \ker(\tilde{\gamma}_*) \) then we’re done: \( u \in \partial(\ker(\tilde{\gamma}_*)) \). If, on the other hand, \( t \notin \ker(\tilde{\gamma}_*) \), then by commutativity of the middle square \( \tilde{\gamma}_* t \in \ker d^n \). Thus, \( \tilde{\gamma}_* t \in E^\infty_{n,0} \). So \( \exists s \in H_n(E) \) s.t. \( \gamma_* (s) = \tilde{\gamma}_* (t) \). By commutativity of the leftmost square, \( \gamma_* j_* (s) = \tilde{\gamma}_* (t) \). Let \( \tilde{t} = t - j_* (s) \). \( \tilde{t} \) is clearly in \( \ker(\gamma_* \gamma) \), and we have \( \partial(\tilde{t}) = \partial(\gamma) = u \). So \( \ker(q) = \partial(\ker(\tilde{\gamma}_*)) \), and the second column of Diagram 6.4 is exact. \( \square \)

Proof of Lemma 6.5. Consider the following diagram, which is simply diagram 6.2 (the definition of \( \gamma \)) modified by the results of Lemmas 6.4 and 6.3:

\[
\begin{array}{cccccc}
 \ker \tilde{\gamma}_* & \rightarrow & H_q(E, F) & \rightarrow & E^n_{n,0} & \\
 \downarrow \partial & & \downarrow \partial & & \downarrow d^n & \\
 \partial(\ker(\gamma_*)) & \rightarrow & H_{q-1}(F) & \rightarrow & E^n_{0,n-1} & \\
\end{array}
\] (6.11)

Thus we see that \( d^n = \gamma \). \( \square \)

Proof of Theorem 6.1. Now refer back to diagram 6.4. Letting \( n = 2 \), using the fact that \( \im(\gamma_* (H_2(E)) \rightarrow H_2(B)) = E^\infty_{2,0} \) as in diagram 6.3, as well as the isomorphism \( E^2_{2,0} \cong H_2(B) \cong H_2((B, *)) \), Theorem 6.1 follows. \( \square \)

Corollary 6.7 (Corollary to Theorem 6.1). Given the sequence \( \pi_1(N_w)_k \twoheadrightarrow \pi_1(N_w) \rightarrow \pi_1(N_w)_k \), form the canonical fibration of classifying spaces,

\[
B (\pi_1(N_w)_k) \twoheadrightarrow B (\pi_1(N_w)) \twoheadrightarrow B \left( \frac{\pi_1(N_w)}{\pi_1(N_w)_k} \right)
\]

Then we get the following commutative diagram (where the homology is to understood to be of the classifying spaces):

\[
\begin{array}{cccccc}
 H_2(\pi_1(N_w)) & \rightarrow & H_2(\pi_1(N_w), \pi_1(N_w)_k) & \rightarrow & H_1(\pi_1(N_w)_k) & \rightarrow & H_1(\pi_1(N_w)) \\
 \| & & \| & & \| & & \| \\
 H_2(\pi_1(N_w)) & \rightarrow & H_2 \left( \frac{\pi_1(N_w)}{\pi_1(N_w)_k} \right) & \rightarrow & H_1(\pi_1(N_w)_k) \pi_1(N) \rightarrow H_1(\pi_1(N_w)) \\
\end{array}
\]
Chapter 7

On the map to
\[ \ker \left( \Omega_3^{spin}(B\frac{F}{F_k}) \rightarrow \Omega_3(B\frac{F}{F_k}) \right) \]

Since \( \Omega_n^{Spin}(X) = H_n(X; \text{MSPIN}) \) and is an additive homology theory, and \( B\frac{F}{F_k} \) is a CW complex, the Leray-Serre-Atiyah-Hirzebruch spectral sequence for the fibration \(* \hookrightarrow B\frac{F}{F_k} \rightarrow B\frac{E}{F_k}\) yields \( H_p(B\frac{E}{F_k}; \Omega_q^{Spin}) \Rightarrow \Omega_{p+q}^{Spin}(B\frac{E}{F_k}) \). We will first examine the \( E^2 \) page of the spectral sequence:

\[
\begin{array}{cccc}
3 & 0 & 0 & 0 \\
2 & \mathbb{Z}_2 & \mathbb{Z}_2^m & \mathbb{Z}_2 + \mathbb{Z}_2 \\
1 & \mathbb{Z}_2 & \mathbb{Z}_2^m & \mathbb{Z}_2^m \otimes \mathbb{Z}_2 \\
0 & \mathbb{Z} & \mathbb{Z}^m & \mathbb{Z}^m \otimes \mathbb{Z} \\
& 0 & 1 & 2 & 3 & 4
\end{array}
\]

\( \Omega_1^{spin} = \mathbb{Z}_2 \), generated by a circle with the nonbounding Lie group framing. \( \Omega_2^{spin} = \mathbb{Z}_2 \), generated by a torus with the Lie framing. Since it is shown that every spin three-manifold spin-bounds, \( \Omega_3^{spin} = 0 \), and the top row of our chart is all zeroes. ([15, Ch. 7])

The terms we are interested in are the components of \( \Omega_3^{spin}(B\frac{F}{F_k}) \), i.e. the terms with index \((p, q)\) such that \( p + q = 3 \), indicated by the boxes. The \((1, 2)\) term corresponds to the classic Arf invariant of the link components. (Recall the isomorphism \( \Omega_2^{spin} = \mathbb{Z}_2 \) induced by the Arf invariant of any Seifert surface for a knotted component. [23])

Since the edge homomorphism from the \((0, 2)\) term, \( \Omega_2^{spin} = \mathbb{Z}_2 \rightarrow \Omega_2^{spin}(B\frac{F}{F_k}) \), is the same as the map induced by the split injection \(* \hookrightarrow B\frac{F}{F_k} \) in our fibration, the differential from the...
(2,1) term, \( d^2 : \mathbb{F}_{k+1} \otimes \mathbb{Z}_2 \to \Omega_2^{\text{spin}} \), indicated by the dotted line, must be zero. [8, Section 9.3]

Suppose we are given an \( m \)-component link, \( L \), representing the class \( [L] \in \mathcal{W}^{\infty}_{2k-2} \). By our construction (cf. Corollary 5.9) we have \( [M_L, \beta, s_0] \in \ker (\Omega_3^{\text{spin}}(B^F_{\mathbb{F}_k}) \to \Omega_3(B^F_{\mathbb{F}_k})) \). However, it remains to show that:

**Theorem 7.1.** The above construction induces a well-defined map

\[
h : \mathcal{W}^{\infty}_{2k-2} \to \ker (\Omega_3^{\text{spin}}(B^F_{\mathbb{F}_k}) \to \Omega_3(B^F_{\mathbb{F}_k})),
\]

well-defined in the sense of being independent of both the link representative and independent of the element of \( \tau^{\infty}_{2k-2} \) mapping to \( \mathcal{W}^{\infty}_{2k-2} \).

We will give two proofs of the independence of link representatives, one directly geometric and the other using facts we will see later (cf. Section 8) concerning the secondary edge invariant of Teichner as it applies to the spectral sequence for \( \Omega_*^{\text{spin}}(B^F_{\mathbb{F}_k}) \). The first proof will automatically imply independence of intersection tree.

**Proof of independence of link representative.** Suppose we are given two \( m \)-component links, \( L \) and \( L' \) which represent the same class in \( \mathcal{W}^{\infty}_{2k-2} \). Recall from Section 2.3 that this means that they are twisted Whitney tower concordant of order \( 2k-1 \): for each \( i \), their \( i \)th components, \( L_{1,i} \subset S^3 \times 0 \) and \( -L_{2,i} \subset S^3 \times 1 \) cobound an immersed annulus \( A_i \subset S^3 \times I \) the collection of which supports a twisted Whitney tower of order \( 2k-1 \). Note that, by removing 3-balls from \( S^3 \) away from the links, the same concordance exists if we view the links in \( B^3 \times I \).

Now put \( L \) and \( -L' \) in separate copies of \( B^3 \subset S^3 = \partial D^4 \), with respective basings \( \delta \) and \( \delta' \). Recall that a twisted Whitney tower of order \( 2k-1 \) is simply a framed Whitney tower of order \( 2k-1 \). Thus, \( L \sqcup -L' \) bound \( m \) immersed annuli \( \{A_i\} \subset D^4 \) supporting a framed Whitney tower of order \( 2k-1 \).

Note that if we were to remove tubular neighborhoods, the boundary becomes \( E_L \# -E_{L'} \), and the separate basing maps on \( E_L \setminus B^3 \) (respectively, \( -E_{L'} \setminus B^3 \)) extend uniquely to basings on the closed-up \( E_L \) (respectively, \( -E_{L'} \)). By the same arguments as before, these basings induce isomorphisms on the nilpotent quotients

\[
\frac{\pi_1(E_L)}{\pi_1(E_L)_n} \to \frac{F}{F_n}, \quad \text{for } n \leq 2k
\]

which, after finishing the zero surgeries, extend to isomorphisms

\[
\frac{\pi_1(M_L)}{\pi_1(M_L)_n} \to \frac{F}{F_n}, \quad \text{for } n \leq 2k-1.
\]

with the same statements applying to \( -L' \), \( -E_{L'} \), and \( -M_{L'} \). Denote the induced maps \( \beta : M_L \to B^F_{\mathbb{F}_k} \) and \( \beta' : M_{L'} \to B^F_{\mathbb{F}_k} \).
CHAPTER 7. ON THE MAP TO \( \ker(\Omega_3^{\text{spin}}(B_{F_k}^E) \to \Omega_3(B_{F_k}^E)) \)

Also note that, since the concording annuli support a framed Whitney tower concordance of order \(2k - 1\), our link meridians \(\{\mu_L\}\) and \(\{\mu_{L'}\}\) map to the same elements of \(F_k^E\), and so the separate basings induce the same map \(\tilde{\beta} : M_L\# - M_{L'} \to B_{F_k}^E\).

We perform the same construction as before, but this time in a framed Whitney tower setting (see Theorem 7.4). This gives us \(D_{L,L'} = D^4\# S^2 \times S^2\). And now slice annuli play the role of slice disks. Performing the Whitney move and removing open neighborhoods of the slice annuli. We now have a 4-manifold, call it \(N_{L,L'}\), whose boundary is \(M_L\# - M_{L'}\).

Although this time we’ve removed annuli rather than disks, the same homology calculations of Sections 4 and 3 apply. For example, analogously to Corollary 4.2,

\[
H_2(D_{L,L'}, N_{L,L'}) \cong H^2(\bigcup_{i=1}^m A_i/\partial A_i) \cong \mathbb{Z}^m,
\]

generated by the dual meridinal disks of the \(A_i\), and as in Corollary 4.3,

\[
H_3(D_{L,L'}, N_{L,L'}) \cong H^1(\bigcup_{i=1}^m A_i/\partial A_i) = 0
\]

By Seifert-van Kampen, the fundamental group of \(N_{L,L'}\) is simply the free product of \(\pi_1(M_L)\) and \(\pi_1(M_{L'})\). Thus, \(\tilde{\beta} : M_L\# - M_{L'} \to B_{F_k}^E\) extends to \(B : N_{L,L'} \to B_{F_k}^E\).

Again, since the concording annuli support a framed Whitney tower concordance of order \(2k - 1\) and since \(\pi_1(N_{L,L'})_{2k-1} \subset [\pi_1(N_{L,L'}), \pi_1(N_{L,L'})]\), the generators of \(H_1(N_{L,L'})\) coming from the meridians of \(L\) are the same as the corresponding generators coming from the meridians of \(L'\). And thus, just as in Lemma 5.1 we have

**Lemma 7.2.** \(H_1(N_{L,L'})\) is freely generated by the meridinal circles dual to the (now removed) \(m\) properly embedded slice annuli, i.e. the meridians of the link components \(\{\mu_L\} = \{\mu_{L'}\}\).

By the corresponding isomorphisms on \(H^1\) the spin structures \(s_0\) and \(s_0'\) uniquely extend to a spin structure \(S\) on \(N_{L,L'}\).

Since, by Theorem 7.4, \(w_2(N_{L,L'}, M_L\# M_{L'}) = 0\), we have

\[
[M_L\# - M_{L'}, \tilde{\beta}] = 0 \in \Omega_3^{\text{spin}}(B_{F_k}^E).
\]

To get \([M_L \sqcup - M_{L'}, \beta \sqcup \beta'] = 0\) in \(\Omega_3^{\text{spin}}(B_{F_k}^E)\), attach a 3-handle to a 2-sphere separating the \(M_L\) and \(-M_L\) in \(\partial N_{L,L'}\). Such a sphere originates—before we do the surgery and remove slice annuli—as a sphere in \(S^3\) separating the copies of \(B^3 \subset \partial D^4\) which contained our links. This sphere does not affect \(\pi_1\), \(H_1\), or the calculations of \(H_2(N_{L,L'})\) and the relative Stiefel-Whitney class, and the map similarly extends.

Also note that since \(L\) is concordant to itself and this argument did not involve any particular choice of Whitney tower, this also shows that our map is independent of the Whitney tower which \(L\) bounds, and thus, independent of which intersection tree maps to \(L\) via \(R_{2k-2}^\omega : T_{2k-2}^\omega \to W_{2k-2}^\omega\). 

\(\square\)
2nd proof of independence of link representative. Suppose we are given two m-component links, L and L' which represent the same class in $\mathcal{W}_{2k-2}$. Since the links are Whitney tower concordant, by [6, Theorem 3.3] we can choose corresponding Whitney towers, $\mathcal{W}$ and $\mathcal{W}'$ with the same intersection invariant, $\tau^\omega(\mathcal{W}) = \tau^\omega(\mathcal{W}') = \tau \in \tau^\omega_{2k-2}$.

We will analyze the components that make up $[M_L, \beta]$ and $[M_L', \beta']$ in $\Omega_3^{spin}(B^4_F)$. From the spectral sequence we see the building blocks of $\Omega_3^{spin}(B^4_F)$ are

\begin{align}
H_3(B^4_F; \Omega_3^{spin}) &\cong H_3(B^4_F; \mathbb{Z}) \\
H_2(B^4_F; \Omega_1^{spin}) &\cong \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2 \\
H_1(B^4_F; \Omega_2^{spin}) &\cong \mathbb{Z}_2^m
\end{align}

(7.3)

Since $M_L$ and $M_L'$ bound oriented 4-manifolds—N and N', respectively—with maps extending to $B^4_F$, the $H_3(B^4_F) = \Omega_3^{SO}(B^4_F)$ component is zero. Since the links are concordant, corresponding link components have the same Arf invariants, and thus $[M_L, \beta]$ and $[M_L', \beta']$ have the same term in $H_1(B^4_F; \mathbb{Z}_2)$. (Which is actually zero if $k > 1$ since the (classic) Arf invariants are obstructions to a link being in $\mathcal{W}_3^\omega$. [25] [7, Corollary 1.16]) Finally, since, in each case, the only class in $H_2$ that doesn’t go to zero in the map to $B^4_F$ is the grope representing $PD(w_2(N, M_L))$ (respectively, $PD(w_2(N', M_L'))$), then, by Theorem 8.1, the $H_2(B^4_F)$ component is $\sec(M_L)$ (respectively, $\sec(M_L')$), which in each case is the image of $b(\tau) \in \frac{F_k}{F_{k+1}} \otimes \mathbb{Z}_2$ (mod $d^2$). Thus, we have $[M_L, \beta] = [M_L', \beta']$ in $\Omega_3^{spin}(B^4_F)$.

Proof of independence of intersection tree. Recall from Section 2.3 that the realization maps, $R_n^\omega : \tau_n^\omega \to \mathcal{W}_n^\omega$, are isomorphisms for $n \equiv 0, 1, or 3$ (mod 4), $R_n^\omega$, and, for $n \equiv 2$ (mod 4), the kernel of $R_n^\omega$ is generated by symmetric $\omega$-trees of the form $(J, J)\omega$ of the appropriate order, i.e. where the order of the rooted tree $J$ is $\frac{n+2}{4}$. But by our construction, a symmetric tree contributes nothing to the image in $H_2(B^4_F)$: spherical classes go to zero in the map to $B^4_F$.

From the above we see that $\beta$ induces a well-defined map from $\mathcal{W}_{2k-2}^\omega$ to $\Omega_3^{spin}(B^4_F)$. Moreover, we have:

**Theorem 7.3.** $\beta$ induces a homomorphism

$$h : \mathcal{W}_{2k-2}^\omega \to \ker(\Omega_3^{spin}(B^4_F) \to \Omega_3(B^4_F))$$

**Proof.** Since $\tau_{L_1} \omega \tau_{L_2} = \tau_{L_1} + \tau_{L_2}$, since the band sum of two links $L_1$ and $L_2$ can be realized via a boundary sum of two copies of $D^4$ away from the links, and since the corresponding Stiefel-Whitney classes sum appropriately, we see that this is in fact a homomorphism.
CHAPTER 7. ON THE MAP TO $\ker (\Omega_3^{\text{spin}}(B_{\mathcal{F}_k}^E) \to \Omega_3(B_{\mathcal{F}_k}^E))$

Theorem 7.4. $W_{2k-2}$ is in the kernel of the homomorphism $h : W_{2k-2}^\infty \to \Omega_3^{\text{spin}}(B_{\mathcal{F}_k}^E)$.

Proof. Since the order $\geq 2$, by Whitney moves (as in [6]) we can arrange that the unpaired intersection point for the top-level Whitney disk is not with a zero-order surface but rather with another Whitney disk. Again, do surgery along the Whitney circle as before, again yielding a pair of spheres $S^2_\alpha$ and the meridinal $S^2_\beta$. This time, however, the intersection form is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since the original disk is framed. Again use the auxiliary disk in $S^2_\alpha$ to do the Whitney move. Upon removal of the zero-order disks, both $S^2_\beta$ and $S^2_\alpha$—tubed via the meridian of the intersecting disk—become gropes in $N_{W}$. The intersection form hasn’t changed. (Homology calculations are similar.) Since $w_2 = 0$, there is no obstruction to extending the spin structure from the boundary $M_L$ to $N_{W}$. Thus, $L \in W_{2k-2}$ implies $M_L = 0$ in $\Omega_3^{\text{spin}}(B_{\mathcal{F}_k}^E)$. \qed
Chapter 8

The secondary edge invariant of Teichner

**Theorem 8.1.** Given any link $L \in \mathcal{W}_{2k-2}^w$, there exists a four-manifold $N$ with boundary the zero-surgery $M_L$, such that the relative Stiefel-Whitney class $w_2(N, M_L)$ maps to the corresponding secondary edge invariant of Teichner [29], $\sec(M_L) \in H_2(B_{\mathcal{W}_k}; \mathbb{Z}_2)/\text{im}d_2$.

**Proof.** Take $N$ to be $N_W$ constructed from the Whitney tower as above. By Lemma 5.8, $\beta : M_L \to B_{\mathcal{W}_k}$ induces $\beta_*[M_L] \in \ker(\Omega^\text{spin}_3(B_{\mathcal{W}_k}) \to \Omega_3(B_{\mathcal{W}_k}))$. Therefore, by [29, Proposition 3.2.3], the obstruction to extending the spin structure from $M_L$ to $N$ maps to the secondary edge invariant, that is,

$$\beta_*(\text{PD}(w_2(N, M_L))) = \sec(M_L, \beta)$$

Note as an aside that, in general, a relative obstruction class $w_2(X, \partial X)$ depends on the choice of a spin structure on $\partial X$. The difference in obstruction determined by different spin structures on $\partial X$ will be determined by coboundaries in $H^2(X, \partial X)$, which come via the exact cohomology sequence for the pair from $H^1(\partial X)$, modulo the image of $H^1(X)$. However, in our case, $w_2(N_W, M_L)$ does not depend on the choice of spin structure, as $H^1(N_W) \cong H^1(M_L)$.

**Corollary 8.2.** Given a link $L \subset \partial B^4$, bounding any twisted order $2k - 2$ Whitney tower, $\mathcal{W} \subset B^4$, and given any basing $\beta : M_L \to B_{\mathcal{W}_k}$, we have $\text{SL}(L) = \sec(M_L, \beta)$, and thus, another proof that the Sato-Levine invariant of a link is independent of the Whitney tower used to compute it.
Bibliography


