Strong Well-foundedness and the Genericity of Countable Sets

by

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Abstract

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In support of the Inner Model Program, we establish that a strengthening of well-foundedness, *strong well-foundedness*, holds for models constructed with iteration trees of certain important types. We subsequently find instances in which this property implies that every countable subset of a model is generic over the model, thereby proving instances of the Genericity Hypothesis. We also place the property of strong well-foundedness in context by proposing how it could be used to establish *iterability*, a key concept of the Inner Model Program, via a new route.
Chapter 1

Introduction

Herein we investigate a problem related to *iterability*: the ability to extend iterations of models of set theory. These iterations have a tree structure, and are generated using *extenders*, combinatorial objects which encode elementary embeddings. We isolate a new class of branches of iteration trees: *strongly well-founded branches*. Such branches necessarily give rise to well-founded models. We prove that under certain conditions trees have strongly well-founded branches, and give instances in which strong well-foundedness implies a useful genericity property.

The motivation for studying iterability comes out of the Inner Model Program, which seeks to define and analyze models of set theory which both contain large cardinals and generalize Gödel’s $L$. Assuming the existence of large cardinals, class-size models containing large cardinals can often be defined in straightforward (though not obvious) ways. But finding class-size models which both contain large cardinals and have “fine-structure” similar to the fine-structure of $L$ discovered by Gödel, Jensen, and others, has required the ability to iterate small structures approximating such models with a substantial amount of freedom.

The main engine for establishing iterability is the Martin-Steel Theorem ([1]) and theorems like it (for example, [4] contains a generalization of it to trees with long extenders). These theorems state that countable iterations trees satisfying certain hypotheses can be extended in useful ways. One of the hypotheses, however, is often quite difficult to establish. The hypothesis is, broadly, that there is no maximal path with well-founded limit off the main branch of limit length less than the length of the tree. For trees of limit length, the theorems guarantee a cofinal, well-founded branch. For trees of successor length, the theorems guarantee that extenders from the final model can applied to previous models of the tree so as to produce a next model, as long as the previous models exhibit sufficient agreement with the final model.

This criterion has been established in some cases, enabling structural analysis of inner models of strong axioms of infinity. The strongest axioms yet reached are “There is a cardinal which is a Woodin limit of Woodin cardinals” and slight strengthenings of it. This analysis is due to Neeman ([2]).

The concept of strong well-foundedness comes from an effort to decouple two aspects of
the main engine of the Neeman's proof in [2], and then to use this new technology to find
a new route to establishing iterability without relying on a theorem like the Martin-Steel
Theorem. A key part of Neeman’s proof establishing the limited form of iterability needed
for his situation is, given a specific kind of iteration tree which only uses short extenders,
constructing with one induction both an iteration-tree-like structure ancillary to the tree and
iterated forcings which give genericity of maps into the models of this tree-like structure over
those models; he then uses this map in the generic extension to show that the original tree
can be extended (later in the paper he also gives a construction which provides genericity
over the models of the original tree of maps into them, also using an entwined induction).
Here we examine trees not covered by Neeman’s proof (ours can be long extenders and,
unlike the extenders in [2], are strongly closed in the model from which they are selected),
and decouple the two parts of the approach in his paper.

First, given a countable subset of a model in the tree, we build an ancillary structure by
induction. Second, we define a forcing from this structure once it is completed. This forcing
gives genericity of the countable subset over the model of the tree. We do not examine the
most general tree structure, but believe that the approach taken here could, with sufficient
effort, be applied to more complex trees as well. We establish strong well-foundedness and the
genericity property for trees useful for Woodin’s project exposited in [4] and its forthcoming
sequels, though not all such trees; the trees we examine are countable strongly closed trees,
and of length $\omega + 1$, such that if we consider the tree with the final model removed, there is a
unique well-founded branch of length $\omega$. We also establish strong well-foundedness, though
not genericity, for linear strongly closed iteration trees of any countable length.

Our work here sheds some light on the concept of strong well-foundedness, but our
proofs which establish it rely heavily on the assumption that the tree has unique well-
founded branches which are maximal in the set of branches of their (limit) length. Though
our work here does not, therefore, directly aid in an attempt to separate iterability from
the Martin-Steel Theorem, we do however establish the genericity of countable subsets of
models in certain iteration trees, both in the manner mentioned above, and abstractly, given
an additional assumption about the tree. In establishing this genericity of countable subsets
we prove instances of Woodin’s Genericity Hypothesis ([4]), which was inspired by Neeman’s
paper:

**Definition 1.1. (Genericity Hypothesis)** Suppose that $(V_\Theta, \delta)$ is a premouse such that
$\Theta$ is a limit of Woodin cardinals. Suppose $M$ is an iterate of $V_\Theta$ by a countable strongly
closed iteration tree on $(V_\Theta, \delta)$ and $a \subset \Theta$ is bounded and countable. Then $a$ is set generic
over $M$.

We will define strongly closed iteration trees and the other technical concepts used in the
paper shortly, in Chapter 2. In Chapter 7 we prove that the Genericity Hypothesis holds for
strongly closed iteration trees of length $\omega + 1$ which have a unique well-founded branch of
length $\omega$. We first show that such trees are strongly-well founded, which we do in Chapter
6. From Chapter 2 to Subchapter 5.1 we build up to Chapter 6; in Subchapter 5.2 we
establish strong well-foundedness for linear strongly closed iteration trees of any countable length. In Chapter 8, we subsequently isolate another tree property, nice tracking. We show that strongly well-founded trees of length $\omega + 1$ whose strong well-foundedness tracks nicely also witness the Genericity Hypothesis. We conclude in Chapter 9, where we form some conjectures, and state some questions.
Chapter 2

Preliminaries

The concepts of extender and iteration tree have become established as key notions in modern set theory. We refer the reader to Chapter 3 of [4] for further background on these; we make very similar definitions. Let us state the precise definitions of extender and iteration tree we will use.

Let \( j : V \rightarrow M \) be an elementary embedding with critical point \( \kappa \). Let \( \eta > \kappa \) and let \( \hat{\eta} \) be least such that \( \eta \leq j(\hat{\eta}) \). For each finite \( s \subseteq \eta \), let \( E_s = \{ A \subseteq [\hat{\eta}]^{|s|} \mid s \in j(A) \} \). Then the extender \( E \) of length \( \eta \) derived from \( j \) is the set

\[
E = \{ (s, A) \mid s \in \eta^\omega \text{ and } A \in E_s \}.
\]

We call \( \eta \) the length of \( E \) (denoted \( \text{LTH}(E) \)), and \( \hat{\eta} \) the support of \( E \) (denoted \( \text{SPT}(E) \)). We let the strength of \( E \) (denoted \( \text{STR}(E) \)) be the greatest ordinal \( \rho \) such that \( V_\rho \subseteq \text{Ult}(V, E) \). If \( \eta \leq j(\kappa) \), then \( \hat{\eta} = \kappa \), i.e., the critical point of \( E \) (\( \text{CRT}(E) \)) equals the support of \( E \). In this case we say that \( E \) is a short extender; otherwise it is a long extender.

Iteration trees use sequences of extenders to build models and embeddings between them. Before we state our definition of iteration tree we must be clear on what we mean by tree order:

**Definition 2.1.** Let \( T \) be a strict partial order on an ordinal \( \gamma \). We call \( T \) a tree order on \( \gamma \) iff:

1) \( \forall \beta < \gamma \) (\( T \) well-orders \{ \( \alpha \mid \alpha T \beta \} \))
2) \( \forall \alpha, \beta \ (\alpha T \beta \Rightarrow \alpha < \beta) \)
3) \( \forall \alpha \ (0 < \alpha \Rightarrow 0 T \alpha) \)
4) \( \forall \alpha \ (\alpha \text{ is a successor ordinal } \iff \alpha \text{ is a successor under } T) \)
5) \( \forall \lambda < \gamma \ (\lambda \text{ is a limit ordinal } \Rightarrow \{ \alpha \mid \alpha T \lambda \} \text{ is } \epsilon \text{-cofinal in } \lambda) \).

We use normal interval notation: if \( \alpha T \beta \),

\[
[\alpha, \beta]_T = \{ \gamma \mid \alpha = \gamma, \alpha T \gamma T \beta, \text{ or } \gamma = \beta \},
\]

with other kinds of intervals defined in the analogous ways.

Now we can state our definition of iteration tree:
Definition 2.2. Let $M$ be a model of ZFC. An iteration tree on $M$ of length $\Omega$ is a tuple 

$$
\tilde{T} = (T, \langle E_\alpha | \alpha + 1 < \Omega \rangle, \langle M_\alpha | \alpha < \Omega \rangle, \langle j_{\alpha,\beta} : M_\alpha \rightarrow M_\beta | \alpha T \beta < \Omega \rangle)
$$

such that $T$ is a tree order on $\Omega$, $M_\alpha$ is a model for $\alpha < \Omega$ and $j_{\alpha,\beta} : M_\alpha \rightarrow M_\beta$ is an elementary embedding defined whenever $\alpha T \beta$, satisfying the following conditions:

1) $M_0 = M$,
2) $E_\alpha \in M_\alpha$ and $M_\alpha = \{E_\alpha \text{ is an extender}\}$,
3) Let $\alpha + 1 < \eta$, and let $\alpha^*$ be the unique predecessor of $\alpha$ under $T$. Then if $\alpha^* < \alpha$, $\SPT(E_\alpha) + 1 \leq \min\{\STR(E_\beta) | \alpha^* \leq \beta < \alpha\}$. Whether or not $\alpha^* < \alpha$, $M_{\alpha+1} = \Ult(M_{\alpha^*}, E_\alpha)$, and $j_{\alpha^*,\alpha+1} : M_{\alpha^*} \rightarrow M_{\alpha+1}$ is the ultrapower embedding
4) If $0 < \beta < \eta$ is a limit ordinal, then $M_\beta$ is the direct limit of the models $M_\alpha$ for which $\alpha T \beta$ relative to the embeddings $j_{\alpha,\gamma}$ for $\alpha T \gamma T \beta$, and for $\alpha T \beta$, $j_{\alpha,\beta}$ is the direct limit embedding.

We will also identify iteration trees solely by their first model, tree order, and sequence of extenders. These fully determine an iteration tree, and it is often convenient to embed the pair $(T, \langle E_\alpha | \alpha < \Omega \rangle)$ into another model so as to produce a new tree iteration tree on that model.

We note that it can be shown that the ultrapower indicated in (3) is defined (as is of course vital).

The structure of the tree $T$ strongly affects the models of the tree. A key structural feature that these trees can exhibit is that of having cofinal well-founded branch. A subset $a$ of $\Omega$ is called a branch of $T$ if it is totally ordered by $T$ and connected with respect to the interval topology on $\Omega$ induced by $T$. A branch $a$ of $T$ is a cofinal well-founded branch if it is cofinal in $\Omega$, and the direct limit of the models indexed by ordinals on $a$ under the tree embeddings is well-founded.

If certain classes of iteration trees have unique well-founded cofinal branches, inner models of strong axioms can be thoroughly analyzed. Any deepening of our understanding of when iteration trees have well-founded cofinal branches, particularly when they have unique cofinal well-founded branches, has important foundational ramifications.

Most research has been directed towards iteration trees with short extenders. For example, in [2] Neeman constructs an inner model of strong axioms (in fact, the strongest with known inner models) using iteration trees which use only a specific kind of short extenders. Our extenders will not be assumed to be short, but they will be assumed to be strongly closed; an extender $E$ is strongly closed if $\STR(E) = \LTH(E)$, and this ordinal is strongly inaccessible (the extenders in Neeman’s paper cannot exhibit either of these properties). We say that an iteration tree $\tilde{T}$ is strongly closed when all its extenders are strongly closed in the models from which they are selected. All our trees will be strongly closed.

We make one more remark about the assumptions we place on trees: our proofs do not require that the critical points along a branch be non-decreasing, so we do not assume that they are (we call a tree $\tilde{T}$ whose branches all have that property weakly non-overlapping; if $\alpha + 1 T \beta + 1$ implies that $\LTH(E_\alpha) \leq \CRT(E_\beta)$ we say that $\tilde{T}$ is non-overlapping). Since
the only situations of interest to the Inner Model Program involve weakly non-overlapping
trees, we could have assumed otherwise. Doing so doesn’t seem to substantially simplify our
proofs however, so as was done in [1], we present our results in more generality than may be
required.

We now proceed to stating some new definitions. Before we do so we remark that we
will at times use the capital letter $J$ to represent elementary embeddings, even though it is
standard practice to reserve this symbol for levels in Jensen’s stratification of $L$. We depart
from this convention here because there are many embeddings present in our constructions,
and it is helpful to use lowercase $j$ and uppercase $J$ to keep track of two related sequences
of these embeddings. We do not make use here of Jensen’s $J$-hierarchy.

Now let us define an important kind of model:

**Definition 2.3.** Let $j$ be an elementary embedding $j : N \rightarrow M$, and let $X \subseteq M$. Then

$$(M[[X]])^j = \{ j(f)(a) \mid f \text{ is a function and } a \text{ is a finite subset of } X \}.$$  

Note that $X \subseteq M[[X]]$. If $N, M \models ZFC$ and $j$ is cofinal in the ordinals of $M$, then $M[[X]] \leq M$.

The superscript in $(M[[X]])^j$ is useful in the case that there is more than one embedding
leading to $M$. If, for example, we have $j : N_1 \rightarrow N_2$ and $k : N_2 \rightarrow N_3$, we may distinguish
between $(N_3[[X]])^{k\circ j}$ and $(N_3[[X]])^k$. In the case that the embedding into $M$ has domain
$V$ and is the only such embedding under consideration we drop the superscript.

Now we can define the key concept:

**Definition 2.4.** **Strong well-foundedness:** Let $j$ be an elementary embedding $j : V \rightarrow M$; then $j$ (or, $(j,M)$) is strongly well-founded iff for all countable $Y \subseteq M$, there is some countable $X \subseteq M$ with $Y \subseteq X$ such that the transitive collapse of $M[[X]]$ is an internal iteration of $V$ by ultrafilters.

Since every such internal iteration by ultrafilters is well-founded, if $j : V \rightarrow M$ is strongly
well-founded, $M$ is necessarily well-founded. This makes strong well-foundedness a very
useful property, and one worth investigating with respect to branch embeddings of iteration
trees.

Let $b$ be a branch of the tree $T$; we call $b$ a strongly well-founded branch if the embedding
determined by the tree from $V$ into the direct limit of the models indexed by elements of $b$
is strongly well-founded.

A candidate for a class of trees for which this property might hold for branch embeddings
is the class of strongly closed trees such that for every limit ordinal less than or equal to
the length of the tree, there is a unique well-founded maximal branch of that length. We show
that, at least in the length $\omega + 1$ case, strong well-foundedness does indeed hold for branch
embeddings of such trees:

**Theorem 6.1.** Let $\tilde{T} = (T, \{ E_n \mid n < \omega \})$ be an iteration tree on $V$ with models $\{ M_n \mid n \leq \omega \}$ and embeddings $\{ J_{n,m} \mid nTm \}$. Assume that for all $n \in \omega$, $M_n \models \text{“} E_n \text{ is strongly closed”}$, and that $T \downarrow \omega$ has a unique cofinal well-founded branch. Then $J_{0,\omega}$ is strongly well-founded.
We will also use our proof of strong well-foundedness to show that for these trees, every countable subset of the limit model is generic over that model. We prove this in Chapter 7. In Chapter 8, we present a strengthening of strong well-foundedness (nicely-tracking strong well-foundedness), which implies this genericity property abstractly (i.e., without recourse to examining a specific proof of strong well-foundedness).

We remark that if \( M \) is countable, \( \Omega < \omega_1 \), and \( M_\Omega \) is the result of iterating \( M \) with an iteration tree, strong well-foundedness of \( M \to M_\Omega \) gives that \( M_\Omega \) is actually the result of an (internal) ultrapower iteration. So, strong well-foundedness for iteration trees on a model in the model doesn’t necessarily go down from a large model to a smaller one which embeds elementarily into it and contains a pull-back of the tree, unlike well-foundedness and continuous ill-foundedness off well-founded branches (defined below).

We will need some lemmas in order to start putting the machinery together in Chapter 5. We state and prove them now, and in doing so shed some light on models of the form \( (M[[X]])^j \):

**Lemma 2.5. The pass-through lemma:** Let \( i : P \to M \), \( j : P \to N \), and \( k : N \to M \) be elementary embeddings with \( N,M \) transitive and \( i = k \circ j \). Let \( X \subseteq N \). Then

\[
\left( (M[[k''X]])^i, \epsilon \upharpoonright (M[[k''X]])^i \right) \cong \left( (N[[X]])^j, \epsilon \upharpoonright (N[[X]])^j \right).
\]

**Proof:**

\( k \) restricted to \( (N[[X]])^j \) is clearly an \( \epsilon \)-monomorphism. But it is also surjective: let \( i(f)(y) \in (M[[k''X]])^i \), with \( y \) a finite subset of \( k''X \). Let \( x \) be the set of pre-images of elements of \( y \) under \( k \); then \( y = k(x) \). But then \( k(j(f))(x) = k(j(f))(k(x)) = i(f)(y) \).

\[\square\]

**Lemma 2.6.** Let \( j : N \to M \) and \( X \subseteq M \). Let

\[
(M[[\text{seq}(X)]]^j) = \{ j(f)(a) \mid f \text{ is a function and } a \in X^{< \omega} \}.
\]

Then \( (M[[X]])^j = (M[[\text{seq}(X)]]^j) \).

**Proof:**

The proof is trivial.

\[\square\]
Lemma 2.7. The middle-man lemma: Let \( j_1 : V \rightarrow N \) and \( j_2 : N \rightarrow M \) be elementary, with \( N, M \) transitive, and \( N = (N[[X]])^{j_1} \) for some \( X \subset N \). Let \( Y \subset M \). Then
\[
(M[[Y]])^{j_2} = (M[[j''_2X \cup Y]])^{j_2 \circ j_1}.
\]

Proof:
We show that
\[
(M[[\text{seq}(Y)]]))^{j_2} = (M[[\text{seq}(j''_2X \cup Y)]])^{j_2 \circ j_1}.
\]

"\( \subseteq \)": Let \( j_2(f)(y) \in (M[[\text{seq}(Y)]]))^{j_2} \) with \( f \in N \) and \( y \) a finite sequence of elements of \( Y \). Let \( k_1 \) be the length of \( y \). As \( N = N[[X]] = N[[\text{seq}(X)]] \) there is some \( g \in V \) and \( x \) a finite sequence of elements of \( X \) such that \( f = j_1(g)(x) \). Let \( k_2 \) be the length of \( x \). Define \( h \in V \) by \( h(z) = g(x)(y) \) if \( z = x^\gamma y \) is a sequence of length \( k_1 + k_2 \), \( x \) is in the domain of \( g \), and \( g(x) \) is a function with \( y \) in the domain of \( g(x) \). Then
\[
j_2(f)(y) = j_2(j_1(g)(x))(y) = (j_2(j_1(g))(j_2(x)))(y) = j_2(j_1(h)(j_2(x)^\gamma y)) \in (M[[\text{seq}(j''_2X \cup Y)]])^{j_2 \circ j_1}.
\]

"\( \supseteq \)": Let \( j_2(j_1(f))(z) \in (M[[\text{seq}(j''_2X \cup Y)]])^{j_2 \circ j_1} \); we may assume that \( z \) is of the form \( j_2(x)^\gamma y \) with \( x \) a finite sequence of elements of \( X \) and \( y \) a finite sequence of elements of \( Y \). Define \( g \) in \( N \) by \( g(s) = j_1(f)(x^\gamma s) \) for such \( s \) that this definition makes sense. Then
\[
j_2(j_1(f))(z) = j_2(j_1(f))(j_2(x)^\gamma y) = j_2(g)(y) \in (M[[\text{seq}(Y)]]))^{j_2}.
\]

Let us prove one more useful lemma before fixing notation.

Lemma 2.8. Let \( E \) be a strongly closed extender, and let \( \gamma < \text{SPT}(E) \). Then \( \text{Ult}(V, E) \) is closed under \( \gamma \)-sequences.

Proof:
We make use of the fact that \( V_{\text{STR}(E)} \subseteq \text{Ult}(V, E) \), and that \( \text{STR}(E) \) is a strongly-inaccessible cardinal. Let \( \{ j(f_\alpha)(a_\alpha) \mid \alpha < \gamma \} \in \text{Ult}(V, E) \), with \( a_\alpha \) a finite subset of \( \text{LTH}(E) \) for all \( \alpha \). Then \( \{ a_\alpha \mid \alpha < \gamma \} \in \text{Ult}(V, E) \). Furthermore, \( \{ j(\alpha) \mid \alpha < \gamma \} \in \text{Ult}(V, E) \).

Let \( j((f_\alpha \mid \alpha < \gamma)) \in \text{Ult}(V, E) \); for let \( j((f_\alpha \mid \alpha < \gamma)) = (g_\alpha \mid \alpha < j(\gamma)) \). Then
\[
(g_{j(\alpha)} \mid \alpha < \gamma) = (j(f_\alpha) \mid \alpha < \gamma) \in \text{Ult}(V, E),
\]
so \( \{ j(f_\alpha)(a_\alpha) \mid \alpha < \gamma \} \in \text{Ult}(V, E) \) as desired.

This concludes the preliminaries.
Chapter 3

Semi-generalized Prikry forcings and ultrafilter iterations

In this Chapter we describe a class of forcings, then show how an internal iteration of a model of set theory of length $\omega$ by countably complete ultrafilters naturally gives rise to such a forcing in the first model. We shift the forcing to the limit model, and show that the sequence of the pushforwards of the kernels of the ultrafilters is a generic object. We will later adapt this proof to show that in certain instances, strong-well-foundedness leads to the genericity of all countable sets over the final model.

A forcing of the following form will be called a semi-generalized Prikry forcing. Let $X$ be a set of finite sequences closed under initial segment, such that every element of $X$ has infinitely many extensions in $X$, and let $F$ be any function defined on $X$ such that for $\bar{x} \in X$, $F(\bar{x})$ is a countably complete ultrafilter on the set of all $y$ such that $\bar{x} \leq y \in X$. Now let

$$\text{Pr}(X,F) = \{(s,f) \mid s \in X, \text{dom}(f) = X \land \forall t \in X(f(t) \in F(t))\}.$$ 

Let $\mathbb{P}(X,F) = (\text{Pr}(X,F), \leq)$ where $(t,g) \leq (s,f)$ iff $t \supseteq s$, $g(\bar{x}) \subseteq f(\bar{x})$ for all $\bar{x} \in X$, and for all $i \geq \text{dom}(s)$, $t(i) \in f(t \upharpoonright i)$. A generic filter $G$ determines an infinite sequence $G$ through $X$, whose range is the union of ranges of the first coordinates of elements of $G$.

Let $M \vDash ZFC$, and let $X,F \in M$ be as above, in the sense of $M$. Woodin has shown ([3]) that if $M \vDash ZFC$ and $Y$ is an infinite path through $X$, then there is some $M$-generic filter $G \in (\mathbb{P}(X,F))^M$ such that $Y = Y_G$ iff for all $f \in M$ such that $\text{dom}(f) = X$ and $\forall t \in X(f(t) \in F(t))$, there is an $m \in \omega$ such that for all $n \geq m$, $Y(n) \in f(Y \downarrow n)$. We refer to such $f$ as a “challenge to genericity”.

Now let us consider the situation we need to analyze. Let $\langle N_n \mid n \leq \omega \rangle$, $\langle i_{n,m} \mid n \leq m \leq \omega \rangle$ be an internal iteration of a transitive model of set theory $N_0$ by countably complete ultrafilters $\langle U_n \mid n \in \omega \rangle$, where $U_n \in N_n$ for $n \in \omega$, and $\langle U_n \mid n \in \omega \rangle \in N_0$. We will find $X,F \in N_0$ such that the sequence of the images in $N_0$ of the kernels of the ultrafilters is generic over $N_0$ for $i_{0,\omega}(\mathbb{P}(X,F))$.

The function $F$ we are going to use will be, roughly, the union of functions representing the ultrafilters $U_n$. We will have to construct it piece by piece.
CHAPTER 3. SEMI-GENERALIZED PRIKRY FORCINGS AND ULTRAFILTER ITERATIONS

We first find its domain, \( X \). We find nice functions representing the sets the ultrafilters are measuring; we will find functions \( \{ h_n^\prime \mid n \in \omega \} \) in \( N_0 \) such that when we take the \( n \)-th function, send it to the \( n \)-th model, and evaluate it at the sequence consisting of the pushforwards of the kernels of the ultrafilters, we get the set measured by \( U_n \). \( X \) will be set of all finite sequences that arise from sequentially applying these functions.

For \( n \in \omega \), let \( D_n \in N_n \) be the set on which \( U_n \) is a countably complete ultrafilter, let \( \text{id}_n \) be the identity function on \( D_n \), and let \( k_{n+1} \in [\text{id}_n] U_n \). Then for all \( y \in N_{n+1} \) there is some function \( f \) on \( D_n \) such that \( y = i_{n,n+1}(f)(k_{n+1}) \).

Let us define a term; let \( f \) be a function. We define a route in \( f \) to be a sequence \( \langle x_1, \ldots, x_k \rangle \) such that for \( j \leq k \),

\[
(\ldots((f(x_1))(x_2))\ldots)(x_j)
\]

is defined.

We show the first few steps of the construction of the functions \( h' \) explicitly, then give the general construction. In addition to the functions \( h'_n \), we will also find functions \( h^n_m \), where \( h^n_m \) is taken from \( N_m \) and represents \( D_n \).

Let \( h^0_0 = \{ (\emptyset, D_0) \} \). Let \( h^0_1 \in N_0 \) be a function on \( D_0 \) representing \( D_1 \). That is, \( [h^0_1]U_0 = D_1 \), or equivalently, \( i_{0,1}(h^0_1)(k_1) = D_1 \). Let

\[
h'_1 = \{ ((x), y) \mid (x, y) \in h^0_1 \}.
\]

Let \( h^1_2 \) be a function in \( N_1 \) on \( D_1 \) such that \( i_{1,2}(h^1_2)(k_2) = D_2 \). Now let \( h^0_2 \) be a function in \( N_0 \) on \( D_0 \) such that \( i_{0,1}(h^0_2)(k_1) = h^1_2 \), and such that \( h^0_2(x) \) is a function on \( h^0_1(x) \) for all \( x \in D_0 \), not just for \( U_0 \)-many such \( x \). This step is vital. Now let

\[
h'_2 = \{ ((x, y), h^0_2(x)(y)) \mid y \in \text{dom}(h^0_2)(x) \}.
\]

Then \( i_{0,2}(h'_2)((i_{1,2}(k_1), k_2)) = D_2 \).

Now we give the general construction.

We construct functions

\[
\{ h^n_m \mid n > 0 \text{ and } 0 < m < n < \omega \}, \text{ and } \{ h'_n \mid n \in \omega \}, \text{ with } h^0_1, h^1_2, h^0_2 \text{ and } h'_1, h'_2 \text{ as above, such that for all } n \in \omega, \text{ for all } m \text{ such that } 0 < m < n,
\]

A) \( h^n_m \in N_m \) and the domain of \( h^n_m \) is \( D_m \).
B) If \( n > 0 \), \( i_{n-1,n}(h^{n-1}_m)(k_n) = D_n \).
C) If \( m < n - 1 \), for all \( x \in D_m \), \( h^n_m(x) \) is a function on \( h^{n-1}_m(x) \), and \( i_{m,m+1}(h^n_m)(k_{n+1}) = h^{m+1}_m \). Also, every maximal route in \( h^n_m \) has length at least \( n - m \) (it can be longer if, for example, \( D_n \) is a set of functions). For every non-maximal route in \( h^n_m \langle x_m, \ldots, x_p \rangle \) with \( p < n - 1 \), \( (\ldots((h^n_m(x_m))(x_{m+1}))(x_{m+2}))\ldots(x_p) \) is a function on \( (\ldots((h^n_m(x_m))(x_{m+1}))\ldots(x_{m+2}))(x_{m+3}))\ldots(x_p) \).

D) The domain of \( h'_n \) consists of those sequences of length \( n \langle x_0, \ldots, x_{n-1} \rangle \) such that

\[
((\ldots((h^0_n(x_0))(x_1))\ldots)(x_{n-1})
\]
makes sense, and in this case

\[ h'_n((x_0, x_1, \ldots x_{n-1})) = ((h^0_n(x_0))(x_1)) \ldots (x_{n-1}). \]

From (A)-(D) it follows that

E) For \( 0 \leq m < p \leq n, \)

\[ i_{p-1,p}((i_{m+1,m+2}((i_{m,m+1}(h^m_n))(k_{m+1})))(k_{m+2})\ldots))(k_p) = \]

\[ \ldots (i_{m,p}(h^m_n)(i_{m+1,p}(k_{m+1})))(i_{m+2,p}(k_{m+2})\ldots))(k_p) = h^p_n, \]

that

F) \( (i_{0,n}(h'_n))((i_{1,n}(k_1), i_{2,n}(k_2), \ldots, i_{n-1,n}(k_{n-1}), k_n)) = D_n, \)

and that

G) the domain of \( h'_{n+1} \) is the set of all sequences of the form

\[ \langle x_0, \ldots, x_{n-1}, y \rangle, \]

where \( \langle x_0, \ldots, x_{n-1} \rangle \) is in the domain of \( h'_n \) and \( y \in h'_n((x_0, \ldots, x_{n-1}). \)

We let

\[ X = \bigcup_{n \in \omega} \text{dom}(h'_n). \]

Now we find the function \( F \) in a similar way, by first finding nice functions representing the ultrafilters. We will find functions \( \{ F'_n \mid n \in \omega \} \) in \( N_0 \) such that when we take the \( n \)-th function, send it to the \( n \)-th model, evaluate it at the sequence consisting of the pushforwards of the kernels of the ultrafilters, we get \( U_n. \) \( F \) will be the union of these functions, and will have \( X \) as its domain.

We show the first few steps of the construction of the functions \( F' \) explicitly, then give the general construction. In addition to the functions \( F'_n \) we will also find functions \( \{ F^m_n \mid 0 \leq m < n \in \omega \} \), where \( F^m_n \) is taken from \( N_m \) and represents \( U_n \). Moreover, \( F^{n-1}_n(x) \) is always a countably complete ultrafilter on \( h^{n-1}_n(x) \), and if \( m < n - 1, \) \( F^m_n(x) \) is a function on \( h^{m}_{n-1}(x). \)

Let \( F'_0 = \{ (\emptyset, U_0) \}. \) Let \( F'_1 \) be a function on \( D_0 \) such that \( i_{0,1}(F'_1)(k_1) = U_1. \) We can assume that \( F'_1(x) \) is a countably complete ultrafilter on \( h^0_1(x) \) for all \( x \in D_0 \). Now let

\[ F'_1 = \{ (\langle x, y \rangle) \mid (x, y) \in F'_1 \}. \]

Let \( F'_2 \) be a function in \( N_1 \) on \( D_1 \) such that \( i_{1,2}(F'_2)(k_2) = U_2. \) We can assume that \( F'_2(x) \) is a countably complete ultrafilter on \( h^0_2(x) \) for all \( x \in D_1. \) Let \( F^0_2 \) be a function on \( D_0 \) such that \( i_{0,1}(F^0_2)(k_1) = F'_2. \) We can assume that for all \( x \in D_0, F^0_2(x) \) is a function on \( h^0_2(x) \) such that for \( y \in h^0_2(x), (F^0_2(x))(y) \) is a countably complete ultrafilter on \( (h^0_2(x))(y) \.

Now in \( V \) define \( F'_2 \) by \( F'_2((x, y)) = (F^0_2(x))(y) \) for \( x, y \) such that \( x \in D_0 \) and \( y \in h^0_2(x). \) Then

\[ (i_{0,2}(h'_2))((i_{1,2}(k_1), k_2))) = U_2. \]
CHAPTER 3. SEMI-GENERALIZED PRIKRY FORCINGS AND ULTRAFILTER ITERATIONS

Let us proceed to the general construction. We construct functions \( \{ F^m_n \mid n > 0 \text{ and } 0 \leq m < n < \omega \} \), and \( \{ F'_n \mid n \in \omega \} \), with \( F^0_1, F^1_2, F^2_2 \) and \( F'_0, F'_1, F'_2 \) as above, such that for all \( n \in \omega \), for all \( m \) such that \( 0 \leq m < n \),

A) \( F^m_n \in N_m \) and the domain of \( F^m_n \) is \( D_m \).
B) If \( n > 0 \), \( i_{n-1,n}(F^{n-1}_n)(k_n) = U_n \), and for all \( x \in D_{n-1} \), \( F^{n-1}_n(x) \) is a countably complete ultrafilter on \( h^{n-1}_m(x) \).
C) If \( m < n \), for all \( x \in D_m \), \( F^m_n(x) \) is a function on \( h^m_{n-1}(x) \), and \( i_{m,m+1}(F^m_n)(k_{m+1}) = F^{m+1}_n \). Also, every maximal route in \( F^m_n \) has length at least \( n - m \) (it can, as above, be longer if, for example, \( D_n \) is a set of functions). For every non-maximal route in \( F^m_n \langle x_m, \ldots, x_p \rangle \) with \( p < n - 1 \),

\[
(...((F^m_n(x_m))(x_{m+1}))\ldots)(x_p)
\]

is a function on

\[
(...((h^m_{p+1}(x_m))(x_{m+1}))\ldots)(x_p).
\]

For every route of length \( n - m \langle x_m, \ldots, x_{n-1} \rangle \), \( (...((F^m_n(x_m))(x_{m+1}))\ldots)(x_{n-1}) \) is a countably complete ultrafilter on \( (...((h^m_n(x_m))(x_{m+1}))\ldots)(x_{n-1}) \).

D) The domain of \( F'_n \) consists of those sequences of length \( n \langle x_0, \ldots, x_{n-1} \rangle \) such that

\[
(...((F^0_n(x_0))(x_1))\ldots)(x_{n-1})
\]

is defined, and in this case

\[
F'_n((x_0, x_1, \ldots, x_{n-1})) = (...(((F^0_n(x_0))(x_1))\ldots)(x_{n-1}).
\]

From (A)-(D) It follows that

E) For \( 0 \leq m < p \leq n \),

\[
(i_{p-1,p}(\ldots(i_{m+1,m+2}(\ldots(i_{m,m+1}(F^m_n)(k_{m+1})))(k_{m+2})\ldots))(k_p) =
\]

\[
(...(i_{m,p}(F^m_n)(i_{m+1,p}(k_{m+1}))(i_{m+2,p}(k_{m+2})\ldots))(k_p) = F^p_n,
\]

that

F) \( (i_{0,n}(F^0_n)((i_{1,n}(k_1), i_{2,n}(k_2), \ldots, i_{n-1,n}(k_{n-1}), k_n)) = U_n \), and that

G) the domain of \( F'_n \) is the domain of \( h'_n \), and for all \( \langle x_0, \ldots, x_{n-1} \rangle \) in the domain of \( F'_n \),

\[
F'_n((x_0, \ldots, x_{n-1}))
\]

is a countably complete ultrafilter on \( h'_n((x_0, \ldots, x_{n-1})) \).

We let \( F = \bigcup_{n \in \omega} F'_n \). Then the domain of \( F \) is \( X \).

**Theorem 3.1.** \( \langle i_{m,\omega}(k_m) \mid 0 < m < \omega \rangle \) is a generic sequence for \( i_{0,\omega}(\mathbb{P}(X, F)) \) over \( N_\omega \).

**Proof:**
Let $f \in N_\omega$ be a “challenge to genericity”, that is, a function on $i_0,\omega(X)$ such that for all $\bar{x} \in X$, $f(\bar{x}) \in (i_\omega(F))(\bar{x})$.

We show that there is a $d \in \omega$ such that for all $n \geq d$,

\[
\begin{align*}
(i_m,\omega(k_m) | 0 < m < \omega)(n) = i_n,\omega(k_n) & \in f((i_m,\omega(k_m) | 0 < m < \omega) \downarrow n) = \\
f((i_m,\omega(k_m) | 0 < m < n)).
\end{align*}
\]

Let $d$ be least such that there is some $g \in N_{d-1}$ such that $i_{d-1,\omega}(g) = f$, let $g$ be a fixed such function, and let $n \geq d$. Let $f' = i_{d-1,n-1}(g)$. Then $f' \in N_{n-1}$ and $i_{n-1,\omega}(f') = f$. We show that

\[
\begin{align*}
\begin{align*}
k_n & \in i_{n-1,n}(f'((i_{m,n-1}(k_m) | 0 < m \leq n - 1))) = \\
i_{n-1,n}(f')(\{i_{m,n}(k_m) | 0 < m \leq n - 1\}).
\end{align*}
\end{align*}
\]

Now $f'$ is according to $i_{0,n-1}(F)$, i.e.,

\[
\begin{align*}
f'(\{i_{m,n-1}(k_m) | 0 < m \leq n - 1\}) & \in (i_{0,n-1}(F))((i_{m,n-1}(k_m) | 0 < m \leq n - 1)) = \\
(i_{0,n-1}(F'_{n-1}))(\{i_{m,n-1}(k_m) | 0 < m \leq n - 1\}) & = U_{n-1}.
\end{align*}
\]

Since

\[
N_{n-1} \models "\forall W \subseteq D_{n-1}(W \in U_{n-1} \iff k_n \in i_{n-1,n}(W)"
\]

we have that

\[
k_n \in i_{n-1,n}(f'(\{i_{m,n-1}(k_m) | 0 < m \leq n - 1\}))
\]

as claimed. So then

\[
i_{n,\omega}(k_n) \in f((i_{m,\omega}(k_m) | 0 < m < n)),
\]

so $d$ is as desired. So

\[
\{i_{m,\omega}(k_m) | 0 < m \omega
\]

is a generic sequence for $i_{0,\omega}(\mathbb{P}(X,F))$ over $N_\omega$. 

\hfill $\square$
Chapter 4

Continuous ill-foundedness

The following is vital. In the non-linear case of Chapter 4, we will crucially use a witness to continuous ill-foundedness off the main branch to give us places to take hulls.

Lemma 4.1. Assume $\tilde{T}$ is a strongly closed iteration tree on $V$ of length $\omega$. Then $\tilde{T}$ is continuously ill-founded off its infinite well-founded branches; that is, there is a sequence

$$(\theta_n \mid n < \omega)$$

such that for $nTm$, if $m \in a$ for some cofinal well-founded branch $a$ of $\tilde{T}$, then $i_{n,m}(\theta_n) = \theta_m$, and otherwise, $i_{n,m}(\theta_n) > \theta_m$.

Proof: This lemma is a scholium of a theorem sketched in Martin-Steel [1]; a result close to it is stated without proof in [2]. It is likely that the authors of those papers were aware of this result but had no need to state it in the generality we do here.

Let $\tilde{T} = (T, (E_n \mid n < \omega))$. Let the models of $\tilde{T}$ be denoted $(M_n \mid n < \omega)$, and let the embeddings of $\tilde{T}$ be denoted $(i_{n,m} \mid nTm)$. For $n \in \omega$ let $n^*$ denote the unique predecessor of $n + 1$ in the tree order.

Let $B$ be the set of all $a \subseteq \omega$ such that $a$ is linearly ordered by $T$, $a$ is maximal, and either $a$ is finite or the direct limit along $a$ is ill-founded. Let $A$ be the set of infinite $a \subseteq \omega$ with $a$ linearly ordered by $T$, with $a$ maximal, and such that the direct limit along $a$ is well-founded.

Let $a \in B$ be infinite; then the direct limit along $a$ is ill-founded. Let this ill-founded model be denoted $M^a_\omega$. Let $i^a_n,\omega$ for $n \in a$ be the direct limit embedding from $M_n$ into $M^a_\omega$ for $n \in a$. Note however that $\text{V}_{\text{CRT}}(i^a_\omega) \subseteq M^a_\omega$, so in particular $\omega$ is in the well-founded part of $M^a_\omega$. Let $\phi_0 > \phi_1 > \ldots > \phi_n > \ldots$ be a decreasing sequence of ordinals in $M^a_\omega$. We can thin the $\phi_p$-sequence to ensure that each $\phi_p$ is a limit ordinal in the sense of $M^a_\omega$ (Proof: for each $p$, the sequence $[\phi_p, \phi_p - 1, \phi_p - 2 \ldots]$ can be constructed in $M^a_\omega$. In $M^a_\omega$, it has finite length. But $M^a_\omega$ labels numbers as finite correctly. So the sequence stops as a limit ordinal $\phi'_p$ such that there are infinitely many $\phi_n < \phi'_p$. We can thus replace $\phi_p$ with $\phi'_p$ and proceed, etc.)
CHAPTER 4. CONTINUOUS ILL-FOUNDEDNESS

Now say that \( \phi_p = i_{n_p,\omega}^p(\tau_p) \), where \( \tau_p \in M_{n_p}, n_p \in a \) for each \( p \). We may assume that the \( n_p \) sequence is strictly increasing, and so we do.

Now for each \( p \), let

\[
m_p = |\{ x \in a \mid n_p < x < n_{p+1} \}|.
\]

For \( k \in a \), let \( q(k) \) be such that \( n_q(k) \leq k < n_{q(k)+1} \), and let

\[
l_k = |\{ x \in a \mid n_q(k) < x \leq k \}|.
\]

Let

\[
\theta^a_k = i_{n_q(k),k}(\tau_q(k)) + (m_q(k) - l_k)
\]

(with, of course, \( i_{n_k,n_k} = \text{id} \).

Then if \( m, n \in a \) and \( mTn, i_{m,n}(\theta^a_m) > \theta^a_n \).

In the case that \( a \in B \) is finite, it is trivial that ordinals \( \theta^a_k \) for \( k \in a \) exist such that if \( m, n \in a \) and \( mTn, i_{m,n}(\theta^a_m) > \theta^a_n \).

Then \( B \) and the function with domain \( B \ (p, a) \mapsto \theta^a_p \) are in \( M_0 = V \).

Also, letting

\[
\theta = \sup \{ \theta^a_0 \mid a \in B \},
\]

\( \theta \in M \).

Let \( X \) be the set of pairs \((p, f)\) such that \( p \in \omega \) and \( f : \{ a \in B \mid p \in a \} \to \theta \). Define a binary relation \( R \) on \( X \) by

\[
(p, f) R (q, g) \iff (qTp \text{ and } \forall a \in \text{dom}(f), f(a) < g(a)).
\]

We claim that \( R \) is “well-founded off of its well-founded cofinal branches” in the following sense: if

\[
\ldots R(n_k, f_k) R(n_{k-1}, f_{k-1}) R \ldots R(n_0, f_0)
\]

is an infinite decreasing sequence, then \([n_0, n_1, \ldots] \subseteq a \) for some \( a \in A \). For if not, the integers \( n_0, n_1, \ldots \) all lie in some \( a \in B \), and then \( a \in \text{dom}(f_k) \) for all \( k \in \omega \). Then \( \{ f_k(a) \mid k \in \omega \} \) is an infinite descending chain of ordinals in \( V \). So the maximal extension of \([n_0, n_1, \ldots] \) is in \( A \).

Of course, \( X, R \in V \).

Note that each \( M_k \) is \( 2^{\alpha_0} \)-closed for each \( k \). In fact they are all \( \gamma \)-closed for all \( \gamma \) less than the smallest support of any extender used.

Now given \( k < \omega \), define \( f_k \), a function with domain \( \{ a \in B \mid k \in a \} \), by

\[
f_k(a) = \theta^a_k.
\]

Then by \( 2^{\alpha_0} \)-closure, \( f_k \in M_k \). In fact,

\[
(k, f_k) \in i_{0,k}(X) = \{ (p, f) \mid p \in \omega, f : \{ a \in B \mid p \in a \} \to \theta_0 \mid a \in B \} \}
\]

\[
M_k.
\]
CHAPTER 4. CONTINUOUS ILL-FOUNDEDNESS

Now if \( k \notin a \) for all \( a \in A \), let \( \theta_k \) be the rank of \( (k, f_k) \) in \( i_{0,k}(R) \) (this ordinal is defined in \( M_k \)); this rank exists because \( R \), and so \( i_{0,k}(R) \), is well-founded off the well-founded branches of \( T \). Let \( \theta_0 \) be greater than all these. Now if \( k \in a \) for some \( a \in A \), let \( \theta_k = i_{0,k}(\theta_0) \).

Then let \( kTq, q \notin a \) for all \( a \in A \); we show \( i_{k,q}(\theta_k) > \theta_q \). If \( k \) is in some \( a \in A \), this is clear. Otherwise, we need to show that the rank of \( i_{k,q}((k, f_k)) \) in \( i_{0,q}(R) \) is greater than the rank of \( (q, f_q) \) in \( i_{0,q}(R) \). It is enough to show that

\[
(q, f_q)(i_{0,q}(R))(k, i_{k,q}(f_k)).
\]

But let \( a \) be in the domain of \( f_q \); then \( a \) is in the domain of \( i_{k,q}(f_k) \). \( f_q(a) = \theta^a_q \), and \( i_{k,q}(f_k)(a) = i_{k,q}(f_k)(i_{k,q}(a)) = i_{k,q}(f_k(a)) = i_{k,q}(\theta^a_k) > \theta^a_q \). So indeed \( i_{k,q}(\theta_k) > \theta_q \).

So the \( \theta_k \)'s witness continuous ill-foundedness off the well-founded branches of \( T \).

We will use this in Chapter 4. But first we show that a linear iteration of \( V \) of length \( \omega + 1 \) by countably-complete extenders is strongly-well-founded: even in the general, non-linear case, we will apply this fact to a linear tree.
Chapter 5

Strong well-foundedness in the linear case

5.1 Length $\omega + 1$

In this Subchapter we prove the following:

Lemma 5.1. Let $\bar{T} = (T, \{ E_n \mid n < \omega \})$ be a linear iteration tree on $V$ with models $\{ M_n \mid n \leq \omega \}$ and embeddings $\{ J_{n,m} \mid nTm \}$. Assume that for all $n \in \omega$, $M_n \models \text{"E}_n\text{ is countably complete"}$. Then $J_{0,\omega}$ is strongly well-founded.

Proof: $M_\omega$ is well-founded: we show $J_{0,\omega}$ is strongly well-founded. Let $S \subset M_\omega$ be countable. Let $\bar{S} = (s_n \mid 0 < n \in \omega)$ be an enumeration. Then $M_\omega[[S \cup \bar{S}]] = M_\omega[[\bar{S}]]$. Let $b : \{ n \in \omega \mid 0 < n \} \to \omega$ be a strictly increasing function such that for all $n \geq 1$, $b(n)$ is some $m \geq n$ such that there exists $x \in M_m$ such that $J_{m,\omega}(x) = s_n$; and let $s'_n \in M_{b(n)}$ be such an $x$. By inserting 0 into the sequence $(s_n \mid n \in \omega)$ as a dummy term where necessary we can force $b(n) = n$. So for each $n$, $s'_n \in M_n$ is such that $J_{n,\omega}(s'_n) = s_n$.

Now for each $n \geq 1$, let $f_n : V \to V$ and $a^m_n \in [\text{LTH}(E_{n-1})]^\omega$ for $0 < m \leq n$ be such that

$s'_n = J_{n-1,n}(J_{n-2,n-1}(\ldots (J_{0,1}(f)(a^1_n) \ldots))(a^{n-1}_n))(a^n_n) = (\ldots ((J_{0,n}(f_n)(J_{1,n}(a^1_n))(J_{2,n}(a^2_n)) \ldots)(J_{n-1,n}(a^{n-1}_n)))(a^n_n))$.

Each $a^m_n$ is used in $M_m$ then sent along its merry way. Let

$\{ J_{m,\omega}(a^m_n) \mid 0 < n < \omega, 0 < m \leq n \} = S'$.

Then $M_\omega[[S \cup \bar{S} \cup S']] = M_\omega[[S']]$ and we will try to capture $S'$ from now on.
CHAPTER 5. STRONG WELL-FOUNDENESS IN THE LINEAR CASE

Now let $\eta$ be an inaccessible cardinal fixed by all the embeddings of $\tilde{T}$ with $\tilde{T}, S \in V_\eta$. Then the tree order and extenders of $\tilde{T}$ induce an iteration tree on $V_\eta$ the models of which are the rank-initial segments of height $\eta$ of the corresponding models of $\tilde{T}$. Call this tree $\tilde{T}'$; we will use the same notation to refer to the embeddings of $\tilde{T}$ and $\tilde{T}'$. The models of $\tilde{T}'$ are $(V_\eta)^{M_n}$ for $n \leq \omega$.

Let $X_0$ be an elementary substructure of $V_\eta$ containing $S'$ and $\tilde{T}'$, and let $R_0$ be its transitive collapse. Let $\pi_0$ invert the collapsing map. Let the models of $\pi_0^{-1}(\tilde{T}')$ be denoted $R_n$ for $n \leq \omega$. Then $\tilde{T}'$ is the tree on $V_\eta$ which comes from copying $\pi_0^{-1}(\tilde{T}')$. Let the embeddings of $\pi_0^{-1}(\tilde{T}')$ be denoted $j_{n,m}$ for $n < m \leq \omega$. Let $\pi_n : R_n \to V_\eta^{M_n}$ for $n \leq \omega$ be the copy map, and let $e_n = \pi_n^{-1}(E_n)$.

Then $S'$ is included in the the range of $\pi_\omega$. This follows from the fact that every element of $S'$ is in the range of $\pi_0 \downarrow ([\text{Ord}]^{<\omega})^{R_0}$, $([\text{Ord}]^{\omega})^{R_0} = ([\text{Ord}]^{<\omega})^{R_\omega}$, and $\pi_\omega = \pi_0 \downarrow R_\omega$. This last fact follows from a simple lemma which always holds in any copy diagram; it is always the case that $\pi_\alpha = \pi_0 \downarrow R_\alpha$.

We now proceed to the task of inserting an internal iteration of $V$ by ultrafilters between the two trees. The models of this iteration will be denoted $N_n$ (with $N_0 = V$). For each $n \geq 1$, let $l_n = \pi_n^{-1}([\text{LTH}(E_{n-1})]^{<\omega})$, we aim to have $\pi_n''l_n$ be a subset of the range of the embedding from $N_n$ into $M_n$ (and since $N$ will be $\omega$-closed this set will also be an element). We can use this fact to define the embeddings needed to produce a commutative diagram. When all is said and done, we will have that $N_\omega$ is the transitive collapse of $M_\omega[[Y]]$, where $Y$ is the set of pushforwards of enumerations of the sets $\pi''_{n}l_n$. We will have that $M_\omega[[S \cup Y]] = M[[Y]]$ so this will give us what we want.

We proceed as follows. For good measure let $N_0 = V$. Now let $l'_1 = \langle b_n | n \in \omega \rangle$ be an enumeration of $\pi''_1l_1$ (as $M_1$ is $\omega$-closed, both are in $M_1$). Let $U_0$ be the ultrafilter on $([\text{SPT}(E_0)]^{<\omega})^{\omega}$ defined by

$$X \in U_0 \leftrightarrow l'_1 \in J_{0,1}(X).$$

Then $U_0$ is $\text{CRT}(E_0)$-complete. Let $N_1 = \text{Ult}(V, U_0)$, and let $i_{0,1} : V \to N_1$ be the ultrapower embedding. Then $N_1$ is $\text{CRT}(E_0)$ closed, and $N_1$ is the transitive collapse of $M_1[[\{l'_1\}]] \supseteq M_1[[\pi''_{l'}l_1]]$. Note that $\eta \in M_1[[\{l'_1\}]]$, because it is fixed by $J_{0,1}$, and so $\eta$ is also fixed by $i_{0,1}$.

Let $\tau_1 : N_1 \to M_1$ invert the transitive collapse: then $\tau_1$ is elementary, and $\tau_1 \downarrow (V_\eta)^{N_1} : (V_\eta)^{N_1} \to (V_\eta)^{M_1}$ is elementary (as the satisfaction relation is $\Delta_1^{ZF}$).

Now let us show that $\pi''_{l'}R_1 \subset M_1[[\{l'_1\}]]$. Let $y = j_{0,1}(f)(a) \in R_1$. Let $n$ be such that $\pi_1(a) = \pi_0(a)$ is the $n$-th element of the $l'_1$ sequence. Define

$$f' : \{ \{x\} | x \in ([\text{SPT}(\pi_0^{-1}(E_0))]^{<\omega})^{\omega} \} \to R_0$$

by letting $f'(\{x\}) = f(x(n))$. Then

$$\pi_1(y) = J_{0,1}(\pi_0(f)(\pi_0(a))) = J_{0,1}(\pi_0(f'))(\{l'_1\}) \in M_1[[\{l'_1\}]].$$
CHAPTER 5. STRONG WELL-FOUNDEDNESS IN THE LINEAR CASE

So we can define an elementary map \( \sigma_1 : R_1 \to (V_n)^{N_1} \) by

\[
\sigma_1(y) = (\tau_1 \downarrow (V_n))^{-1}(\pi_1(y)).
\]

Then \( \pi_1 = \tau_1 \circ \sigma_1 \), and \( N_1 = N_1[[\{\sigma_1^m l_i\}]] \).

This is the basic part of the construction. To proceed from \( n \) to \( n + 1 \), we produce a commuting diagram using \( \sigma_n(\pi_n^{-1}(E_n)) \), and in \( N_n \) do the same process we did with \( V \), \( M_1 \) before, but now with \( N_n, \text{Ult}(N_n, \sigma_n(\pi_n^{-1}(E_n))) \):

For each \( 2 \leq n < \omega \), let \( l'_n \) be an enumeration of \( \pi_n^m l_n \). We construct sequences of ultrafilters \( (U_n \mid n \in \omega) \), models \( (N_n \mid n \in \omega) \), elementary embeddings \( \tau_n : N_n \to M_n \mid n \in \omega \), elementary embeddings \( \sigma_n : R_n \to (V_n)^{N_n} \mid n \in \omega \), and elementary embeddings \( i_{p,q} : N_p \to N_q \mid 0 \leq p < q < \omega \) such that \( N_0 = V \), \( U_0 \) is as above, \( \tau_0 \) is the identity, \( \sigma_0 = \pi_0 \), and for \( 0 < n \) and \( 0 \leq p \leq q \leq n \) we have the following properties:

A) \( U_{n-1} \in N_{n-1} \) and \( N_n = \text{Ult}(N_{n-1}, U_{n-1}) \),
B) \( i_{p,q} : N_p \to N_q \) is the canonical embedding,
C) \( N_n \) is the transitive collapse of \( M_n[[\{J_{m,n}({l'}_m) \mid 0 < m \leq n \}]] \), and \( \tau_n \) is the inverse of the collapsing map,
D) \( \sigma_q \circ j_{p,q} = i_{p,q} \circ \sigma_p \),
E) \( \tau_q \circ i_{p,q} = J_{p,q} \circ \tau_p \),
F) \( \pi_n = \tau_n \circ \sigma_n \),
G) \( l'_n \) has a preimage \( l''_n \) under \( \tau_n \), and
H) \( N_n = (N_n[[\{i_{m,n}({l''}_m) \mid 0 < m \leq n \}]])^{i_{0,n}} \).

Assume that we have constructed such objects through stage \( n \): we show how to proceed to stage \( n + 1 \). That is, we produce \( U_n, \tau_{n+1}, \pi_{n+1} \) while maintaining properties (A)-(H).

Let \( Q_{n+1} \) be the (internal) ultrapower of \( N_n \) by \( \sigma_n(e_n) = \tau_n^{-1}(E_n) \). Let \( i'_{n,n+1} : N_n \to Q_{n+1} \) be the ultrapower embedding, and let \( \sigma'_{n+1} : R_{n+1} \to (V_n)^{Q_{n+1}} \), \( \tau'_{n+1} : Q_{n+1} \to M_{n+1} \) be the copy maps. As \( Q_{n+1} \) is \( \omega \)-closed and \( l''_n \) is contained in the range of \( \tau'_{n+1} \), \( l''_{n+1} \) has a preimage under \( \tau'_{n+1} \); call it \( l'''_{n+1} \). Then \( (Q_{n+1})[[\{l'''_{n+1}\}]])^{\tau_{n+1}} \simeq (M_{n+1})[[\{i'(l''_{n+1})\}]]^{\pi_{n+1}} \). Let \( U_{n+1} \) be the ultralimit on \( (\text{SPT}((\sigma(E_n)))^{\omega})^{\omega} \) defined by \( X \in U_n \leftrightarrow l'''_{n+1} \in i'_{n,n+1}(X) \) and let \( N_{n+1} = \text{Ult}(N_n, U_{n+1}) \). Then \( N_{n+1} \) is the transitive collapse of both \( (Q_{n+1})[[\{l'''_{n+1}\}]])^{\tau_{n+1}} \) and \( (M_{n+1})[[\{i'(l''_{n+1})\}]]^{\pi_{n+1}} \), and the canonical embedding from \( N_n \) to \( M_{n+1} \) is the inverse of the transitive collapse. Let \( \tau_{n+1} \) be that embedding, and let \( l''_{n+1} \) be the preimage of \( l''_n \) under \( \tau_{n+1} \). As in the \( n = 1 \) case, \( R_{n+1} \subseteq (Q_{n+1})[[\{l'''_{n+1}\}]]^{\tau_{n+1}} \), so \( R_{n+1} \) embeds elementarily into \( N_{n+1} \) via the inverse of the transitive collapse composed with \( \sigma'_{n+1} \); let \( \sigma_{n+1} \) be that embedding.

Then \( N_{n+1} = (N_{n+1}[[\{l'''_{n+1}\}]])^{i_{n,n+1}} \). So since

\[
N_n = (N_n[[\{i_{m,n}({l''}_m) \mid 0 < m \leq n \}]])^{i_{0,n}} \]

by the middle-man lemma,

\[
N_{n+1} = (N_{n+1}[[\{i_{m,n+1}({l''}_m) \mid 0 < m \leq n + 1 \}]])^{i_{0,n+1}}.
\]
CHAPTER 5. STRONG WELL-FOUNDEDNESS IN THE LINEAR CASE

But
\[ \{ J_{m,n+1}(\{ l'_m \}) \mid 0 < m \leq n + 1 \} = \tau''_{n+1} \{ i_{m,n+1}(\{ l''_m \}) \mid 0 < m \leq n + 1 \}, \]
and so by the pass-through lemma, \( N_{n+1} \) is the transitive collapse of
\[ (M_{n+1}[\{ J_{m,n+1}(\{ l'_m \}) \mid 0 < m \leq n + 1 \}])^{\tau''_{n+1} \circ i_{m,n+1}} = M_{n+1}[\{ J_{m,n+1}(\{ l'_m \}) \mid 0 < m \leq n + 1 \}] \]
as desired.

Define all the other objects so as to make the diagram commutative. Now let \( N_\omega \) be the direct limit of the \( N_n \)'s under the embeddings \( i_{m,n} \). It follows from property (H) that
\[ N_\omega = (N_\omega[\{ i_{m,\omega}(\{ l'_m \}) \mid 0 < m < \omega \}])^{i_{m,\omega}}. \]
But then by the pass-through lemma, \( N_\omega \) is isomorphic to and hence is the transitive collapse of
\[ (M_\omega[\{ J_{m,\omega}(\{ l'_m \}) \mid 0 < m < \omega \}])^{J_{m,\omega}} = (M_\omega[\{ S \cup J_{m,\omega}(\{ l'_m \}) \mid 0 < m < \omega \}])^{J_{m,\omega}} = M_\omega[\{ S \cup J_{m,\omega}(\{ l'_m \}) \mid 0 < m < \omega \}] \]
as desired.

5.2 The transfinite linear case

In this Subchapter we show how to extend the construction of the previous Subchapter.

**Theorem 5.2.** Let \( \tilde{T} = (T, (E_\alpha \mid \alpha < \Omega)) \) be a linear iteration tree on \( V \) with models \( (M_\alpha \mid \alpha \leq \Omega) \) and embeddings \( (J_{\alpha,\beta} \mid \alpha \leq \beta) \), with \( \Omega < \omega_1 \). Assume that for all \( \alpha \in \Omega \), \( M_\alpha \models "E_\alpha \text{ is strongly closed}". Assume further that if \( \alpha < \beta \) then \( \text{SPT}(e_\alpha) \leq \text{SPT}(e_\beta) \), and if \( \beta \) is a limit ordinal,
\[ \text{SPT}(E_\beta) > \sup(\{ J_{\alpha,\beta}(\text{SPT}(E_\alpha)) \mid \alpha < \beta \}). \]
Then \( J_{0,\Omega} \) is strongly well-founded.

**Proof:** Assume the objects and notation from the previous chapter, *mutatis mutandis* (including particularly that \( \Omega + 1 \subseteq X_0 \)).

First assume that \( \Omega = \omega + 1 \). The problem we face is that the models \( M_\omega, N_\omega \) are not \( \omega \)-closed. But the range of \( \sigma_\omega \) can be covered by a set of size less than \( \text{SPT}(\sigma_\omega(e_\omega)) \) and we can use this. Let
\[ \lambda = \sup(\{ i_{\alpha,\omega}(\text{SPT}(\sigma_\alpha(e_\alpha))) \mid \alpha < \omega \}). \]
\( \lambda \) has a pre-image in \( R_0 \) under \( \sigma_\omega \circ j_{0,\omega} \); this pre-image is a continuity point of \( \pi_\omega = \pi_0 \upharpoonright R_\omega \).
Let
\[ F_R = \{ \pi_0(f) \mid f \in R_0 \land f \text{ is a function} \}. \]
Now let \( \Psi = \{ g(a) \mid g \in i_0(\omega(F_R) \land a \in [\! [\lambda] \! ]^{\omega}_\omega) \}. \) It is clear that the range of \( \pi_\omega = \tau_\omega \circ \sigma_\omega \) is included in \( \tau_\omega(\Psi_\omega) \) and that \( \tau_\omega(\lambda) < \text{SPT}(E_\omega) \); so then the range of \( \sigma_\omega \) is included in \( \Psi_\omega \), and
\[
|\Psi_\omega| \leq 80 \times |[[[\lambda]^{\omega}_\omega]| = |\lambda| < \text{SPT}(\sigma_\omega(e_\omega)),
\]
and this also holds in \( N_\omega \).

Now in \( N_\omega \), let \( B'_\omega \) be a Skolem hull in \( (V_\theta)^{N_\omega} \) of \( \Psi_\omega \). Then \( (|B'_\omega|)^{N_\omega} < \text{SPT}(\sigma_\omega(e_\omega)) \) as well and \( \sigma_\omega \) is an elementary embedding from \( R_\omega \) to \( B'_\omega \). Let \( B_\omega \) be the transitive collapse of \( B'_\omega \) and let \( \xi_\omega \) be the inverse of the collapsing embedding.

Now let \( B'_{\omega+1} = \{ x \mid (B'_\omega, \in \cap B'_\omega) = x \in \text{Ult}(V, \sigma_\omega(e_\omega)^{\omega_\omega}) \}. \) Then \( B'_{\omega+1} \in N_\omega \). Let \( B_{\omega+1} \) be the transitive collapse of \( B'_{\omega+1} \), and let \( \xi_{\omega+1} \) be the inverse of the collapsing embedding.

In \( N_\omega \), let \( l''_{\omega+1} \) be an enumeration of the set \( \{ (\iota'_{\omega+1}(x), x) \mid x \in B'_{\omega+1} \} \) of length \( \nu_{\omega+1} = |B'_{\omega+1}|^{N_\omega} < \text{SPT}(\sigma_\omega(e_\omega)) \). \( N_\omega = \text{“} \sigma_\omega(e_\omega) \text{ is strongly closed”} \) so by lemma 1.6, \( l''_{\omega+1} \in Q_{\omega+1} \). We show \( B'_{\omega+1} \subseteq \text{Ult}(Q_{\omega+1}, \{ [l''_{\omega+1}] \})^{\text{SPT}(\xi_{\omega+1})} \); let \( x \in B'_{\omega+1} \). In \( N_\omega \) let \( f \) be defined by the property \( f(X) = y \) iff for some \( \alpha, (\alpha, (x, y)) \in X \). Then \( \iota'_{\omega+1}(f(x))(l''_{\omega+1}) = x \).

In fact, \( B'_{\omega+1} \in (Q_{\omega+1}[[[l''_{\omega+1}]])^{\xi_{\omega+1}} \). Define \( F \) in \( N_\omega \) so that
\[
F(X) = \{ y \mid \exists \alpha, x((\alpha, (x, y)) \in X) \}.
\]

Then \( \iota'_{\omega+1}(F)(l''_{\omega+1}) = B'_{\omega+1} \).

Note also that \( R_{\omega+1} \) embeds into \( B_{\omega+1} \) elementarily via \( (\xi_{\omega+1})^{-1} \circ \alpha'_{\omega+1} \).

Let \( N_{\omega+1} \) be the transitive collapse of \( (Q_{\omega+1}[[[l''_{\omega+1}]])^{\xi_{\omega+1}} \). Let \( \chi_{\omega+1} : N_{\omega+1} \to Q_{\omega+1} \) be the inverse of the collapsing map, and let \( \tau_{\omega+1} = \chi'_{\omega+1} \circ \chi_{\omega+1} \), where \( \tau_{\omega+1} \) is the canonical copy map from \( Q_{\omega+1} \) into \( M_{\omega+1} \). Let \( \xi_{\omega+1} : B_{\omega+1} \to N_{\omega+1} = \chi_{\omega+1}^{-1} \circ \xi_{\omega+1} \), and let
\[
\phi_{\omega+1} : R_{\omega+1} \to B_{\omega+1} = (\xi_{\omega+1})^{-1} \circ \alpha'_{\omega+1}.
\]

Let the range of \( \xi_{\omega+1} \) be denoted by \( B''_{\omega+1} \). \( B''_{\omega+1} \) is in \( N_{\omega+1} \), being equal to \( \chi_{\omega+1}^{-1}(B_{\omega+1}) \). Moreover, and as is crucial in order to proceed,
\[
N_{\omega+1} = \text{“} |B''_{\omega+1}| < \text{SPT}(\sigma_\omega(e_\omega)) \text{”}.
\]

Otherwise,
\[
M_{\omega+1} = \tau'_{\omega+1} \circ \chi_{\omega+1}(\chi_{\omega+1}^{-1}(\nu_{\omega+1})) \geq \text{SPT}(E_{\omega+1}),
\]
but \( \tau_{\omega+1} = \tau_\omega \downarrow Q_{\omega+1} \) so we would also then have that \( \tau_\omega(\nu_{\omega+1}) \geq \text{SPT}(E_{\omega+1}) \), which violates our condition that the supports of the extenders used are nondecreasing, since \( \nu_{\omega+1} \leq \text{SPT}(\sigma_\omega(e_\omega)) \).

So now the situation is reset, and we can repeat the process at each succeeding successor. At the next limit stage, we create a new \( B \) model. Let us lay out the blueprint for the full construction. The inductive proof of its validity proceeds exactly in line with the arguments above.
CHAPTER 5. STRONG WELL-FOUNDEDNESS IN THE LINEAR CASE

We recursively construct sequences of models

\[ \{ N_\alpha \mid \alpha \leq \Omega \} \]

and

\[ \{ B_\alpha \mid \omega \leq \alpha \leq \Omega \} \]

and embeddings

\[ \{ \tau_\alpha : N_\alpha \rightarrow M_\alpha \mid \alpha \leq \Omega \}, \]
\[ \{ \sigma_\alpha : R_\alpha \rightarrow N_\alpha \mid \alpha \leq \Omega \}, \]
\[ \{ \phi_\alpha : R_\alpha \rightarrow B_\alpha \mid \omega \leq \alpha \leq \Omega \}, \]

and

\[ \{ \xi_\alpha : B_\alpha \rightarrow N_\alpha \mid \omega \leq \alpha \leq \Omega \} \]

with the following properties:

A) \( \xi_\alpha \circ \phi_\alpha = \sigma_\alpha \)
B) \( \tau_\alpha \circ \sigma_\alpha = \pi_\alpha \)
C) If \( \alpha = \beta + 1 \) for some \( \beta \), then \( B_\alpha \) is the transitive collapse of

\[ B'_\alpha = \{ x \mid (B_\beta, e \cap B_\beta) = \text{Ult}(V, \phi_\beta(e_\beta)) \} \]

and \( N_\alpha \) is the transitive collapse of \( \text{Ult}(N_\beta, \sigma_\beta(e_\beta))[[l''_\alpha]]^{i'_{\beta, R+1}} \), where \( l''_\alpha \) is an enumeration in \( N_\beta \) of

\[ \{ (i'_{\beta, R+1}(x), x) \mid x \in B'_\omega \} \]

of length \( u_{\beta+1} = |B'_{\beta+1}|^{N_\beta} < \text{SPT}(\sigma_\beta(e_\beta)) \). \( N_\beta \) is “\( \sigma_\beta(e_\beta) \) is strongly closed” so by lemma 1.6, \( l''_{\beta+1} \in Q_{\beta+1} = \text{Ult}(N_\beta, \sigma_\beta(e_\beta)) \). We let \( \chi_\alpha : N_\alpha \rightarrow Q_\alpha \) invert the transitive collapse. Furthermore \( B'_\alpha \subseteq (Q_\alpha[[l''_\alpha]])^{i'_{\beta, R+1}}, B'_\alpha \in (Q_\alpha[[l''_\alpha]])^{i'_{\beta, R+1}}, \) and \( N_\alpha = \{ \xi_\alpha(x) \mid x \in B_\alpha \} < \text{SPT}(\sigma_\alpha(e_\alpha)) \). The last statement follows from the above, as \( \tau'_\alpha = \tau_\beta \upharpoonright Q_\alpha \).

D) If \( \alpha \) is a limit ordinal, \( N_\alpha \) is the direct limit of the models \( \{ N_\beta \mid \beta < \alpha \} \). Furthermore, letting \( i_{\beta, \alpha} : N_\beta \rightarrow N_\alpha \) be the canonical embedding and letting

\[ \lambda_\alpha = \sup(\{ i_{\beta, \alpha}(\text{SPT}(\sigma_\beta(e_\beta))) \mid \beta < \alpha \} ), \]

\( B_\alpha \) is the transitive collapse of a skolem hull in \( (V_\eta)^{N_\alpha} \) of

\[ \{ g(a) \mid g \in i_{0, \alpha}(F_R) \land a \in [\lambda_\alpha]^{<\omega} \} . \]

\( \sigma_\alpha \) is the direct limit of \( \{ \sigma_\beta \mid \beta < \alpha \} \), \( \tau_\alpha \) is the direct limit of \( \{ \tau_\beta \mid \beta < \alpha \} \), and \( \xi_\alpha : B_\alpha \rightarrow N_\alpha \) is the inverse of the transitive collapse.

If \( \Omega \) is a limit ordinal it follows, as in the above Subchapter, that

\[ N_\Omega = (N_\Omega[[\{ i_{\beta+1, \Omega}^{-1}(l''_{\beta+1}) \} \mid 0 \leq \beta < \Omega]})^{i_{0, \Omega}}, \]
and so $N_\Omega$ is the transitive collapse of

$$
(M_\Omega[[\{ J_{\beta+1,\Omega}(\{ \tau_{\beta+1}(x_{\beta+1}^{-1}(i_{\beta+1}^{\mu})) | 0 \leq \beta < \Omega \}) \}])^{J_0,\Omega}
$$

$$
= (M_\Omega[[S \cup \{ J_{\beta+1,\Omega}(\{ \tau_{\beta+1}(x_{\beta+1}^{-1}(l_{\beta+1}^{\mu})) | 0 \leq \beta < \Omega \}) \}])^{J_0,\Omega}.
$$

Similarly if $\Omega$ is a successor, it follows that $N_\Omega$ is the transitive collapse of

$$
(M_\Omega[[\{ J_{\beta+1,\Omega}(\{ \tau_{\beta+1}(x_{\beta+1}^{-1}(i_{\beta+1}^{\mu})) | 0 \leq \beta < \Omega \}) \}])^{J_0,\Omega}
$$

$$
= (M_\Omega[[S \cup \{ J_{\beta+1,\Omega}(\{ \tau_{\beta+1}(x_{\beta+1}^{-1}(l_{\beta+1}^{\mu})) | 0 \leq \beta < \Omega \}) \}])^{J_0,\Omega}.
$$

So we are done. \qed
Chapter 6

Strong well-foundedness in the non-linear case

Now we proceed to our main objective, which is to show that if $\tilde{T}$ is a strongly closed iteration tree on $V$ of length $\omega + 1$ with a unique well-founded branch of length $\omega$, then its final model is strongly well-founded. Assume throughout that there is a proper class of inaccessible cardinals in $V$. In the following, if $\varphi_1, \varphi_2$ are maps with domains $N_1, N_2$, respectively, then we say that $\varphi_1$ and $\varphi_2$ agree through $\xi$ iff $\varphi_1\upharpoonright (V_\xi)^{N_1} = \varphi_2\upharpoonright (V_\xi)^{N_2}$. Similarly, we say two models $N_1, N_2$ agree through $\xi$ iff $(V_\xi)^{N_1} = (V_\xi)^{N_2}$.

Theorem 6.1. Let $\tilde{T} = (T, \langle E_n | n < \omega \rangle)$ be an iteration tree on $V$ with models $\langle M_n | n \leq \omega \rangle$ and embeddings $\langle J_{n,m} | n T m \leq \omega \rangle$. Assume that for all $n \in \omega$, $M_n \models " E_n \text{ is strongly closed} "$, and that $T \upharpoonright \omega$ has a unique cofinal well-founded branch. Then $J_{0,\omega}$ is strongly well-founded.

Proof:

Let

$$\tilde{T} = (T, \langle E_n | n < \omega \rangle).$$

Let the models of $\tilde{T}$ be denoted $\langle M_n | n \leq \omega \rangle$, and let the embeddings of $\tilde{T}$ be denoted $\langle J_{n,m} | n T m \leq \omega \rangle$. For $n \in \omega$ let $n^*$ denote the unique predecessor of $n + 1$ in the tree order. Since $\tilde{T} \upharpoonright \omega$ has a unique well-founded branch of length $\omega$, it is continuously ill-founded off this branch. Let $\langle \gamma_n | n \in \omega \rangle$ witness this. Let $f$ be a function defined on $\gamma_0 + 1$ such that $f$ is a strictly increasing function with only limit ordinals in the range. Let $\theta_n = J_{0,n}(f)(\gamma_n)$. Then $\langle \theta_n | n \in \omega \rangle$ is also a witness to continuous ill-foundedness, and all the ordinals $\theta_n$ are limit ordinals.

Let $S \subseteq M_\omega$ be countable. Let $\eta$ be an inaccessible cardinal such that $\tilde{T}, S, \langle \theta_n | n \in \omega \rangle \in V_\eta$. Let $R_0$ be the transitive collapse of an elementary substructure of $V_\eta$ containing $\tilde{T}, S$, and $\langle \theta_n | n \in \omega \rangle$. Let $\pi_0$ be the inverse of the collapsing map, $\pi_0 : R_0 \to V_\eta$. Then $\tilde{T}$ is the tree we get by copying $\pi_0^{-1}(\tilde{T})$; let $\langle e_n | n \in \omega \rangle$ denote the extenders of the latter tree, let $\langle R_n | n \leq \omega \rangle$ be its models, and let $\langle j_{n,m} | n T m \rangle$ be its embeddings.

We aim to insert between $\tilde{T}$ and $\pi_0^{-1}(\tilde{T})$ something like a $g$-enlargement of $\pi_0^{-1}(\tilde{T})$. We will call it an insertion. It will consist of a sequence of models and vertical embeddings from
the \(R\) sequence to it and from it to the \(M\) sequence, but horizontal embeddings will only exist between inserted models indexed by numbers on the main branch of \(T\). The models of this insertion will all be \(\omega\)-closed, and the insertion will have one cofinal branch which is isomorphic to an internal linear iteration of \(V\) by \(\omega\)-closed extenders. We will then be able to use our result about such trees to achieve the desired result.

More precisely we construct a sequence of models \(\{P_n \mid n \leq \omega\}\) where if \(n \notin [0, \omega)_T\) then \(P_n \in P_{n-1}\), sequences of embeddings \(\alpha_n : R_n \rightarrow (V_{\alpha_n})^{P_n} \mid n \leq \omega\), \(\beta_n : P_n \rightarrow (V_{\beta_n})^{M_n} \mid n \leq \omega\)

(with the ordinals \(\alpha_n\) and \(\beta_n\) yet to be defined, and the ordinals \(\beta_n\) only existing off the main branch, so effectively \(\beta_n = \text{Ord}\) for \(n\) on the main branch) and embeddings \(i_{n,m} : P_n \rightarrow P_m\) defined when \(n \leq m \in [0, \omega)_T\), and extenders \(\{e'_n \mid n \in \omega\} = \{\sigma_n(e_n) \mid n \in \omega\} = \{\tau_n^{-1}(E_n) \mid n \in \omega\}\), such that, for all \(n \in \omega\), the following hold:

A) For all \(p \leq r \leq q \leq n\), if \(p, r, q \in [0, \omega)_T\), \(i_{p,q} : P_p \rightarrow P_q = i_{r,q} \circ i_{p,r}\),
B) For all \(p \leq q \leq n\), if \(i_{p,q}\) exists, \(\sigma_q \circ j_{p,q} = i_{p,q} \circ \sigma_p\),
C) For all \(p \leq q \leq n\), if \(i_{p,q}\) exists, \(\tau_q \circ i_{p,q} = j_{p,q} \circ \tau_p\),
D) \(\tau_n = \alpha_n \circ \sigma_n\),
E) If \(0 < n\), \(P_{n-1}\) and \(P_n\) agree through \(\text{STR}(e'_n)\),
F) If \(0 < n\), \(\tau_{n-1}\) and \(\tau_n\) agree through \(\text{STR}(e'_n)\),
G) If \(0 < n\), \(\tau_{n-1}\) and \(\tau_n\) agree through \(\text{STR}(e_n)\),
H) \(P_n \models \text{“there are in order type at least } \tau_n^{-1}(\theta_n)\text{ many inaccessibles greater than } \alpha_n\”\),
I) If \(n \notin [0, \omega)_T\), \(P_n \in P_{n-1}\), and
J) \(P_n\) is closed under \(\omega\)-sequences.

In the above and the following, \(\text{STR}(e_n)\) denotes the strength of \(e_n\) as computed in \(R_n\), \(\text{STR}(e'_n)\) denotes the strength of \(e'_n\) as computed in \(P_n\), and \(\text{STR}(E_n)\) denotes the strength of \(E_n\) as computed in \(M_n\).

Now to the construction. Let \(P_0 = V\), \(\alpha_0 = \eta\), \(\sigma_0 = \pi_0\), and \(\tau_0 = \text{id}\). Then all our conditions are satisfied at this stage.

Now assume that we have constructed the models and embeddings as above through stage \(n\); we show what to do next.

First we take the ultrapower of \(P_n\) by \(e'_n\). Let us show that this is possible.

\(P_n^*\) and \(P_n\) agree through \(\text{min}(\{\text{STR}(e'_m) \mid n^* \leq m < n\})\). Let this minimum be attained at \(m\). We have that \(\sigma_n^*, \sigma_m,\) and \(\sigma_n\) agree through \(\text{STR}(e_m)\). \(\text{SPT}(e_n) + 1 \leq \text{STR}(e_m)\), but since \(\text{STR}(e_m)\) is a limit ordinal, the inequality is strict. Since \(\text{SPT}(e_n) + 1 \in R_m\) we can apply \(\sigma_m\) to \(\text{SPT}(e_n) + 1\) and get that \(\sigma_m(\text{SPT}(e_n) + 1) < \sigma_m(\text{STR}(e_m)) = \text{STR}(e'_m)\). But since \(\text{SPT}(e_n) + 1 \leq \text{STR}(e_m)\) we have that \(\sigma_m(\text{SPT}(e_n) + 1) = \text{SPT}(e_n) + 1 = \text{SPT}(e'_n)\). So the hypotheses of the shift lemma (page 148 of [4]) are met.

And so indeed \(\text{Ult}(P_{n^*}, e'_n)\) exists. Call it \(Q_{n+1}\), and let \(i'_{n^*+1} : P_{n^*} \rightarrow Q_{n+1}\) be the ultrapower embedding. Since \(P_{n^*}\) and \(P_n\) agree through \(\text{SPT}(e'_n) + 1\), \(\text{Ult}(P_n, e'_n)\) and \(Q_{n+1}\) agree through \(\text{STR}(e'_n) + 1\). But \(P_n \models \text{“} e'_n \text{ is strongly closed”} \), so \(P_n\) and \(\text{Ult}(P_n, e'_n)\) agree through \(\text{STR}(e'_n)\). So \(P_n\) and \(Q_{n+1}\) agree through \(\text{STR}(e'_n)\). Furthermore the copy map \(\tau'_{n+1} : Q_{n+1} \rightarrow M_{n+1}\) agrees with \(\tau_n\) through \(\text{STR}(e'_n)\). The existence of \(\tau'_{n+1}\) shows that \(Q_{n+1}\)
is well-founded. Let $\sigma' : R_{n+1} \to (V_{\alpha^*_n,n+1}(\alpha_n))^Q_{n+1}$ be the canonical copy map. Then $\sigma_n, \sigma'$ agree through STR$(e_n)$.

We also have that $Q_{n+1}$ is countably closed; since $P_n$ is $\omega$-closed, every $\omega$-sequence of the generators of $e'_n$ is in $P_n$. As $P_n = \text{"e}'_n$ is strongly closed", every such sequence is also in Ult$(P_n, e'_n)$, and therefore in $Q_{n+1}$. Then since $P_{n+1}$ is $\omega$-closed, so is $Q_{n+1}$.

As $P_{n+1} = \text{"e}'_n$ there are in order type at least $\tau_{n+1}^{-1}(\theta_{n+1})$ many inaccessibles greater than $\alpha_n$. $\forall n+1 \Rightarrow \exists \tau_{n+1} \forall \theta_{n+1} \in (\tau_{n+1})^{-1}(\theta_{n+1})$ many inaccessibles greater than $\theta$.

So since $(\tau_{n+1})^{-1}(J_{\alpha^*_n,n+1}(\theta_{n+1}))$ is a limit ordinal greater than or equal to $(\tau_{n+1})^{-1}(\theta_{n+1})$ (depending on whether or not $n+1$ is on the main branch), $Q_{n+1} = \text{"e}'_n$ there are in order type at least $(\tau_{n+1})^{-1}(\theta_{n+1})$ many inaccessibles greater than $\theta$.

Note that since $\theta_{n+1}$ is in the range of $\pi_{n+1}$ it has a preimage under $\tau_{n+1}$.

To proceed further we need to look at the two cases:

Case 1: $n+1 \in [0, \omega]_T$. In this case set $P_{n+1} = Q_{n+1}$, and let $i_{n^*,n+1} = i_{n^*,n+1} = i_{n^*,n+1}$. For all $mTn^*$, set $i_{m^*,n+1} = i_{n^*,n+1} \circ i_{m^*,n}$. Let $\sigma_{n+1} = \sigma_{n+1}'$, let $\tau_{n+1} = \tau_{n+1}'$, and let $\alpha_{n+1} = i_{n^*,n+1}(\alpha_n)$.

So in the case that $n+1 \in [0, \omega]_T$, (A)-(J) are maintained. Note that $P_{n+1}$ is formed by taking an internal ultrapower (if $n^*$ is $n$ this is clear, otherwise $e'_n \in P_n \in P_{n-1} \in \ldots \in P_n$).

Case 2: $n+1 \notin [0, \omega]_T$. In this case, we will form a countably-closed substructure of a rank $\alpha$-sequence of $Q_{n+1}$ and let $P_{n+1}$ be its transitive collapse. The rank will be specified by the witness $(\theta_n : n \in \omega)$. Note that since $n+1$ is off the main branch, $(\tau_{n+1})^{-1}(J_{\alpha^*_n,n+1}(\theta_{n+1}))$ is a limit ordinal strictly greater than $\alpha$. So $Q_{n+1} = \text{"e}'_n$ there are in order type at least $(\tau_{n+1})^{-1}(\theta_{n+1}) + 1$ many inaccessibles greater than $\theta$.

Let $\theta'$ be that ordinal $\alpha$, which $Q_{n+1}$ specifies as being the inaccessible such that the order type of the inaccessibles greater than $\theta$ is $(\tau_{n+1})^{-1}(\theta_{n+1})$ and less than $\alpha$ is $(\tau_{n+1})^{-1}(\theta_{n+1})$.

Since $Q_{n+1}$ is countably closed, $\sigma' \in Q_{n+1}$, and in fact $\sigma' \in (V_{\theta'})_{n+1}$. Let $B = (V_{\text{STR}(e'_n)^{Q_{n+1}}} \supset \{\sigma, i_{n^*,n+1}(\alpha_n), (\tau_{n+1})^{-1}(\theta_{n+1})\})$.

Working in $Q_{n+1}$ we can form a countably closed elementary substructure $X$ of $(V_{\theta'})_{n+1}$ of size $(|V_{\text{STR}(e'_n)^{Q_{n+1}}}|)Q_{n+1}$ such that $B \subset X$; as

$$Q_{n+1} \models \text{"e}'_n \in [0, \omega]_T \Rightarrow \exists [B, |B|, |B|, |B|, \omega|, |B|, |B|, \omega_1 = (V_{\text{STR}(e'_n)^{Q_{n+1}}}||B|)$$

this is possible. Since $Q_{n+1}$ is closed under $\omega$-sequences, so is $X$.

Note that $X$ = “there are in order type at least $(\tau_{n+1})^{-1}(\theta_{n+1})$ many inaccessibles greater than $\theta$. Furthermore, $R_{n+1}$ embeds elementarily into $(V_{\alpha^*_n,n+1}(\alpha_n))^X$ via $\sigma'$.

Let $P_{n+1}$ be the transitive collapse of $X$. Let $\varepsilon$ invert the transitive collapse. Let $\beta_{n+1} = \tau_{n+1}(\theta')$. Let $\alpha_{n+1} = \varepsilon^{-1}(i_{n^*,n+1}(\alpha_n))$. Set $\tau_{n+1} : P_{n+1} \rightarrow (V_{\beta_{n+1}})^{M_{n+1}} = \tau_{n+1} \circ \varepsilon$, and let $\sigma_{n+1} : R_{n+1} \rightarrow (V_{\alpha_{n+1}})^{P_{n+1}} = \varepsilon^{-1} \circ \sigma'$. Since $X$ is closed under $\omega$-sequences, so is $P_{n+1}$.

As

$$Q_{n+1} \models |P_{n+1}|^{Q_{n+1}} = \text{STR}(e'_n),$$

$P_{n+1}$ is coded by a subset of $(V_{\text{STR}(e'_n)^{Q_{n+1}}}$. But $Q_{n+1}$ and Ult$(P_n, e'_n)$ agree through STR$(e'_n)$, so then

$$P_{n+1} \in \text{Ult}(P_n, e'_n).$$
and so \( P_{n+1} \in P_n \) as required.

It is also true that \( P_{n+1} \) and \( P_n \) agree through \( \text{STR}(e'_n) \), since \( P_n, \text{Ult}(P_n, e'_n) \) do, as do \( \text{Ult}(P_n, e'_n), Q_{n+1}, \text{and } Q_{n+1}, P_{n+1} \). Since \( \tau_{n+1} \) agrees with \( \tau'_{n+1} \) through \( \text{STR}(e'_n) \), and \( \tau_n \) and \( \tau'_{n+1} \) agree through \( \text{STR}(e'_n) \), \( \tau_n \) and \( \tau_{n+1} \) agree through \( \text{STR}(e'_n) \) as well.

Since \( \sigma_n, \sigma' \) agree through \( \text{STR}(e_n) \), so do \( \sigma_n, \sigma_{n+1} \). Furthermore, \( P_{n+1} \models \text{"there are in order type at least } (\tau_{n+1})^{-1}(\theta_{n+1}) \text{ many inaccessibles greater than } \alpha_{n+1}.\)"

So (A)-(J) are maintained. This completes the construction for \( n < \omega \).

Let \( P_\omega \) be the direct limit of the models \( \langle P_n \mid n \in [0, \omega) \rangle \) under the embeddings \( \langle i_{n,m} \mid 0 \leq n \leq m \in [0, \omega) \rangle \), and let \( \sigma_\omega, \tau_\omega \) be defined in the canonical way; then \( \tau_\omega \circ \sigma_\omega = \pi_\omega. \) So since \( S \) is included in the range of \( \pi_\omega, S \) is included in the range of \( \tau_\omega; \) let \( Y \) be the set of pre-images of its elements under \( \tau_\omega. \)

For \( n^*, n + 1 \) on the main branch, since \( P_{n+1} = \text{Ult}(P_n, e'_n) \) is \( \omega \)-closed and \( e'_n \in P_{n^*}, P_{n^*} \models \text{"e'} \text{ is a countably closed extender on } V." \) So we can define a linear internal iteration of \( V \) by countably closed extenders by letting its models and extenders be defined from the \( P_n, e_n \) sequences in the obvious way. The final model on this linear tree is of course \( P_\omega \), where \( P_\omega \) is as above, and the canonical embedding from \( V \) into \( P_\omega \) in this tree equals \( i_{0,\omega}. \)

Then by our earlier result, \( (i_{0,\omega}, P_\omega) \) is strongly well-founded, and so there is some \( Y' \supseteq Y \) such that the transitive collapse of \( P_\omega[[Y']] \) is an internal iteration of \( V \) by ultrafilters. Then \( S \subseteq \tau''_\omega Y'. \) But by the pass-through lemma, the transitive collapse of \( M_\omega[[\tau''_\omega Y']] \) and the transitive collapse of \( P_\omega[[Y']] \) are equal, and so we are done.

What we are in effect doing in the above construction is using a countable system as an anchor to make sure that the linear iteration we produce embeds into the original tree along the main branch, in such a way that \( S \) is included in the range of the final upward embedding. Then we jettison this system and use a new one to complete the construction from Chapter 3, but of course with our linear tree, not \( \tilde{T}. \) Then the final model of this tree, \( P_\omega \), includes a preimage of \( S \), and since \( J_{0,\omega} = \tau_\omega \circ i_{0,\omega}, \) we can push this instance of strong well-foundedness up to \( (J_{0,\omega}, M_\omega). \)
Chapter 7

An instance of the Genericity Hypothesis

In this Chapter we show that the construction above also leads to genericity over $M_\omega$ of every countable subset of $M_\omega$:

**Theorem 7.1.** Let $\tilde{T} = (T, \{ E_n \mid n < \omega \})$ be an iteration tree on $V$ with models $\{ M_n \mid n \leq \omega \}$. Assume that for all $n \in \omega$, $M_n = \text{"E}_n$ is strongly closed", and that $T$ has a unique well-founded branch of length $\omega$. Let $S \subseteq M_\omega$ be countable. Then $S$ is generic over $M_\omega$.

Before we proceed to the proof of Theorem 7.1, we need a technical lemma which will enable us to adapt the proof of Theorem 3.1 to this more complex situation. Let $\varsigma : \omega \to [0, \omega)_T$ be the strictly increasing bijection. For $n \in \omega$, let $P'_n = P_{\varsigma(n)}$, and let $\iota_n : P'_n \to P_{\varsigma(n)}$ be the identity. Let $\{ N_n \mid n \in \omega \}$ be the internal iteration by ultrafilters we get by applying the construction from Chapter 4.1 to $\{ P'_n \mid n \in \omega \}$ and $Y$; as at the end of Chapter 3 let $Y' \supseteq Y$ be countable, $Y' \subseteq P'_\omega$ such that the transitive collapse of $P'_\omega[[Y']]$ is $N_\omega$. For $i \leq k \leq \omega$ let $h_{i,k}$ be the canonical map $N_i \to N_k$, and for $n \leq \omega$ let $\phi_n : N_n \to P'_n$ be the map specified by the construction. Let $F, X$ be as above, constructed relative to the models $N_n$ and the ultrafilters $U_n$ (which are defined mutatis mutandis). So to recap, the embeddings between the $M$ models are labelled by $J$, the embeddings between the $P$ models will not be needed, the embeddings between the $P'$ models will be labelled by $i$, the embeddings between the $N$ models are labelled by $h$, and the embeddings between the (new) $R$ models will be labelled $j$. It is important to note, however, that the $R$ models we are using here are the second set produced. Furthermore let the upward embeddings be labelled as follows: $\sigma_n : R_n \to N_n$, $\phi_n : N_n \to P'_n$, $\iota_n : P'_n \to P_{\varsigma(n)}$, and $\tau_{\varsigma(n)} : P_{\varsigma(n)} \to M_{\varsigma(n)}$. Let $\pi_n : R_n \to M_{\varsigma(n)}$. The same extender is indexed differently with respect to the insertion and the linear iteration; we will notate this using superscripts, e.g. $E'_{n_\upsilon} = E_{\varsigma(n_\upsilon+1)-1}$. We will refer to extenders of the $M$ sequence as elements of the range of a $\tau$ embedding, e.g. $\tau_m(E'_m)$ and as $E'_{n_\upsilon}^M$.

For each $n < \omega$ let $l'_{n+1} \in P'_{n+1}$ be the enumeration of $(\phi_{n+1} \circ \sigma_{n+1})''(\text{STR}(e_n))$ used in the construction of the $N$ sequence. For $n < \omega$ let $l'_{\varsigma(n)} = (\tau_{\varsigma(n)} \circ \iota_n)(l'_n)$. We show that

\[
\{ J_{\varsigma(n), \omega}(l'_{\varsigma(n)}) \mid 0 < n < \omega \}
\]
is generic for $J_{0,\omega}(F)$ (remember that $N_0 = P'_0 = P_0 = M_0 = V$) over $M_\omega$. Say $l'_{n+1} = [a, f]_{E_k^{P'_n}}$. Then $l^M_{(n+1)\varsigma} = [\tau_{(n+1)\varsigma}(a), \tau_{(n)\varsigma}(f)]_{E_k^{P'_n}}$. The problem we have here in applying the proof of the above Subchapter is that $F$ is defined using ultrafilters defined using extenders from the linear tree, but getting from one model on the main branch of the $M$ sequence to the next is not effected by applying the push-up of the corresponding extender in the insertion. This will be made more clear in the proof of the following:

**Lemma 7.2.** Let

$$W \in J_{0,\varsigma(n)}(F)(( J_{\varsigma(m),\varsigma(n)}(l^M_{\varsigma(m)}))\ 0 < m \leq n)).$$

Then $l^M_{\varsigma(n+1)} \in J_{\varsigma(n),\varsigma(n+1)}(W)$.

**Proof:**

This lemma confronts the fact that $\tau_{\varsigma(n)} \circ \iota_n(E_n^{P'}) \neq E_{\varsigma(n+1)\varsigma}^{M}$. Let us start down in the $P'$ and work our way up.

$$P'_n \models \forall Z(Z \in (i_{0,n}(F))(( i_{k,n}(l'_k)|0 < k \leq n)) \implies \{ y \in (\text{SPT}(E_n^{P'}))[a] | f(y) \in Z \} \in (E_n^{P'})_a) .$$

Now, for all $k \leq n$, $\tau_{\varsigma(n)} \circ \iota_n \circ i_{k,n} = J_{\varsigma(k),\varsigma(n)}(\tau_{\varsigma(k)} \circ \iota_k)$, so

$$M_{\varsigma(n)} \models \forall Z(Z \in (J_{0,\varsigma(n)}(F))(( \tau_{\varsigma(k),\varsigma(n)}(l^M_{\varsigma(k)}))|0 < k \leq n)) \implies \{ y \in (\text{SPT}(\tau_{\varsigma(n)}(f)) \circ \iota_n(E_n^{P'}))[\tau_{\varsigma(n)}(a)] | (\tau_{\varsigma(n)}(f))(y) \in Z \}

\epsilon \tau_{\varsigma(n)}(\circ \iota_n((E_n^{P'}))a)) .$$

So

$$\{ y \in (\text{SPT}(\tau_{\varsigma(n)}(f)) \circ \iota_n(E_n^{P'})) | (\tau_{\varsigma(n)}(f))(y) \in W \}

\epsilon \tau_{\varsigma(n)}(\circ \iota_n((E_n^{P'}))a)) .$$

Note that $E_n^{P'} = E_{\varsigma(n+1)\varsigma}^{P}$. But $\tau_{\varsigma(n)}$ and $\tau_{\varsigma(n+1)\varsigma}$ agree past

$$\text{SPT}(E_{\varsigma(n+1)\varsigma}^{P}) + 2,$$

so

$$\tau_{\varsigma(n)} \circ \iota_n((E_n^{P'}))a) = \tau_{\varsigma(n+1)\varsigma}((E_{\varsigma(n+1)\varsigma}^{P})a).$$

Call this ultrafilter $U$. $\tau_{\varsigma(n)}$ and $\tau_{\varsigma(n+1)\varsigma}$ do not necessarily agree on $a$; while

$$M_{\varsigma(n)} \models U = (\tau_{\varsigma(n)}(E_n^{P'}))_{\tau_{\varsigma(n)}(a)};$$

on the other hand

$$M_{\varsigma(n+1)\varsigma} \models U = (\tau_{\varsigma(n+1)\varsigma}(E_{\varsigma(n+1)\varsigma}^{P}))_{\tau_{\varsigma(n+1)\varsigma}(a)}.$$
CHAPTER 7. AN INSTANCE OF THE GENERICITY HYPOTHESIS

But it is certainly true that

\[
(SPT(\tau_s(n) \circ t_n(E^P_n)))^{\tau_s(n) \circ t_n(a)} = (SPT(\tau_s(n+1) - 1(E^P_{\varsigma(n+1)-1})))^{\tau_s(n+1)-1(a)}.
\]

Of course \(\tau_s(n+1)-1(E^P_{\varsigma(n+1)-1}) = E^M_{\varsigma(n+1)-1}\). So we have that

\[
\left\{ y \in (SPT(E^M_{\varsigma(n+1)-1}))^{\tau_s(n+1)-1(\alpha)} \big| (\tau_s(n) \circ t_n(f))(y) \in W \right\}
\]

\(\in (E^M_{\varsigma(n+1)-1})^{\tau_s(n+1)-1(\alpha)}\),

i.e.

\[
l^M_{\varsigma(n+1)} = \left[ \tau_s(n+1)(\alpha), \tau_s(n)(f) \right]^{M(\alpha)}_{E^M_{\varsigma(n+1)-1}} \in J_{\varsigma(n),\varsigma(n+1)}(W).
\]

We can now give the proof of Theorem 7.1:

**Proof of Theorem 7.1:**

With Lemma 7.2 in hand, we can easily adapt the proof of Theorem 3.1 to show that

\(J_{\varsigma(n),\omega}(l^M_{\varsigma(n)})\) is generic over \(M_\omega\) for \(J_{0,\omega}(F)\) and so \(S\) is generic over \(M_\omega\), as desired.
Chapter 8

Genericity from nice strong well-foundedness

The above genericity result relies strongly on the assumption that $\bar{T} \upharpoonright \omega$ has a unique cofinal well-founded branch, and any extension of the constructions above would probably require a similar assumption at every limit stage as in [2]. Moreover, genericity has been obtained here through careful examination of the proof of strong well-foundedness.

It may be more useful, however, to demonstrate that whenever a particular strengthening of strong well-foundedness holds, genericity follows abstractly. We would then have a property which is possibly easier to establish as holding for branch embeddings of relevant trees at every limit ordinal than uniqueness of the well-founded branch cofinal in that ordinal, and which implies the very useful property of genericity. Here is an example of such a property, in the length $\omega + 1$ case:

**Definition 8.1. Nice tracking:** Let $\bar{T} = (T, \{E_\alpha | \alpha < \omega\})$ be an iteration tree with models $\langle M_n | n \leq \omega \rangle$ and embeddings $\langle j_{n,m} | nTm \leq \omega \rangle$ such that $j_{0,\omega}$ is strongly well-founded. We say that the strong well-foundedness of $\bar{T}$ *tracks nicely* if, for each countable subset $Y$ of $M_\omega$, there is a countable subset $X \supseteq Y$ such that the transitive collapse of $M_\omega[[X]]$ is an internal iteration of $V$ by ultrafilters such that, letting $\langle N_n | n \leq \omega \rangle$ be the models and $\langle i_{n,m} | n \leq m \leq \omega \rangle$ be the embeddings of this internal iteration, and letting $\tau$ be the embedding from $N_\omega$ into $M_\omega$ we get from strong well-foundedness (that is, $\tau$ is the isomorphism between $N_\omega$ and $(M_\omega[[X]])^{m_\omega}$), and letting $\lambda$ be the supremum of $\{ j_{n+1,\omega}(LTH(E_n)) | n + 1 \in [0, \omega]_T \}$, then if $\kappa$ is the liminf of $\{ CRT(i_{n,n+1}) | n \in \omega \}$, $\tau(\kappa) > \lambda$.

We show this property implies the genericity hypothesis for strongly closed, strongly well-founded trees of length $\omega + 1$. We first point out that in the case of a tree not of length $\omega + 1$, the natural definition of nice-tracking would be more general; the linear interal iteration by ultrafilters need not be of the same length as the branch embedding of the tree. However the linear internal iteration should emulate a given extender’s contribution to the countable subset of the final model by using only finitely many ultrafilters.
Theorem 8.2. Let \( \hat{T} = (T, \{ E_n \mid n < \omega \}) \) be a strongly closed, strongly well-founded iteration tree on \( V \) with models \( \{ M_n \mid n \leq \omega \} \) and embeddings

\[
\{ j_{n,m} \mid nTm \leq \omega \},
\]
such that the strong well-foundedness tracks nicely, and the associated internal iteration of \( V \) by ultrafilters has length \( \omega + 1 \). Then every countable subset of \( M_\omega \) is generic over \( M_\omega \).

Proof: Let \( \lambda \) be the supremum of \( \{ j_{n+1,\omega}(LTH(E_n)) \mid n + 1 \in [0, \omega)_T \} \). Let \( Y \) be a countable subset of \( M_\omega \), and let \( X \supseteq Y \cup \{ \lambda \} \) be countable such that the transitive collapse of \( M_\omega[[X]] \) equals the internal iteration of \( V \) of length \( \omega \) by ultrafilters \( \{ U_n \mid n \in \omega \} \), with the latter iteration witnessing nice tracking. Let \( \langle N_n \mid n \leq \omega \rangle \) be the models of the ultrapower iteration and let \( \{ i_{n,m} \mid n \leq m \leq \omega \} \) be its embeddings. For \( n < \omega \), let \( \kappa_n \) be the critical point of \( i_{n,n+1} \). Let \( \tau_\omega \) be the embedding from \( N_\omega \) into \( M_\omega \) we get from strong well-foundedness (that is, \( \tau_\omega \) is the isomorphism between \( N_\omega \) and \( (M_\omega[[X]])^{j_\omega} \)). Let \( k \) be the liminf of \( \{ \kappa_n \mid n \in \omega \} \). Then \( \tau_\omega(k) \succ \lambda \).

For \( n \geq 1 \), let \( k_n \) be the kernel of \( U_{n-1} \) (i.e., for \( A \in \bigcup U_{n-1} \), \( A \ni U_{n-1} \iff k_n \in i_{n-1,n}(A) \)). Let \( X, F \) be the sets we get by applying the construction of Chapter 3 to the iteration \( \{ N_n \mid n \leq \omega \}, \{ U_n \mid n \in \omega \} \), and let \( \mathbb{P}(X, F) \in V \) be the semi-generalized Prikry forcing we get from \( X, F \). We show that

\[
\{ \tau_\omega(i_{n,\omega}(k_n)) \mid 0 \leq n < \omega \}
\]
is generic over \( M_\omega \) for \( j_{0,\omega}(\mathbb{P}(X, F)) \). Since \( M_\omega \) contains all the reals, this suffices by the pass-through lemma.

Let us show that every “challenge to genericity” in \( M_\omega \) is met by

\[
\{ \tau_\omega(i_{n,\omega}(k_n)) \mid 0 \leq n < \omega \}.
\]

Let \( g \) be any such challenge. Let \( \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \in \lambda^{<\omega}, f \in V \) be such that

\[
g = (j_{0,\omega}(f))(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n).
\]

We can assume that the range of \( f \) consists exclusively of challenges to genericity for \( \mathbb{P}(X, F) \). Let \( l \in \{ \inf(\{ k_p \mid p \geq r \}) \mid r \in \omega \} \) be such that

\[
\tau_\omega(\kappa_i) > \operatorname{sup}(\{ \max(\bar{a}_q) \mid 1 \leq q \leq n \}).
\]

Let \( m \geq l \) be a natural number. Then

\[
N_m = \forall \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \in \kappa_i^{<\omega},
\]

\[
k_{m+1} \in i_{m,m+1}(((i_{0,m}(f))(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n))(i_{1,m}(k_1), i_{2,m}(k_2), \ldots, k_m)) =
\]

\[
(((i_{0,m+1}(f))(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n))(i_{1,m+1}(k_1), i_{2,m+1}(k_2), \ldots, i_{m,m+1}(k_m))).
\]
But this is a $\Sigma^0_0$ statement about elements of $N_{m+1} \subseteq N_m$, so

$$N_{m+1} \models \forall \bar{b}_1, \bar{b}_2, \ldots \bar{b}_n \in \kappa_l^{<\omega},$$

$$k_{m+1} \in ((i_{0,m+1}(f)) (\bar{b}_1, \bar{b}_2, \ldots \bar{b}_n)) (i_{1,m+1}(k_1), i_{2,m+1}(k_2), \ldots i_{m,m+1}(k_m)).$$

We can shift this all to $N_\omega$ and then to $M_\omega$ and obtain the desired result: $\kappa_l$ may be moved by $i_{m+1,\omega}$ (if $k_m = k_{m+1} = k_l$), but from the above we certainly get that

$$N_\omega \models \forall \bar{b}_1, \bar{b}_2, \ldots \bar{b}_n \in \kappa_l^{<\omega},$$

$$i_{m+1,\omega}(k_{m+1}) \in ((i_{0,\omega}(f)) (\bar{b}_1, \bar{b}_2, \ldots \bar{b}_n)) (i_{1,\omega}(k_1), i_{2,\omega}(k_2), \ldots i_{m,\omega}(k_m)).$$

So

$$M_\omega \models \forall \bar{b}_1, \bar{b}_2, \ldots \bar{b}_n \in \tau_\omega(\kappa_l)^{<\omega},$$

$$\tau_\omega(i_{m+1,\omega}(k_{m+1})) \in ((\tau_\omega(i_{0,\omega}(f)) (\bar{b}_1, \bar{b}_2, \ldots \bar{b}_n))(\tau_\omega(i_{1,\omega}(k_1)), \tau_\omega(i_{2,\omega}(k_2)), \ldots \tau_\omega(i_{m,\omega}(k_m))),$$

so

$$M_\omega \models \tau_\omega(i_{m+1,\omega}(k_{m+1})) \in ((\tau_\omega(i_{0,\omega}(f)) (\bar{a}_1, \bar{a}_2, \ldots \bar{a}_n))(\tau_\omega(i_{1,\omega}(k_1)), \tau_\omega(i_{2,\omega}(k_2)), \ldots \tau_\omega(i_{m,\omega}(k_m))),$$

so

$$M_\omega \models \tau_\omega(i_{m+1,\omega}(k_{m+1})) \in g(\tau_\omega(i_{1,\omega}(k_1)), \tau_\omega(i_{2,\omega}(k_2)), \ldots \tau_\omega(i_{m,\omega}(k_m)))$$

as desired. \hfill \Box
Chapter 9

Conclusion

Extending the construction of Chapter 6 into the transfinite seems possible, under assumptions similar to those in Subchapter 5.2. A potential approach would be to insert a connected sequences of enlargements, like those constructed in section 3.5 of [4]. If this is accomplished, extending the constructions (and definitions) of Chapters 3 and 6 should enable a transfinite version of Theorem 7.1. We sum this up with two conjectures:

**Conjecture 8.1:** Let $M$ be an iterate of $V$ by a countable strongly closed iteration tree $\tilde{T}$ such that, letting $\Omega < \omega_1$ be the length of $\tilde{T}$, for every limit ordinal $\gamma \leq \Omega$, $[0, \gamma)_{\tilde{T}}$ is the unique cofinal well-founded branch through $\tilde{T} \upharpoonright \gamma$. Assume further that if $\alpha < \beta < \Omega$ then $\text{SPT}(e_\alpha) \leq \text{SPT}(e_\beta)$, and if $\beta < \Omega$ is a limit ordinal,

$$\text{SPT}(E_\beta) > \sup(\{ J_{\alpha,\beta}(\text{SPT}(E_\alpha)) \mid \alpha < \beta \}).$$

Then $M$ is strongly well-founded.

**Conjecture 8.2:** Let $M$ be an iterate of $V$ by a countable strongly closed iteration tree $\tilde{T}$ such that, letting $\Omega < \omega_1$ be the length of $\tilde{T}$, for every limit ordinal $\gamma \leq \Omega$, $[0, \gamma)_{\tilde{T}}$ is the unique cofinal well-founded branch through $\tilde{T} \upharpoonright \gamma$. Assume further that if $\alpha < \beta < \Omega$ then $\text{SPT}(e_\alpha) \leq \text{SPT}(e_\beta)$, and if $\beta < \Omega$ is a limit ordinal,

$$\text{SPT}(E_\beta) > \sup(\{ J_{\alpha,\beta}(\text{SPT}(E_\alpha)) \mid \alpha < \beta \}).$$

Then if $M$ is strongly well-founded, every countable subset of $M$ is set generic over $M$.

The assumption of unique cofinal branches at limit stages is crucial to both the constructions in Chapters 6 and 7 and those of [2] (Neeman only requires the assumption at limits less than the length of the tree). As others have likely found before in similar situations, we see no way that the proof in Chapter 6 can be simply adapted to the case that the tree has, for example, exactly two well-founded branches. The ability to convert models off the main branch into submodels which form a descending $\in$-chain relies on the witness to continuous ill-foundedness off this one branch. If, however, genericity results similar to ours could be achieved without branch uniqueness assumptions, then the line of inquiry initiated in [2]
and continued here may lead to iterability theorems which do not hinge on theorems like the Martin-Steel Theorem, thereby leading to new advances of the Inner Model Program.

We reflect on this state of affairs by asking the following questions about iteration trees on $V$:

**Q1.** Does strong well-foundedness imply genericity?

**Q2.** Does strong well-foundedness for strongly closed countable iteration trees imply genericity?

**Q3.** Does strong well-foundedness for strongly closed, non-overlapping, countable iteration trees imply genericity?

**Q4.** Can there be an iteration tree with two strongly well-founded branches?

**Q5.** Does a Martin-Steel theorem for strong well-foundedness give iterability?

**Q6.** Given a countable strongly closed tree with a strongly well-founded branch, is that the only well-founded branch the tree has?

**Q7.** We have different flavors of well-foundedness: simple well-foundedness on one end of the spectrum and continuous well-foundedness off well-founded branches on the other. Where does strong well-foundedness lie in this spectrum?
Bibliography


