The Quantum Focussing Conjecture and Quantum Null Energy Condition

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Abstract

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by

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Evidence has been gathering over the decades that spacetime and gravity are best understood as emergent phenomenon, especially in the context of a unified description of quantum mechanics and gravity. The Quantum Focussing Conjecture (QFC) and Quantum Null Energy Condition (QNEC) are two recently-proposed relationships between entropy and geometry, and energy and entropy, respectively, which further strengthen this idea.

In this thesis, we study the QFC and the QNEC. We prove the QNEC in a variety of contexts, including free field theories on Killing horizons, holographic theories on Killing horizons, and in more general curved spacetimes. We also consider the implications of the QFC and QNEC in asymptotically flat space, where they constrain the information content of gravitational radiation arriving at null infinity, and in AdS/CFT, where they are related to other semiclassical inequalities and properties of boundary-anchored extremal area surfaces. It is shown that the assumption of validity and vacuum-state saturation of the QNEC for regions of flat space defined by smooth cuts of null planes implies a local formula for the modular Hamiltonian of these regions. We also demonstrate that the QFC as originally conjectured can be violated in generic theories in $d \geq 5$, which led the way to an improved formulation subsequently suggested by Stefan Leichenauer.
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Chapter 1
Introduction

Quantum gravity remains the final frontier of fundamental physics. Attempts to quantize the gravitational field directly lead to inconsistencies away from the regime in which perturbation theory provides a good approximation.

There is increasing evidence that a fundamentally different approach is needed. One such approach is to take seriously the possibility that spacetime and gravity are not fundamental, but instead emerge from a microscopic description which is quantum mechanical but not gravitational. The celebrated AdS/CFT correspondence provides an explicit example of this for the case of spacetimes with asymptotically negative curvature. In more general spacetimes we must rely on less direct evidence.

A hint comes from thought experiments involving black holes. In classical general relativity, the dynamics of black holes can naturally be recast in a form which resembles the laws of thermodynamics, in which the entropy is the area of the event horizon in Planck units:

\[ S_{BH} = \frac{A}{4G\hbar} . \] (1.1)

This association of a thermodynamical entropy with a geometrical quantity was historically the first hint that gravity and spacetime might be an emergent phenomenon, in the same way thermodynamics emerges from statistical mechanics. When quantum effects are included in the description, black holes become able to radiate and evaporate. Hence the similarity to thermodynamics becomes more than just mathematical analogy, provided that one include the entropy of both black holes and matter (including the radiation):

\[ S_{gen} = S_{BH} + S_{out} , \] (1.2)

where \( S_{out} \) is the entropy of matter outside of the black hole. This \textit{generalized entropy} is really just the total entropy including the entropy of black holes and matter, but is referred to as the generalized entropy for historical reasons.

If gravity and spacetime are in some sense a reorganization of the degrees of freedom of a more fundamental microscopic description, two immediate questions present themselves:
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first, what is this fundamental microscopic description, and second, how do gravity and spacetime emerge? This thesis is concerned primarily with the second question.

Any relationship in semiclassical gravity which relates geometry to entropy provides a clue to this emergence. A few such relationships have been uncovered. One example is the quantum focussing conjecture (QFC), which is a monotonicity condition on an appropriately defined second derivative of the generalized entropy. It reads, schematically,

\[ \frac{d^2 S_{\text{gen}}}{d\lambda^2} \leq 0. \]  

This conjecture – if true – leads to a constraint on the energy density of matter fields which can be perturbatively coupled to gravity, the quantum null energy condition (QNEC):

\[ \langle T_{ab} \rangle k^a k^b \geq \frac{\hbar}{2\pi} \frac{d^2 S_{\text{out}}}{d\lambda^2}. \]  

The QFC and QNEC are the two principles at the heart of this thesis. They are studied both in order to understand the contexts in which they are true, and to understand their implications for quantum gravity.

**Summary** We begin in Chapter 1, where we more carefully define the QNEC (Eq. (8.1)) and prove it for free fields quantized on spacetimes containing Killing horizons. The proof uses the technique of null quantization, which is only known to apply to free field theories. This chapter is based on Ref. [33].

In Chapter 2 we prove the QNEC for quantum field theories in flat space which have a holographic dual, using AdS/CFT as a tool. We use the Hubeny-Rangamani-Takayanagi prescription [152, 153, 101] and the usual holographic dictionary to relate the QNEC to a particular property of boundary-anchored extremal area surfaces in the bulk asymptotically AdS spacetime. This chapter is based on Ref. [115].

In Chapter 3 we consider the implications of the QNEC and other relationships between entropy and energy near null infinity in asymptotically flat spacetimes. We test the bounds in classical general relativity, and find that they are consistent with the equivalence principle both with and without the presence of gravitational radiation arriving at null infinity. We further consider the recently suggested possibility that Minkowski space has an infinite vacuum degeneracy, and conclude that this is inconsistent with the equivalence principle. This chapter is based on Ref. [30].

In Chapter 4 we consider the implications of the QNEC, QFC, and other semiclassical inequalities in the context of the AdS/CFT correspondence. The gravitational inequalities have implications for semiclassical gravity in the bulk, and their non-gravitational limits are connected to properties of the von Neumann entropy of regions of the boundary theory, an example of which was given in Chapter 1. These two sets of inequalities (bulk and boundary) are connected through geometrical statements about the behavior of boundary-anchored extremal area surfaces. This chapter is based on Ref. [4].
In Chapter 5 we use the QNEC to derive a local expression for the modular Hamiltonian of a region bounded by a smooth cut of a null plane, generalizing the class of regions for which such an expression is known. The result holds in any theory in which the QNEC is both true and saturated in the vacuum state. We discuss the validity of these assumptions in free theories and holographic theories to all orders in $1/N$. This chapter is based on Ref. [116].

In Chapter 6 we point out that the QFC as explicitly conjectured in Ref. [34] can be violated in theories containing a perturbative Gauss-Bonnet gravitational coupling – as any generic effective field theory of gravity will – in dimensions $d \geq 5$. This chapter is based on Ref. [81]. This work led to an improved formulation of the QFC which avoids this issue [123].

Finally, in Chapter 7 we conclude by discussing the status of the QNEC in holographic theories in curved spacetimes. We identify a set of sufficient restrictions on the spacetime geometry and surface for the QNEC to hold in $d \leq 5$. In $d \geq 6$, we find that these conditions are not sufficient. This chapter is based on Ref. [80].
Chapter 2

Proof of the Quantum Null Energy Condition

2.1 Introduction

The null energy condition (NEC) states that $T_{kk} \equiv T_{ab}k^a k^b \geq 0$, where $T_{ab}$ is the stress tensor and $k^a$ is a null vector. This condition is satisfied by most reasonable classical matter fields. In Einstein’s equation, it ensures that light-rays are focussed, never repelled, by matter. The NEC underlies the area theorems [91, 28] and singularity theorems [146, 93, 172], and many other results in general relativity [135, 79, 66, 167, 92, 144, 171, 145, 84].

However, quantum fields violate all local energy conditions, including the NEC [65]. The energy density $\langle T_{kk} \rangle$ at any point can be made negative, with magnitude as large as we wish, by an appropriate choice of quantum state. In a stable theory, any negative energy must be accompanied by positive energy elsewhere. Thus, positive-definite quantities linear in the stress tensor that are bounded below may exist, but must be nonlocal. For example, a total energy may be obtained by integrating an energy density over all of space; an “averaged null energy” is defined by integrating $\langle T_{kk} \rangle$ along a null geodesic [22, 173, 114, 169, 87, 98]. In some field theories, “quantum energy inequalities” have also been shown, in which an integral of the stress-tensor need not be positive, but is bounded below [76].

In this article, we will consider a new type of lower bound on $\langle T_{kk} \rangle$ at a single point $p$. Here the bound itself is computed from a nonlocal object: the von Neumann entropy $S_{\text{out}}[\Sigma] \equiv - \text{Tr}(\rho \ln \rho)$ of the quantum fields restricted to some finite or infinite spatial region whose boundary $\Sigma$ contains $p$, is normal to $k^a$, and has vanishing null expansion at $p$. (There are infinitely many ways of choosing such $\Sigma$ for any $(p, k^a)$. ) Then a lower bound is given by the second derivative of $S_{\text{out}}$, under deformations of an infinitesimal area element $A$ of $\Sigma$ in the $k^a$ direction at $p$ (see Figure 2.1):

$$\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi A} S''_{\text{out}}[\Sigma].$$

We call (2.1) the Quantum Null Energy Condition (QNEC) [34]. The quantity $S_{\text{out}}$ is
divergent but its derivatives are finite. (A more rigorous formulation in terms of functional derivatives will be given in the main text.) Note that the right hand side can have any sign. If it is positive, then the QNEC is stronger than the NEC; but since it can be negative, it can accommodate situations where the NEC would fail. By integrating the QNEC along a null generator, we can obtain the ANEC, in situations where the boundary term $S_{\text{out}}'$ vanishes at early and late times.

Intriguingly, the QNEC—an intrinsically field theoretic statement—was recognized by studying conjectured properties of the generalized entropy,

$$S_{\text{gen}}[\Sigma] = \frac{A[\Sigma]}{4G\hbar} + S_{\text{out}}[\Sigma], \quad (2.2)$$

a key concept arising in quantum gravity [13, 14, 15]. Here $\Sigma$ is a codimension-2 surface which divides a Cauchy surface in two, $A[\Sigma]$ is its area and $S_{\text{out}}$ is the von Neumann entropy of the matter fields on one side of $\Sigma$.

The generalized second law (GSL) is the conjecture [13] that the generalized entropy cannot decrease as $\Sigma$ is moved up along a causal horizon. Equation (2.1) first appeared as a sufficient condition for the GSL, satisfied by a nontrivial class of states of a 1+1 dimensional CFT [179]. The QNEC emerged as a general constraint on quantum field theories when it was noted that the Quantum Focussing Conjecture (QFC) implies (2.1) in an appropriate limit [34]. We will briefly describe the QFC and outline how the QNEC arises from it.

A generalized entropy can be ascribed not only to horizon slices, but to any surface that splits a Cauchy surface [180, 64, 19, 136, 69]. Moreover, one can define a quantum expansion $\Theta[\Sigma; y_1]$, the rate (per unit area) at which the generalized entropy changes when the infinitesimal area element of $\nu$ at a point $y_1$ is deformed in one of its future orthogonal null directions [34] (see Fig. 2.1). This quantity limits to the classical (geometric) expansion as $\hbar \to 0$. The QFC states that the quantum expansion $\Theta[\Sigma; y_1]$ will not increase under any second variation of $\Sigma$ along the same future congruence, be it at $y_1$ or at some other point $y_2$ [34].

The QFC, in turn, was proposed as a quantum version of the covariant entropy bound (Bousso bound) [23, 25, 72], a quantum gravity conjecture which bounds the entropy on a nonexpanding null surface in terms of the difference between its initial and final area. The QFC implies the Bousso bound; but because the generalized entropy appears to be insensitive to the UV cutoff [165, 105, 161], the QFC remains well-defined in more general settings. (The QFC is distinct from the quantum Bousso bound of [32, 31], which defines the entropy by vacuum subtraction [45], a procedure applicable if the gravitational effects of matter are negligible.)

In the case where $y_1 \neq y_2$, it can be shown [34] that the QFC follows from strong subadditivity, an entropy inequality which all quantum systems must obey.\(^1\) For $y_1 = y_2$,

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\(^1\)Some recent articles [17, 121] considered a different type of second derivative of the entropy in 1+1 field theory. These inequalities involve varying the two endpoints of an interval independently, and therefore follow from strong subadditivity alone, without making reference to the stress-tensor.
the QFC remains a conjecture in general, but in special cases it can be proven. The QFC constrains a combination of “geometric” terms proportional to $G^{-1}$ that stem from the classical expansion, as well as “matter entropy” terms that stem from $S_{\text{out}}$ and do not involve Newton’s constant. The classical expansion is governed by Raychaudhuri’s equation, $\theta' = -\theta^2/2 - \sigma^2 - 8\pi G \langle T_{kk} \rangle$. If the expansion $\theta$ and the shear $\sigma$ vanish at $y_1$, then the rate of change of the expansion is governed by a term proportional to $G$. In this case, all $G$’s cancel in the terms of the QFC, and (2.1) emerges as an apparently nongravitational statement.

Outline In this paper, we will prove the QNEC in a broad arena. Our proof applies to free or superrenormalizable, massive or massless bosonic fields, in all cases where the surface $\Sigma$ lies on a stationary null hypersurface (one with everywhere vanishing expansion). The most important example is Minkowski space, with $\Sigma$ lying on a Rindler horizon. Such a horizon exists at every point $p$, with every orientation $k^a$, so the QNEC constrains all null components of the stress tensor everywhere in Minkowski space.

A similar situation arises in a de Sitter background, where $p$ and $k^a$ specify a de Sitter horizon, and in Anti-de Sitter space, where they specify a Poincaré horizon. Other examples

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Figure 2.1: The spatial surface $\Sigma$ splits a Cauchy surface, one side of which is shown in yellow. The generalized entropy $S_{\text{gen}}$ is the area of $\Sigma$ plus the von Neumann entropy $S_{\text{out}}$ of the yellow region. The quantum expansion $\Theta$ at one point of $\Sigma$ is the rate at which $S_{\text{gen}}$ changes under a small variation $d\lambda$ of $\Sigma$, per cross-sectional area $A$ of the variation. The Quantum Focussing Conjecture states that the quantum expansion cannot increase under a second variation in the same direction. If the classical expansion and shear vanish (as they do for the green null surface in the figure), the Quantum Null Energy Condition is implied as a limiting case. Our proof involves quantization on the null surface; the entropy of the state on the yellow spacelike slice is related to the entropy of the null quantized state on the future (brighter green) part of the null surface.

2Raychaudhuri’s equation immediately implies that, in cases where the classical geometrical terms dominate, the QFC is true iff the classical spacetime obeys the null curvature condition.
include an eternal Schwarzschild or Kerr black hole, but in this case our proof applies only
to points on the horizon, with \( k^a \) tangent to the horizon generators. These should all be
viewed as fixed background spacetimes with no dynamical gravity; our proof establishes that
free scalar field theory on these backgrounds satisfies (2.1).

We give a brief review of the formal statement of the QNEC in Sec. 2.2. We then set
up the calculation of all relevant terms in Sec. 2.3. In Sec. 2.3, we review the null surface
quantization of the theory, on the particular null surface \( N \) that is orthogonal to \( \Sigma \) with
tangent vector \( k^a \). Null quantization has the remarkable feature that the vacuum state
factorizes in the transverse spatial directions. This reduces any purely kinematic problem
(such as ours) to the analysis of a large number of copies of the free chiral scalar CFT in
1+1 dimensions. We then restrict attention to the particular chiral CFT on the infinitesimal
pencil that passes through the point \( p \) where \( \Sigma \) is varied. The state on this pencil is entangled
with an auxiliary quantum system which contains both the information crossing the other
generators of \( N \), and the information that does not fall across \( N \) at all.

In the 1+1 chiral CFT, the pencil state is very close to the vacuum, but not so close that
the QNEC would be trivially saturated by application of the first law of the entanglement
entropy. To constrain the second order variations of \( S_{\text{out}} \) (the Fisher information), we must
keep track of the deviation of the pencil state from the vacuum to second order. We discuss
the appropriate expansion of the overall state in Sec. 2.3. We write the state in terms of
operators inserted on the Euclidean plane corresponding to the pencil and expand in a basis
of the auxiliary system. Then in Sec. 2.3, we expand the entropy and identify the parts of
our expansion enter into the second derivative.

In Sec. 2.4, we compute the sign of \( \langle T_{kk} \rangle - \frac{\hbar}{2\pi A} S''_{\text{out}} \). In Sec. 2.4 we review the replica trick
for computing the von Neumann entropy by the analytic continuation of Renyi entropies. We
extract two terms relevant to the QNEC, which are computed in Sec. 2.4 and 2.4 respectively.
The most subtle part of the calculation is the analytic continuation of the second of these
terms, in Sec. 2.4. In Sec. 2.4, we combine the terms and conclude that the QNEC holds for
all states.

In Sec. 2.5, we extend our result to establish the QNEC also for superrenormalizable
scalar fields, and for bosonic fields of higher spin. We also discuss the extension to interacting
theories. We expect that the proof we have given can be extended to fermionic fields, but
we leave this task for the future.

Discussion  Our result establishes a new and surprising link between quantum information
and a more familiar physical quantity, the stress tensor. The QNEC identifies the “accel-
eration” of information transfer as a lower bound on the energy density. Equivalently, the
stress tensor can be viewed as imposing a constraint on the second derivative of the von
Neumann entropy. The latter can be difficult to calculate but plays an important role in
quantum information theory, condensed matter, and high energy physics.

Our proof of the QNEC requires no assumptions beyond the known properties of free
quantum fields, but it is quite lengthy and somewhat involved. Yet, the QNEC follows
almost trivially from a statement involving gravity, the Quantum Focussing Conjecture. This perplexing situation is somewhat reminiscent of the proof of the quantum Bousso bound [32], particularly in the interacting case [31]. It is intriguing that the study of quantum gravity can lead us to simple conjectures such as (2.1) which can be proven entirely within the nongravitational sector, where they are far from obvious—so far, indeed, that they had not been recognized until they emerged as implications of holographic entropy bounds or of properties of the generalized entropy.

It is becoming clear that the structure of known quantum field theories carries a deep imprint of causal and information theoretic properties ultimately dictated by quantum gravity. This adds to the evidence that “quantizing gravity” has nothing to do with the inclusion of one last force in a quantization program. It would be interesting to try to formulate models of quantum gravity in which focussing of the entropy occurs naturally.

Remarkably, the QNEC does not seem to follow from any of the standard identities that apply purely at the level of quantum information. Our proof did involve additional structure supplied by quantum field theory. The QNEC is related to the relative entropy $S(\rho|\sigma) = \text{Tr}(\rho \ln \rho) - \text{Tr}(\rho \ln \sigma)$, which equals $-S_{\text{gen}}$ (up to a constant) when $\sigma$ is taken to be the vacuum state. The relative entropy satisfies positivity, which guarantees that $S_{\text{gen}}(\rho)$ is less than in the vacuum state. It also enjoys monotonicity, which implies that $S_{\text{gen}}$ is increasing under restrictions; this constrains the first derivative, which is the GSL [174]. It may appear that the QNEC can be proven using properties of the relative entropy. But the QNEC is a statement about the \textit{second derivative} of the generalized entropy. It is possible that the QNEC hints at more general quantum information inequalities, which are yet to be discovered. It is interesting that a recently proposed new GSL, which applies in strongly gravitating regions such as cosmology, also can be shown to follow from the QFC [27].

2.2 Statement of the Quantum Null Energy Condition

The statement of the QNEC involves the choice of a point $p$ a null vector $k^a$ at $p$, and a smooth codimension-2 surface $\Sigma$ orthogonal to $k^a$ at $p$ such that $\Sigma$ splits a Cauchy surface into two portions. The null vector $k^a$ is a member of a vector field orthogonal to $\Sigma$ defined in a neighborhood of $p$, $k^a(y)$. Here and below we use $y$ as a coordinate label on $\Sigma$, also called the “transverse direction.” We can consider a family of surfaces $\Sigma[\lambda(y)]$ obtained by deforming $\Sigma$ along the null geodesics generated by $k^a(y)$ by the affine parameters $\lambda(y)$.

The deformed surfaces will also be Cauchy-splitting [29]. This allows us to define a family of entropies $S_{\text{out}}[\lambda(y)]$, which are the von Neumann entropies of the quantum fields restricted to the Cauchy surface on one side of $\Sigma[\lambda(y)]$. The choice of Cauchy surface is unimportant, since by unitarity the entropy will be independent of that choice. The choice of side of $\Sigma[\lambda(y)]$ also does not matter, because the QNEC is symmetric with respect to $k^a \rightarrow -k^a$.

Once we have defined $S_{\text{out}}[\lambda(y)]$, we can consider its functional derivatives. In general, the second functional derivative will contain diagonal and off-diagonal terms (present because $S_{\text{out}}$ is a non-local functional), and the diagonal terms will be proportional to a $\delta$-function.
We define the second functional derivative at coincident points by factoring out that δ-function:

$$\frac{\delta^2 S_{\text{out}}}{\delta \lambda(y) \delta \lambda(y')} = \frac{\delta^2 S_{\text{out}}}{\delta \lambda(y)^2} \delta(y - y') + \text{off-diagonal.} \quad (2.3)$$

Then if the expansion and the shear of $k^a(y)$ vanish at $p$, we have the general conjecture

$$\langle T_{kk}(p) \rangle \geq \frac{\hbar}{2\pi} \frac{\delta^2 S_{\text{out}}}{\sqrt{h(p)}} \delta \lambda(p)^2 \bigg|_{\lambda(y)=0}, \quad (2.4)$$

where $h$ is the determinant of the induced metric on $\Sigma$ and $T_{kk} \equiv T_{ab} k^a k^b$. We will find it convenient below to work with a discretized version of the functional derivative, obtained by dividing $\Sigma$ into regions of small area $A$ and considering variations locally constant in those regions. Then (2.4) reduces to the form advertised in (2.1):

$$\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi A} S''_{\text{out}}. \quad (2.5)$$

### 2.3 Reduction to a 1+1 CFT and Auxiliary System

#### Null Quantization

The proof that follows applies when $\Sigma$ is a section of a general stationary null surface $N$ in $D > 2$ (the case $D = 2$ will be treated separately, in section 2.5). We consider deformations of $\Sigma$ along $N$ toward the future, so the deformation vector $k^a$ is future-directed, and we choose to take the “outside” direction to be the side towards which $k^a$ points. As mentioned above, a proof of this case automatically implies a proof for the opposite choice of outside. By unitary time evolution of the spacelike Cauchy data, we can consider the state to be defined on the portion of $N$ in the future of $\Sigma$ together with a portion of future null infinity.

We rely on null quantization on $N$, which requires that $N$ be stationary [174]. Null quantization is simplest if we first discretize $N$ along the transverse direction into regions of small transverse area $A$. These regions, which are fully extended in the null direction, are called pencils. Ultimately we will take the continuum limit $A \to 0$, and the QNEC will be shown to hold in this limit. At intermediate stages, $A$ acts as a small expansion parameter.\(^3\) This is the reason why we are restricting ourselves to $D > 2$ spacetime dimensions for now: without a transverse direction to discretize, there would be no small expansion parameter.

Also, while logically independent from the discretization used to define the QNEC in (2.1), we will take these two discretizations to be the same. That is, we will consider deformations of the surface $\Sigma$ which are localized to the same regions of size $A$ that define the discretized null quantization.

---

\(^3\) The dimensionless expansion parameter is $A$ in units of a characteristic length scale of the state we are interested in, e.g., the wavelength of typical excitations. The state remains fixed as $A \to 0$. 
There is a distinguished pencil that contains the point $p$; this is the pencil on which we will perform our deformations. The total Hilbert space of the system can be decomposed as $\mathcal{H} = \mathcal{H}_{\text{pen}} \otimes \mathcal{H}_{\text{aux}}$, where $\mathcal{H}_{\text{pen}}$ refers to the fields on the distinguished pencil and $\mathcal{H}_{\text{aux}}$ is everything else. “Everything else” includes both the remaining pencils on $N$ restricted to the future of $\Sigma$, as well as the relevant portion of null infinity. We do not have to be specific about the exact structure of the auxiliary system; our proof does not assume anything about it other than what is implied by quantum mechanics. Beginning with a density matrix on $\mathcal{H}$, we obtain a one-parameter family of density matrices $\rho(\lambda)$ by tracing out the part of the pencil in the past of affine parameter $\lambda$. When $\lambda \to -\infty$ the pencil is fully extended, and when $\lambda \to +\infty$ the entire pencil has been traced out. $\lambda = 0$ corresponds to no deformation of the original surface.

When restricted to $N$, the theory decomposes into a product of 1+1-dimensional free chiral CFTs, with one CFT associated to each pencil of $N$. In particular, this means that the vacuum state factorizes with respect to the pencil decomposition of $N$ [174].

Crucially, when $A$ is small, the state of the pencil is near the vacuum. This can be seen as follows. For a region of small size $A$, the amplitude to have $n$ particles on the pencil scales like $A^{n/2}$ (so the probability is appropriately extensive), and therefore the coefficient of $|n\rangle\langle m|$ in the pencil Fock basis expansion of the state scales like $A^{(n+m)/2}$. Hence for small $A$ we can write the state as

$$\rho(\lambda) = \rho_{\text{pen}}^{(0)}(\lambda) \otimes \rho_{\text{aux}}^{(0)} + \sigma(\lambda), \quad (2.6)$$

where $\rho_{\text{pen}}^{(0)}(\lambda)$ is the vacuum state density matrix on the part of the pencil with affine parameter greater than $\lambda$, $\rho_{\text{aux}}^{(0)}$ is some state in the auxiliary system (not necessarily the vacuum), and the perturbation $\sigma(\lambda)$ is small: the largest terms are obtained by taking the partial trace of $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$ in the pencil Fock basis, and these terms have coefficients which scale like $A^{1/2}$. Entanglement between the pencil and the auxiliary system is also present in $\sigma$; we will explore the form of $\sigma$ in more detail in the following section.

**Expansion of the State**

As discussed above, the pencil state can be described in terms of a 1+1-dimensional free chiral CFT, with fields that depend only on the coordinate $z = x + t$. In this notation, translations along the Rindler horizon in the 1+1 CFT are translations in $z$, and are generated by $\partial \equiv \frac{\partial}{\partial z}$. In a chiral theory, this is equivalent to translations in the spatial coordinate $x$. Therefore the shift in affine parameter $\lambda$ of the previous section can be replaced by a shift in the spatial coordinate for the purposes of the CFT calculation. In addition, quantization on a surface of constant Euclidean time $\tau = it = 0$ in a chiral theory is equivalent to quantization on the Rindler horizon. Thus when we construct the state we can use standard Euclidean methods for two-dimensional CFTs.
We have argued that, at order $A^{1/2}$, the perturbation $\sigma$ on the full pencil must be of the schematic form $|0\rangle\langle 1|$ (plus Hermitian conjugate). So on the full pencil, we have the state
\[ \rho = \rho(-\infty) = |0\rangle\langle 0| \otimes \rho^{(0)}_{\text{aux}} + A^{1/2} \sum_{ij} (|0\rangle\langle \psi_{ij}| + |\psi_{ji}\rangle\langle 0|) \otimes |i\rangle\langle j| + \cdots, \] (2.7)
where $|i\rangle\langle j|$ is a basis of operators in the auxiliary system and "\cdots" denotes terms which vanish more quickly as $A \to 0$. We will argue in Sec. 2.3 that those terms are not relevant for the QNEC, and so we will ignore them from now on. For later convenience, we will take the basis $|i\rangle$ in the auxiliary system to be the one in which $\rho^{(0)}_{\text{aux}}$ is diagonal. The states $|\psi_{ij}\rangle$ are single-particle states in the CFT, and we have ensured that the state is Hermitian. The CFT part of the state can be constructed by acting on the vacuum with a single copy of the field operator. In a Euclidean path integral picture, we can get the most general single-particle state by allowing arbitrary single-field insertions on the Euclidean plane. This is shown in Fig. 2.2.

To obtain the state at a finite value of $\lambda$, we need to take the trace of (2.7) over the region $x < \lambda$. Alternatively, we can hold fixed the inaccessible region, $x < 0$, but translate the field operators used to construct the state by $\lambda$. From this point of view the vacuum is independent of $\lambda$ and we write it as
\[ \rho^{(0)}_{\text{pen}} = e^{-2\pi K_{\text{pen}}}, \] (2.8)
where, up to an additive constant, the modular Hamiltonian $K_{\text{pen}}$ coincides with the Rindler boost generator for the CFT [168, 20]. Specializing to the case of a single chiral scalar field (extensions will be discussed in Sec. 2.5), the trace of (2.7) becomes
\[ \rho(\lambda) = e^{-2\pi K_{\text{pen}}} \otimes \rho^{(0)}_{\text{aux}} + A^{1/2} \sum_{ij} \left(e^{-2\pi K_{\text{pen}}} \int dr d\theta \ f_{ij}(r, \theta) \partial \Phi(\sqrt{r^2 + \theta^2} - \lambda)\right) \otimes |i\rangle\langle j|, \] (2.9)
CHAPTER 2. PROOF OF THE QUANTUM NULL ENERGY CONDITION

where \( \partial \Phi(z) \) is now a holomorphic local operator on a two-dimensional Euclidean plane\(^4\) and \((r, \theta)\) are polar coordinates on that plane, with \( z = re^{i\theta} \). Rotations in \( \theta \) are generated by \( K_{\text{pen}} \). Thus the operator \( \partial \Phi \) is defined by\(^5\)

\[
\partial \Phi(re^{i\theta}) = e^{-i\theta} e^{i\theta K_{\text{pen}}} \partial \Phi(r)e^{-i\theta K_{\text{pen}}}.
\] (2.10)

All of the operators in (2.9) are manifestly operators on the Hilbert space corresponding to \( x > 0, \tau = 0 \). We are taking \( \Phi \) to be a real scalar field, so in particular \( \partial \Phi \) is a Hermitian operator for real arguments. Then in order for \( \rho \) to be Hermitian, we must have

\[
f_{ij}(r, \theta) = f_{ji}(r, 2\pi - \theta)^*.
\] (2.11)

Aside from this reality condition, letting \( f \) be completely general gives all possible single particle states.

To facilitate our later calculations, we will modify (2.9) in order to put the auxiliary system on equal footing with the CFT. To that end, define \( K_{\text{aux}} \) through the equation

\[
\rho(0)_{\text{aux}} = \exp(-2\pi K_{\text{aux}}).
\]

We can invent a coordinate \( \theta \) for the auxiliary system and declare that evolution in \( \theta \) is generated by \( K_{\text{aux}} \). Then define the operators

\[
E_{ij}(\theta) \equiv e^{\theta K_{\text{aux}}} |i\rangle \langle j| e^{-\theta K_{\text{aux}}} = e^{\theta(K_i - K_j)} |i\rangle \langle j|.
\] (2.12)

Since \( K_{\text{aux}} \) is diagonal in the \( |i\rangle \) basis, with eigenvalues \( K_i \), \( E_{ij}(\theta) \) is just a rescaled \( |i\rangle \langle j| \). More generally, multiplying \( |i\rangle \langle j| \) on either side by arbitrary functions of \( K_{\text{aux}} \) results in the same operator up to an \((i,j)\)-dependent numerical factor. So by making the replacement

\[
f_{ij}(r, \theta) \rightarrow e^{(2\pi - \theta)K_i} e^{\theta K_j} f_{ij}(r, \theta),
\] (2.13)

which does not alter the reality condition on \( f \), we can write

\[
\rho(\lambda) = e^{-2\pi K_{\text{tot}}} + A^{1/2} e^{-2\pi K_{\text{tot}}} \sum_{ij} \int dr d\theta f_{ij}(r, \theta) \partial \Phi(re^{i\theta} - \lambda) \otimes E_{ij}(\theta),
\] (2.14)

where \( K_{\text{tot}} \equiv K_{\text{pen}} + K_{\text{aux}} \). From now on, we will simply write \( K \) for \( K_{\text{tot}} \).

Below it will be useful to write \( \sigma(\lambda) \) as

\[
\sigma(\lambda) \equiv A^{1/2} \rho^{(0)} \mathcal{O}(\lambda).
\] (2.15)

Thus comparing with (2.14), we find

\[
\mathcal{O}(\lambda) = \sum_{ij} \int dr d\theta f_{ij}(r, \theta) \partial \Phi(re^{i\theta} - \lambda) \otimes E_{ij}(\theta).
\] (2.16)

As a side comment, we note that one could prepare the state (2.14) via a Euclidean path integral over the entire plane with an insertion of \( \mathcal{O} \) and boundary field configurations defined at \( \theta = 0^+ \) and \( \theta = (2\pi)^- \).

\(^4\)We insert \( \partial \Phi \) instead of \( \Phi \) in order to remove any zero-mode subtleties. We have checked that the proof still works formally if one inserts \( \Phi \) instead of \( \partial \Phi \), and in fact continues to work when an arbitrary number of derivatives, \( \partial^l \Phi \), are used. This latter fact is not surprising since insertions of \( \Phi \) alone (or \( \partial \Phi \) if we drop the zero mode) are sufficient to generate all single particle states. See [36, 174] for details on the zero-mode.

\(^5\)Here \( \theta \) is restricted to be in the range \([0, 2\pi)\).
Expansion of the Entropy

In the previous sections we saw that null quantization gives us a state of the form
\[ \rho(\lambda) = \rho_{\text{pen}}^{(0)}(\lambda) \otimes \rho_{\text{aux}}^{(0)} + \sigma(\lambda), \]  
(2.17)
where \( \rho_{\text{pen}}^{(0)}(\lambda) \) is the vacuum state reduced density matrix on the part of the pencil with affine parameter greater than \( \lambda \), \( \rho_{\text{aux}}^{(0)} \) is an arbitrary state in the auxiliary system, and the perturbation \( \sigma \) is proportional to the small parameter \( A^{1/2} \). In this section, we will expand the entropy perturbatively in \( \sigma \) and show that the QNEC reduces to a statement about the contributions of \( \sigma \) to the entropy. We will assume that both \( \rho(\lambda) \) and \( \rho_{\text{pen}}^{(0)}(\lambda) \equiv \rho_{\text{pen}}^{(0)}(\lambda) \otimes \rho_{\text{aux}}^{(0)} \) are properly normalized density matrices, so \( \text{Tr}(\sigma) = 0 \).

The von Neumann entropy of \( \rho(\lambda) \) is \( S_{\text{out}}(\lambda) \). We will expand it as a perturbation series in \( \sigma(\lambda) \):
\[ S_{\text{out}}(\lambda) = S^{(0)}(\lambda) + S^{(1)}(\lambda) + S^{(2)}(\lambda) + \cdots \]  
(2.18)
where \( S^{(n)}(\lambda) \) contains \( n \) powers of \( \sigma(\lambda) \). At zeroth order, since \( \rho^{(0)}(\lambda) \) is a product state, we have
\[ S^{(0)}(\lambda) = -\text{Tr} \left[ \rho^{(0)}(\lambda) \log \rho^{(0)}(\lambda) \right] = -\text{Tr} \left[ \rho_{\text{pen}}^{(0)}(\lambda) \log \rho_{\text{pen}}^{(0)}(\lambda) \right] - \text{Tr} \left[ \rho_{\text{aux}}^{(0)} \log \rho_{\text{aux}}^{(0)} \right]. \]  
(2.19)
The first term on the right-hand side is independent of \( \lambda \) because of null translation invariance of the vacuum: all half-pencils have the same vacuum entropy. The second term is manifestly independent of \( \lambda \). So \( S^{(0)} \) is \( \lambda \)-independent and does not play a role in the QNEC.

Now we turn to \( S^{(1)}(\lambda) \):
\[ S^{(1)}(\lambda) = -\text{Tr} \left[ \sigma(\lambda) \log \rho^{(0)}(\lambda) \right] = -\text{Tr} \left[ \sigma(\lambda) \log \rho_{\text{pen}}^{(0)}(\lambda) \right] - \text{Tr} \left[ \sigma(\lambda) \log \rho_{\text{aux}}^{(0)} \right]. \]  
(2.20)
Once again, the second term is \( \lambda \)-independent, which we can see by evaluating the trace over the pencil subsystem:
\[ \text{Tr} \left[ \sigma(\lambda) \log \rho_{\text{aux}}^{(0)} \right] = \text{Tr}_{\text{aux}} \left[ \text{Tr}_{\text{pen}} \sigma(\lambda) \log \rho_{\text{pen}}^{(0)} \right] = \text{Tr}_{\text{aux}} \left[ \sigma(\infty) \log \rho_{\text{aux}}^{(0)} \right]. \]  
(2.21)
To evaluate the first term, we use the fact that \( \rho_{\text{pen}}^{(0)}(\lambda) \) is thermal with respect to the boost operator on the pencil. Then we have
\[ -\text{Tr} \left[ \sigma(\lambda) \log \rho_{\text{pen}}^{(0)}(\lambda) \right] = \frac{2\pi A}{\hbar} \int_{\lambda}^{\infty} d\lambda' (\lambda' - \lambda) \langle T_{kk}(\lambda') \rangle, \]  
(2.22)
where the integral is along the generator which defines the pencil and the expectation value is taken in the excited state. This is the first \( \lambda \)-dependent term we have in the perturbative expansion of \( S(\lambda) \). Taking two derivatives and evaluating at \( \lambda = 0 \) gives the identity
\[ (S^{(0)} + S^{(1)})'' = \frac{2\pi A}{\hbar} \langle T_{kk} \rangle. \]  
(2.23)
Subtracting $S''_{\text{out}}$ from both sides of this equation shows that
\[
\frac{\hbar}{2\pi A} S''_{\text{out}} - \langle T_{kk} \rangle = \frac{\hbar}{2\pi A} \left( S_{\text{out}} - S^{(0)} - S^{(1)} \right)'' = \frac{\hbar}{2\pi A} S^{(2)''} + \cdots , \tag{2.24}
\]
where “\cdots” contains terms higher than quadratic order in $\sigma$. The QNEC (equation (2.1)) is the statement that this quantity is negative in the limit $A \to 0$. Earlier we showed that $\sigma$ was proportional to $A^{1/2}$. Then $S^{(2)}$ is proportional to $A$, and we must check that $S^{(2)''}$ is negative. However, the higher order terms $S^{(\ell)}$ for $\ell > 2$ vanish more quickly with $A$ and therefore drop out in the limit $A \to 0$.

We have shown that the QNEC reduces to the statement that $S^{(2)''} \leq 0$ for perturbations from the vacuum. In fact, we have shown something a little stronger. In general, the perturbation $\sigma$ will have terms proportional to $A^{n/2}$ for all $n \geq 1$. Our arguments show that only the term proportional to $A^{1/2}$ matters for the QNEC, and furthermore that this term is off-diagonal in the single-particle/vacuum subspace. So we can simplify matters by considering states which contain only such a term proportional to $A^{1/2}$ and no higher powers of $A$. In other words, we can take the state to be of the form in (2.7) with the unwritten “\cdots” terms set equal to zero. Now we only need to show that $S^{(2)''} \leq 0$ for such states.

### 2.4 Calculation of the Entropy

#### The Replica Trick

The replica trick prescription is to use the following formula for the von Neumann entropy [38]:
\[
S_{\text{out}} = - \text{Tr}[\rho \log \rho] = (1 - n \partial_n) \log \text{Tr}[\rho^n] \bigg|_{n=1} . \tag{2.25}
\]

This can be written as
\[
S_{\text{out}} = D \log \tilde{Z}_n \tag{2.26}
\]
where $\tilde{Z}_n \equiv \text{Tr}[\rho^n]^6$ and the operator $D$ is defined by
\[
D f(n) \equiv (1 - n \partial_n) f(n) \bigg|_{n=1} \tag{2.27}
\]
where $f(n)$ is some function of $n$. Since $\tilde{Z}_n$ is only defined for integer values of $n$, we first must analytically continue to real $n > 0$ in order to apply the $D$ operator. The analytic continuation step is in general quite tricky, and will require care in our calculation. (Our analytic continuation is performed in Section 2.4.)

---

6In the replica trick one often works with the partition function $Z_n$, in terms of which $\tilde{Z}_n = Z_n / (Z_1)^n$. Choosing $Z_n$ over $\tilde{Z}_n$ is equivalent to choosing a different normalization for $\rho$, but we find it convenient to keep $\text{Tr} \rho = 1$. 

On general grounds discussed above, we must study the second-order term in a perturbative expansion of the entropy about the state $\rho^{(0)}$. Suppressing all $\lambda$ dependence, we have

$$\tilde{Z}_n = \text{Tr} \left[ (\rho^{(0)} + \sigma)^n \right].$$

(2.28)

Expanding $\tilde{Z}_n$ to quadratic order to isolate $S^{(2)''}$, we have

$$\tilde{Z}_n = \text{Tr} \left[ (\rho^{(0)})^n \right] + n \text{Tr} \left[ \sigma (\rho^{(0)})^{n-1} \right] + \frac{n}{2} \sum_{k=0}^{n-2} \text{Tr} \left[ (\rho^{(0)})^k \sigma (\rho^{(0)})^{n-k-2} \right] + \cdots.$$  

(2.29)

Using the notation introduced in (2.15) we can write

$$\tilde{Z}_n = \text{Tr} \left[ (\rho^{(0)})^n \right] + n \text{Tr} \left[ \mathcal{O} (\rho^{(0)})^n \right] + \frac{n}{2} \sum_{k=1}^{n-1} \text{Tr} \left[ (\rho^{(0)})^{-k} \mathcal{O} (\rho^{(0)})^k \mathcal{O} (\rho^{(0)})^n \right] + \cdots.$$  

(2.30)

We denote by $\mathcal{O}^{(k)}$ the operator $\mathcal{O}$ conjugated by $(\rho^{(0)})^k$:

$$\mathcal{O}^{(k)} \equiv (\rho^{(0)})^{-k} \mathcal{O} (\rho^{(0)})^k$$

$$= e^{2\pi k K} \mathcal{O} e^{-2\pi k K}. \quad (2.31)$$

This is equivalent to a Heisenberg evolution of $\mathcal{O}$ in the angle $\theta$ by an amount $2\pi k$. Since $\mathcal{O}$ is the integral of operators with angles $0 \leq \theta < 2\pi$, it follows that $\mathcal{O}^{(k)}$ will be an integral over operators with angles $2\pi k < \theta < 2\pi (k + 1)$.

Furthermore, since rotations by $2\pi k$ commute with translations by $\lambda$, we can obtain $\mathcal{O}^{(k)}$ from $\mathcal{O}$ simply by letting the range of integration that defines $\mathcal{O}$ shift from $[0, 2\pi]$ to $[2\pi k, 2\pi (k + 1)]$, as long as we define $f_{ij}(r, \theta)$ to be periodic in $\theta$ with period $2\pi$.

It will also be convenient to introduce an angle-ordered expectation value, defined as

$$\langle \cdots \rangle_n \equiv \frac{\text{Tr}[(\rho^{(0)})^n T[\cdots]]}{\text{Tr}[(\rho^{(0)})^n]},$$

(2.33)

where $T[\cdots]$ is $\theta$-ordering. Then (2.30) can be written

$$\tilde{Z}_n = \text{Tr} \left[ (\rho^{(0)})^n \right] \left( 1 + n \langle \mathcal{O} \rangle_n + \frac{n}{2} \sum_{k=1}^{n-1} \langle \mathcal{O}^{(k)} \mathcal{O} \rangle_n \right) + \cdots.$$  

(2.34)

Taking the logarithm of $\tilde{Z}_n$ and extracting the part quadratic in $\sigma$ gives

$$\log \tilde{Z}_n \supset \frac{n}{2} \sum_{k=1}^{n-1} \langle \mathcal{O}^{(k)} \mathcal{O} \rangle_n - \frac{n^2}{2} \langle \mathcal{O} \rangle_n^2,$$  

(2.35)

7One could worry that the phase factor in (2.10) spoils this relation, but notice that the phase has period $2\pi$ in $\theta$ and so does not appear when shifting by $2\pi k$. 
where we have kept only the part quadratic in $O$. The contribution of the second term to the entanglement entropy will be proportional to $\langle O \rangle$, which vanishes because of the tracelessness of $\sigma$. Therefore we only need to consider the first term.

Since we are considering angle-ordered expectation values, we have the identity

$$\left\langle \left( \sum_{k=0}^{n-1} O^{(k)} \right)^2 \right\rangle_n = n \sum_{k=0}^{n-1} \langle O^{(k)} O \rangle_n,$$

and so from the first term in (2.35) the relevant part of $\log \tilde{Z}_n$ can be written as

$$\log \tilde{Z}_n \supset -\frac{n}{2} \langle OO \rangle_n + \frac{1}{2} \left\langle \left( \sum_{k=0}^{n-1} O^{(k)} \right)^2 \right\rangle_n.$$

(2.37)

Restoring the $\lambda$ dependence and taking $\lambda$ derivatives gives

$$S^{(2)''} = \left. \frac{\partial^2}{\partial \lambda^2} \right|_{\lambda=0} D \log \tilde{Z}_n(\lambda)$$

$$= D \left[ -\frac{n}{2} \langle OO \rangle_n'' + \frac{1}{2} \left\langle \left( \sum_{k=0}^{n-1} O^{(k)} \right)^2 \right\rangle_n'' \right].$$

(2.39)

The $\langle \ldots \rangle_n''$ notation means take two $\lambda$ derivatives and then set $\lambda = 0$. In the following sections we will compute these two terms separately.

We note that the two terms in (2.39) are analogous to $\delta S_{EE}^{(1)}$ and $\delta S_{EE}^{(2)}$ of Ref. [67], where a similar perturbative computation of the entropy was performed. Though the details of the two calculations differ (in particular we have an auxiliary system as well as a CFT), it would be interesting to explore further the connection between our present work and that of Ref. [67].

**Evaluation of Same-Sheet Correlator**

In this section we consider the term $\langle OO \rangle_n''$ appearing in (2.39). The analytic continuation of this term in $n$ is straightforward. We first apply $D$:

$$D \left[ -\frac{n}{2} \langle OO \rangle_n'' + \frac{1}{2} \left\langle \left( \sum_{k=0}^{n-1} O^{(k)} \right)^2 \right\rangle_n'' \right]$$

$$= -\pi \langle OO \Delta K \rangle$$

(2.41)

where $\Delta K \equiv K - \langle K \rangle$ is the vacuum-subtracted modular Hamiltonian. When an expectation value $\langle \ldots \rangle$ appears without a subscript it is understood to refer to the normalized expectation value $\langle \ldots \rangle_n$ with $n = 1$, i.e., the angle-ordered expectation value with respect to $\rho^{(0)}$. Also
note that $K$ appears outside of the angle-ordering in the trace form of the expectation value, which is formally equivalent to being inserted at $\theta = 0$.

We now consider the $\lambda$ dependence. Recall that $K$ is defined to be $\lambda$-independent, and the $\lambda$-dependence of $\mathcal{O}$ enters through a shift in the coordinate insertion of $\partial \Phi$ (see (2.16)). We first split $\Delta K$ into $\Delta K_{\text{pen}}$ and $\Delta K_{\text{aux}}$. The expectation value involving $\Delta K_{\text{aux}}$ will be independent of $\lambda$ because of translation invariance of the CFT, and so can be ignored. Since $K_{\text{pen}}$ is the CFT boost generator on the half-line $x > 0$, $\Delta K_{\text{pen}}$ has a well-known expression in terms of the energy-momentum tensor of the CFT [168, 20]:

$$\Delta K_{\text{pen}} = A \int_0^\infty dx \, x \, T_{kk}(x) = -\frac{1}{2\pi} \int_0^\infty dx \, x \, T(x).$$

(2.42)

Therefore the correlation function (2.41) is expressed in terms of the correlation functions $\langle \partial \Phi(z - \lambda) \partial \Phi(w - \lambda) T(x) \rangle$, which are the same as $\langle \partial \Phi(z) \partial \Phi(w + \lambda) T(x + \lambda) \rangle$ by translation invariance. This makes the $\lambda$-derivatives easy to evaluate. We find

$$\mathcal{D} \frac{-n}{2} \langle \mathcal{O} \mathcal{O} \rangle'' = \frac{1}{2} \langle \mathcal{O} \mathcal{O} T(0) \rangle.$$ 

(2.43)

Inserting the explicit form of $\mathcal{O}$ gives

$$\langle \mathcal{O} \mathcal{O} T(0) \rangle = \frac{1}{(2\pi)^2} \sum_{i,j,i',j'} \int dr \, dr' \, d\theta \, d\theta' \left( f_{ij}^{(m)}(r) f_{ij'}^{(m')} (r') e^{-im\theta} e^{-im'\theta'} \right. 
\times \langle \partial \Phi(r e^{i\theta}) \partial \Phi(r' e^{i\theta'}) T(0) \rangle \langle E_{ij}(\theta) E_{ij'}(\theta') \rangle \right),$$

(2.44)

where we have introduced Fourier representations of $f_{ij}(r, \theta)$ defined by

$$f_{ij}(r, \theta) = \frac{1}{2\pi} \sum_{m=\infty}^{\infty} f_{ij}^{(m)}(r) e^{-im\theta}.$$ 

(2.45)

The correlation functions we need are evaluated in the appendix. Plugging equation (A.12) with $n = 1$ and equation (A.6) into equation (2.44) yields

$$\langle \mathcal{O} \mathcal{O} T(0) \rangle$$

$$= \frac{-2}{(2\pi)^2} \sum_{i,j,i',j'} \int \frac{dr \, dr' \, d\theta \, d\theta'}{(rr')^2} f_{ij}^{(m)}(r) f_{ij'}^{(m')} (r') e^{-\pi(K_i + K_j)} \frac{\sinh \pi \alpha_{ij}}{ip + \alpha_{ij}} e^{i\theta(p-m'-2)} e^{i\theta'(p-m-2)},$$

$$= \frac{1}{\pi} \sum_{i,j,m} \int \frac{dr \, dr'}{(rr')^2} f_{ij}^{(m-2)}(r) f_{ji}^{(m-2)} (r') e^{-\pi(K_i + K_j)} \frac{\sinh \pi \alpha_{ij}}{im - \alpha_{ij}},$$

(2.46)

where we used the Kronecker deltas coming from the $\theta$ integration and redefined the dummy variable $m \to m - 2$, and $\alpha_{ij} = K_i - K_j$ is the difference between two eigenvalues of $K_{\text{aux}}$. 
CHAPTER 2. PROOF OF THE QUANTUM NULL ENERGY CONDITION

Note that we reserve the letters $p$ and $q$ throughout to denote integers divided by $n$, but in this case $n = 1$ and so $p$ ranges over the integers. Substituting equation (2.46) into equation (2.43), we find

$$D \frac{-n}{2} \langle \mathcal{O} \rangle''_n = \frac{1}{2\pi} \sum_{i,j,m} \int \frac{dr \, dr'}{(rr')^2} f_{ij}^{(m-2)}(r) f_{ji}^{(m-2)}(r') e^{-\pi(K_i + K_j)} \sinh \frac{\pi \alpha_{ij}}{im - \alpha_{ij}}. \quad (2.47)$$

Evaluation of Multi-Sheet Correlator

We now turn to the second term in (2.39),

$$\frac{1}{2} D \left\langle \left( \sum_{k=0}^{n-1} \mathcal{O}^{(k)} \right)^2 \right\rangle''_n. \quad (2.48)$$

The analytic continuation of this term to real $n$ will turn out to be much more challenging than that of the first term of (2.39), because $n$ appears in the upper summation limit.

Using (2.16), can write the sum over replicas in (2.48) as follows:

$$\left\langle \left( \sum_{k=0}^{n-1} \mathcal{O}^{(k)} \right)^2 \right\rangle''_n = \left\langle \left( \sum_{i,j} \int_0^{2\pi n} dr d\theta \ f_{ij}(r, \theta) \partial \Phi(r, \theta; \lambda) \otimes E_{ij}(\theta) \right)^2 \right\rangle''_n. \quad (2.49)$$

This equality comes from interpreting $\mathcal{O}^{(k)}$ as $\mathcal{O}$ inserted on the $(k+1)$th replica sheet (see (2.31)). Summing over sheets and integrating $\theta \in [0, 2\pi]$ on each one is equivalent to just integrating $\theta \in [0, 2\pi n]$, which covers the entire replicated manifold. The definition of $\partial \Phi$ for angles greater than $2\pi$ is given by the Heisenberg evolution rule, the right hand side of (2.10). The field is still holomorphic, but it would be misleading to write it as a function of $re^{i\theta}$ since it is not periodic in $\theta$ with period $2\pi$.

Because the $f_{ij}(r, \theta)$ are not dynamical, they should be identical on each sheet. In the Fourier representation as in (2.45), this means keeping the Fourier coefficients fixed and keeping the $m$ parameters integer. Thus we have

$$\frac{1}{2} D \left\langle \left( \sum_{k=0}^{n-1} \mathcal{O}^{(k)} \right)^2 \right\rangle''_n = \frac{1}{2(2\pi)^2} \sum_{i,j,i',j'} \sum_{m,m'} \int dr \, dr' \, d\theta \, d\theta' \ f_{ij}^{(m)}(r) f_{i'j'}^{(m')} (r') e^{-im\theta} e^{-im'\theta'} \times \langle \partial \Phi(r, \theta) \partial \Phi(r', \theta') \rangle''_n \langle E_{ij}(\theta) E_{i'j'}(\theta') \rangle_n. \quad (2.50)$$

The CFT two point function is calculated in Appendix A.1:

$$\langle \partial \Phi(z) \partial \Phi(w) \rangle''_n = \frac{1}{n(zw)^2} \sum_{|q| < 1} \text{sign}(q) q (q^2 - 1) \left( \frac{w}{z} \right)^q \quad (2.51)$$

$$= \frac{1}{n(rr')^2} \sum_{|q| < 1} \text{sign}(q) P(q, r, r') e^{i\theta(-q-2)} e^{i\theta'(q-2)} \quad (2.52)$$
where \( q \) takes values in the integers divided by \( n \), and

\[
P(q, r, r') \equiv q(q^2 - 1) \left( \frac{r'}{r} \right)^q.
\]

(2.53)

When \( n = 1 \) there are no nonzero terms in the sum, but when \( n > 1 \) the answer is nonzero. For future convenience, we separated the parts which depend on \( \theta \) from those that do not.

The auxiliary system two point function is calculated in Appendix A.1:

\[
\langle E_{ij}(\theta)E_{i'j'}(\theta') \rangle_n = \delta_{ij}\delta_{i'j'}e^{-2\pi n K_i} \frac{1}{\pi n \tilde{Z}_{aux}^n} \sum_p e^{-ip(\theta-\theta')} \sinh \frac{n\pi \alpha_{ij} e^{n\pi \alpha_{ij}}}{ip + \alpha_{ij}},
\]

where \( p \) is also an integer divided by \( n \) and \( \tilde{Z}_{aux}^n \equiv \text{Tr} \left[ e^{-2\pi n K_{aux}} \right] \) is a normalization factor. Substituting this equation as well as (2.52) into (2.50) gives

\[
\mathcal{D} \frac{1}{n^2 (2\pi)^3 \tilde{Z}_{aux}^n} \sum_{i,j,p} \frac{dr dr' d\theta d\theta'}{(rr')^2} f_{ij}^{(m)}(r) f_{ji}^{(m')}(r') e^{ip(\theta - \theta')} e^{i\theta (-q - p - 2 - m)} e^{i\theta (q + p - 2 - m')}
\]

\[
\times \frac{\sinh \pi n \alpha_{ij} e^{-n(\pi K_i + K_j)}}{ip + \alpha_{ij}} \sum_{|q| < 1} \text{sign}(q) P(q, r, r').
\]

(2.54)

The angle integrations give Kronecker deltas multiplied by \( 2\pi n \). The result is

\[
\mathcal{D} \frac{i}{2\pi \tilde{Z}_{aux}^n} \sum_{i,j,m} \left| \int \frac{dr dr'}{(rr')^2} f_{ij}^{(m-2)}(r) f_{ji}^{(m-2)}(r') \sinh \pi n \alpha_{ij} e^{-\pi n(K_i + K_j)} \right| \sum_{|q| < 1} \frac{\text{sign}(q) P(q, r, r')}{q + m + i\alpha_{ij}}
\]

\[
= \frac{i}{2\pi} \sum_{i,j,m} \left| \int \frac{dr dr'}{(rr')^2} f_{ij}^{(m-2)}(r) f_{ji}^{(m-2)}(r') \sinh \pi \alpha_{ij} e^{-\pi(K_i + K_j)} \mathcal{D} \right| \sum_{|q| < 1} \frac{\text{sign}(q) P(q, r, r')}{q + m + i\alpha_{ij}}.
\]

(2.55)

In going to the last line, we used the fact that the sum in brackets vanishes when \( n = 1 \) and that, for any two functions \( f(n), g(n) \) such that \( f(1) \) and \( \left[ \frac{d}{dn} f(n) \right]_{n=1} \) are finite and \( g(1) = 0 \), the following relation holds:

\[
\mathcal{D} (f(n) g(n)) = f(1) \mathcal{D} g(n).
\]

(2.56)

We now turn to the analytic continuation and application of \( \mathcal{D} \) on the term in brackets in (2.56). We will take care of the awkward \( \text{sign}(q) \) by writing the \( q \)-dependent part of the sum as two sums with positive argument. We will suppress the \((r, r')\) dependence for the rest of the calculation:

\[
\sum_{|q| < 1} \frac{\text{sign}(q) P(q)}{q + m + i\alpha_{ij}} = \sum_{0 < q < 1} \frac{P(q)}{q + m + i\alpha_{ij}} + \frac{P(-q)}{q - m - i\alpha_{ij}}.
\]

(2.57)
Now we write \( q = k/n \) to turn this into a sum over integers:

\[
\sum_{0 < q < 1} \left( \frac{P(q)}{q + m + i\alpha_{ij}} + \frac{P(-q)}{q - m - i\alpha_{ij}} \right) = \sum_{k=1}^{n-1} \left( \frac{P(k/n)}{k/n + m + i\alpha_{ij}} + \frac{P(-k/n)}{k/n - m - i\alpha_{ij}} \right). \tag{2.59}
\]

In the next section we will see how to evaluate and analytically continue such sums quite generally.

**Analytic Continuation**

We need to evaluate

\[
\mathcal{D} \sum_{k=1}^{n-1} \left( \frac{P(k/n)}{k/n - z} + \frac{P(-k/n)}{k/n + z} \right), \tag{2.60}
\]

where \( P(z) \) is given by (2.52). However, for the remainder of this section, we will consider \( P(z) \) to be an arbitrary analytic function whose functional form is independent of \( n \). We will specialize to the form given by (2.52) in section 2.4.

We start by writing the sum in (2.60) as

\[
\mathcal{D} \sum_{k=1}^{n-1} \left( \frac{P(k/n) - P(z)}{k/n - z} + \frac{P(-k/n) - P(z)}{k/n + z} \right) + \mathcal{D} \sum_{k=1}^{n-1} \left( \frac{P(z)}{k/n - z} + \frac{P(z)}{k/n + z} \right) \tag{2.61}
\]

and then we evaluate the terms separately. Consider the first term in the first set of parenthesis. Because \( P(z) \) is analytic, we can expand it in a power series with positive powers of \( z \): \( P(z) = \sum_{r=0}^\infty a_r z^r \). This gives

\[
\mathcal{D} \sum_{k=1}^{n-1} \sum_{r=1}^{\infty} a_r \frac{(k/n)^r - z^r}{k/n - z}. \tag{2.62}
\]

We can simplify the fraction using polynomial division; for \( r \geq 1 \),

\[
\frac{(k/n)^r - z^r}{k/n - z} = \sum_{s=0}^{r-1} z^{r-s-1} \left( \frac{k}{n} \right)^s, \tag{2.63}
\]

which means the first term in the first set of parenthesis in (2.61) is

\[
\mathcal{D} \sum_{k=1}^{n-1} \frac{P(k/n) - P(z)}{k/n - z} = \sum_{r=1}^{\infty} \sum_{s=0}^{r-1} a_r z^{r-s-1} \mathcal{D} \sum_{k=1}^{n-1} \left( \frac{k}{n} \right)^s. \tag{2.64}
\]

The advantage of writing it this way is that it isolates the \( n \) dependence into something which can be easily analytically continued. First, recall that overall factors of powers of \( n \)}
don’t matter if the expression they multiply vanishes at \( n = 1 \), as in (2.57). Next, note that the resulting expression is actually a polynomial in \( n \). It can be expressed this way using Faulhaber’s formula:

\[
\sum_{k=1}^{n-1} k^s = \frac{1}{s+1} \sum_{j=0}^{s} (-1)^j \binom{s+1}{j} B_j (n-1)^{s-j+1},
\]

where \( B_s \) is the \( j \)-th Bernoulli number in the convention that \( B_1 = -1/2 \). This makes application of \( D \) straightforward:

\[
\mathcal{D} \sum_{k=1}^{n-1} \left( \frac{k}{n} \right)^s = \mathcal{D} \sum_{k=1}^{n-1} k^s = (-1)^s B_s.
\]

Thus for the first term in (2.61) we have

\[
\mathcal{D} \sum_{k=1}^{n-1} \frac{P(k/n) - P(z)}{k/n - z} = - \sum_{r=1}^{\infty} \sum_{s=0}^{r-1} a_r z^{-r-s-1} (-1)^s B_s.
\]

The second term follows completely analogously:

\[
\mathcal{D} \sum_{k=1}^{n-1} \frac{P(-k/n) - P(z)}{k/n + z} = \sum_{r=1}^{\infty} \sum_{s=0}^{r-1} a_r z^{-r-s-1} B_s.
\]

Combining these results, the first set of large parenthesis in (2.61) is

\[
- \sum_{r=1}^{\infty} \sum_{s=0}^{r-1} a_r z^{-r-s-1} B_s \left[ (-1)^s - 1 \right].
\]

For even \( s \) this is zero. For odd \( s > 1 \), \( B_s = 0 \), and so only \( s = 1 \) can contribute. Substituting \( B_1 = -1/2 \) gives

\[
- \frac{P(z)}{z^2} + \frac{a_1}{z} + \frac{a_0}{z^2}.
\]

We now turn to the second set of parenthesis in (2.61). These two terms can be evaluated simultaneously. First, we can multiply through by \( n/n \) to give an overall factor of \( n \) (which is irrelevant) and convert the denominators to \( k - zn \) and \( k + zn \). We also pull \( P(z) \) through \( \mathcal{D} \) because it is independent of \( n \):

\[
P(z) \mathcal{D} \sum_{k=1}^{n-1} \left( \frac{1}{k - zn} + \frac{1}{k + zn} \right).
\]
This sum can be evaluated in terms of the digamma function \( \psi^{(0)}(w) \), which is defined in terms of the Gamma function \( \Gamma(w) \):

\[
\psi^{(0)}(w) \equiv \frac{\Gamma'(w)}{\Gamma(w)} = -\gamma + \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{k+w} \right). \tag{2.72}
\]

By manipulating the sum, one can show

\[
\sum_{k=1}^{n-1} \frac{1}{k-w} = \psi^{(0)}(n-w) - \psi^{(0)}(1-w). \tag{2.73}
\]

Thus the second set of parenthesis in (2.61) is equal to

\[
P(z)\mathcal{D} \left[ \psi^{(0)}(n-zn) - \psi^{(0)}(1-zn) + \psi^{(0)}(n+zn) - \psi^{(0)}(1+zn) \right]. \tag{2.74}
\]

We cannot naively apply \( \mathcal{D} \) yet. We first have to select the correct analytic continuation to real positive \( n \) from the many possible analytic continuations of integer \( n \) data. This is known to be a challenging problem in general.\(^8\) Nevertheless, in our context the correct analytic continuation prescription is clear.

The digamma function has poles in the complex plane at zero and all negative real integers. Recall that we are ultimately interested in plugging in \( z_m \equiv -m - i\alpha_{ij} \). Thus if we are not careful, for certain values of \( m \), the digamma functions in (2.74) will blow up when \( \alpha_{ij} \to 0 \) near \( n = 1 \). On the other hand, on physical grounds we expect our result to be perfectly well-behaved when \( \alpha_{ij} \to 0 \), which simply corresponds to a degeneracy in the auxiliary system. The way we avoid the poles of the digamma function near \( n = 1 \) as \( \alpha_{ij} \to 0 \) is by using the reflection formula

\[
\psi^{(0)}(1-w) = \psi^{(0)}(w) + \pi \cot \pi w, \tag{2.75}
\]

which produces different analytic continuations given the same integer data. These observations lead to the following prescription: for each value of \( m \), use the reflection formula (2.75) to avoid the poles of the digamma function near \( n = 1 \) as \( \alpha_{ij} \to 0 \).

As an example, consider the term \( \psi^{(0)}(1 - z_m n) = \psi^{(0)}(1 + mn + \alpha_{ij} n) \) in (2.74). When \( \alpha_{ij} = 0 \), this has a pole when \( nm \leq 1 \). Thus for a given \( m \leq 1 \), we cannot expect to have a smooth \( n \)-derivative at \( n = 1 \). The resolution is to use (2.75) to get

\[
\psi^{(0)}(1 + mn + \alpha_{ij} n) = \psi^{(0)}(-mn - \alpha_{ij}) - \pi \cot \pi (mn + \alpha_{ij}) \tag{2.76}
\]

\[
= \psi^{(0)}(-mn - \alpha_{ij}) - \pi \cot \pi \alpha_{ij}, \tag{2.77}
\]

\(^8\)See Ref. [67] for a recent discussion of the difficulties of the analytic continuation. Ref. [67] also contains another method for computing the entropy perturbatively that does not rely on the replica trick. Such a method avoids the need to analytically continue, and applying it to the present calculation would serve as a check of our analytic continuation prescription. We leave that check to future work.
where the last equality is only true for integer \( n \). The remaining digamma term is now free of poles for \( mn \leq 1 \), which is precisely when there was a problem before the application of the reflection formula, and \( D \) can now be easily applied. This example illustrates how the correct analytic continuation depends on the value of \( m \). We must apply this reasoning separately to each term in (2.74). After applying this procedure to each digamma function as needed to avoid the poles, it will turn out that all of the extra cotangent terms cancel against each other.

There is another way to motivate this prescription. Even for small but finite \( \alpha_{ij} \), the analytic continuations picked out by our prescription can be seen to be qualitatively better than the one obtained by using (2.74) directly, as illustrated in Figure 2.3. Notice that while both curves match for integer \( n \), the curve obtained by applying the prescription outlined above is the only one which smoothly interpolates between the integers. The oscillations of the “wrong” curves get larger and larger as \( \alpha_{ij} \) is reduced or \( m \) is increased.

Applying our prescription to (2.74), there are three expressions depending on the value of \( m \). We are focussing on the quantity in brackets in (2.74):

\[
\left\{ \begin{array}{ll}
\psi^{(0)}(1 - n - nz_m) - \psi^{(0)}(-nz_m) + \psi^{(0)}(n - nz_m) - \psi^{(0)}(1 - nz_m) & m > 0 \\
\psi^{(0)}(n + nz_m) - \psi^{(0)}(1 + nz_m) + \psi^{(0)}(n - nz_m) - \psi^{(0)}(1 - nz_m) & m = 0 \\
\psi^{(0)}(n + nz_m) - \psi^{(0)}(1 + nz_m) + \psi^{(0)}(1 - n + nz_m) - \psi^{(0)}(nz_m) & m < 0
\end{array} \right. 
\] (2.78)

Now we are ready to apply \( D \). The digammas \( \psi^{(0)}(w) \) will turn into polygammas \( \psi^{(1)}(w) \equiv \frac{d}{dw} \psi^{(0)}(w) \), which obey the recurrence relation

\[
\psi^{(1)}(w + 1) = \psi^{(1)}(w) - \frac{1}{w^2}. 
\] (2.79)
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This recurrence relation simplifies the result for $m > 0$ and $m < 0$ while the recurrence relation along with the reflection formula simplifies the result for $m = 0$. The result for the second set of parenthesis in (2.61) with $z = z_m$ is

$$
\frac{P(z_m)}{z_m^2} + \delta(m) P(z_m) \frac{\pi^2}{\sinh^2 \pi \alpha_{ij}} .
$$

We are now ready to give the final expression for (2.60). Adding (2.70) with $z = z_m$ and (2.80) we find

$$
D \sum_{k=1}^{n-1} \left( \frac{P(k/n)}{z - z_m} + \frac{P(-k/n)}{z + z_m} \right) = a_1 + \frac{a_0}{z_m^2} + \delta(m) P(-i \alpha_{ij}) \frac{\pi^2}{\sinh^2 \pi \alpha_{ij}}
$$

(2.81)

for arbitrary analytic $P(z_m)$.

Completing the Proof

Now we specialize to the form of $P(z)$ needed for our calculation which came from the particular $\langle \partial \Phi \partial \Phi \rangle^{''}$ two-point function we were computing ((2.52) and (2.53)):

$$
P(z) = z(z^2 - 1)e^{z \log (r'/r)} .
$$

(2.82)

Thus $a_0 = 0$, and $a_1 = -1$. Using (2.81) gives

$$
D \sum_{k=1}^{n-1} \left( \frac{P(k/n)}{z - z_m} + \frac{P(-k/n)}{z + z_m} \right) = i \frac{m - \alpha_{ij}}{im - \alpha_{ij}} + \delta(m) i \pi \frac{2 \sinh^2 \pi \alpha_{ij}}{\sinh^2 \pi \alpha_{ij}} \alpha_{ij}(\alpha_{ij}^2 + 1) \left( \frac{r'}{r} \right)^{-i \alpha_{ij}} .
$$

(2.83)

Plugging this into (2.56) and plugging that into (2.49) gives the term from (2.39) that we have been focussing on in this section:

$$
D \frac{1}{2} \left\langle \left( \sum_{k=0}^{n-1} O^{(k)} \right)^2 \right\rangle_n^{''} = -\frac{1}{2\pi} \sum_{i,j,m} \int \frac{dr dr'}{(rr')^2} f^{(m-2)}_{ij}(r)f^{(-m-2)}_{ji}(r') \sinh \pi \alpha_{ij} e^{-\pi (K_i + K_j)}

\times \left[ \frac{1}{im - \alpha_{ij}} + \delta(m) \frac{\alpha_{ij}}{\sinh^2 \pi \alpha_{ij}} \alpha_{ij}(\alpha_{ij}^2 + 1) \left( \frac{r'}{r} \right)^{-i \alpha_{ij}} \right] .
$$

(2.84)

Notice that the first term in this expression exactly cancels the contribution to $S^{(2)''}$ coming from the first term in (2.39), presented in (2.47). We now consider the second term, and define the manifestly positive quantity $M_{ij} \equiv e^{-\pi (K_i + K_j)} \pi^2 (\alpha_{ij}^2 + 1)$ to clean up the notation. Then we have

$$
S^{(2)''} = -\frac{1}{2\pi} \sum_{i,j} \int \frac{dr dr'}{(rr')^2} f^{(-2)}_{ij}(r)f^{(-2)}_{ji}(r') \left( \frac{r'}{r} \right)^{-i \alpha_{ij}} \frac{\alpha_{ij}}{\sinh \pi \alpha_{ij}} M_{ij} .
$$

(2.85)
The integrals over \( r, r' \) factorize, giving
\[
S^{(2)''} = -\frac{1}{2\pi} \sum_{i,j} \left[ \int_0^\infty dr \ r^{i\alpha_{ij}-1} f_{ij}^{(-2)}(r) \right] \left[ \int_0^\infty dr \ r^{-i\alpha_{ij}-1} f_{ji}^{(-2)}(r) \right] \frac{\alpha_{ij}}{\sinh \pi \alpha_{ij} M_{ij}}. \tag{2.86}
\]

Recall the constraint on the test functions derived previously by requiring the density matrix be Hermitian (equation (2.11)): \( f_{ij}(r, \theta) = f_{ji}(r, 2\pi - \theta)^\ast \). In Fourier space, this implies \( f_{ji}^{(m)}(r) = f_{ij}^{(m)}(r)^\ast \). Inserting this into (2.86) we see that the factors in brackets are complex-conjugates of each other. Furthermore, because \( \sinh \pi \alpha_{ij} \) always has the same sign as \( \alpha_{ij} \), the overall sign of the entire term is negative and so we find
\[
S^{(2)''} \leq 0. \tag{2.87}
\]

As discussed after (2.24), this proves the QNEC.

### 2.5 Extension to \( D = 2 \), Higher Spin, and Interactions

In \( D = 2 \), there are no transverse directions, and so it is not possible to use the fact that the state is very close to the vacuum. Nevertheless, once one has proven the QNEC for a free scalar field in \( D > 2 \), one can use dimensional reduction to prove it for free scalar fields in \( D = 2 \). Let \( \Phi(z, y) \) be the chiral scalar on \( N \) in \( D > 2 \), where \( y \) labels the \( D - 2 \) transverse coordinates. One can isolate a single transverse mode by integrating \( \Phi(z, y) \) against a real transverse wavefunction, and this defines an effective two-dimensional field:
\[
\Phi_{2D}(z) \equiv \int dy \psi(y)\Phi(z, y), \tag{2.88}
\]
where \( \psi \) is normalized such that \( \int \psi^2 = 1 \). Correlation functions of \( \Phi_{2D} \) and its derivatives exactly match those of a two-dimensional chiral scalar, and so our dimensional reduction is defined by the subspace of the \( D \)-dimensional theory obtained by acting on the vacuum with \( \Phi_{2D} \). In any such state, one can integrate the \( D \)-dimensional QNEC along the transverse direction to find
\[
\int dy \langle \mathcal{T}_{kk}(y) \rangle \geq \frac{1}{2\pi} \int dy \frac{\delta^2 S_{\text{out}}}{\delta \lambda(y)^2}. \tag{2.89}
\]

Here we have suppressed the value of the affine parameter as a function of the transverse direction. The effective two-dimensional change in the entropy is defined by considering a total variation in all of the generators which is uniform in the transverse direction. For such a variation we have
\[
S''_{2D} = \int dy dy' \frac{\delta^2 S_{\text{out}}}{\delta \lambda(y) \delta \lambda(y')} \leq \int dy \frac{\delta^2 S_{\text{out}}}{\delta \lambda(y)^2}, \tag{2.90}
\]
where the the inequality comes from applying strong subadditivity to the off-diagonal second derivatives [34]. The two-dimensional energy momentum tensor is defined in terms of the
normal ordered product of the two-dimensional fields, $T_{2D} =: \partial \Phi_{2D} \partial \Phi_{2D} :$. However, using Wick’s theorem one can easily check that $T_{2D}$ acts on the dimensionally reduced theory in the same way as the integrated $D$-dimensional $T_{kk}$:

$$\langle T_{2D}(w)\Phi_{2D}(z_1)\cdots \Phi_{2D}(z_n) \rangle = \int dy \langle T_{kk}(w,y)\Phi_{2D}(z_1)\cdots \Phi_{2D}(z_n) \rangle.$$  \hfill (2.91)

Therefore the QNEC holds for a free scalar field in two dimensions:

$$\langle T_{2D} \rangle = \int dy \langle T_{kk}(y) \rangle \geq \frac{1}{2\pi} \int dy \frac{\delta^2 S_{out}}{\delta \lambda(y)^2} \geq \frac{1}{2\pi} S''_{2D}. \hfill (2.92)$$

The extension to bosonic fields with spin is trivial, as these simply reduce on $N$ to multiple copies of the 1+1 chiral scalar CFT, one for each polarization. These facts are reviewed in [174]. Similarly, fermionic fields reduce to the chiral 1+1 fermion CFT; we expect that there is a similar proof in this case.

Astute readers may have noticed that the mass term of the higher dimensional field theory plays no role in our analysis. Since it does not contribute to the commutation relations on $N$ or to $T_{kk}$, it plays no role in our analysis. Regardless of whether the $D$ dimensional theory has a mass, the 1 + 1 chiral theory is massless. In a sense, null surface quantization is a UV limit of the field theory. One might therefore expect that the addition of interactions with positive mass dimension (superrenormalizable couplings) will also not change the algebra of observables on $N$. So long as this is the case, the extension to theories with superrenormalizable interactions is trivial.

One argument that superrenormalizable interactions are innocuous proceeds in two stages [174]. First, one considers the direct effects of adding interaction terms to the Lagrangian; for example a scalar field potential $V(\phi)$. So long as these interaction terms contain no derivatives (or are Yang-Mills couplings), they do not contribute to the commutation relations of fields restricted to the null surface, or to $T_{kk}$. (So far, the interaction could be of any scaling dimension, so long as one avoids derivative couplings.)

Next, one considers loop corrections due to renormalization. In the case of a marginally renormalizable, or nonrenormalizable theory, these loop corrections normally require the addition of counterterms containing derivatives (for example, field strength renormalization), spoiling the null surface formulation. On the other hand, in a superrenormalizable theory, only couplings with positive mass dimension require counterterms. For a standard QFT consisting of scalars, spinors, and/or gauge fields, none of these superrenormalizable interactions include the possibility derivative couplings. Thus one expects that loop corrections do not spoil the algebra of observables on the null surface. However, superrenormalizable theories are difficult to construct except when $D < 4$. (For example, the $\phi^3$ theory is superrenormalizable in $D < 6$, but is unstable.)

It is an open question whether the QNEC is valid for non-Gaussian $D = 2$ CFT’s in states besides conformal vacua, or more generally for QFT’s in any dimension which flow...
to a nontrivial UV fixed point. Nor have we carefully considered the effects of making the scalar field noncompact. QCD in $D = 4$ is a borderline case; the coupling flows to zero, but slowly enough that there is an infinite field strength renormalization. Strictly speaking this makes null surface quantization invalid, yet it is still a useful numerical technique for studying hadron physics [36]. However, we conjecture that the QNEC will be true in every QFT satisfying reasonable axioms.

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9In more than 2 dimensions, interacting CFTs appear to have no nontrivial observables on the horizon[174, 31], so the current proof cannot be extended to this situation.
Chapter 3

Holographic Proof of the Quantum Null Energy Condition

3.1 Introduction

The Null Energy Condition (NEC), \( T_{kk} \equiv T_{ij}k^{i}k^{j} \geq 0 \), is ubiquitous in classical physics as a signature of stable field theories. In General Relativity it underlies many results, such as the singularity theorems [146, 93, 172] and area theorems [91, 28]. In AdS/CFT, imposing the NEC in the bulk has several consequences for the field theory at leading order in large-\( N \), including the holographic c-theorems [137, 138, 77] and Strong Subadditivity of the covariant holographic entanglement entropy [177]. Yet ultimately the NEC, interpreted as a local bound on the expectation value \( \langle T_{kk} \rangle \), is known to fail in quantum field theory [65].

The Quantum Null Energy Condition (QNEC) was proposed in [34] as a correction the NEC which holds true in quantum field theory. In the QNEC, \( \langle T_{kk} \rangle \) at a point \( p \) is bounded from below by a nonlocal quantity constructed from the von Neumann entropy of a region. Suppose we divide space into two regions, one of which we call \( R \), with the dividing boundary \( \Sigma \) passing through \( p \). We compute the entropy of \( R \), and consider the second variation of the entropy as \( \Sigma \) is deformed in the null direction \( k^{i} \) at \( p \). Call this second variation \( S'' \) (a more careful construction of \( S'' \) is given in below in Section 3.2). Then the QNEC states that

\[
\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi \sqrt{\hbar}} S'',
\]

where \( \sqrt{\hbar} \) is the determinant of the induced metric on \( \Sigma \) at the location \( p \).\(^1\) The QNEC has its origins in quantum gravity: it arose as a consequence of the Quantum Focussing Conjecture (QFC), proposed in [34], but is itself a statement about quantum field theory alone.

\(^1\)In general, there may be ambiguities in the definition of \( T_{kk} \) because of “improvement terms.” It is plausible that a similar ambiguity in the definition of \( S \) leaves the QNEC unaffected by these issues [49, 3, 97, 122].


In [33], the QNEC was proved for the special case of free (or superrenormalizable) bosonic field theories for certain surfaces $\Sigma$. Here we will prove the QNEC for a completely different class of field theories, namely those which have a good gravity dual, at leading order in the large-$N$ expansion. We will consider any theory obtained from such a large-$N$ UV CFT by a scalar relevant deformation. We will also assume that the bulk theory is an Einstein gravity theory, so that the leading order part of the entropy is given by the area of an extremal surface in the bulk in Planck units:

$$S = \frac{A(m)}{4G_N\hbar},$$

(3.2)

where $A(m)$ is the area of a bulk codimension-two surface $m$ which is homologous to $R$ and is an extremum of the area functional in the bulk [152, 153, 101]. Computing that change in the extremal area as the surface $\Sigma$ is deformed is then a simple task in the calculus of variations.2

A key property is that the change in area of an extremal surface under deformations is due entirely to the near-boundary asymptotic region, where a general analytic computation is possible.

Our proof method involves tracking the motion of $m$ as $\Sigma$ is deformed. The “entanglement wedge” proposal for the bulk region dual to $R$, together with bulk causality, suggests that $m$ should move in a spacelike way as we deform $\Sigma$ in our chosen null direction [53, 96], and a theorem of Wall [177] shows that this is, in fact, correct.3 We construct a bulk vector $s^\mu$ in the asymptotic bulk region which points in the direction of the deformation of $m$, and since $s^\mu$ is spacelike we have $s^\mu s_\mu \geq 0$. Holographically, $\langle T_{kk}\rangle$ is encoded in the near-boundary expansion of the bulk metric, and therefore enters into the expression for $s^\mu s_\mu$. We will see that the inequality $s^\mu s_\mu \geq 0$ is precisely the QNEC.4

The remainder of the paper is organized as follows. In Section 3.2 we will give a careful account of the construction of $S''$ and the statement of the QNEC. In Section 3.3 we prove the QNEC at leading order in large-$N$ using holography. In Section 3.3 we recall the asymptotic expansions of the bulk metric and extremal surface embedding functions that we will use for the rest of our proof. In Section 3.3 we discuss the fact that null deformations of $\Sigma$ on the boundary induce spacelike deformations of $m$ in the bulk and define the spacelike vector $s^\mu$. In Section 3.3 we construct $s^\mu$ in the asymptotic region and calculate its norm, thereby proving the QNEC. Then in Section 3.3 we specialize to CFTs and examine the QNEC in

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2There can be phase transitions in the holographic entanglement entropy where $S'$ is discontinuous at leading order in $N$. This happens when there are two extremal surfaces with areas that become equal at the phase transition. Since we are instructed to use the minimum of the two areas to compute the entropy, the entropy function is always concave in the vicinity of the phase transition. Therefore $S'' = -\infty$ formally, so the QNEC is satisfied. Thus it is sufficient to assume that no phase transitions are encountered in the remainder of the paper.

3We would like to thank Zachary Fisher, Mudassir Moosa, and Raphael Bousso for discussions about the spacelike nature of these deformations, as well as bringing the theorem of [177] to our attention.

4Relations between the boundary energy-momentum tensor and a coarse-grained entropy were studied using holography in [35]. The entropy we consider in this paper is the fine-grained von Neumann entropy.
CHAPTER 3. HOLOGRAPHIC PROOF OF THE QUANTUM NULL ENERGY CONDITION

3.2 Statement of the QNEC

In this section we will give a careful statement of the QNEC. Consider an arbitrary quantum field theory in \( d \)-dimensional Minkowski space. The QNEC is a pointwise lower bound on the expectation value of the null-null component of the energy-momentum tensor, \( \bar{T}_{kk} \equiv \langle T_{ij} \rangle k^i k^j \), in any given state. Let us choose a codimension-2 surface \( \Sigma \) which contains the point of interest, is orthogonal to \( k^i \), and divides a Cauchy surface into two regions. We can assign density matrices to the two regions of the Cauchy surface and compute their von Neumann entropies. In a pure state these two entropies will be identical, but we do not necessarily have to restrict ourselves to pure states. So choose one of the two regions, which we will call \( \mathcal{R} \) for future reference, and compute its entropy \( S \). If we parameterize the surface
Σ by a set of embedding functions $X^i(y)$ (where $y$ represents $d - 2$ internal coordinates), then we can think of the entropy as a functional $S = S[X^i(y)]$.

Our analysis is centered around how the functional $S[X^i(y)]$ changes as the surface $Σ$ (and region $R$) is deformed.\(^5\) Introducing a deformation $δX^i(y)$, we can define variational derivatives of $S$ through the equation

$$\Delta S = \int dy \frac{δS}{δX^i(y)} δX^i(y) + \frac{1}{2} \int dy dy' \frac{δ^2S}{δX^i(y) δX^j(y')} δX^i(y) δX^j(y') + \cdots. \quad (3.3)$$

One might worry that the functional derivatives $δS/δX^i(y)$, $δ^2S/δX^i(y) δX^j(y')$, and so on are unphysical by themselves because we cannot reasonably consider deformations of the surface on arbitrarily fine scales. But the functional derivatives are a useful tool for compactly writing the QNEC, and we can always integrate our expressions over some small region in order to get a physically well-defined statement. Below we will do precisely that to obtain the global version of the QNEC from the local version.

The QNEC relates $T_{kk}$ to the second functional derivative of the entropy under null deformations, i.e., the second term in (3.3) in the case where $δX^i(y) = k^i(y)$ is an orthogonal null vector field on $Σ$. Let $λ$ be an affine parameter along the geodesics generated by $k^i(y)$; it will serve as our deformation parameter. Then we can isolate the second variation of the entropy by taking two derivatives with respect to $λ$:\(^6\)

$$\frac{D^2 S}{Dλ^2} = \int dy dy' \frac{δ^2S}{δX^i(y) δX^j(y')} k^i(y) k^j(y'). \quad (3.4)$$

It is important that $k^i(y)$ also satisfies a global monotonicity condition: the domain of dependence of $R$ must be either shrinking or growing under the deformation. In other words, the domain of dependence of the deformed region must either contain or be contained in the domain of dependence of the original region. By exchanging the role played by $R$ and its complement, we can always assume that the domain of dependence is shrinking. In this case the deformation has a nice interpretation in the Hilbert space in terms of a continuous tracing out of degrees of freedom. Then consider the following decomposition of the second variation of $S$ into a “diagonal” part, proportional to a $δ$-function, and an “off-diagonal” part:

$$\frac{δ^2S}{δX^i(y) δX^j(y')} k^i(y) k^j(y') = S''(y) δ(y - y') + \text{(off-diagonal)}. \quad (3.5)$$

Our notation for the diagonal part, $S''(y)$, suppresses its dependence on the surface $Σ$, but it is still a complicated non-local functional of the $X^i$. Because of the global monotonicity property of $k^i(y)$, one can show using Strong Subadditivity of the entropy that the “off-diagonal” terms are non-positive [34]. We will make use of this property below to transition from the local to the global version of the QNEC.

\(^{5}\)Deformations of $Σ$ induce appropriate deformations of $R$ [28].

\(^{6}\)We use capital-$D$ for ordinary derivatives to avoid any possible confusion with the $S''$ notation. $D$-derivatives are defined by (3.4), while $S''$ is defined by (3.5).
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For a generic point on a generic surface, \( S'' \) will contain cutoff-dependent divergent terms. It is easy to see why: the cutoff-dependent terms in the entropy are proportional to local geometric integrals on the entangling surface, and the second variation of such terms is present in \( S'' \).\(^7\) By restricting the class of entangling surfaces we consider, we can guarantee that the cutoff-dependent parts of the entropy have vanishing second derivative. In the course of our proof (see section 3.3), we will find that a sufficient condition to eliminate all cutoff-dependence in \( S'' \) is that \( k_i K_{ab}^i = 0 \) in a neighborhood of the location where we wish to bound \( T_{kk} \), where \( K_{ab}^i \) is the extrinsic curvature tensor of \( \Sigma \) (also known as the second fundamental form).\(^8\) The locality of this statement should be emphasized: away from the point where we wish to bound \( T_{kk} \), \( \Sigma \) can be arbitrary.

Finally, we can state the QNEC. When \( k^i(y) \) satisfies the global monotonicity constraint and \( k_i K_{ab}^i = 0 \) in a neighborhood of \( y = y_0 \), we have

\[
T_{kk} \geq \frac{1}{2\pi \sqrt{h}} S''
\]

(3.6)

where \( \sqrt{h} \) is the surface volume element of \( \Sigma \) and all terms are evaluated at \( y = y_0 \). A few remarks are in order. In \( d = 2 \), the requirement \( k_i K_{ab}^i = 0 \) is trivial. In that case we are also able to prove the stronger inequality

\[
T_{kk} \geq \frac{1}{2\pi} \left[ S'' + \frac{6}{c} (S')^2 \right].
\]

(3.7)

Here \( S' \equiv k^i \delta S/\delta X^i \) and \( c \) is the central charge of the UV fixed point of the theory. This stronger inequality in \( d = 2 \) is actually implied by the weaker one in the special case of a CFT by making use of the conformal transformation properties of the entropy [179], though here we will prove it even when the theory contains a relevant deformation. One can use similar logic in \( d > 2 \) to generalize the statement of the QNEC when applied to a CFT. By Weyl transformation, we can transform a surface that has \( k_i K_{ab}^i = 0 \) to one where \( k_i K_{ab}^i h^{ab} \neq 0 \), though the trace-free part still vanishes. In that case, we will find

\[
T_{kk} - A^{(T)}_{kk} \geq \frac{1}{2\pi \sqrt{h}} \left[ (S_{\text{fin}} - A^{(S)})'' + \frac{2\theta}{d-2} \left( S_{\text{fin}} - A^{(S)} \right)' \right]
\]

(3.8)

for CFTs in \( d > 2 \), where \( \theta \equiv -k_i K_{ab}^i h^{ab} \) is the expansion in the \( k^i \) direction, and \( A^{(T)}_{kk} \) and \( A^{(S)} \) are anomalous shifts in \( T_{kk} \) and \( S \), respectively [86]. The two anomalies are both zero in odd dimensions, and \( A^{(T)}_{kk} \) is zero for global conformal transformations in Minkowski space. \( A^{(S)} \) is a local geometric functional of \( \Sigma \), and may be non-zero even when \( A^{(T)}_{kk} \) vanishes. The finite part of the entropy appears in this equation because we are starting with the

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\(^7\)Although it is the case that all of the cutoff-dependence in the second variation of the entropy is contained in the diagonal part, which we have called \( S'' \), it is still true that \( S'' \) contains finite terms as well. If it did not, the QNEC would be the same as the NEC.

\(^8\)\( K_{ab}^i \) is defined as \( D_a D_b X^i \), where \( D_a \) is the induced covariant derivative on \( \Sigma \).
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finite inequality (3.6). The Weyl-transformed surface violates the condition \( k_i K^i_{ab} = 0 \), so the divergent parts of the variation of \( S \) do not automatically vanish. We will discuss this inequality in more detail in Section 3.3.

Before continuing on with the proof of the QNEC, we should discuss briefly the integrated version. Suppose that \( k^i K^i_{ab} = 0 \) on all of \( \Sigma \) (which we can always enforce by setting \( k^i = 0 \) on some parts of \( \Sigma \)). Then we can integrate (3.6) to obtain

\[
2\pi \int dy \sqrt{h} T_{kk} \geq \frac{D^2 S}{D\lambda^2}.
\] (3.9)

Here we made use of (3.4) and (3.5), and also the fact that the “off-diagonal” terms in (3.5) are non-positive [34]. This is a global version of the QNEC, but it is actually equivalent to the local version. By considering the limiting case of a vector field \( k^i(y) \) with support concentrated around \( y = y_0 \), we can obtain (3.6) from (3.9).

3.3 Proof of the QNEC

Setup: Asymptotic Expansions

Our proof of the QNEC relies on the form of the bulk metric and extremal surface near the AdS boundary. In this section, we review the Fefferman-Graham expansion of the bulk metric and the analogous expansion of the extremal embedding functions, recalling the relevant properties of each.

Metric Expansion We are only interested in QFTs formulated on \( d \)-dimensional Minkowski space. Through order \( z^d \), the asymptotic expansion of the metric near the AdS boundary takes the form

\[
ds^2 = \frac{L^2}{z^2} \left( dz^2 + \left[ f(z) \eta_{ij} + \frac{16\pi G_N}{d L^{d-1}} z^d t_{ij} \right] dx^i dx^j + o(z^d) \right) = G_{\mu\nu} dx^\mu dx^\nu.
\] (3.10)

Here \( L \) is the AdS length, \( f(z) \) only contains powers of \( z \) less than \( d \) (and possibly a term proportional to \( z^d \log z \)) and satisfies \( f(0) = 1 \). The exact form of \( f(z) \) will depend on the theory; in a CFT \( f(z) = 1 \) but we are free to turn on relevant deformations which can modify it. We are assuming that only Poincare-invariant theories are being considered; this is why \( \eta_{ij} \) is the only tensor appearing up to order \( z^d \).

The tensor \( t_{ij} \), defined by its appearance in (3.10) as the coefficient of \( z^d \), is not necessarily the same as \( T_{ij} \). In a CFT on Minkowski space they are equal, but in the presence of a relevant deformation one has to carefully define the renormalized energy-momentum tensor of the new theory.\(^9\) In particular, \( t_{ij} \) may not vanish in the vacuum state of the deformed

\(^9\)See [89] for example.
theory. However, the difference \( T_{ij} - t_{ij} \) is proportional to \( \eta_{ij} \). Therefore \( t_{kk} = T_{kk} \), which is all we will need.

The \((d+1)\)-dimensional bulk metric is denoted by \( G_{\mu\nu} \), but we will also find it convenient to define the rescaled metric

\[
g_{\mu\nu} \equiv \frac{z^2}{L^2} G_{\mu\nu}. \tag{3.11}
\]

**Embedding Functions**

The embedding of the \((d-1)\)-dimensional extremal surface \( m \) in the \((d+1)\)-dimensional bulk can be described by specifying the bulk coordinates as a function of \( z \) and \((d-2)\) intrinsic coordinates \( y^a, \bar{X}^\mu = X^\mu(y^a, z) \). These functions are called the “embedding functions.”

The induced metric on \( m \) is given by

\[
\bar{H}_{\alpha\beta} \equiv \partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu g_{\mu\nu}[\bar{X}], \tag{3.12}
\]

where \( g_{\mu\nu} \) is the bulk metric. Instead of \( \bar{H}_{\alpha\beta} \), it is often more convenient to use a rescaled surface metric:

\[
\bar{h}_{\alpha\beta} \equiv \partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu g_{\mu\nu}[\bar{X}] = \frac{z^2}{L^2} \bar{H}_{\alpha\beta}, \tag{3.13}
\]

where \( \bar{h}_{\alpha\beta} = (z^2/L^2)G_{\mu\nu} \) as defined above. Our internal coordinates for the surface are chosen so that \( \bar{H}_{az} = \bar{h}_{az} = 0 \) and \( \bar{X}^z = z \).

The embedding functions satisfy an equation of motion coming from extremizing the total area. In terms of this induced metric, this can be written as [103]

\[
\frac{1}{\sqrt{\bar{H}}} \partial_\alpha \left( \sqrt{\bar{H}} \bar{H}_{\alpha\beta} \partial_\beta \bar{X}^\mu \right) + \bar{H}_{\alpha\beta} \Gamma^\mu_{\nu\sigma} \partial_\alpha \bar{X}^\nu \partial_\beta \bar{X}^\sigma = 0, \tag{3.14}
\]

where \( \Gamma^\mu_{\nu\sigma} \) is the bulk Christoffel symbol constructed with the bulk metric (3.10) and \( \bar{H} \equiv \det \bar{H}_{\alpha\beta} \). The embedding functions have an asymptotic expansion near the boundary with a structure very similar to that of the bulk metric. There are two solutions, with the state-independent solution containing lower powers of \( z \) than the state-dependent solution. The state-independent solution only contains terms of lower order than \( z^d \), and only depends on the state-independent part of the bulk metric (3.10). If we only include the terms in (3.14) relevant for the terms of lower order than \( z^d \), we find

\[
z^{d-1} \partial_z \left( z^{1-d} \sqrt{\bar{h} \bar{h}^{zz}} f \partial_z \bar{X}^i \right) + \partial_a \left( f \sqrt{\bar{h} \bar{h}^{ab}} \partial_b \bar{X}^i \right) = 0. \tag{3.15}
\]

where \( \bar{h} \equiv \det \bar{h}_{ab} \). The solution to this equation can be found algebraically order-by-order in \( z \) up to \( z^d \). The expansion reads

\[
X^i(y^a, z) = X^i(y^a) + \frac{1}{2(d-2)} z^2 K^i(y^a) + \cdots + \frac{1}{d} z^d \left( V^i(y^a) + W^i(y^a) \log z \right) + o(z^d). \tag{3.16}
\]

\(^{10}\)The difference should be proportional to the relevant coupling \( \phi_0 \), and dimensional analysis dictates that the only possibility is \( \phi_0 \mathcal{O} \eta_{ij} \) where \( \mathcal{O} \) is the relevant operator.

\(^{11}\)Our index conventions are described at the end of the Introduction.
Here $K^i$ is the trace of the extrinsic curvature tensor of the entangling surface $\Sigma$. Since the background geometry is flat, this can be written as

$$K^i = \frac{1}{\sqrt{h}} \partial_a \left( \sqrt{h} h^{ab} \partial_b X^i \right).$$  \hspace{1cm} (3.17)

The omitted terms “…” contain powers of $z$ between 2 and $d$. In a CFT there would be only even powers, but with a relevant deformation odd or fractional powers are allowed depending the dimension of the relevant operator. These terms, as well as the logarithmic term $W^i$, are all state-independent,\textsuperscript{12} and are local functions of geometric invariants of the entangling surface \cite{103}. These geometric invariants are formed from contractions of the extrinsic curvature and its derivatives, and will vanish if the surface is flat: if $K^i$ vanishes in some neighborhood on the surface, then $\bar{X}^i = X^i + V_i^j z^j$ satisfies the equation of motion up to that order in $z$. The logarithmic coefficient $W^i$ is only present in when $d$ is even for a CFT, but it may also show up in odd dimensions if relevant operators of particular dimensions are turned on.

The state-dependent part of the solution starts at order $z^d$, and the only term we have shown in (3.16) is $V^i$. We will find below that this term encodes the variation of the entropy that enters into the QNEC.

### Extremal Surface Area Asymptotic Expansion

With $\bar{H}_{\alpha\beta} = \partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu G_{\mu\nu}$ the induced metric on the extremal surface, the area functional is

$$A = \int dz d^{d-2} y \sqrt{\bar{H}[\bar{X}]}.$$  \hspace{1cm} (3.18)

We are interested in variations of the extremal area when the entangling surface $\Sigma$ is deformed. That is, when the boundary embedding functions $X^i$ are varied. The variation of the area is not guaranteed to be finite: divergences will be regulated by a cutoff surface at $z = \epsilon$. A straightforward exercise in the calculus of variations shows that

$$\delta A = -\frac{L^{d-1}}{z^{d-1}} \int d^{d-2} y \sqrt{\bar{h}} \left( \frac{g_{ij} \partial_z \bar{X}^i}{\sqrt{1 + g_{lm} \partial_z X^l \partial_z X^m}} \right) \delta \bar{X}^j \Bigg|_{z=\epsilon}. $$  \hspace{1cm} (3.19)

Each factor in this expression (including $\delta \bar{X}^j$) should be expanded in powers of $z$ and evaluated at $z = \epsilon$. Making use of (3.10) and (3.16), we find

$$\frac{1}{L^{d-1} \sqrt{\bar{h}}} \left( \frac{\delta A}{\delta \bar{X}^i} \right) = -\frac{1}{(d-2)\epsilon^{d-2}} K^i + \text{(power law)} - W_i \log \epsilon - V_i + \text{(finite state-independent)}.$$  \hspace{1cm} (3.20)

\textsuperscript{12}They are only state-independent if there are no scalar operators of dimension $\Delta < d/2$. For the case of operators with $d/2 > \Delta > (d-2)/2$, see Appendix A.2.
The most divergent term goes like $e^{2-d}$, and is the variation of the usual area-law term expected in any quantum field theory. The logarithmically divergent term is directly determined in terms of the logarithmic term in the expansion of the embedding functions in (3.16). The remaining terms, including both the lower-order power law divergences and the state-independent finite terms, are determined in terms of the “...” of (3.16). Their precise form is not important, but our analysis later will depend on the fact that they are built out of local geometric data on $\Sigma$, and that they vanish when $K_{ab}^i = 0$ locally. That is, if $K_{ab}^i$ and its derivatives vanish at a point $y$, then these terms are zero at that point.

Elimination of Divergences  Now we will illustrate that the condition $k_i K_{ab}^i = 0$ in the neighborhood of a point is enough to remove divergences in $S'$. First we note that the condition $k_i K_{ab}^i = 0$ is robust under null deformations in the $k^i$ direction. That is, if it is satisfied initially then it remains satisfied for all values of $k^i$. To see this, we use the identity\(^{14}\)

$$k_i K_{ab}^i = k_i \partial_a \partial_b X^i = -\partial_a k_i \partial_b X^i \quad (3.21)$$

and take a $\lambda$-derivative to get

$$\partial_\lambda (k_i K_{ab}^i) = -\partial_a k_i \partial_b k^i = -(k_i K_{ac}^i) h^{cd} (k_j K_{db}^j). \quad (3.22)$$

For the last equality we used the fact that $k_i \partial_a k^i = 0$, so the inner product could be evaluated by first projecting onto the tangent space of $\Sigma$. This shows that $k_i K_{ab}^i$ remains zero if it is initially zero, and so all of our remaining results hold even as we deform $\Sigma$.

We claim when $k_i K_{ab}^i = 0$ locally, the expansion (3.16) reduces to

$$\tilde{X}^i(y,z) = X^i(y) + B(y,z) k^i(y) + \frac{1}{d} V^i(y) z^d + o(z^d). \quad (3.23)$$

Here $B(y,z)$ is a function which vanishes at $z = 0$ and contains powers of $z$ less than $d$, and possibly a term proportional to $z^d \log z$. The nontrivial claim here is that the leading $z$ terms up to $z^d$ are all proportional to $k^i$. We will now prove this claim.

We know from the equations of motion that the terms of in the embedding function expansion at orders lower than $z^d$ are determined locally in terms of the geometry of the entangling surface. This means they can only depend on $\eta_{ij}$, $\partial_a X^i$, $K_{ab}^i$, and finitely many derivatives of $K_{ab}^i$ in the directions tangent to $\Sigma$. If $K_{ab}^i$ is proportional to $k^i$, the same is true for its derivatives. To see this, we only need to show that $\partial_a k^i$ is proportional to $k^i$. Since $k^i$ is null, we have $k_i \partial_a k^i = 0$. Therefore $\partial_a k^i$ does not have any components in the null direction opposite to $k^i$ (which we will call $\bar{l}^i$ below). We can also compute its components in the tangent directions:

$$\partial_b X^i \partial_a k_i = -k_i \partial_a \partial_b X^i = -k_i K_{ab}^i = 0. \quad (3.24)$$

---

\(^{13}\)In the remainder of proof we assume $k^i(y) \neq 0$. That is, we are only considering regions of the entangling surface which are actually being deformed.

\(^{14}\)The extrinsic curvature is often defined as $K_{ab}^i = \partial_a X^i \partial_b X^m \nabla_i h_{mn}^i$. “Differentiating by parts” and restricting to Minkowski space gives the first equality of equation (3.21).
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Hence $\partial_a k^i \propto k^i$, and so all of the tangent derivatives of $K_{ab}^i$ are proportional to $k^i$.

Now, one can check that if $K_{ab}^i$ and all of its derivatives are zero then (3.23) with $B = 0$ solves the equation of motion up to order $z^d$. This means that at least one power of $K_{ab}^i$ (or its derivatives) must appear in each of the terms in the expansion of $\tilde{X}^i$ of lower order than $z^d$ beyond zeroth order. But this means that at least one power of $k^i$ appears, and there are no tensors available to give nonzero contractions with $k^i$. Hence each of these terms must be proportional to $k^i$, and this is the claim of (3.23). We emphasize that this expansion is valid in any state of the theory, even in the presence of a relevant deformation.

An analogous result holds for the expansion of the entropy variation, which means that (3.20) reduces to

$$\frac{\delta A}{\delta X^i} = C(y, \epsilon) k^i(y) - L^{d-1} \sqrt{h(y)} V_i(y),$$

(3.25)

where $C(y, \epsilon) k^i(y)$ represents the local terms (both divergent and finite) in (3.20). But now we see that all divergent terms are absent in null variations of the area: by contracting (3.25) with $k^i$ we see that the only non-zero contribution is the finite state-dependent term $k^i V_i$.

**Proof Strategy: Extremal Surfaces are Not Causally Related**

The QNEC involves the change in the von Neumann entropy of a region $\mathcal{R}$ under the local transport of a portion of the entangling surface $\Sigma$ along null geodesics (see Figure 3.1). The entropy $S(\mathcal{R})$ is computed as the area of the extremal surface $m(\mathcal{R})$ in the bulk, and so we need to analyze the behavior of extremal surfaces under boundary deformations. Our analysis is rooted in the following Fact: for any two boundary regions $A$ and $B$ with domain of dependence $D(A)$ and $D(B)$ such that $D(A) \subset D(B)$, $m(B)$ is spacelike- or null-separated from $m(A)$. This result is proved as theorem 17 in [177] and relies on the null curvature condition in the bulk, which in Einstein gravity is equivalent to the bulk (classical) NEC.\(^{15}\)

Even though this Fact can be proved based on properties of extremal area surfaces, it is useful to understand the intuition behind why it should be true. The idea, first advocated in [53], is that associated to the domain of dependence $D(A)$ of any region $A$ in the field theory should be a region $w(A)$ of the bulk, which in [96] was dubbed the “entanglement wedge.” The extremal surface $m(A)$ is the boundary of the entanglement wedge. Consider two regions $A$ and $B$ satisfying $D(A) \subset D(B)$, and consider also the complement of region $B$, $\bar{B}$. Assume for simplicity that $m(B) = m(\bar{B})$. If some part of $m(A)$ were timelike-separated from some part of $m(B)$, then that part of $m(A)$ would also be timelike-separated from $w(\bar{B})$. But the entanglement wedge proposal dictates that (unitary) field theory operators acting in $\bar{B}$ can influence the bulk state anywhere in $w(\bar{B})$, and so by bulk causality could influence the extremal surface $m(A)$ and thereby alter the entropy $S(A)$. But a unitary

\(^{15}\)Strictly speaking, theorem 17 in [177] concludes that $m(A)$ and $m(B)$ are spacelike-separated, because the bulk null generic condition is assumed. However, special regions and special states will have null separation. For example, in the vacuum any region in $d = 2$ as well as spherical regions and half-spaces in arbitrary dimension have this property. This observation is used for spherical regions in section 3.3
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Figure 3.2: The surface $\mathcal{M}$ in the bulk (shaded green) is the union of all of the extremal surfaces anchored to the boundary that are generated as we deform the entangling surface. The null vector $k^i$ (solid arrow) on the boundary determines the deformation, and the spacelike vector $s^\mu$ (dashed arrow) tangent to $\mathcal{M}$ is the one we construct in our proof. The QNEC arises from the inequality $s^\mu s_\mu \geq 0$.

operator acting on $\bar{B}$ leaves the density matrix of $B$ invariant, and therefore also the density matrix of $A$, and therefore also $S(A)$.

Based on this heuristic argument, one expects that a similar spacelike-separation property should exist for the boundaries of the entanglement wedges of $D(A)$ and $D(B)$ in any holographic theory, not just one where those boundaries are given by extremal area surfaces. For this reason, we are optimistic about the prospects for proving the QNEC using the present method beyond Einstein gravity, though we leave the details for future work.

Let $\Sigma$ be the boundary of the region $\mathcal{R}$. We consider deformations of $\Sigma$ by transporting it along orthogonal null geodesics generated by the orthogonal vector field $k^i$ on $\Sigma$, thus giving us a one-parameter family of entangling surfaces $\Sigma(\lambda)$ which bound the regions $\mathcal{R}(\lambda)$, where $\lambda$ is an affine parameter of the deformation. We also obtain a one-parameter family of extremal surfaces $m(\mathcal{R}(\lambda))$ in the bulk whose areas compute the entropies of the regions. Recall the global monotonicity constraint on $k^i$: we demand that the domain of dependence of $\mathcal{R}(\lambda)$ is either shrinking or growing as a function of $\lambda$. In other words, we have either $D(\mathcal{R}(\lambda_1)) \subset D(\mathcal{R}(\lambda_2))$ or $D(\mathcal{R}(\lambda_2)) \subset D(\mathcal{R}(\lambda_1))$ for every $\lambda_1 < \lambda_2$. Then, by the Fact quoted above, the union $\mathcal{M}$ of all of the $m(\mathcal{R}(\lambda))$ is an achronal hypersurface in the bulk (see Figure 3.2). That is, all tangent vectors on $\mathcal{M}$ are either spacelike or null.\(^{16}\) We will see that the QNEC is simply the non-negativity of the norm of a certain vector $s^\mu$ tangent to $\mathcal{M}$: $g_{\mu\nu} s^\mu s^\nu \geq 0$.

Since $\mathcal{M}$ is constructed as a one-parameter family of extremal surfaces (indexed by $\lambda$), we can take as a basis for its tangents space the vectors $\partial_\lambda \bar{X}^\mu$, $\partial_z \bar{X}^\mu$, and $\partial_\lambda \bar{X}^\mu$. The first

\(^{16}\)Part of theorem 17 in [177] is that the extremal surfaces associated to all the $\mathcal{R}(\lambda)$ lie on a single bulk Cauchy surface. $\mathcal{M}$ is just a portion of that Cauchy surface.
two are tangent to the extremal surface at each value of $\lambda$, while the third points in the direction of the deformation. One can check that the optimal inequality is given by choosing $s^\mu$ to be normal to the extremal surface $m(R)$. Thus we can simply define $s^\mu$ as the normal part of $\partial_\lambda \bar{X}^\mu$.

It turns out to be algebraically simplest to construct a null basis of vectors normal to the extremal surface at fixed $\lambda$ and then find the linear combination of them which is tangent to $\mathcal{M}$. We begin with the null vectors $k^i$, $l^i$ on the boundary which are orthogonal to the entangling surface. $k^i$ is the null vector which generates our deformation, and $l^i$ is the other linearly-independent orthogonal null vector, normalized so that $l^i k_i = 1$. We now define the null vectors $\bar{k}^\mu$ and $\bar{l}^\mu$ in the bulk which are orthogonal to the extremal surface and limit to $k^i$ and $l^i$, respectively, as $z \to 0$. $\bar{k}^\mu$ and $\bar{l}^\mu$ can be expanded in $z$ just like $\bar{X}^\mu$, and the expansion coefficients for $\bar{k}^\mu$ and $\bar{l}^\mu$ can be solved for in terms of those for $\bar{X}^\mu$. We will perform this expansion explicitly in the next section.

Once we have constructed $\bar{k}^\mu$ and $\bar{l}^\mu$, we write

$$s^\mu = \alpha \bar{k}^\mu + \beta \bar{l}^\mu.$$  \hfill (3.26)

The coefficients $\alpha$ and $\beta$ are determined by the requirement that $s^\mu$ be tangent to $\mathcal{M}$. This is achieved by setting

$$\alpha = g_{\mu \nu} \bar{l}^\nu \partial_\lambda \bar{X}^\nu, \quad \beta = g_{\mu \nu} \bar{k}^\nu \partial_\lambda \bar{X}^\nu.$$  \hfill (3.27)

Then the inequality $g_{\mu \nu} s^\mu s^\nu \geq 0$ becomes

$$\alpha \beta \geq 0.$$  \hfill (3.28)

Now, $\partial_\lambda \bar{X}^\mu \to \delta^i_\mu k^i$ as $z \to 0$, which implies that $\alpha \to 1$ and $\beta \to 0$ in that limit. This means that the coefficient of the most slowly-decaying term of $\beta$ is non-negative. Below we will compute $g_{\mu \nu} \bar{k}^\mu \partial_\lambda \bar{X}^\nu$ perturbatively in $z$ to derive the QNEC.

**Derivation of the QNEC**

In this section we derive the QNEC by explicitly constructing a perturbative expansion for the null vector field $\bar{k}^\mu$ orthogonal to the extremal surface and compute $g_{\mu \nu} \bar{k}^\mu \partial_\lambda \bar{X}^\nu$. This requires knowledge of the asymptotic expansion of the embedding functions $\bar{X}^\mu(y, z)$ and the metric $g_{\mu \nu}$ up to the order $z^d$. Using the assumption $k_i K^i_{ab} = 0$, which we imposed to eliminate divergences in the entropy, we have the simple expression (3.23) for $\bar{X}^i$, which we reproduce here,

$$\bar{X}^i(y, z) = X^i(y) + B(y, z) k^i(y) + \frac{1}{d} V^i(y) z^d + o(z^d),$$  \hfill (3.29)

it is straightforward to construct the vector $\bar{k}^\mu$. We use the ansatz

$$\bar{k}^\mu(y^a, z) = \delta^a_\mu k^a(y^a, z) + \delta^\mu_i \left( k^i(y^a) + z^d \Delta k^i(y^a) \right),$$  \hfill (3.30)
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where

\[
(k^z)^2 + \left(\frac{16\pi G_N}{dL^{d-1}} T_{kk} + 2k_i \Delta k^i \right) z^d = o(z^d)
\]  
(3.31)

ensures that $\vec{k}^\mu$ is null to the required order. We demand that $\vec{k}^\mu$ is orthogonal to both $\partial_a \vec{X}^\mu$ and $\partial_z \vec{X}^\mu$, which for $d > 2$ results in the two conditions

\[
0 = \partial_a X_i \Delta k^i + \frac{1}{d} k_i \partial_a V^i + \frac{16\pi G_N}{dL^{d-1}} t_{ij} \partial_a X^i k^j,
\]  
(3.32)

\[
0 = k_i \Delta k^i + \frac{8\pi G_N}{dL^{d-1}} T_{kk}.
\]  
(3.33)

For $d = 2$ we instead have

\[
0 = k_i \Delta k^i + \frac{1}{2} (k_i V^i)^2 + \frac{4\pi G_N}{L} T_{kk}.
\]  
(3.34)

Together these equations determine $\Delta k^i$ up to the addition of a term proportional to $k_i$. This freedom in $\Delta k^i$ is an expected consequence of the non-uniqueness of $\vec{k}^\mu$, but the inequality we derive is independent of this freedom. Notice that the function $B$ plays no role in defining $\vec{k}^\mu$. This is because we are only ever evaluating our expressions up to order $z^d$, and since $k^i$ is null and orthogonal to $\Sigma$ there are no available vectors at low enough order to contract with $Bk^i$ which could give a nonzero contribution.

Now we take the inner product of $\vec{k}^\mu$ with $\partial_\lambda \vec{X}^\nu$ to get

\[
g_{\mu\nu} \vec{k}^\mu \partial_\lambda \vec{X}^\nu = \left( k_i \Delta k^i + \frac{1}{d} k_i \partial_\lambda V^i + \frac{16\pi G_N}{dL^{d-1}} T_{kk} \right) z^d + o(z^d).
\]  
(3.35)

Here we used the geodesic equation, $\partial_\lambda k^i = 0$, in order to find once more that the $Bk^i$ term in (3.29) drops out. Using our constraint on $\Delta k^i$ and the inequality (3.28) gives us the inequality

\[
\frac{8\pi G_N}{L} T_{kk} \geq -k_i \partial_\lambda V^i
\]  
(3.36)

for $d > 2$ and the inequality

\[
\frac{8\pi G_N}{L} T_{kk} \geq -k_i \partial_\lambda V^i + (k_i V^i)^2
\]  
(3.37)

for $d = 2$.

The RHS of these equations can be related to variations of the entropy using (3.25), which we reproduce here:

\[
\frac{\delta A}{\delta X^i(y)} = C(y, \epsilon) k_i(y) - L^{d-1} \sqrt{h(y)} \ V_i(y).
\]
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To convert from extremal surface area to the entropy we only need to divide by $4G_N$. Then applying (3.25) to (3.36) and (3.37) immediately yields

$$T_{kk} \geq \frac{1}{2\pi \sqrt{\hbar}} k^i \frac{D}{D\lambda} \frac{\delta S}{\delta X^i}$$  \hspace{1cm} \text{(3.38)}$$

for $d > 2$ and

$$T_{kk} \geq \frac{1}{2\pi} \left[ k^i \frac{D}{D\lambda} \frac{\delta S}{\delta X^i} + \frac{4G_N}{L} \left( k^i \frac{\delta S}{\delta X^i} \right)^2 \right]$$  \hspace{1cm} \text{(3.39)}$$

for $d = 2$. The explicit factor $4G_N/L$ should be re-interpreted in the field theory language in terms of the number of degrees of freedom. For a CFT, we have $4G_N/L = 6/c$. When a relevant deformation is turned on, we have to use the central charge associated with the ultraviolet fixed point, $c_{UV}$. This is the appropriate quantity because our derivation takes place in the asymptotic near-boundary geometry, which is dual to the UV of the theory. In other words, $L$ here refers to the effective AdS length in the near-boundary region.

To complete the proof, we can simply restrict the support of $k^i$ to an infinitesimal neighborhood of the point $y$, in which case we have

$$k^i(y) \frac{D}{D\lambda} \frac{\delta S}{\delta X^i}(y) \rightarrow S''(y),$$  \hspace{1cm} \text{(3.40)}$$

where we recall the definition (3.5) of $S''$. Then (3.38) and (3.39) imply the advertised forms of the QNEC, (3.6):

$$T_{kk} \geq \frac{1}{2\pi \sqrt{\hbar}} S''$$

in $d > 2$ and (3.7):

$$T_{kk} \geq \frac{1}{2\pi} \left[ S'' + \frac{6}{c} (S')^2 \right]$$

in $d = 2$ dimensions. Following the arguments given in Section 3.2, we also have the integrated form of the QNEC, (3.9):

$$2\pi \int dy \sqrt{\hbar} T_{kk} \geq \int dy S'' \geq \frac{D^2 S}{D\lambda^2},$$

as well as the analogous integrated version of (3.7).

**Generalizations for CFTs**

In this section we turn off our relevant deformation, restricting to a CFT in $d > 2$. Suppose we perform a Weyl transformation, sending $\eta_{ij} \rightarrow \hat{\eta}_{ij} = e^{2\Upsilon} \eta_{ij}$. To find a new inequality valid for the new conformal frame, we can simply take the QNEC, (3.6),

$$T_{kk} \geq \frac{1}{2\pi \sqrt{\hbar}} S'',$$
and apply the Weyl transformation laws to $T_{kk}$ and $S''$.

The effect of the Weyl transformation on $T_{ij}$ is well-known. In odd dimensions, it transforms covariantly with weight $d - 2$, while in even dimensions there is an anomalous additive shift for Weyl transformations that are not part of the global conformal group. In general then

$$T_{ij} = e^{(d-2)\Upsilon} \left( \hat{T}_{ij} - A^{(T)}_{ij} \right),$$

(3.41)

where $A^{(T)}_{ij}$ is the anomaly which depends on $\Upsilon$ [42].

The effect of the Weyl transformation on the entropy is entirely encoded in the cutoff dependence of the divergent terms. This is especially clear in the holographic context: a Weyl transformation is simply a change of coordinates in the bulk, so the extremal surface $m$ is the same before and after. The only difference is that we now regulate the IR divergences by terminating the surface on $\hat{z} = \epsilon$ with a new coordinate $\hat{z}$. Graham and Witten considered the transformation of such surface variables under Weyl transformations [86]. The divergent parts all transform with different weights (and shifts), so the transformation of $S$ as a whole is complicated. But the QNEC already isolates the finite part of the entropy, $S_{\text{fin}}$, so we need only ask how it transforms. Graham and Witten have shown that $S_{\text{fin}}$ is invariant when $d$ is odd and has an anomalous shift when $d$ is even [86]:

$$S_{\text{fin}} = \hat{S}_{\text{fin}} - A^{(S)}.$$

(3.42)

The anomalous shift $A^{(S)}$ depends on the surface $\Sigma$ as well as $\Upsilon$, and will generically be nonzero even when $A^{(T)}_{ij}$ vanishes. For a surface with $k_iK_{ab}^i = 0$ prior to the Weyl transformation, the anomaly is [86]

$$A^{(S)} = \frac{1}{8} \int dy \sqrt{h} \left[ \hat{k}^i \hat{K}_i + 2 \partial_\alpha \Upsilon \partial^\alpha \Upsilon \right].$$

(3.43)

Finally, we must say how $S''_{\text{fin}}$ transforms. These derivatives are with respect to the affine parameter $\lambda$ which labels the flow along the geodesics generated by $k^i$. The vector tangent to the same geodesic but affinely-parametrized with respect to the new metric is $\hat{k}^i = e^{-2\Upsilon} k^i$. Acting on a scalar function $S$, the second derivative operator becomes

$$k^i \partial_i (k^j \partial_j S) = e^{2\Upsilon} \hat{k}^i \partial_i \left( e^{2\Upsilon} \hat{k}^j \partial_j S \right) = e^{4\Upsilon} \left( \hat{k}^i \partial_i \left( \hat{k}^j \partial_j S \right) + 2(\hat{k}^i \partial_i \Upsilon)(\hat{k}^j \partial_j S) \right).$$

(3.44)

Then we have, in total,

$$S''_{\text{fin}} = e^{4\Upsilon} \left[ (\hat{S}_{\text{fin}} - A^{(S)})'' + 2(\hat{k}^i \partial_i \Upsilon)(\hat{S}_{\text{fin}} - A^{(S)})' \right],$$

(3.45)

where on the right-hand side we are careful to compute derivatives using the correctly-normalized $\hat{k}^i$. We also note that the expansion in the $\hat{k}^i$ direction is no longer zero after Weyl transformation, and is instead given by

$$\hat{\theta} = \hat{k}^i \partial_i \log \sqrt{h} = (d - 2) \hat{k}^i \partial_i \Upsilon.$$  

(3.46)
Putting these equations together, and dropping hats on the variables, we find that for metrics of the form $e^{2\Upsilon}\eta_{ij}$ we have a “conformal QNEC”:

$$T_{kk} - A_{kk}^{(T)} = \frac{1}{2\pi \sqrt{\hbar}} \left[ (S_{\text{fin}} - A^{(S)})'' + \frac{2}{d-2} \theta (S_{\text{fin}} - A^{(S)})' \right].$$  \hspace{1cm} (3.47)

This is a local inequality that applies to all surfaces $\Sigma$ which are shearless in the $k^i$ direction. This bound can of course be integrated to yield an inequality corresponding to finite deformations.

**Special case: spherical entangling regions**

The entanglement entropy across spheres has special properties compared to regions with less symmetry. Spheres minimize the entanglement entropy among all continuously-connected shapes with the same entangling surface area [9, 5], which has led to the entropy of a sphere being used as a c-function [47, 137, 46, 118]. Spheres also play a special role because the form of their modular Hamiltonian is known explicitly [48, 106, 70].

Spheres are special in the context of our analysis as well. Consider the integrated version of the conformal QNEC (3.47) specialized to the case where $\Sigma$ is a sphere in flat space. This can be obtained by a special conformal transformation from a planar entangling region (so $A_{kk}^{(T)} = 0$). We will also choose $k^i$ to be uniform and directed radially inward around the sphere, so that $\theta = -(d-2)/R$, where $R$ is the sphere radius. Then we have the inequality

$$2\pi R^{d-2} \int d\Omega T_{kk}(\Omega) \geq \frac{D^2}{D\lambda^2} (S_{\text{fin,vac}} - A^{(S)}) - \frac{2}{R} \frac{D}{D\lambda} (S_{\text{fin,vac}} - A^{(S)}).$$  \hspace{1cm} (3.48)

For this setup, we also know that the QNEC should be exactly saturated in the vacuum state. This is because the extremal surface corresponding to a sphere on the boundary in vacuum AdS is just the boundary of the causal wedge, and uniformly transporting the sphere inward in a null direction just transports the extremal surface along the causal wedge. In other words, we know that $s^\mu$ is null, implying saturation of the inequality (3.48):\(^{17}\)

$$0 = \frac{D^2}{D\lambda^2} (S_{\text{fin,vac}} - A^{(S)}) - \frac{2}{R} \frac{D}{D\lambda} (S_{\text{fin,vac}} - A^{(S)}),$$  \hspace{1cm} (3.49)

where we used $T_{kk} = 0$ in the vacuum. We could use this to compute $A^{(S)}$ given the known result for $S_{\text{fin,vac}}$. But we could just as easily subtract this equation from the previous inequality to obtain

$$2\pi R^{d-2} \int d\Omega T_{kk}(\Omega) \geq \frac{D^2}{D\lambda^2} (S - S_{\text{vac}}) - \frac{2}{R} \frac{D}{D\lambda} (S - S_{\text{vac}}),$$  \hspace{1cm} (3.50)

\(^{17}\)If the QNEC is saturated for a particular entangling surface, the conformal QNEC will be saturated for the conformally transformed surface. We can always think of this transformation as a passive Weyl transformation, which doesn’t change the bulk geometry; $s^\mu s_\mu$ is the same in all boundary conformal frames. So saturation of the conformal QNEC for a sphere in the vacuum is equivalent to saturation of the QNEC for a plane in the vacuum.
which is an inequality involving the vacuum-subtracted entropy of a sphere in an excited state of a CFT. Note that we no longer have to specify the finite piece of $S$ because the vacuum subtraction automatically cancels the divergent pieces.

### 3.4 Discussion

#### Potential Extensions

The structure of our proof was very simple, and we expect that a similar proof could extend the results beyond the regime of validity presented here. Let us review the key ingredients:

- It was important that the entropy was computable in terms of a surface observable which was an extremal value, in this case the area. This allowed us to focus on the near-boundary behavior of the surfaces as we made deformations of $\Sigma$, which is the only way we were able to have analytic control of the problem.

- We had to know that the extremal surfaces moved in a spacelike way in the bulk as $\Sigma$ was deformed. In our specific case, theorem 17 of [177] provided the rigorous proof of this fact, but as discussed in Section 3.3 this is should be a general property of the bulk entanglement wedge that is enforced by causality. Thus we expect that an analogous theorem can be proved in other contexts.

- When we performed our near-boundary expansions of $s^\mu$ and $S$, we needed to find the appropriate cancellations down to order $z^d$, where the energy-momentum tensor of the field theory appeared. This cancellation was enforced by a simple geometric requirement on $\Sigma$, namely $k_i K_{ab}^{i} = 0$. It may have seemed miraculous that this happened in our holographic calculation, since it seemed to rely on special properties of the asymptotic expansions of the bulk metric and embedding functions. But cancellation of this type was expected and predicted from field theory arguments alone. Namely, these lower-order terms are the ones that determine the divergent parts of the entropy, and in general the divergent parts of the entropy are local geometric functionals which are state-independent. This means that a local geometric condition on $\Sigma$ should be enough to eliminate them, and all of the “miraculous” properties we found stemmed from that.

**Higher-Curvature Theories**  The proof given in this paper was set in the context of boundary theories dual to Einstein gravity. From the boundary theory point of view there is nothing particularly special about these theories, and thus if the QNEC is at all universal one would expect that the current proof could be modified to include higher-curvature theories in the bulk.

Of the three points discussed above, the first is the most troubling. It is not known in general if the field theory entropy in an arbitrary higher-derivative theory of gravity is
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obtained by extremization of a local functional on a surface, though it has been shown for Lovelock and four-derivative gravity theories [54]. If this is not the case in general, then the proof of the QNEC would have to change dramatically for these other theories.

Next Order in $1/N$ It will likely be much more difficult to extend the proof to include finite-$N$ corrections. Finite-$N$ corresponds to quantum effects in the bulk. At the next order, $N^0$, the inclusion of quantum effects require the addition of the bulk entanglement entropy across the extremal area surface $m$ to the area of $m$ when computing the boundary entropy [69]. It has been suggested that the correct procedure to all orders is to extremize the bulk generalized entropy ($A + S_{\text{bulk}}$) instead of the area [64], but for the first correction we can continue to determine $m$ by extremizing the area alone.

The difficulty in extending our proof to the next order is that, while the surface $m$ is still determined by extremizing a local functional, the entropy itself is not given by the value of that functional. So while we still have (3.36), which is an inequality involving $V^i$, the coefficient of the $z^4$ term in the expansion of the embedding functions, we cannot identify $V^i$ with the variation of the entropy. Instead, the variation of the entropy is given by

$$k^i \frac{\delta S}{\delta X^i} = \frac{1}{4G_N^0} k^i \frac{\delta A}{\delta X^i} + k^i \frac{\delta S_{\text{bulk}}}{\delta X^i} = \frac{\sqrt{h}}{4G_N^0} k^i V_i + k^i \frac{\delta S_{\text{bulk}}}{\delta X^i}.$$  

(3.51)

Applying this result to (3.36), we find that a sufficient (but not necessary) condition for the QNEC to hold at order $N^0$ is

$$\frac{D}{D\lambda} \frac{\delta S_{\text{bulk}}}{\delta X^i} k^i \leq 0. \quad (3.52)$$

Intriguingly, this is almost the QNEC applied in the bulk, except for two things. Notice that the variation $\delta S_{\text{bulk}}/\delta X^i$ is a global variation of $S_{\text{bulk}}$, not a local one. We could re-expand it in terms of a local variation integrated over all of $m$. But the variation of $m$ is spacelike over most of the surface, even though it becomes null at infinity. The integrated QNEC does not apply when the variation is spacelike in some places. We would also expect that the bulk stress tensor should play some role in any bulk entropy inequality.

Curved Backgrounds A straightforward generalization of this proof is the extension to field theories on a curved background. The main problem is that the state-independent terms in the asymptotic metric expansion would not be proportional to the metric and thus would not vanish when contracted with the deforming null vector $k^i$. For example, for arbitrary bulk gravity theories dual to $d = 4$ CFTs the first two terms in the metric expansion read [104]

$$g_{ij}(x, z) = g^{(0)ij} + \frac{z^2}{2} \left[ R_{ij} - \frac{1}{2(d - 1)} R \ g^{(0)ij} \right] + \cdots,$$  

(3.53)
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where $g(0)_{ij}$ is the boundary metric. The $R_{ij}$ term will interfere with the proof if $R_{kk} \neq 0$. But there is another aspect of the curved-background setup which may help: the geometrical condition we have to impose on $\Sigma$ to eliminate divergences is not just $k_i K^i = 0$. The second variation of the area law term in the entropy, for instance, is proportional to the derivative of the geometric expansion of a null geodesic congruence, $\theta$, and by Raychaudhuri’s equation this depends on $R_{kk}$. So it may be that the condition which guarantees the absence of divergences in the QNEC in a curved-background is also strong enough to deal with all the background geometric terms which can show up to ruin the proof.

Quantum Focussing Conjecture  We have discussed at length the restriction to surfaces satisfying $k_i K^i_{ab} = 0$ as a way to eliminate divergences in the variation of the von Neumann entropy. But the original motivation for the QNEC, the Quantum Focussing Conjecture (QFC), was made in the context of quantum gravity, where the von Neumann entropy is finite (and is usually referred to as the generalized entropy). Instead of an area law divergence, the generalized entropy contains a term $A/4G_N$, and instead of subleading divergences there are terms involving (properly renormalized) higher curvature couplings. The QFC is an analogue of the QNEC for the generalized entropy, and simply states $S''_{\text{gen}} \leq 0$. When applied to a surface satisfying $k_i K^i_{ab} = 0$ it reduces to the QNEC, but when applied to a surface where $k_i K^i_{ab} \neq 0$, it has additional terms involving the gravitational coupling constants of the theory.

Using our present method of proof, we could potentially study these additional gravitational terms, and hence prove some version of the QFC. The idea is to consider an induced gravity setup in AdS/CFT, where the field theory lives not on the asymptotic boundary but on a brane located at some finite position. As is well-known, the CFT becomes coupled to a $d$-dimensional graviton in this setup [149, 148]. Furthermore, it has been shown that the area of an extremal surface anchored to the brane and extending into the bulk computes $S_{\text{gen}}$ for the CFT+gravity theory on the brane [19, 136].

For a brane which is close to the boundary, we can essentially apply all of the methodology of our current proof to this situation. The only difference is that, since we are not taking $z \to 0$, we do not have to worry about setting $k_i K^i_{ab} = 0$ to kill the divergences. And when we compute $s^\mu s_\mu$ without the condition $k_i K^i_{ab} = 0$, there will be additional terms that would have dominated in the $z \to 0$ limit. Schematically, we will have

$$0 \leq s^\mu s_\mu = z^2 \left( \text{non-vanishing when } k_i K^i_{ab} \neq 0 \right) + \cdots + z^d \left( T_{kk} - S'' \right). \quad (3.54)$$

Since $z$ is left finite and is related to the finite gravitational constant of the braneworld gravity, these terms have exactly the expected form of terms in the QFC. It remains to be seen if the QFC as conjectured is correct, or if there are other corrections to it. This method should tell us the answer either way, and we will investigate it in future work.

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18Update in version 2: Using a generalization of the method used in this paper, one can show that the QNEC holds when applied to Killing horizons of boundary theories living on arbitrary curved geometries [80].
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Connections to Other Work

Relation to studies of shape-dependence of entanglement entropy The shape-dependence of entanglement entropy in the vacuum state of a quantum field theory has recently been an active area of research.\(^{19}\) Recent studies have focused on the explicit calculation of the “off-diagonal” parts of the second variation of the entropy, sometimes known as the “entanglement density” \([141, 142, 17]\). These terms play no role in the local version of the QNEC, which only involves the diagonal part. For an integrated version of the QNEC, it is sufficient that the off-diagonal terms are negative, a result which can be proven via strong subadditivity alone, as discussed above \([34, 17]\). It would be interesting to see if any of the methods applied to the study of the entanglement density could be applied to the diagonal part of the second variation to study the QNEC for interacting theories without using holography.

Other energy conditions A number of non-local conditions on the stress tensor in quantum field theory have been suggested over the years, some more exotic than others. These include the average null energy condition (ANEC) \([93]\), as well as the more recent “quantum inequalities” (QIs) \([74, 75]\) which imply the “quantum interest conjecture” \([76]\). The motivation for non-local energy conditions in quantum field theory naturally comes from the fact that quantum fields violate all local energy conditions defined at a single point \([65]\).

It would be interesting to understand the relation between these inequalities, and to see which ones imply or are implied by the others. It was pointed out in \([74]\) that the QIs imply the ANEC in Minkowski space, and by integrating the QNEC along a null generator one can obtain the ANEC in situations where the boundary term \(S'\) vanishes at early and late times \([33]\). But does the QNEC imply a null limit of the QI?\(^{20}\) Or can the QI be shown to imply the QNEC? One might expect that the QNEC should be the more general statement, simply because of the huge freedom in the choice of region used to define the entropy.

Semiclassical generalizations of classical proofs from NEC \(\rightarrow\) QNEC Many proofs of theorems in classical gravity rely on the assumption of the Null Energy Condition (NEC) \([91, 28, 146, 93, 172, 135, 79, 66, 167, 92, 144, 171, 145, 84]\). In the context of AdS/CFT, the large-\(N\) limit of the boundary theory is dual to classical gravity in the bulk, and thus the NEC can be used to derive theorems about the AdS/CFT correspondence in this regime (e.g. \([177, 84, 137, 138, 96, 35]\), as well as many others). One wonders about the fate of these results away from the strictly classical limit, because the NEC is known to be violated by quantum fields \([65]\).

As shown in this paper and \([33]\), the QNEC is a generalization of the NEC which holds in several nontrivial examples of fully quantum theories. It would be interesting to try to replace the assumption of the NEC with the assumption of the QNEC to generalize classical proofs

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\(^{19}\)See e.g. \([150, 151, 44, 68, 5]\).

\(^{20}\)In \([74]\) it is mentioned that a QI can be derived for null geodesics for 1+1-dimensional Minkowski space, but that it is not known if an analogous statement holds in higher dimensions.
in gravity to the semi-classical regime. While the introduction of entropy into gravitational theorems may be a non-trivial modification, a similar program of replacing the NEC with the GSL for causal horizons [14, 174] has already had success in various cases [180, 64]. Replacing the NEC with the QNEC could potentially be even more powerful, as the QNEC holds at any point in spacetime without the need for a causal horizon.
Chapter 4

Information Content of Gravitational Radiation and the Vacuum

4.1 Introduction

Entropy bounds control the information flow through any light-sheet [23], in terms of the area difference between two cuts \( \sigma_1, \sigma_2 \) of the light-sheet:

\[
S \leq \frac{A[\sigma_1] - A[\sigma_2]}{4G\hbar}.
\]

(4.1)

A light-sheet is a null hypersurface consisting of null geodesics orthogonal to \( \sigma_1 \) that are nowhere expanding. A cut is a spatial cross-section of the light-sheet.

In simple settings, one can take \( S \) to be the thermodynamic entropy of isolated systems crossing the light-sheet. More generally, the definition of \( S \) is subtle, because in field theory there are divergent contributions from vacuum entanglement across \( \sigma_1 \) and \( \sigma_2 \). Precise definitions of \( S \) were found only recently, leading to rigorous proofs of two different field theory limits (\( G \to 0 \)) of Eq. (4.1). The proofs apply to free [32, 33] and interacting [31] scalar fields. Entropy bounds have also been verified [31] or proven [115] holographically for interacting gauge fields with a gravity dual.

Gravitational waves heat water, so they can be used to send information. In general, it is challenging to distinguish between gravitational waves and a curved spacetime. This can be done approximately in a setting where the wavelength of the radiation is small compared to other curvature radii in the geometry. A more rigorous notion of gravitational radiation is the “Bondi news,” which is defined in terms of an asymptotic expansion of the metric of asymptotically flat spacetimes [21, 154].

The Bondi news corresponds to gravitational radiation that reaches distant regions (see Fig. 4.1). It has been observed by monitoring test masses far from the source [1, 2]. Its definition contains a rescaling by a factor of the radius, so that it remains finite as the radiation is diluted and weakened. Ultimately, it can be thought of as a spin-2 degree of freedom on future null infinity, \( \mathcal{I}^+ \).
Figure 4.1: Penrose diagram of an asymptotic flat spacetime. Gravitational radiation (i.e., Bondi news) arrives on a bounded portion of $I^+$ (red). The asymptotic regions before and after this burst (blue) are Riemann flat. The equivalence principle requires that observers with access only to the flat regions cannot extract classical information; however, an observer with access to the Bondi news can receive information (see Sec. 4.3). We find in Sec. 4.4 that the asymptotic entropy bounds of Sec. 4.2 are consistent with these conclusions.

Recently, bounds on the entropy of arbitrary subregions of $I^+$ were obtained as the limit of known bulk entropy bounds [24]. These bounds constrain both the vacuum-subtracted entropy of states reduced to the subregion, and its derivatives as the subregion is varied. However, only nongravitational fields were treated rigorously. In this paper, we show how to incorporate gravitational radiation into the asymptotic entropy bounds.
The bulk entropy bounds that formed the starting point of Ref. [24] have been proven for certain fields [174, 32, 31, 33, 115]. Unless there is a discontinuity in the asymptotic limit, we expect these proofs to apply to the asymptotic bounds as well. Explicit proofs have not yet been given for a spin-2 field, however. To be conservative, the asymptotic bounds on gravitational radiation should be regarded as a conjecture.

Therefore, we will perform a simple consistency check: we ask whether the bounds are compatible with the equivalence principle. We take this principle to be the statement that an empty, Riemann-flat spacetime region contains no classical information. (By this we mean a subset of Minkowski space, not of a Riemann-flat spacetime with nontrivial topology. In this paper, “flat” will always mean Riemann-flat and devoid of matter.) In particular, the classical information of the spacetime geometry is contained only in its Riemann curvature, and not, for example, in the choice of coordinates.

The simplest setting is empty Minkowski space. In any subregion of $I^+$, our upper bounds vanish, implying that the vacuum-subtracted entropy is nonpositive and independent of the subregion. (In particular, the upper bounds do not depend on a “choice of vacuum” of Minkowski space.) This is consistent with the equivalence principle, which tells us that no classical information is present.

The asymptotic metric of Minkowski space, written in Bondi coordinates, is not uniquely fixed by fall-off conditions. One can freely choose the $1/r$ correction to the shape of spheres specified by setting the coordinates $u$ and $r$ to constants. (Note that this correction describes the shape of an embedded surface, whose location is determined by an arbitrary coordinate choice. Its shape is not indicative of any actual curvature of Minkowski space, which is manifestly Riemann-flat.) The freedom corresponds to a choice of a single real function $c(\Omega)$ of the coordinates on the sphere.

Recently, this degeneracy in the choice of Bondi coordinates has been interpreted as a degeneracy of the actual vacuum state of Minkowski space [162, 94]. We take no position on the formal convenience of elevating a classical coordinate choice to a degeneracy of the vacuum.

However, the equivalence principle rules out the possibility that a coordinate choice in Minkowski space has any measurable consequences. Therefore, $c(\Omega)$ must be unobservable. This is consistent with the fact that our bounds are insensitive to $c(\Omega)$ and vanish identically in Minkowski space.

We also consider a classical gravitational wavepacket with finite support, which arrives at $I^+$ as Bondi news. In portions of $I^+$ where the news has no support, our upper bounds vanish. This is consistent with the absence of classical information according to the equivalence principle: distant regions without gravitational radiation are Riemann-flat, so their geometry cannot be distinguished from Minkowski space.

In Bondi coordinates, the Bondi news does change the function $c(\Omega)$, by an integral of the news [162, 164, 94]. Since the news can be measured, this integral can be measured; for example, it results in a permanent displacement of physical detectors. Thus, in a nonvacuum spacetime, $c(\Omega)$ is a coordinate choice only in that it can be picked freely either before or after the burst. The difference—the gravitational memory—is invariant and physical. The
equivalence principle, and our bounds, constrain how the memory can be observed: namely, only by recording the news with physical detectors (which must be present during the burst). The memory cannot be measured by merely probing the asymptotic vacuum regions before and after the burst.

Outline In Sec. 4.2, we review the derivation of asymptotic entropy bounds of Ref. [24] (Sec. 4.2), and we show that they respond to gravitational radiation through the square of the Bondi news (Sec. 4.2).

In Sec. 4.3, we discuss implications of the equivalence principle. In Sec. 4.3, we consider the term $C_{AB}(\infty)$ that appears in the asymptotic (Bondi) metric of asymptotically flat spacetimes. This term can be nonvanishing even in Minkowski space and has been interpreted as labelling degenerate vacua [162, 7, 94]. Since it corresponds to a coordinate choice in Minkowski space, the equivalence principle demands that $C_{AB}$ be unobservable in any experiment. In Sec. 4.3, we consider gravitational memory (the integral of Bondi news). The equivalence principle implies that the memory can only be measured by an observer or apparatus that has access to all the Bondi news that produces the memory. In Sec. 4.3, we discuss “soft” gravitons and gravitational waves, by which we mean waves with long wavelength compared to some other time scale in the process that produces them. Given enough time, such excitations can be distinguished from the vacuum and so their information content is unconstrained by the equivalence principle.

In Sec. 4.4, we discuss implications of the entropy bounds of Sec. 4.2, in the same settings considered in Sec. 4.3. In Minkowski space, there is no news, and all our upper bounds vanish. We also consider a classical probabilistic ensemble (i.e., a mixed state) of classical gravitational wave bursts. We find that our bounds permit an observer to distinguish between different classical messages if and only if the observer has access to the news. Thus the implications of our entropy bounds are consistent with the conclusions we draw from the equivalence principle in Sec. 4.3.

In Appendix A.3, we discuss an asymptotic entropy bound proposed by Kapec et al. [110]. We focus on the case of empty Minkowski space. Whether this bound differs from (a special case of) ours depends on the definition of the entropy, which was not fully specified in Ref. [110]. We argue that consistency with the equivalence principle requires a choice under which the bounds agree. We clarify that the extra term in the upper bound of Ref. [110] originates from a difference in how the relevant null surfaces are constructed before the asymptotic limit is taken.

In Appendix A.4, we apply the bounds of Sec. 4.2 to a single graviton wavepacket. This case is not obviously constrained by the equivalence principle and so lies outside the main line of argument pursued here. We find that our bounds have implications similar to those derived for the classical Bondi news in Sec. 4.4.
4.2 Asymptotic Entropy Bounds and Bondi News

In Ref. [24], entropy bounds were applied to a distant planar light-sheet. The bounds can be expressed in terms of the stress tensor of matter crossing the light-sheet, and the square of the shear of the light-sheet. It was shown that the matter contribution is independent of the orientation of the light-sheet in the asymptotic limit. However, this was not proven for the contribution from the shear. Here we fill this gap by demonstrating that the shear term contributes to the upper bounds as the square of the Bondi news. Thus, it is associated with gravitational radiation reaching the boundary. In particular, this implies that the asymptotic bounds of Ref. [24] are fully independent of the orientation of the light-sheets used to derive them.

Asymptotic Entropy Bounds

In this subsection we briefly review the derivation and formulation of the asymptotic entropy bounds of Ref. [24]. Expectation value brackets are left implicit throughout.

We consider entropy bounds [23, 72, 34] in the general form of Eq. (4.1). In the weak-gravity limit, Newton’s constant $G$ is taken to become small, and a light-sheet is chosen that consists of initially parallel light-rays ($\theta_0 = 0$). An example of this is a null plane $t - z = \text{const}$ in Minkowski space. The effects of matter on the light-sheet are computed to leading nontrivial order in $G$, from the focussing equation [172]

$$- \frac{d\theta}{dw} = 8\pi G T_{ab} k^a k^b + \varsigma_{ab} \varsigma^{ab}. \quad (4.2)$$

Here $T_{ab}$ is the matter stress tensor, $k^a$ is the tangent vector to the light-rays that comprise the light-sheet, $w$ is an affine parameter and $\varsigma$ is the shear (defined by Eq. 4.20). The expansion $\theta$ is the logarithmic derivative of the area of a cross-section spanned by infinitesimally nearby light-rays.

By Eq. (4.1), the upper bound is given by the total area loss between two cross-sections of the light-sheet. It can be computed by integrating Eq. (4.2) twice along the light-rays, and then across the transverse directions. If the shear scales as $G^{1/2}$, the area loss will scale as $G$, so Newton’s constant drops out in Eq. (4.1). The resulting bound involves only Planck’s constant $\hbar$, so it can be viewed as a pure field theory statement.

Near the boundary of an asymptotically flat spacetime, the matter stress tensor falls off as $r^{-2}$ and the shear associated with gravitational radiation falls off as $r^{-1}$, so the above argument can be carried out at finite $G$, as an expansion in $G/r^2$. In particular, one can work on a Minkowski background,

$$ds^2 = -du^2 - 2du \, dr + r^2 d\Omega^2, \quad (4.3)$$

and compute area differences at order $G/r^2$, by integrating the focussing equation (4.2).

Keeping the radiation under consideration fixed, the area of the radiation front increases in the asymptotic limit as the local stress tensor decreases. It is convenient to rescale
CHAPTER 4. INFORMATION CONTENT OF GRAVITATIONAL RADIATION AND THE VACUUM

both [24], and formulate asymptotic entropy bounds directly in terms of finite quantities on $I^+$. The asymptotic energy flux is the energy arriving on $I^+$ per unit advanced time and unit solid angle:

\[ \hat{T} = \hat{T}_{uu} + \hat{\varsigma}_a\hat{\varsigma}^a, \tag{4.4} \]

The first term is the energy flux of nongravitational radiation,

\[ \hat{T}_{uu} = \lim_{r \to \infty} r^2 T_{uu}, \tag{4.5} \]

The second term is set by the shear of the light-sheet and will be defined in Sec. 4.2. It will be shown to correspond to the energy delivered by gravitational waves.

In Ref. [24], the basic tool for deriving the asymptotic entropy bounds is the notion of a distant planar light-sheet. Let $p \in I^+$ be a point at affine time $u_p$ and angle $\vartheta_p = \pi$. Let $H(u_p)$ be the boundary of the past of $p$:

\[ H(u_p) \equiv \hat{I}^-(p), \quad p \in I^+. \tag{4.6} \]

As discussed in Ref. [24], $H(u_p)$ is a null hypersurface. At $O(G/u_p^2)^0$, it is the null plane $t + z = u_p$ in Minkowski space, with affine parameter $w \equiv t - z$ and tangent vector $k^\mu = dx^\mu/dw$. In the \{u, r, \vartheta, \phi\} coordinates, $k^\mu$ has components

\[ k^u = \cos^2(\vartheta/2) \tag{4.7} \]
\[ k^r = -(\cos \vartheta)/2 \tag{4.8} \]
\[ k^\vartheta = \sin \vartheta \cos^2(\vartheta/2)/(u_p - u) \tag{4.9} \]
\[ k^\phi = 0. \tag{4.10} \]

In Ref. [24] it was shown that a number of known weak-gravity entropy bounds apply on $H(u_p)$. Cuts on different $H(u_p)$ were identified for different $u_p$ by using the same function $u(\Omega)$ to define each cut; this function also defines a cut on $I^+$. Bulk entropy bounds were applied to subregions defined by the cuts. The limit as $u_p \to \infty$ was taken and the bulk entropy bounds were re-expressed in terms of the asymptotic energy flux $\hat{T}$. We will now list these results; see Ref. [24] for details.

From the Quantum Null Energy Condition [34, 33] (QNEC) on $H(u_p)$, one obtains the Boundary QNEC,

\[ \frac{1}{\delta \Omega} \frac{d^2}{du^2} \hat{S}_{\text{out}}[\hat{\sigma}, \Omega] \leq \frac{2\pi}{\hbar} \hat{T}. \tag{4.11} \]

Here $\delta \Omega$ is a small solid angle element near a null geodesic at angle $\Omega$ on $I^+$. The second derivative is computed as this element is pushed to larger $u$, starting a given cut $\hat{\sigma}$ of $I^+$. The limit as $\delta \Omega \to 0$ is implicit. The entropy $\hat{S}_{\text{out}}$ is the von Neumann entropy of the state of the subregion of $I^+$ above the cut. That is,

\[ \hat{S}_{\text{out}} \equiv -\text{tr}_{> \hat{\sigma}} \rho \log \rho. \tag{4.12} \]

\[ ^1\text{We will generally refer to boundary versions of bulk quantities by adding a hat.} \]
The reduced state in the region above the cut \( \hat{\sigma} \) is defined by
\[
\rho = \text{tr}_{< \hat{\sigma}} \rho_g ,
\]
(4.13)
where \( \rho_g \) is the global state on \( I^+ \). We need not include future timelike infinite since we assume that all matter decays to radiation at sufficiently late times. Note that all cuts of \( I^+ \) have the same intrinsic and extrinsic geometry. Therefore, divergent terms in \( S_{\text{out}} \) drop out when differences are computed, or when derivatives are taken (also below). With the above definition, the QNEC has been proven for free scalar fields, and also for interacting gauge fields with a gravity dual [115].

From the differential, weak gravity Generalized Second Law (GSL) on \( H(u_p) \) [13, 174], or by integrating Eq. (4.11), one obtains the Boundary GSL in differential form
\[
-\frac{1}{\partial \Omega} \frac{d}{du} \hat{S}_{\text{out}}[\hat{\sigma} ; \Omega] \leq \frac{2\pi}{\hbar} \int_\hat{\sigma}^\infty du \; \hat{T} .
\]
(4.14)

From the integrated weak-gravity GSL on \( H(u_p) \), or by integrating Eq. (4.14), one obtains the Boundary GSL in integral form,
\[
\hat{S}_{\text{out}}[\hat{\sigma}_2] - \hat{S}_{\text{out}}[\infty] \leq \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^\infty d^2\Omega du [u - u_2(\Omega)] \hat{T} ,
\]
(4.15)
where \( \hat{S}_{\text{out}}[\infty] \) is to be understood in a limiting sense.

Finally, from the Quantum Bousso Bound (QBB) [32, 31] on finite “slabs” of \( H(u_p) \) one obtains a Boundary QBB,
\[
\hat{S}[\hat{\sigma}_1, \hat{\sigma}_2] \leq \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\hat{\sigma}_1} d^2\Omega du \; \hat{g}(u) \; \hat{T}(u, \Omega) .
\]
(4.16)

The weighting function \( \hat{g} \) is different for free and interacting bulk fields [32, 31]. Since fields become free asymptotically, we expect that it is given by the free field expression \( \hat{g}(u) = (u_1 - u)(u - u_2)/(u_1 - u_2) \).

In Eq. (4.16), \( \hat{S}_C \) is the vacuum-subtracted entropy [129, 45] of a finite affine interval on \( I^+ \). It is defined directly on the finite portion of the light-sheet between \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \), as the difference of two von Neumann entropies
\[
\hat{S}_C[\hat{\sigma}_1, \hat{\sigma}_2] = -\text{tr} \rho \log \rho + \text{tr} \chi \log \chi .
\]
(4.17)

Here the density operator \( \rho \) is obtained from the global quantum state \( \rho_g \) by tracing out the exterior of the region between \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \); and \( \chi \) is similarly obtained from the global vacuum state.\(^2\) The ultraviolet contributions from vacuum entanglement are the same in

\(^2\)As discussed in the introduction, the equivalence principle requires that the reduced vacuum state is unique, so it implies that the definition of the vacuum-subtracted entropy is unambiguous. We return to this point in App. A.3.
both reduced states, so they cancel out [129, 45, 32]. With this definition, the QBB has been proven both for free and interacting scalar fields. It has also been verified for gauge fields with gravity duals [31].

For free theories, the algebra of operators factorizes over the null geodesics that generate the light-sheet [174]. We expect that this case applies to $I^+$. Then the von Neumann entropy of the vacuum state restricted to the semi-infinite region above a cut $\hat{\sigma}$ is independent of the cut. Therefore, we have

$$\hat{S}_{out}[\hat{\sigma}_2] - \hat{S}_{out}[\hat{\sigma}_1] = \hat{S}_C[\hat{\sigma}_2] - \hat{S}_C[\hat{\sigma}_1],$$

(4.18)

where $\hat{S}_C[\hat{\sigma}]$ is now computed on the semi-infinite regions above $\hat{\sigma}_1$ and $\hat{\sigma}_2$. Thus we can also express other bounds, Eqs. (4.11) and (4.14), in terms of derivatives and differences of the manifestly finite quantity $\hat{S}_C$, instead of $\hat{S}_{out}$.

In particular, we can write the integrated Boundary GSL, Eq. (4.15), as

$$\hat{S}_C[\hat{\sigma}_2] \leq \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} d^2\Omega \, du \left[u - u_2(\Omega)\right] \hat{T}. $$

(4.19)

We have used the fact that the reduced density matrix of any physical state above a cut at sufficiently large $u$ is that of the vacuum restricted to the same region, and thus $\hat{S}_C[\infty] = 0$. We will use this form of the integrated Boundary GSL in Sec. 4.4.

**Bondi News as Shear on Distant Light-Sheets**

Let $\varsigma_{ab}$ be the shear tensor on $H(u_p)$, defined as the tracefree part of the extrinsic curvature:

$$\varsigma_{ab} = B_{ab} - \frac{1}{2} \theta q_{ab},$$

(4.20)

where $B_{ab} = g_a^c q_b^d \nabla_c k_d$, and $q_{ab}$ is the metric on the cuts $w = \text{const}$. One could choose different cuts, but some foliation of $H(u_p)$ into cuts has to be chosen in order to discuss the evolution of the shear. The shear tensor has only transverse components, so its information is fully captured by the lower-dimensional tensor

$$\varsigma_{AB} \equiv \varsigma_{ab} e^a_A e^b_B. $$

(4.21)

The $D - 2$ orthonormal vectors $e^a_A$ are tangent to the cut. Below we will denote any projection with the $e^a_A$ by capital indices placed on higher-dimensional tensors.

The evolution equation for the shear is [172, 147]

$$\frac{d}{dw} \varsigma_{AB} = W_{AB} - \theta \varsigma_{AB},$$

(4.22)

where

$$W_{AB} = -C_{abcd} e^a_A k^b_k e^c_B k^d_k \equiv -C_{AbCd} k^b_k k^d_k,$$

(4.23)
and $C_{abcd}$ is the Weyl tensor. We now recall that at fixed $(u, \Omega)$ there is no difference between expansions in inverse powers of $u_p$ and $r$, since [24]

$$r = \frac{u_p - u}{2 \cos^2(\vartheta/2)}.$$  

The asymptotic behavior of the Weyl tensor is [73]

$$C_{\bar{A}B} \sim O(r^{-1}) ,$$  

$$C_{\bar{A}uB} \sim O(r^{-3}) ,$$  

$$C_{ArB} \sim O(r^{-4}) ,$$  

$$C_{\bar{A}\varphi B} \sim O(r^{-3}) ,$$  

$$C_{A\varphi B} \sim O(r^{-1}) ;$$

and from Eqs. (4.9) and (4.10)

$$k^\vartheta \sim O(r^{-1}) , \quad k^\phi = 0 .$$

Hence we have

$$W_{\bar{A}B} = - C_{\bar{A}uB}(k^u)^2 + O(u_p^{-2}) .$$

These Weyl components are related to the Bondi news, $N_{AB}$ [73]:

$$C_{\bar{A}uB} = - \frac{1}{2r} \frac{d}{du} N_{AB} + O(r^{-2}) .$$

We have introduced an unbarred basis defined by $e^a_A = re^a_{\bar{A}}$, with the feature that in this basis boundary quantities such as $C_{AB}$ and $N_{AB}$ are independent of $r$. Unbarred capital indices will be raised and lowered with the unit two sphere metric, $h_{AB}$.

Since the expansion of $H(u_p)$ is of order $G/r^2$, the $\theta \varsigma_{\bar{A}B}$ term in Eq. (4.22) is always subleading in our analysis. Because the Bondi news and the shear of $H(u_p)$ both vanish in the far future, Eq. (4.22) implies

$$\varsigma_{\bar{A}B} = \frac{1}{2r} N_{AB} \cos^2(\vartheta/2) + O(r^{-2}) ,$$

where we have used $d^2u/du^2 \sim O(r^{-1})$.

On the other hand, the “boundary shear tensor” appearing in Eq. (4.4) was defined in Ref. [24] as

$$\hat{\varsigma}_{ab}(u, \vartheta, \phi) \equiv \frac{1}{\sqrt{8\pi G}} \lim_{r \to \infty} \frac{r \varsigma_{ab}(u, r, \vartheta, \phi)}{\cos^2(\vartheta/2)} .$$

Comparing the previous two equations and using Eq. (4.21), we recognize that the boundary shear is the Bondi news, up to an $O(1)$ rescaling:

$$\hat{\varsigma}_{AB} = \frac{N_{AB}}{\sqrt{32\pi G}} .$$

3In the Newman-Penrose formalism, the Bondi news is commonly identified with the $u$-derivative of the shear of the family of outgoing null congruences specified by $u = \text{const}$ [139]. Here we relate the news to the shear of ingoing null congruences.
The factor of $G^{-1/2}$ ensures that $\varsigma^2$ has the dimension of an energy flux.

Returning to the definition of the total asymptotic energy flux, Eq. (4.4), we can now write $\mathcal{T}$ in terms of the Bondi news:

$$\mathcal{T} = \dot{T}_{uu} + \frac{1}{32\pi G} N_{AB} N^{AB},$$

(4.36)

Note that the definition of the boundary shear $\varsigma_{AB}$ was tied to a family of null planes $H(u_p)$ whose orientation picks out a special point on the sphere. Since the Bondi news admits an independent definition that does not require us to pick such a point, it follows that the asymptotic bounds derived in Ref. [24] are independent of the orientation of the $H(u_p)$.

In the remainder of this paper, we will specialize to the case where all outgoing radiation is gravitational. Then $\dot{T}_{uu} = 0$ and $\mathcal{T} = N_{AB} N^{AB}/32\pi G$. We see that the square of the Bondi news controls the entropy flux of gravitational radiation.

### 4.3 Implications of the Equivalence Principle

In this section, we consider classical aspects of gravitational radiation. We derive consequences of the equivalence principle: the hypothesis that no subset of Minkowski space contains any measurable classical information. Since we use the notion of classical information throughout this and the following sections, we begin with a simple example of such information and its description, in Sec. 4.3.

It is possible to find nonvacuum quantum states whose effective stress tensor (analogous to Eq. (4.36)) vanishes in a bounded region. The geometry in this region could be Riemann-flat, yet the region could contain quantum information. Here we only assume the absence of classical information in Minkowski space. In particular we assume that no observable is associated with a coordinate choice in Minkowski space.

The geometry of Minkowski space is trivial, but of course the coordinates are arbitrary. So the matrix of metric components can take many different forms, both generally and in the asymptotic region. Restricting to Bondi coordinates does not fully fix this ambiguity. The equivalence principle implies that any parameters of the Bondi metric that are not unique in Minkowski space must be unobservable, or else that parameter would constitute measurable information. There is no $\hbar$ in the metric of Minkowski space in any Bondi gauge, so the corresponding coordinate information would be classical information.

This includes in particular a parameter $C_{AB}$ (defined below) that has been interpreted [7, 162, 94] as labelling degenerate vacua (Sec. 4.3). Indeed, no observation of this parameter has yet been made, and we are not aware of a proposal for how it could be measured.

A key consequence of the equivalence principle is that the gravitational memory created by Bondi news can be measured only by recording the news. It cannot be measured by probing the vacuum before and after the news (Sec. 4.3). Finally, we note that the equivalence principle does not preclude soft gravitational radiation from carrying information, if “soft” is understood in the physically relevant sense of a small expansion parameter (Sec. 4.3).
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These conclusions are in harmony with our findings in Sec. 4.4, where we apply the bounds of Sec. 4.2 to constrain the information content of gravitons and of the vacuum.

Classical Information

A simple example is a classical \( n \)-bit message written by Alice and delivered to Bob, say as a sequence of red and blue balls shot across space. Before Bob looks at the balls, he is ignorant of their state. Thus he can describe it as a density operator in a \( 2^n \) dimensional Hilbert space, which is diagonal in the \{red, blue\} \(^n\) basis, with equal probability \( 2^{-n} \) for each possible message. The Shannon and von Neumann entropies are both

\[
-\sum_{i=1}^{2^n} p_i \log p_i = -\text{Tr} \rho \log \rho = n \log 2 .
\]

This is an incoherent superposition, or classical probabilistic ensemble (not to be confused with a coherent quantum superposition of ball sequences).

By looking at the balls, Bob learns Alice’s message. Alice cannot send Bob more information than the maximum entropy of the system that carries the message. Since we can express Bob’s initial ignorance as a density operator, quantum entropy bounds limit classical communication, as a special case.

Of course, the full quantum Hilbert space is much larger due to the internal degrees of freedom of the balls. And even in the tiny subfactor spanned by \{red, blue\} \(^n\), more general states are possible at the quantum level, which are not product states of the individual balls.

But for classical messages represented by a quantum density operator \( \rho_i \), the ensemble interpretation [140] implies that the full density operator can be written as

\[
\rho = \sum_{i=1}^{2^n} \rho_i .
\]

Since the \( \rho_i \) are classically distinguishable—and therefore mutually orthogonal—states, there is an irreducible uncertainty in the von Neumann entropy: the entropy cannot be parametrically less than the classical value, \( n \log 2 \). At the field theory level, this will remain true for the vacuum-subtracted von Neumann entropy: it must be parametrically at least \( n \log 2 \) (assuming the region contains all balls), since the vacuum entanglement is an ultraviolet quantum property shared by all the classical states.

In this paper we often consider the equivalence principle: the statement that Minkowski space, and any subset of it, contain no classical information. It is worth reflecting on what it would mean if empty Riemann flat space did contain measurable information. In that case it could be used by Alice to communicate a message to Bob.

To be concrete, consider an arbitrarily large patch of flat space (say, the interior of a falling elevator, or a large void in our universe). For it to contain information in an operationally meaningful sense, Alice would have to be capable of “preparing” this region, perhaps by
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sending a certain sequence of gravitational waves through it. Later, long after those waves
have left the region and it is again empty and Riemann-flat, Bob would have to be capable
of reading out the message that Alice “left behind”, by examining only this patch.

Specifically, if \( c(\Omega) \) was observable, then independent observers with access only to the
flat space region, would all come to the same conclusion as to which coordinates should
be used to label its spacetime points. More precisely, up to corrections subleading in \( 1/r \),
such observers would uniquely identify topological spheres on which the Bondi coordinates
\( u \) and \( r \) must be constant, thus partially fixing the chart. This would indeed be a textbook
violation of the equivalence principle.

Empty Space Has No Classical Information

Let us consider the asymptotic metric of an asymptotically flat spacetime, in standard re-
tarded Bondi coordinates (see, e.g., [21, 154, 7, 162, 164, 73]):

\[
ds^2 = - \left( 1 - \frac{2m_B(u, \Omega)}{r} \right) du^2 - 2 du dr \\
+ r^2 \left( h_{AB} + \frac{C_{AB}(u, \Omega)}{r} \right) \times \\
\times \left( d\theta^A + \frac{D_C C^{AC}}{2r^2} du \right) \left( d\theta^B + \frac{D_C C^{CB}}{2r^2} du \right) \\
+ \ldots
\]

where \( m_B \) is the Bondi mass aspect, and the ellipses indicate terms subleading in \( r \). Here,
\( C_{AB}(u, \Omega) \) appears as the \( 1/r \) correction to the round two-sphere metric \( h_{AB} \). It satisfies
\( h^{AB} C_{AB} = 0 \) and \( C_{AB} = C_{BA} \). The Bondi news is defined by

\[
N_{AB} = \partial_u C_{AB} .
\]

In Minkowski space, the news vanishes. However, the asymptotic metric of Minkowski
space can be written in the form of Eq. (4.39), with \( m_B \equiv 0 \) and any \( u \)-independent choice
of a tracefree symmetric \( C_{AB}(\Omega) \) satisfying

\[
C_{AB} = (2D_A D_B - h_{AB} D_C D^C) c(\Omega)
\]

for some function \( c \) on the sphere. But of course, the geometry is always the same, no matter
how we label its points. There is no curvature of any kind, whatever value we choose for
\( c(\Omega) \). By the equivalence principle, this implies that \( c \) and \( C_{AB} \) cannot be measured.

\( C_{AB} \) does transform nontrivially under large diffeomorphisms of the asymptotic met-
ic [162, 12, 11, 73]; indeed, this is one way to see that it is non-unique in Minkowski space.
Under a BMS supertranslation, \( u \to u + f(\Omega) \), one has

\[
C_{AB} \to C_{AB} + (2D_A D_B - h_{AB} D_C D^C) f(\Omega)
\]
in regions where $N_{AB} = 0$. This corresponds to a well-defined change in the shape of a large coordinate sphere at constant $u, r$. It affects all such spheres equally; for example $C_{AB}(\infty)$ and $C_{AB}(-\infty)$ will change by the same amount under a supertranslation. Of course, this does not imply that $C_{AB}$ is observable. A coordinate sphere is not a physical object but a collection of spacetime points. Its initial shape before the transformation is set by a coordinate choice.

The transformation properties of $C_{AB}$ under supertranslations have been interpreted as an infinite “vacuum degeneracy” of Minkowski space [162, 7, 94]. Each “vacuum” is labeled by the function $c(\Omega)$ in Eq. (4.41). We conclude that the equivalence principle precludes any observable consequences of this degeneracy.\footnote{Note that the equivalence principle only precludes diffeomorphisms from transforming the classical vacuum into a physically distinct configuration. The equivalence principle does not imply that large diffeomorphisms always act trivially. When acting on an excited state, a supertranslation generically produces a distinct excited state, for example with a different relative timing of the Bondi news arriving at different angles.} (Refs. [10, 62] give an argument that the vacua are indistinguishable starting from different assumptions.)

**Gravitational Memory**

In nonvacuum spacetimes, $C_{AB}$ need not be constant in $u$, and differences between $C_{AB}$ at different cuts are observable as “gravitational memory.” However, the value of $C_{AB}$ at any one cut (or its zero-mode) must be unobservable in any asymptotically flat spacetime, or else the equivalence principle would be violated in regions where no news arrives. We will now discuss this.

Suppose that some process (a binary inspiral, say) produces gravitational radiation, and that the corresponding Bondi news arrives entirely between the cuts $\hat{\sigma}_1$ and $\hat{\sigma}_2$ of $\mathcal{I}^+$. The integral of the news along the null direction is called the gravitational memory produced by the process,

$$\Delta C_{AB}(\Omega) \equiv \int_{\hat{\sigma}_1}^{\hat{\sigma}_2} du N_{AB}(u, \Omega)$$ (4.43)

By Eq. (4.43), the production of memory requires nonzero flux of radiation, $N_{AB}$. Hence memory production occurs only in excited states, not in the vacuum. For example, a graviton wavepacket can produce memory; but then the global state is not the vacuum, but a one-particle state. This qualitative fact continues to hold invariantly in the “soft limit,” as the wavepacket is taken to have arbitrarily large wavelength.

What is the physical manifestation of $\Delta C_{AB}$, or equivalently, how can it be measured? In Sec. 4.2, we showed that $N_{AB}$ is proportional to the shear of a planar null congruence $H(u_p)$ near $\mathcal{I}^+$. Hence the gravitational memory is related to the integrated shear, i.e., the resulting strain of the congruence. The displacement vector $\eta^A$ of two infinitesimally nearby null geodesics will change by

$$\Delta \eta^A = \Delta C^A_{\ B} \eta^B \frac{1}{2r}$$ (4.44)
between \( \sigma_1 \) and \( \sigma_2 \). This can be measured by setting up (before \( \sigma_1 \)) a collection of physical, massless particles propagating along the null geodesics that constitute \( H(u) \), and observing their transverse location on a screen that they hit after \( \sigma_2 \). \( \Delta C_{AB} \) can also be measured using an array of timelike detectors distributed over a large sphere. The displacement of any two detectors similarly suffers an overall change given by Eq. (4.44).

In general, the memory captures only a small fraction of the information that arrives in distant regions: the integral of the Bondi news. It would certainly be nice to measure this component using gravitational wave detectors \([50, 99]\). Such a measurement would not take infinite time, and it would not be conceptually distinct from any other measurement of the outgoing radiation.

By Eq. (4.40) we can write the gravitational memory, Eq. (4.43), as a difference of the metric quantity \( C_{AB} \) evaluated at the two cuts,

\[
\Delta C_{AB}(\Omega) = C_{AB}[\sigma_2] - C_{AB}[\sigma_1].
\]  

(4.45)

If \( C_{AB} \) is interpreted as labelling a vacuum, the creation of gravitational memory by news could be described as a “transition” between two such vacua. However, according to the equivalence principle this language is misleading, because \( C_{AB} \) cannot be observed at a local cut. A “vacuum” in the above sense is a coordinate label that contains no physical information.

Only the difference \( \Delta C_{AB} \) is invariant (up to Lorentz transformations \([73]\)) and so can be observed. \( \Delta C_{AB} \) is nonzero only in global states which are not the vacuum, and it is fully determined by the integral of the Bondi news. So the function \( C_{AB}(u, \Omega) \) contains no physical information beyond what is already in its \( u \)-derivative, the news \( N_{AB} \). The observable memory, \( \Delta C_{AB} \), captures a subset of the information in the news.

By the equivalence principle, \( \Delta C_{AB} \) can only be measured by an observer who has access to the entire region in which news arrives. For example, if physical test particles are introduced into the asymptotic region, and their initial position at \( \sigma_1 \) is recorded, then the memory \( \Delta C_{AB} \) can be measured at \( \sigma_2 \) by observing the new location of these physical objects. This is an integrated measurement of the Bondi news, with the dynamics of the test masses doing the integration.

Formally, it can be convenient to consider the “zero mode” of the news,

\[
C_{AB}(\Omega, \infty) - C_{AB}(\Omega, -\infty) \equiv \int_{-\infty}^{\infty} du \, N_{AB}(\Omega, u).
\]  

(4.46)

This quantity represents the total amount of memory produced in an asymptotically flat spacetime. As written, it is not observable, since no experiment began in the infinite past and will end in the infinite future. Fortunately, in any physical process or sequence of processes, the production of news will have a beginning and an end. So one can record the entire memory in a finite-duration experiment, corresponding to a sufficiently large finite range of integration.

To summarize both this and the previous subsection, the value of \( C_{AB} \) at any one cut can be changed by a global change of coordinates. By the equivalence principle, \( C_{AB} \) cannot be
observed and contains no physical information. Therefore, in particular, we cannot measure the gravitational memory, $\Delta C_{AB}$ by observing $C_{AB}$ locally at $\hat{\sigma}_1$ and $\hat{\sigma}_2$ and computing the difference. Rather, physical test masses are essential for recording the news and integrating it to obtain $\Delta C_{AB}$ between the two cuts. If we forgot to introduce real test masses at $\hat{\sigma}_1$, we cannot look at empty space at $\hat{\sigma}_2$ and learn anything from it.

**Soft Gravitons**

A soft particle is an excitation of a massless field whose characteristic wavelength, or inverse frequency, is large compared to some dynamical timescale that otherwise characterizes a problem. For example, consider a binary system composed of neutron stars or black holes. They orbit each other with some frequency $\omega$, which varies slowly as they approach, until they eventually merge. The system will emit “hard” gravitational waves with frequency of order $\omega$. The overall duration of the inspiral process is much greater than $\omega^{-1}$; it is characterized by a second time scale $\tau \gg \omega^{-1}$. Or consider a black hole emitting Hawking radiation. The wavelength of the “hard part” of the radiation is of order the black hole radius, $\omega^{-1} \sim O(R)$, which changes slowly. Nonetheless, the overall process takes a much longer time, $\tau \sim O(R^3/G\hbar)$.

Because the emission of “hard” radiation slowly transports gravitating energy from the center to distant regions, the gravitational field will vary not only with characteristic frequency $\omega$, but also over the timescale $\tau$. Therefore, signals with characteristic frequency as low as $\tau^{-1}$ are produced in the above processes. Such signals are referred to as “soft”. (Often the term “soft graviton” is used, even when the signal is classical.)

This terminology is convenient when we wish to distinguish particles associated with different timescales in a given problem. Useful results can be obtained by expanding in ratios of such timescales [182]. It can also be convenient to idealize soft particles by taking a $\tau \rightarrow \infty$ limit, for the purposes of making such expansions sharp. It is worth stressing, however, that infinite-duration experiments are not actually needed to produce and measure a soft particle. (If they were, soft particles would have no physical relevance.) The larger time scale $\tau$ is necessarily finite in any physical process.

Moreover, the production of observable radiation comes at a nonzero energy cost. If a soft graviton were added to the vacuum, one would obtain an excited state orthogonal to the vacuum, not a new vacuum. This is a qualitative statement, and independent of $\tau$. Thus, there is no fundamental difference between soft particles and any other form of radiation that arrives in distant regions.

Correspondingly, when we apply the boundary entropy bounds of Sec. 4.2 in Sec. 4.4, all Bondi news can be treated on the same footing. For example, if the interval under consideration in Eq. (4.19) or Eq. (4.16) is large enough to contain a news wavepacket (hard or soft), we will find that this graviton will contribute to the energy side, and generically also to the entropy side of the inequality.
4.4 Entropy Bounds on Gravitational Wave Bursts and the Vacuum

In this section, we compute the upper bounds of Sec. 4.2 in simple asymptotically flat spacetimes: Minkowski space, and a burst of Bondi news that creates gravitational memory. We show that the upper bounds are consistent with constraints derived in the previous section from the equivalence principle.

Vacuum

Let us apply the bounds of Sec. 4.2 to empty Minkowski space: the Boundary QNEC, Eq. (4.11); the Boundary GSL in integrated and differential form, Eqs. (4.14) and (4.19); and the Boundary QBB, Eq. (4.16). All of these bounds are linear in the boundary stress tensor $\hat{T}$, i.e., quadratic in the Bondi news. Since $\hat{T} = 0$ in Minkowski space, the upper bounds all vanish.

The Boundary QBB implies that the vacuum-subtracted entropy is nonpositive in any finite subregion of $I^+$. The Boundary GSL implies that it is nonpositive for any semi-infinite region above a cut, and independent of the choice of region or deformations of the cut. The Boundary QFC implies (redundantly with the above) that the second derivative under deformations also vanishes.

These upper bounds are consistent with all implications of the equivalence principle described in the previous section: no subset of Minkowski space contains any classical information. Moreover, both the bounds and the equivalence principle are consistent with the simplest possibility for the quantum description of Minkowski space: that the ground state is unique, and that the vacuum-subtracted entropy precisely vanishes on any subregion of $I^+$.

Classical Bondi News

For simplicity, we will consider a single wave packet of gravitational radiation, of characteristic wavelength $\lambda$ in the $u$-direction. The wave packet is roughly centered on $u = 0$ and delocalized on the sphere. The wave packet can be used to send a message to an observer at $I^+$, for example by encoding it in its polarization, its shape, its direction (the angle $\Omega$ at which it arrives), or the time of arrival, within a finite discrete set of $N$ possible choices.

For concreteness, let us encode the information in the energy of the wavepacket. We take the energy to be of order $E$ for any message, but with a grading into $N$ different values.

A single graviton has energy of order $\hbar/\lambda$. Since we wish to work in the classical regime, the grading must be much coarser than that, so the number of distinct classical states will satisfy

$$N \ll \frac{E\lambda}{\hbar}, \quad (4.47)$$
We assume that any of the distinct classical signals arrives with equal probability $1/N$. The classical Shannon entropy is thus $\log N$.

If we apply the Boundary QBB, Eq. (4.16), or the Boundary GSL, Eq. (4.19), to the region occupied by the wavepacket, we obtain

$$\hat{S}_C \lesssim \frac{E\lambda}{\hbar}$$

(4.48)

This is consistent: in our example, the vacuum-subtracted entropy need not be much greater than the Shannon entropy $\log N$, which is much smaller than $N$ and hence, by Eq. (4.47), much smaller than the upper bound. Thus, the asymptotic bounds of Sec. 4.2 easily accommodate the classical information contained in the Bondi news.

On the other hand, if we apply the same bounds to a region that fails to overlap with the wavepacket, then the upper bound vanishes:

$$\hat{S}_C \leq 0$$

(4.49)

This is consistent with the absence of classical information in asymptotic regions that do not contain news, as required by the equivalence principle.

In particular, the bounds are consistent with our conclusion in Sec. 4.3 that gravitational memory can only be measured by an observer who has access to the news that creates the memory. In our present example, the news is featureless but for its overall energy. So its integral, the memory, contains the same amount of information as the news, $\log N$. [We have $T \sim N_{AB}N^{AB}/G \sim E/\lambda$, so the memory will be of order $\Delta C_{AB} \sim N_{AB}\lambda \sim (G\lambda)^{1/2}$.] By Eq. (4.49), this information is unavailable to an observer who cannot access the news.
Chapter 5

Geometric Constraints from Subregion Duality Beyond the Classical Regime

5.1 Introduction

AdS/CFT implies constraints on quantum gravity from properties of quantum field theory. For example, field theory causality requires that null geodesics through the bulk are delayed relative to those on the boundary. Such constraints on the bulk geometry can often be understood as coming from energy conditions on the bulk fields. In this case, bulk null geodesics will always be delayed as long as there is no negative null energy flux [84].

In this paper, we examine two constraints on the bulk geometry that are required by the consistency of the AdS/CFT duality. The starting point is the idea of subregion duality, which is the idea that the state of the boundary field theory reduced to a subregion \( A \) is itself dual to a subregion of the bulk. The relevant bulk region is called the entanglement wedge, \( \mathcal{E}(A) \), and consists of all points spacelike related to the extremal surface anchored on \( \partial A \), on the side towards \( A \) [53, 96]. The validity of subregion duality was argued [55, 88] to follow from the Ryu-Takayanagi-FLM formula [152, 153, 101, 69, 126, 56], and the consistency of subregion duality immediately implies two constraints on the bulk geometry.

The first constraint, which we call Entanglement Wedge Nesting (EWN), is that if a region \( A \) is contained in a region \( B \) on the boundary (or more generally, if the domain of dependence of \( A \) is contained in the domain of dependence of \( B \)), then \( \mathcal{E}(A) \) must be contained in \( \mathcal{E}(B) \). This condition was previously discussed in [53, 177].

The second constraint is that the set of bulk points \( I^-(D(A)) \cap I^+(D(A)) \), called the causal wedge \( \mathcal{C}(A) \), is completely contained in the entanglement wedge \( \mathcal{E}(A) \). We call this \( \mathcal{C} \subseteq \mathcal{E} \). See [53, 177, 64, 100, 96] for previous discussion of \( \mathcal{C} \subseteq \mathcal{E} \).

We refer to the delay of null geodesics passing through the bulk relative to their boundary counterparts [84] as the Boundary Causality Condition (BCC), as in [63]. These three
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conditions, and their connections to various bulk and boundary inequalities relating entropy and energy, are the primary focus of this paper.

In section 5.2 we will spell out the definitions of EWN and \( \mathcal{C} \subseteq \mathcal{E} \) in more detail, as well as describe their relations with subregion duality. Roughly speaking, EWN encodes the fact that subregion duality should respect inclusion of boundary regions. \( \mathcal{C} \subseteq \mathcal{E} \) is the statement that the bulk region dual to a given boundary region should at least contain all those bulk points from which messages can be both received from and sent to the boundary region.

Even though EWN, \( \mathcal{C} \subseteq \mathcal{E} \), and the BCC are all required for consistency of AdS/CFT, part of our goal is to investigate their relationships to each other as bulk statements independent of a boundary dual. As such, we will demonstrate that EWN implies \( \mathcal{C} \subseteq \mathcal{E} \), and \( \mathcal{C} \subseteq \mathcal{E} \) implies the BCC. Thus EWN is in a sense the strongest statement of the three.

Though this marks the first time that the logical relationships between EWN, \( \mathcal{C} \subseteq \mathcal{E} \), and the BCC have been been independently investigated, all three of these conditions are known in the literature and have been proven from more fundamental assumptions in the bulk [177, 100]. In the classical limit, a common assumption about the bulk physics is the Null Energy Condition (NEC). However, the NEC is known to be violated in quantum field theory. Recently, the Quantum Focussing Conjecture (QFC), which ties together geometry and entropy, was put forward as the ultimate quasi-local “energy condition” for the bulk, replacing the NEC away from the classical limit [34].

The QFC is currently the strongest reasonable quasi-local assumption that one can make about the bulk dynamics, and indeed we will show below that it can be used to prove EWN. Additionally, there are other, weaker, restrictions on the bulk dynamics which follow from the QFC. The Generalized Second Law (GSL) of horizon thermodynamics is a consequence of the QFC. In [64], it was shown that the GSL implies what we have called \( \mathcal{C} \subseteq \mathcal{E} \). Thus the QFC, the GSL, EWN, and \( \mathcal{C} \subseteq \mathcal{E} \) form a square of implications. The QFC is the strongest of the four, implying the three others, while the \( \mathcal{C} \subseteq \mathcal{E} \) is the weakest. This pattern continues in a way summarized by Figure 5.1, which we will now explain.

The QFC, the GSL, and the Achronal Averaged Null Energy Condition (AANEC) reside in the first column of Fig. 5.1. As we have explained, the QFC is the strongest of these three, while the AANEC is the weakest [178]. In the second column we have EWN, \( \mathcal{C} \subseteq \mathcal{E} \), and the BCC. In addition to the relationships mentioned above, it was shown in [84] that the ANEC implies the BCC, which we extend to prove the BCC from the AANEC.

The third column of Figure 5.1 contains “boundary” versions of the first column: the Quantum Null Energy Condition (QNEC) [34, 33, 115], the Quantum Half Averaged Null Energy Condition (QHANEC), and the boundary AANEC. These are field theory statements which can be viewed as nongravitational limits of the corresponding inequalities in the first column. The QNEC is the strongest, implying the QHANEC, which in turn implies the AANEC. All three of these statements can be formulated in non-holographic theories, and

1See [96] for a related classical analysis of bulk constraints from causality, including \( \mathcal{C} \subseteq \mathcal{E} \).

2For simplicity we are assuming throughout that the boundary theory is formulated in Minkowski space. There would be additional subtleties with all three of these statements if the boundary were curved.
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Figure 5.1: The logical relationships between the constraints discussed in this paper. The left column contains semi-classical quantum gravity statements in the bulk. The middle column is composed of constraints on bulk geometry. In the right column is quantum field theory constraints on the boundary CFT. All implications are true to all orders in $G\hbar \sim 1/N$. We have used dashed implication signs for those that were proven to all orders before this paper.

all three are conjectured to be true generally. (The AANEC was recently proven in [71] as a consequence of monotonicity of relative entropy and in [90] as a consequence of causality.)

In the case of a holographic theory, it was shown in [115] that EWN in the bulk implies the QNEC for the boundary theory to leading order in $G\hbar \sim 1/N$. We demonstrate that this relationship continues to hold with bulk quantum corrections. Moreover, in [112] the BCC in the bulk was shown to imply the boundary AANEC. Here we will complete the pattern of implications by showing that $C \subseteq \mathcal{E}$ implies the boundary QHANEC.

In the classical regime, the entanglement wedge is defined in terms of a codimension-2 surface with extremal area [101, 69, 96, 56]. It has been suggested that the correct quantum generalization should be defined in terms of the “quantum extremal surface”: a Cauchy-splitting surface which extremizes the generalized entropy to one side [64]. Indeed, we find that the logical structure of Fig. 5.1 persists to all orders in perturbation theory in $G\hbar \sim 1/N$.
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if and only if the entanglement wedge is defined in terms of the quantum extremal surface. This observation provides considerable evidence for prescription of [64].

The remainder of this paper is organized as follows. In Section 5.2 we will define all of the statements we set out to prove, as well as establish notation. Then in Sections 5.3 and 5.4 we will prove the logical structure encapsulated in Figure 5.1. Several of these implications are already established in the literature, but for completeness we will briefly review the relevant arguments. We conclude with a discussion in Section 8.4.

5.2 Glossary

Regime of Validity Quantum gravity is a tricky subject. We work in a semiclassical (large-\(N\)) regime, where the dynamical fields can be expanded perturbatively in \(G\hbar \sim 1/N\) about a classical background [174]. For example, the metric has the form

\[
g_{ab} = g^0_{ab} + g^{1/2}_{ab} + O((G\hbar)^{3/2}),
\]

where the superscripts denote powers of \(G\hbar\). In the semi-classical limit — defined as \(G\hbar \to 0\) — the validity of the various inequalities we consider will be dominated by their leading non-vanishing terms. We assume that the classical \(O((G\hbar)^0)\) part of the metric satisfies the null energy condition (NEC), without assuming anything about the quantum corrections. For more details on this type of expansion, see Wall [178].

We primarily consider the case where the bulk theory can be approximated as Einstein gravity with minimally coupled matter fields. In the semiclassical regime, bulk loops will generate Planck-suppressed higher derivative corrections to the gravitational theory and the gravitational entropy. We will comment on the effects of these corrections throughout.

We consider a boundary theory on flat space, possibly deformed by relevant operators. When appropriate, we will assume the null generic condition, which guarantees that every null geodesic encounters matter or gravitational radiation.

Geometrical Constraints

There are a number of known properties of the AdS bulk causal structure and extremal surfaces. At the classical level (i.e. at leading order in \(G\hbar \sim 1/N\)), the NEC is the standard assumption made about the bulk which ensures that these properties are true [177]. However, some of these are so fundamental to subregion duality that it is sensible to demand them

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\[3\] The demensionless expansion parameter would be \(G\hbar/\ell^{D-2}\), where \(\ell\) is a typical length scale in whatever state we are considering. We will leave factors of \(\ell\) implicit.

\[4\] Such corrections are also necessary for the generalized entropy to be finite. See Appendix A of [34] for details and references. Other terms can be generated from, for example, stringy effects, but these will be suppressed by the string length \(\ell_s\). For simplicity, we will not separately track the \(\ell_s\) expansion. This should be valid as long as the string scale is not much different from the Planck scale.
and to ask what constraints in the bulk might ensure that these properties hold, even under quantum corrections. Answering this question is one key focus of this paper.

In this section, we review three necessary geometrical constraints. In addition to defining each of them and stating their logical relationships (see Figure 5.1), we explain how each is critical for subregion duality.

**Boundary Causality Condition (BCC)**

A standard notion of causality in asymptotically-AdS spacetimes is the condition that *the bulk cannot be used for superluminal communication relative to the causal structure of the boundary*. More precisely, any causal bulk curve emanating from a boundary point \( p \) and arriving back on the boundary must do so to the future of \( p \) as determined by the boundary causal structure.

This condition, termed “BCC” in [63], is known to follow from the averaged null curvature condition (ANCC) [84]. Engelhardt and Fischetti have derived an equivalent formulation in terms of an integral inequality for the metric perturbation in the context of linearized perturbations to the vacuum [63].

Microcausality in the boundary theory requires that the BCC hold. If the BCC were violated, a bulk excitation could propagate between two spacelike-separated points on the boundary leading to nonvanishing commutators of local fields at those points. In Sec. 5.4 we will show that BCC is implied by \( \mathcal{C} \subseteq \mathcal{E} \). Thus BCC is the weakest notion of causality in holography that we consider.

\[ \mathcal{C} \subseteq \mathcal{E} \]

Consider the domain of dependence \( D(A) \) of a boundary region \( A \). Let us define the causal wedge of a boundary region \( A \) to be \( \mathcal{C}(A) \equiv I^-(D(A)) \cap I^+(D(A)) \).

By the Ryu-Takayanagi-FLM formula, the entropy of the quantum state restricted to \( A \) is given by the area of the extremal area bulk surface homologous to \( A \) plus the bulk entropy in the region between that surface and the boundary [152, 153, 101, 69, 126, 56]. This formula was shown to hold at \( O((1/N)^0) \) in the large-\( N \) expansion. In [64], Engelhardt and Wall proposed that the all-orders modification of this formula is to replace the extremal area surface with the Quantum Extremal Surface (QES), which is defined as the surface which extremizes the generalized entropy: the surface area plus the entropy in the region between the surface and \( A \). Though the Engelhardt-Wall prescription remains unproven, we will assume that it is the correct all-orders prescription for computing the boundary entropy of \( A \). We denote the QES of \( A \) as \( e(A) \).

The entanglement wedge \( \mathcal{E}(A) \) is the bulk region spacelike-related to \( e(A) \) on the \( A \) side of the surface. This is the bulk region believed to be dual to \( A \) in subregion duality [53].

\[ J^\pm(S) \] represent respectively the chronological future and past of the set \( S \). The causal wedge was originally defined in [96] in terms of the causal future and past, \( J^\pm(S) \), but for our purposes the chronological future and past are more convenient.
Dong, Harlow and Wall have argued that this is the case using the formalism of quantum error correction [55, 88].

$C \subseteq E$ is the property that the entanglement wedge $E(A)$ associated to a boundary region $A$ completely contains the causal wedge $C(A)$ associated to $A$. $C \subseteq E$ can equivalently be formulated as stating that $e(A)$ is out of causal contact with $D(A)$, i.e. $e(A) \cap (I^+(D(A) \cup I^-(D(A))) = \emptyset$. In our proofs below we will use this latter characterization.

Subregion duality requires $C \subseteq E$ because the bulk region dual to a boundary region $A$ should at least include all of the points that can both send and receive causal signals to and from $D(A)$. Moreover, if $C \subseteq E$ were false then it would be possible to use local unitary operators in $D(A)$ to send a bulk signal to $e(A)$ and thus change the entropy associated to the region [53, 177, 64, 96]. That is, of course, not acceptable, as the von Neumann entropy is invariant under unitary transformations.

This condition has been discussed at the classical level in [96, 177]. In the semiclassical regime, Engelhardt and Wall [64] have shown that it follows from the generalized second law (GSL) of causal horizons. We will show in Sec. 5.4 that $C \subseteq E$ is also implied by Entanglement Wedge Nesting.

**Entanglement Wedge Nesting (EWN)**

The strongest of the geometrical constraints we consider is EWN. In the framework of subregion duality, EWN is the property that a strictly larger boundary region should be dual to a strictly larger bulk region. More precisely, for any two boundary regions $A$ and $B$ with domain of dependence $D(A)$ and $D(B)$ such that $D(A) \subset D(B)$, we have $E(A) \subset E(B)$.

This property was identified as important for subregion duality and entanglement wedge reconstruction in [53, 177], and was proven by Wall at leading order in $\mathcal{G}\hbar$ assuming the null curvature condition [177]. We we will show in Sec. 5.4 that the Quantum Focussing Conjecture (QFC) [34] implies EWN in the semiclassical regime assuming the generalization of HRT advocated in [64].

**Constraints on Semiclassical Quantum Gravity**

Reasonable theories of matter are often assumed to satisfy various energy conditions. The least restrictive of the classical energy conditions is the null energy condition (NEC), which states that

$$T_{kk} \equiv T_{ab} k^a k^b \geq 0 ,$$

for all null vectors $k^a$. This condition is sufficient to prove many results in classical gravity. In particular, many proofs hinge on the classical focussing theorem [172], which follows from the NEC and ensures that light-rays are focussed whenever they encounter matter or gravitational radiation:

$$\theta' \equiv \frac{d}{d\lambda} \theta \leq 0 ,$$
where $\theta$ is the expansion of a null hypersurface and $\lambda$ is an affine parameter.

Quantum fields are known to violate the NEC, and therefore are not guaranteed to focus light-rays. It is desirable to understand what (if any) restrictions on sensible theories exist in quantum gravity, and which of the theorems which rule out pathological phenomenon in the classical regime have quantum generalizations. In the context of AdS/CFT, the NEC guarantees that the bulk dual is consistent with boundary microcausality \cite{84} and holographic entanglement entropy \cite{177, 37, 95, 96}, among many other things.

In this subsection, we outline three statements in semiclassical quantum gravity which have been used to prove interesting results when the NEC fails. They are presented in order of increasing strength. We will find in sections 5.3 and 5.4 that each of them has a unique role to play in the proper functioning of the bulk-boundary duality.

**Achronal Averaged Null Energy Condition**

The achronal averaged null energy condition (AANEC) \cite{173} states that

$$\int T_{kk} \, d\lambda \geq 0 ,$$

where the integral is along a complete achronal null curve (often called a “null line”). Local negative energy density is tolerated as long as it is accompanied by enough positive energy density elsewhere. The *achronal* qualifier is essential for the AANEC to hold in curved spacetimes. For example, the Casimir effect as well as quantum fields on a Schwarzschild background can both violate the ANEC \cite{114, 170} for chronal null geodesics. An interesting recent example of violation of the ANEC for chronal geodesics in the context of AdS/CFT was studied in \cite{83}.

The AANEC is fundamentally a statement about quantum field theory formulated in curved backgrounds containing complete achronal null geodesics. It has been proven for QFTs in flat space from monotonicity of relative entropy \cite{71}, as well as causality \cite{90}. Roughly speaking, the AANEC ensures that when the backreaction of the quantum fields is included it will focus null geodesics and lead to time delay. This will be made more precise in Sec. 5.4 when we discuss a proof of the boundary causality condition (BCC) from the AANEC.

**Generalized Second Law**

The generalized second law (GSL) of horizon thermodynamics states that the generalized entropy (defined below) of a causal horizon cannot decrease in time.

Let $\Sigma$ denote a Cauchy surface and let $\sigma$ denote some (possibly non-compact) codimension-2 surface dividing $\Sigma$ into two distinct regions. We can compute the von Neumann entropy of
the quantum fields on the region outside of $\sigma$, which we will denote $S_{\text{out}}$. The generalized entropy of this region is defined to be

$$S_{\text{gen}} = S_{\text{grav}} + S_{\text{out}}$$

(5.5)

where $S_{\text{grav}}$ is the geometrical/gravitational entropy which depends on the theory of gravity. For Einstein gravity, it is the familiar Bekenstein-Hawking entropy. There will also be Planck-scale suppressed corrections, denoted $Q$, such that it has the general form

$$S_{\text{grav}} = \frac{A}{4G\hbar} + Q$$

(5.6)

There is mounting evidence that the generalized entropy is finite and well-defined in perturbative quantum gravity, even though the split between matter and gravitational entropy depends on renormalization scale. See the appendix of [34] for details and references.

The quantum expansion $\Theta$ can be defined (as a generalization of the classical expansion $\theta$) as the functional derivative per unit area of the generalized entropy along a null congruence [34]:

$$\Theta[\sigma(y); y] \equiv \frac{4G\hbar \delta S_{\text{gen}}}{\sqrt{h} \delta \sigma(y)}$$

(5.7)

$$= \theta + \frac{4G\hbar}{\sqrt{h}} \delta Q \delta \sigma(y) + \frac{4G\hbar}{\sqrt{h}} \delta S_{\text{out}} \delta \sigma(y)$$

(5.8)

where $\sqrt{h}$ denotes the determinant of the induced metric on $\sigma$, which is parametrized by $y$. These functional derivatives denote the infinitesimal change in a quantity under deformations of the surface at coordinate location $y$ along the chosen null congruence. To lighten the notation, we will often omit the argument of $\Theta$.

A future (past) causal horizon is the boundary of the past (future) of any future-infinite (past-infinite) causal curve [108]. For example, in an asymptotically AdS spacetime any collection of points on the conformal boundary defines a future and past causal horizon in the bulk. The generalized second law (GSL) is the statement that the quantum expansion is always nonnegative towards the future on any future causal horizon

$$\Theta \geq 0,$$

(5.9)

with an analogous statement for a past causal horizon.

---

6The choice of “outside” is arbitrary. In a globally pure state both sides will have the same entropy, so it will not matter which is the “outside.” In a mixed state the entropies on the two sides will not be the same, and thus there will be two generalized entropies associated to the same surface. The GSL, and all other properties of generalized entropy, should apply equally well to both.

7There will also be stringy corrections suppressed by $\alpha'$. As long as we are away from the stringy regime, these corrections will be suppressed in a way that is similar to the Planck-suppressed ones, and so we will not separately track them.
In the semiclassical $G\hbar \to 0$ limit, Eq. (5.7) reduces to the classical expansion $\theta$ if it is nonzero, and the GSL becomes the Hawking area theorem [91]. The area theorem follows from the NEC.

Assuming the validity of the GSL allows one to prove a number of important results in semiclassical quantum gravity [180, 64]. In particular, Wall has shown that it implies the AANEC [178], as we will review in Section 5.3, and $C \subseteq E$ [64], reviewed in Section 5.4 (see Fig. 5.1).

Quantum Focussing Conjecture

The Quantum Focussing Conjecture (QFC) was conjectured in [34] as a quantum generalization of the classical focussing theorem, which unifies the Bousso Bound and the GSL. The QFC states that the functional derivative of the quantum expansion along a null congruence is nowhere increasing:

$$\frac{\delta \Theta[\sigma(y_1); y_1]}{\delta \sigma(y_2)} \leq 0.$$  \hspace{1cm} (5.10)

In this equation, $y_1$ and $y_2$ are arbitrary. When $y_1 \neq y_2$, only the $S_{\text{out}}$ part contributes, and the QFC follows from strong subadditivity of entropy [34]. For notational convenience, we will often denote the “local” part of the QFC, where $y_1 = y_2$, as

$$\Theta'[\sigma(y); y] \leq 0.$$  \hspace{1cm} (5.11)

Note that while the GSL is a statement only about causal horizons, the QFC is conjectured to hold on any cut of any null hypersurface.

If true, the QFC has several non-trivial consequences which can be teased apart by applying it to different null surfaces [34, 27, 64]. In Sec. 5.4 we will see that EWN can be added to this list.

Quantum Null Energy Condition

When applied to a locally stationary null congruence, the QFC leads to the Quantum Null Energy Condition (QNEC) [34, 115]. Applying the Raychaudhuri equation and Eqs. (5.5), (5.7) to the statement of the QFC (5.10), we find

$$0 \geq \Theta' = -\frac{\theta^2}{D-2} - \sigma^2 - 8\pi G T_{kk} + \frac{4G\hbar}{\sqrt{h}} (S''_{\text{out}} - S'_\text{out} \theta)$$  \hspace{1cm} (5.12)

where $S''_{\text{out}}$ is the local functional derivative of the matter entropy to one side of the cut. If we consider a locally stationary null hypersurface satisfying $\theta^2 = \sigma^2 = 0$ in a small neighborhood,

\hspace{1cm} \text{Strictly speaking, we should factor out a delta function $\delta(y_1 - y_2)$ when discussing the local part of the QFC [33, 115]. Since the details of this definition are not important for us, we will omit this in our notation.}
this inequality reduces to the statement of the Quantum Null Energy Condition (QNEC) [34]:

\[ T_{kk} \geq \frac{\hbar}{2\pi \sqrt{h}} S''_{out} \quad (5.13) \]

It is important to notice that the gravitational coupling \( G \) has dropped out of this equation. The QNEC is a statement purely in quantum field theory which can be proven or disproven using QFT techniques. It has been proven for both free fields [33] and holographic field theories at leading order in \( G\hbar [115]. \)

In Section 5.4 of this paper, we generalize the holographic proof to all orders in \( G\hbar \). These proofs strongly suggest that the QNEC is a true property of general quantum field theories.\(^9\) In the classical \( \hbar \to 0 \) limit, the QNEC becomes the NEC.

**Quantum Half-Averaged Null Energy Condition**

The quantum half-averaged energy condition (QHANEC) is an inequality involving the integrated stress tensor and the first null derivative of the entropy on one side of any locally-stationary Cauchy-splitting surface subject to a causality condition (described below):

\[
\int_{\lambda}^{\infty} T_{kk} \, d\tilde{\lambda} \geq -\frac{\hbar}{2\pi \sqrt{h}} S'_{\lambda},
\]

Here \( k^a \) generates a null congruence with vanishing expansion and shear in a neighborhood of the geodesic and \( \lambda \) is the affine parameter along the geodesic. The geodesic thus must be of infinite extent and have \( R_{ab}k^ak^b = C_{abcd}k^ak^c = 0 \) everywhere along it. The aforementioned causality condition is that the Cauchy-splitting surfaces used to define \( S(\lambda) \) should not be timelike-related to the half of the null geodesic \( T_{kk} \) is integrated over. Equivalently, \( S(\lambda) \) should be well-defined for all \( \lambda \) from the starting point of integration all the way to \( \lambda = \infty \).

The causality condition and the stipulation that the null geodesic in (5.14) be contained in a locally stationary congruence ensures that the QHANEC follows immediately from integrating the QNEC (Eq. (5.13)) from infinity (as long as the entropy isn’t evolving at infinite affine parameter, i.e., \( S'(\infty) = 0 \)). Because the causality condition is a restriction on the global shape of the surface, there will be situations where the QNEC holds locally but we cannot integrate to arrive at a QHANEC.

The QHANEC appears to have a very close relationship to monotonicity of relative entropy. Suppose that the modular Hamiltonian of the portion of a null plane above an arbitrary cut \( u = \sigma(y) \) (where \( u \) is a null coordinate) is given by

\[
K[\sigma(y)] = \int d^{d-2}y \int_{\sigma(y)}^{\infty} d\lambda \, (\lambda - \sigma(y)) \, T_{kk}
\]

\(^9\)There is also evidence [82] that the QNEC holds in holographic theories where the entropy is taken to be the casual holographic information [100], instead of the von Neumann entropy.

\(^{10}\)The free-field proof of [33] was for arbitrary cuts of Killing horizons. The holographic proof of [115] (generalized in this paper) showed the QNEC for a locally stationary (\( \theta = \sigma = 0 \)) portion of any Cauchy-splitting null hypersurface in flat space.
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Then (5.45) becomes monotonicity of relative entropy. As of yet, there is no known general proof in the literature of (5.15), though for free theories it follows from the enhanced symmetries of null surface quantization [174]. Eq. (5.15) can be also be derived for holographic field theories [116]. It has also been shown that linearized backreaction from quantum fields obeying the QHANEC will lead to a spacetime satisfying the GSL [174].11

In Sec. 5.4, we will find that \( C \subseteq E \) implies the QHANEC on the boundary.

5.3 Relationships Between Entropy and Energy Inequalities

The inequalities discussed in the previous section are not all independent. In this section we discuss the logical relationships between them.

GSL implies AANEC

Wall has shown [178] that the GSL implies the AANEC in spacetimes which are linearized perturbations of classical backgrounds, where the classical background obeys the null energy condition (NEC). Here, we point out that this proof is sufficient to prove the AANEC from the GSL in the semi-classical regime, to all orders in \( \sqrt{G} \) (see Sec. 5.2).

Because the AANEC is an inequality, in the semi-classical \( \sqrt{G} \to 0 \) limit its validity is determined by the leading non-zero term in the \( \sqrt{G} \) expansion. Suppose that this term is order \( \sqrt{G}^m \). Suppose also that at order \( \sqrt{G}^{m-1} \) the metric contains a complete achronal null geodesic \( \gamma \), i.e. a null geodesic without a pair of conjugate points. (If at this order no such geodesics exist, the AANEC holds trivially at this order as well as all higher orders, as higher-order contributions to the metric cannot make a chronal geodesic achronal.) Achronality guarantees that \( \gamma \) lies in both a future and past causal horizon, \( \mathcal{H}^\pm \).

Wall’s proof required that, in the background spacetime, the expansion and shear vanish along \( \gamma \) in both \( \mathcal{H}^+ \) and \( \mathcal{H}^- \). Wall used the NEC in the background spacetime to derive this, but here we note that the NEC is not necessary given our other assumptions. The “background” for us is the \( O(\sqrt{G}^{m-1}) \) part of the metric. Consider first the past causal horizon, \( \mathcal{H}^- \), which must satisfy the boundary condition \( \theta(-\infty) \to 0 \). Since \( \gamma \) is achronal, the expansion \( \theta \) of \( \mathcal{H}^- \) cannot blow up to \( -\infty \) anywhere along \( \gamma \). As \( \lambda \to \infty \), \( \theta \) can either remain finite or blow up in the limit. Suppose first that \( \theta \) asymptotes to a finite constant as \( \lambda \to \infty \). Then \( \lim_{\lambda \to \infty} \theta' = 0 \). Assuming the matter stress tensor dies off at infinity (as it must for the AANEC to be well-defined), Raychaudhuri’s equation gives \( \lim_{\lambda \to \infty} \theta' = -\theta^2/(D-2) - \sigma^2 \),

11It has been shown [35] that holographic theories also obey the QHANEC when the causal holographic information [100] is used, instead of the von Neumann entropy. This implies a second law for the causal holographic information in holographic theories.
the only solution to which is\(^{12}\)
\[
\lim_{\lambda \to \infty} \theta = \lim_{\lambda \to \infty} \sigma = 0 .
\] (5.16)

Similar arguments apply to \(\mathcal{H}^+\). This also implies that \(\mathcal{H}^+ = \mathcal{H}^-\). The rest of the proof follows [178]. This proves the AANEC at order \((G\hbar)^m\).

The alternative case is that \(|\theta| \to \infty\) as \(\lambda \to \infty\). But if \(T_{kk}\) dies off at infinity, then for large enough \(\lambda\) we have \(\theta' < -\theta^2/(D-2) + \epsilon\) for some \(\epsilon\). Then a simple modification of the standard focusing argument shows that \(\theta\) goes to \(-\infty\) at finite affine parameter, which is a contradiction.

**QFC implies GSL**

In a manner exactly analogous to the proof of the area theorem from classical focusing, the QFC can be applied to a causal horizon to derive the GSL. Consider integrating Eq. 5.10 from future infinity along a generator of a past causal horizon:

\[
\int d^{d-2}y \sqrt{h} \int_{\lambda}^\infty d\bar{\lambda} \Theta'[\sigma(y, \bar{\lambda}); y] \leq 0
\] (5.17)

Along a future causal horizon, \(\theta \to 0\) as \(\lambda \to \infty\), and it is reasonable to expect the matter entropy \(S_{\text{out}}\) to stop evolving as well. Thus \(\Theta \to 0\) as \(\lambda \to \infty\), and the integrated QFC then trivially becomes

\[
\Theta[\sigma(y); y] \geq 0
\] (5.18)

which is the GSL.

**QHANEC implies AANEC**

In flat space, all achronal null geodesics lie on a null plane. Applying the QHANEC to cuts of this null plane taking \(\lambda \to -\infty\) produces the AANEC, Eq. (5.4).

### 5.4 Relationships Between Entropy and Energy Inequalities and Geometric Constraints

In this section, we discuss how the bulk generalized entropy conditions reviewed in Sec. 5.2 imply the geometric conditions EWN, \(\mathcal{C} \subseteq \mathcal{E}\) and BCC (described in Sec. 5.2). We also explain how these geometric conditions imply the boundary QNEC, QHANEC and AANEC.
The causal relationship between $e(A)$ and $D(A)$ is pictured in an example space-time that violates $C \subseteq E$. The boundary of $A$’s entanglement wedge is shaded. Notably, in $C \subseteq E$ violating spacetimes, there is necessarily a portion of $D(A)$ that is timelike related to $e(A)$. Extremal surfaces of boundary regions from this portion of $D(A)$ are necessarily timelike related to $e(A)$, which violates EWN.

**EWN implies $C \subseteq E$ implies the BCC**

**EWN implies $C \subseteq E$**

The $C \subseteq E$ and EWN conditions were defined in Sec. 5.2. There, we noted that $C \subseteq E$ can be phrased as the condition that the extremal surface $e(A)$ for some boundary region $A$ lies outside of timelike contact with $D(A)$. We will now prove that EWN implies $C \subseteq E$ by proving the contrapositive: we will show that if $C \subseteq E$ is violated, there exist two boundary regions $A, B$ with nested domains of dependence, but whose entanglement wedges are not nested.

Consider an arbitrary region $A$ on the boundary. $C \subseteq E$ is violated if and only if there exists at least one point $p \in e(A)$ such that $p \in I^+(D(A)) \cup I^-(D(A))$, where $I^+$ ($I^-$) denotes the chronological future (past). Then violation of $C \subseteq E$ is equivalent to the existence of a timelike curve connecting $e(A)$ to $D(A)$. Because $I^+$ and $I^-$ are open sets, there exists an open neighborhood $O \subset D(A)$ such that every point of $O$ is timelike related to $e(A)$ (see Figure 5.2). Consider a new boundary region $B \subset O$. Again by the openness of $I^+$ and $I^-$, the corresponding entanglement wedge $E(B)$ also necessarily contains points that are timelike related to $e(A)$. Since $E(A)$ is defined to be all points *spacelike-related* to $e(A)$ on the side towards $A$, $E(B) \not\subseteq E(A)$. But by construction $D(B) \subseteq D(A)$, and thus EWN is violated.

---

**Footnote:** We absorb the graviton contribution to the shear into the stress tensor.
Figure 5.3: The boundary of a BCC-violating spacetime is depicted, which gives rise to a violation of $\mathcal{C} \subseteq \mathcal{E}$. The points $p$ and $q$ are connected by a null geodesic through the bulk. The boundary of $p$’s lightcone with respect to the AdS boundary causal structure is depicted with solid black lines. Part of the boundary of $q$’s lightcone is shown with dashed lines. The disconnected region $A$ is defined to have part of its boundary in the timelike future of $q$ while also satisfying $p \in D(A)$. It follows that $e(A)$ will be timelike related to $D(A)$ through the bulk, violating $\mathcal{C} \subseteq \mathcal{E}$.

violated.

In light of this argument, we have an additional characterization of the condition $\mathcal{C} \subseteq \mathcal{E}$: $\mathcal{C} \subseteq \mathcal{E}$ is what guarantees that $\mathcal{E}(A)$ contains $D(A)$, which is certainly required for consistency of bulk reconstruction.

\textbf{$\mathcal{C} \subseteq \mathcal{E}$ implies the BCC}

We prove the contrapositive. If the BCC is violated, then there exists a bulk null geodesic from some boundary point $p$ that returns to the boundary at a point $q$ not to the future of $p$ with respect to the boundary causal structure. Therefore there exist points in the timelike future of $q$ that are also not to the future of $p$.

If $q$ is not causally related to $p$ with respect to the boundary causal structure, we derive a contradiction as follows. Define a boundary subregion $A$ with two disconnected parts: one that lies entirely within the timelike future of $q$ but outside the future of $p$, and one composed of all the points in the future lightcone of $p$ on a boundary timeslice sufficiently close to $p$ such that $A$ is completely achronal. By construction, $p \in D(A)$. Moreover, because $\partial A$ includes points timelike related to $q$, $e(A)$ includes points timelike related to $q$ and by


extension \( p \). Therefore \( A \) is an achronal boundary subregion whose extremal surface contains points that are timelike related to \( D(A) \). See Figure 5.3.

If \( q \) is in the past of \( p \), then a contradiction is reached more easily. Define a boundary subregion \( A \) as the intersection of \( p \)’s future lightcone with any constant time slice sufficiently close to \( p \), chosen so that \( e(A) \) is not empty. Then \( p \) is in \( D(A) \) and \( q \) is in \( I^-(A) \) according to the boundary causal structure (though according to the bulk causal structure it is in \( J^+(p) \)). Hence \( e(A) \) is timelike related to \( D(A) \) in the bulk causal structure, which is the sought-after contradiction.

**Semiclassical Quantum Gravity Constraints Imply Geometric Constraints**

**Quantum Focussing implies Entanglement Wedge Nesting**

Consider a boundary region \( A \) associated with boundary domain of dependence \( D(A) \). As above, we denote the quantum extremal surface anchored to \( \partial A \) as \( e(A) \). For any other boundary region, \( B \), such that \( D(B) \subset D(A) \), we will show that \( E(B) \subset E(A) \) assuming the QFC.

Since we are treating quantum corrections perturbatively, every quantum extremal surface is located near a classical extremal area surface.\(^{13}\) Wall proved in [177] that \( E(B) \subset E(A) \) is true at the classical level if we assume classical focussing. Thus to prove the quantum statement within perturbation theory we only need to consider those (nongeneric) cases where \( e(B) \) happens to intersect the boundary of \( E(A) \) classically.\(^{14}\) In such a case, one might worry that a perturbative quantum correction could cause \( e(B) \) to exit \( E(A) \). We will now argue that this does not happen.\(^{15}\)

First, deform the region \( B \) slightly to a new region \( B' \subset A \) such that \( e(B') \) lies within \( E(A) \) classically. Then, since perturbative corrections cannot change this fact, we will have \( E(B') \subset E(A) \) even at the quantum level. Now, following [64], we show that in deforming \( B' \) back to \( B \) we maintain EWN.

The QFC implies that the null congruence generating the boundary of \( I^\pm(e(A)) \) satisfies \( \dot{\Theta} \leq 0 \). Combined with \( \Theta = 0 \) at \( e(A) \) (from the definition of quantum extremal surface), this implies that every point on the boundary of \( E(A) \) satisfies \( \Theta \leq 0 \). Therefore the boundary of \( E(A) \) is a quantum extremal barrier as defined in [64], and so no continuous family of quantum extremal surfaces can cross the boundary of \( E(A) \). Thus, as we deform \( B' \) back into \( B \), the quantum extremal surface is forbidden from exiting \( E(A) \). Therefore \( e(B) \subset E(A) \), and by extension \( E(B) \subset E(A) \).

\(^{13}\)Another possibility is that quantum extremal surfaces which exist at finite \( G\hbar \) move off to infinity as \( G\hbar \to 0 \). In that case there would be no associated classical extremal surface. If we believe that the classical limit is well-behaved, then these surfaces must always be subdominant in the small \( G\hbar \) limit, and so we can safely ignore them.

\(^{14}\)The only example of this that we are aware of is in vacuum AdS where \( A \) is the interior of a sphere on the boundary and \( B \) is obtained by deforming a portion of the sphere in an orthogonal null direction.

\(^{15}\)For now we ignore the possibility of phase transitions. They will be treated separately below.
Finally we will take care of the possibility of a phase transition. A phase transition occurs when there are multiple quantum extremal surfaces for each region, and the identity of the one with minimal generalized entropy switches as we move within the space of regions. This causes the entanglement wedge to jump discontinuously, and if it jumps the "wrong way" then EWN could be violated. Already this would be a concern at the classical level, but it was shown in [177] that classically EWN is always obeyed even accounting for the possibility of phase transitions. So we only need to convince ourselves that perturbative quantum corrections cannot change this fact.

Consider the infinite-dimensional parameter space of boundary regions. A family of quantum (classical) extremal surfaces determines a function on this parameter space given by the generalized entropy (area) of the extremal surfaces. A phase transition occurs when two families of extremal surfaces have equal generalized entropy (or area), and is associated with a codimension-one manifold in parameter space. In going from the purely classical situation to the perturbative quantum situation, two things will happen. First, the location of the codimension-one phase transition manifold in parameter space will be shifted. Second, within each family of extremal surfaces, the bulk locations of the surfaces will be perturbatively shifted. We can treat these two effects separately.

In the vicinity of the phase transition (in parameter space), the two families of surfaces will be classically separated in the bulk and classically obey EWN, as proved in [177]. A perturbative shift in the parameter space location of the phase transition will not change whether EWN is satisfied classically. That is, the classical surfaces in each extremal family associated with the neighborhood of quantum phase transition will still be separated classically in the bulk. Then we can shift the bulk locations of the classical extremal surfaces to the quantum extremal surfaces, and since the shift is only perturbative there is no danger of introducing a violation of EWN.

It would be desirable to have a more unified approach to this proof in the quantum case that does not rely so heavily on perturbative arguments. We believe that such an approach is possible, and in future work we hope to lift all of the results of [177] to the quantum case by the replacement of "area" with "generalized entropy" without having to rely on a perturbative treatment.

**Generalized Second Law implies** $\mathcal{C} \subseteq \mathcal{E}$

This proof can be found in [64], but we elaborate on it here to illustrate similarities between this proof and the proof that QFC implies EWN.

**Wall’s Lemma** We remind the reader of a fact proved as Theorem 4 in [177]. Let two boundary anchored co-dimension two, spacelike surfaces $M$ and $N$, which contain the point $p \in M \cap N$ such that they are also tangent at $p$. Both surfaces are Cauchy-splitting in the bulk. Suppose that $M$ lies completely to one side of $N$. In the classical regime, Wall shows

---

16Wall’s Lemma is a significant part of the extremal surface barriers argument in [64].
Figure 5.4: The surface $M$ and $N$ are shown touching at a point $p$. In this case, $\theta_M < \theta_N$. The arrows illustrate the projection of the null orthogonal vectors onto the Cauchy surface.

that either there exists some point $x$ in a neighborhood of $p$ where

$$\theta_N(x) > \theta_M(x)$$

or the two surfaces agree everywhere in the neighborhood. These expansions are associated to the exterior facing, future null normal direction.

In the semi-classical regime, this result can be improved to bound the quantum expansions

$$\Theta_1(x) > \Theta_2(x)$$

where $x$ is some point in a neighborhood of $p$. The proof of this quantum result requires the use of strong sub-additivity, and works even when bulk loops generate higher derivative corrections to the generalized entropy [180].

We now proceed to prove $C \subseteq E$ from the GSL by contradiction. Suppose that the causal wedge lies at least partly outside the entanglement wedge. In this discussion, by the “boundary of the causal wedge,” we mean the intersection of the past of $I^-(\partial D(A))$ with the Cauchy surface on which $e(A)$ lies. Consider continuously shrinking the boundary region associated to the causal wedge. The causal wedge will shrink continuously under this deformation. At some point, $C(A)$ must shrink inside $E(A)$. There exists some Cauchy surface such that its intersection with the boundary of the causal wedge touches the original extremal surface as depicted in Figure 5.4. There, $M$ is the intersection of the boundary of the causal wedge of the shrunken region with the Cauchy surface and $N$ is $e(A)$.

Assuming genericity of the state, the two surfaces cannot agree in this neighborhood. At this point, by the above lemma, the quantum expansions should obey

$$\Theta_e(x) > \Theta_c(x)$$

for $x$ in some neighborhood of $p$. The Wall-Engelhardt prescription tells us that the entanglement wedge boundary should be given by the quantum extremal surface [64] and so

$$\Theta_e(x) = 0 > \Theta_c(x)$$

Thus, the GSL is violated at some point along this causal surface, which draws the contradiction.
AANEC implies Boundary Causality Condition

The Gao-Wald:1984rg proof of the BCC [84] uses the fact — which follows from their assumptions of the NEC and null generic condition (discussed below) — that all complete null geodesics through the bulk contain a pair of conjugate points.\footnote{Intuitively speaking, points $p$ and $q$ along a geodesic $\gamma$ are conjugate if an infinitesimally nearby geodesic intersects $\gamma$ at both $p$ and $q$. This can be shown to be equivalent to the statement that the expansion of a congruence through $p$ approaches $-\infty$ at $q$. See e.g. [172] for details.} Here, we sketch a slight modification of the proof which instead assumes the achronal averaged null energy condition (AANEC).

We prove that the AANEC implies BCC by contradiction. Let the spacetime satisfy the null generic condition \cite{172}, so that each null geodesic encounters at least some matter or gravitational radiation.\footnote{Mathematically, each complete null geodesic should contain a point where $k^ak^bk^ck^dR_{abcd} \neq 0$.} Violation of the BCC implies that the "fastest" null geodesic $\gamma$ between two boundary points $p$ and $q$ lies in the bulk. Such a null geodesic is necessarily complete and achronal, as $p$ and $q$ are not timelike related. As explained in Sec. 5.3, an achronal null geodesic in an AANEC satisfying spacetime is contained in a congruence that is both a past and future causal horizon. Integrating Raychaudhuri’s equation along the entire geodesic then gives $0 \leq -\int (\theta^2 + \sigma^2)$, which implies $\theta = \sigma = 0$ everywhere along the geodesic, and hence $\theta' = 0$. Raychaudhuri’s equation then says $\theta' = -T_{kk} = 0$ everywhere, which contradicts the generic condition.

Geometric Constraints Imply Field Theory Constraints

The geometrical constraints EWN, $\mathcal{C} \subseteq \mathcal{E}$, and BCC have non-trivial implications for the boundary theory. We derive them in this section, which proves the three implications connecting columns two and three of Fig. 5.1. The key idea behind all three proofs is the same: express the geometrical constraints in terms of bulk quantities near the asymptotic boundary, and then use near-boundary expansions of the metric and extremal surfaces to convert them into field theory statements.

Entanglement Wedge Nesting implies the Boundary QNEC

At leading order in $G\hbar \sim 1/N$, this proof is the central result of [115]. There the boundary entropy was assumed to be given by the RT formula without the bulk entropy corrections. We give a proof here of how the $1/N$ corrections can be incorporated naturally. We will now show, in a manner exactly analogous to that laid out in [115], that EWN implies the boundary QNEC. In what follows, we will notice that in order to recover the boundary QNEC, we must use the quantum extremal surface, not just the RT surface with FLM corrections \cite{64}.

The quantum extremal surface (QES) prescription, as first introduced in [64], is the following. To find the entropy of a region $A$ in the boundary theory, first find the minimal codimension-2 bulk surface homologous to $A$, $e(A)$, that extremizes the bulk generalized

\begin{align*}
\theta' &= -T_{kk} = 0 \\
\theta &= \sigma = 0
\end{align*}
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entropy on the side of $A$. The entropy of $A$ is then given by

$S_A = S_{\text{gen}}(e(A)) = \frac{A_{\text{QES}}}{4G\hbar} + S_{\text{bulk}}$ \hspace{1cm} (5.23)

Entanglement Wedge Nesting then becomes a statement about how the quantum extremal surface moves under deformations of the boundary region. In particular, for null deformations of the boundary region, EWN states that $e(A)$ moves in a spacelike (or null) fashion.

To state this more precisely, we can set up a null orthogonal basis about $e(A)$. Let $k^\mu$ be the inward-facing, future null orthogonal vector along the quantum extremal surface. Let $\ell^\mu$ be its past facing partner with $\ell \cdot k = 1$. Following the prescription in [115], we denote the locally orthogonal deviation vector of the quantum extremal surface by $s^\mu$. This vector can be expanded in the local null basis as

$s^\mu = \alpha k^\mu + \beta \ell^\mu$ \hspace{1cm} (5.24)

The statement of entanglement wedge nesting then just becomes the statement that $\beta \geq 0$.

In order to find how $\beta$ relates to the boundary QNEC, we would like to find its relation to the entropy. We start by examining the expansion of the extremal surface solution in Fefferman-Graham coordinates. Note that the quantum extremal surface obeys an equation of motion including the bulk entropy term as a source

$K^\mu = -4G\hbar \frac{\delta S_{\text{bulk}}}{\sqrt{H}} \delta X^\mu$ \hspace{1cm} (5.25)

Here, $K^\mu = \theta_k \ell^\mu + \theta_\ell k^\mu$ is the extrinsic curvature of the QES. As discussed in [115], solutions to (5.25) without the bulk source take the form

$\bar{X}^i_{\text{HRT}}(y^a, z) = X^i(y^a) + \frac{1}{2(d-2)} z^2 K^i(y^a) + ... + \frac{z^d}{d} (V^i(y^a) + W^i(y^a) \log z) + o(d^d)$ \hspace{1cm} (5.26)

We now claim that the terms lower order than $z^d$ are unaffected by the presence of the source. More precisely

$\bar{X}^i_{\text{QES}}(y^a, z) = X^i(y^a) + \frac{1}{2(d-2)} z^2 K^i(y^a) + ... + \frac{z^d}{d} (V_{\text{QES}}^i + W^i(y^a) \log z) + o(d^d)$ \hspace{1cm} (5.27)

This expansion is found by examining the leading order pieces of the extremal surface equation. First, expand (5.25) to derive

$z^{d-1} \partial_z \left( z^{1-d} f \sqrt{h} \bar{h}^{zz} \partial_z X^i \right) + \partial_a \left( \sqrt{h_{ab}} \bar{h}^{ab} f \partial_b \bar{X}^i \right) = -z^{d-1} 4Gf \hbar f \frac{\delta S_{\text{bulk}}}{\sqrt{h}} g^{ji}$ \hspace{1cm} (5.28)

Here we are parameterizing the near-boundary AdS metric in Fefferman-Graham coordinates by

$ds^2 = \frac{1}{z^2} \left( dz^2 + g_{ij} dx^i dx^j \right)$

$= \frac{1}{z^2} \left( dz^2 + \left[ f(z) \eta_{ij} + \frac{16\pi G_N}{d} z^d t_{ij} \right] dx^i dx^j + o(z^d) \right)$ \hspace{1cm} (5.29)

...
The function \( f(z) \) encodes the possibility of relevant deformations in the field theory which would cause the vacuum state to differ from pure AdS. Here we have set \( L_{\text{AdS}} = 1 \).

One then plugs in (5.27) to (5.28) to see that the terms lower order than \( z^d \) remain unaffected by the presence of the bulk entropy source as long as \( \delta S_{\text{bulk}} / \delta X^i \) remains finite at \( z = 0 \). We discuss the plausibility of this boundary condition at the end of this section.

For null deformations to locally stationary surfaces on the boundary, one can show using (5.27) that the leading order piece of \( \beta \) in the Fefferman-Graham expansion is order \( z^{d-2} \). Writing the coordinates of the boundary entangling surface as a function of some deformation parameter - \( X^i(\lambda) \) - we find that [115],

\[
\beta \propto z^{d-2} \left( T_{kk} + \frac{1}{8\pi G_N} k_i \partial_i V^i_{\text{QES}} \right). \tag{5.31}
\]

We will now show that \( V^i_{\text{QES}} \) is proportional to the variation in \( S_{\text{gen}} \) at all orders in \( 1/N \), as long as one uses the quantum extremal surface and assumes mild conditions on derivatives of the bulk entropy. The key will be to leverage the fact that \( S_{\text{gen}} \) is extremized on the QES. Thus, its variation will come from pure boundary terms. At leading order in \( z \), we will identify these boundary terms with the vector \( V_{\text{QES}} \).

We start by varying the generalized entropy with respect to a boundary deformation

\[
\delta S_{\text{gen}} = \int_{\text{QES}} \frac{\delta S_{\text{gen}}}{\delta \bar{X}^i} \delta \bar{X}^i dz d^{d-2}y - \int_{z=\epsilon} \left( \frac{\partial S_{\text{gen}}}{\partial (\bar{X}^i)} + \ldots \right) \delta \bar{X}^i d^{d-2}y \tag{5.32}
\]

where the boundary term comes from integrating by parts when deriving the Euler-Lagrange equations for the functional \( S_{\text{gen}}[\bar{X}] \). The ellipse denotes terms involving derivatives of \( S_{\text{gen}} \) with respect to higher derivatives of the embedding functions \( \partial S_{\text{gen}} / \partial (\partial^2 X), \ldots \) These boundary terms will include two types of terms: one involving derivatives of the surface area and one involving derivatives of the bulk entropy.

The first area term was already calculated in [115]. There it was found that

\[
\frac{\partial A}{\partial (\bar{X}^i)} = -\frac{1}{z^{d-1}} \int d^{d-2}y \sqrt{h} \frac{g_{ij} \partial_j \bar{X}^i}{\sqrt{1 + g_{lm} \partial_j \bar{X}^l \partial_k \bar{X}^m}} \delta \bar{X}^j |_{z=\epsilon} \tag{5.33}
\]

One can use (5.27) to expand this equation in powers \( \epsilon \), and then contract with the null vector \( k \) on the boundary in order to isolate the variation with respect to null deformations. For boundary surfaces which are locally stationary at some point \( y \), one finds that all terms lower order than \( z^d \) vanish at \( y \). In fact, it was shown in [115] that the right hand side of (5.33), after contracting with \( k^i \), is just \( k^i V_i \) at first non-vanishing order. Finally, we assume that all such derivatives of the bulk entropy in (5.32) vanish as \( z \to 0 \). This is similar to the reasonable assumption that entropy variations vanish at infinity, which should be true in a state with finite bulk entropy. It would be interesting to classify the pathologies of states which violate this assumption. Thus, the final result is that

\[
k^i V^i_{\text{QES}} = -\frac{1}{\sqrt{h}} k^i \frac{\delta S_{\text{gen}}}{\delta X^i}. \tag{5.34}
\]
The quantum extremal surface prescription says that the boundary field theory entropy is equal to the generalized entropy of the QES \[64\]. Setting \( S_{\text{gen}} = S_{\text{dry}} \) in (5.34) and combining that with (5.31) shows that the condition \( \beta \geq 0 \) is equivalent to the QNEC. Since EWN guarantees that \( \beta \geq 0 \), the proof is complete.

We briefly comment about the assumptions used to derive (5.34). The bulk entropy should - for generic states - not depend on the precise form of the region near the boundary. The intuition is clear in the thermodynamic limit where bulk entropy is extensive. As long as we assume strong enough fall-off conditions on bulk matter, the change in the entropy will have to vanish as \( z \to 0 \).

Note here the importance of using the quantum extremal surface. Had we naively continued to use the extremal area prescription, but still assumed \( S_A = S_{\text{bulk}}(e(A)) + \frac{A}{4G_N} \), we would have discovered a correction to the boundary QNEC from the bulk entropy. The variation of the bulk extremal surface area would be given by a pure boundary term, but the QNEC would take the erroneous form

\[
T_{kk} \geq \frac{1}{2\pi\sqrt{\hbar}} (S''_A - S''_{\text{bulk}}(e(A))).
\]

In other words, if one wants to preserve the logical connections put forth in Figure 5.1 while accounting for \( 1/N \) corrections, the use of quantum extremal surfaces is necessary.

We discuss the effects of higher derivative terms in the gravitational action coming from loop corrections at the end of this section.

**C \subseteq E** implies the QHANEC

We now examine the boundary implication of \( C \subseteq E \). As before, this proof will hold to all orders in \( G\hbar \), again assuming proper fall-off conditions on derivatives of the bulk entropy.

The basic idea will be to realize that general states in AdS/CFT can be treated as perturbations to the vacuum in the limit of small \( z \). Again, we will consider the general case where the boundary field theory includes relevant deformations. Then, near the boundary, the metric can be written

\[
ds^2 = \frac{1}{z^2} \left( dz^2 + \left[ f(z) \eta_{ij} + \frac{16\pi G_N}{d} z^d t_{ij} \right] dx^i dx^j + o(z^d) \right),
\]

where \( f(z) \) encodes the effects of the relevant deformations. In this proof we take the viewpoint that the order \( z^d \) piece of this expansion is a perturbation on top of the vacuum. In other words

\[
g_{ab} = g_{ab}^{\text{vac}} + \delta g_{ab}.
\]

Of course, this statement is highly coordinate dependent. In the following calculations, we treat the metric as a field on top of fixed coordinates. We will have to verify the gauge-independence of the final result, and do so below.

For this proof we are interested in regions \( A \) of the boundary such that \( \partial A \) is a cut of a null plane. In null coordinates, that would look like \( \partial A = \{(u = U_0(y), v = 0)\} \). These
regions are special because in the vacuum state $e(A)$ lies on the past causal horizon generated by bulk geodesics coming from $(u = \infty, v = 0)$. This can be shown using Lorentz symmetry as follows:

An arbitrary cut of a null plane can be deformed back to a flat cut by action with an infinite boost (since boosts act by rescalings of $u$ and $v$). Such a transformation preserves the vacuum, and so the bulk geometry possesses an associated Killing vector. The past causal horizon from $(u = \infty, v = 0)$ is a Killing horizon for this boost, and by symmetry the quantum extremal surface associated to the flat cut will be the bifurcation surface of the Killing horizon. Had $e(A)$ for the arbitrary cut left the horizon, then it would have been taken off to infinity by the boost and not ended up on the bifurcation surface.

We can construct an orthogonal null coordinate system around $e(A)$ in the vacuum. We denote the null orthogonal vectors by $k$ and $\ell$ where $k^z = 0 = \ell^z$ and $k^x = k^t = 1$ so that $k \cdot \ell = 1$. Then the statement of $C \subseteq E$ becomes

$$\tag{5.38} k \cdot (\eta - \bar{X}_{SD}) \geq 0$$

Here we use $\eta, \bar{X}_{SD}$ to denote the perturbation of the causal horizon and quantum extremal surface from their vacuum position, respectively. The notation of $\bar{X}_{SD}$ is used to denote the state-dependent piece of the embedding functions for the extremal surface. Over-bars will denote bulk embedding functions of $e(A)$ surface and $X^a$ will denote boundary coordinates. The set up is illustrated in Figure 5.5.

Just as in the previous section, for a locally stationary surface (such as a cut of a null plane), one can write the embedding coordinates of $e(A), \bar{X}$, as an expansion in $z$ [115]:

$$\bar{X}^i(y^a, z) = X^i(y^a) + \frac{1}{2(d-2)} z^2 K^i(y^a) + ... + \frac{z^d}{d} (V^i + W^i(y^a) \log z) + o(z^d) \quad (5.39)$$

where $V^i$ is some local “velocity” function that denotes the rate at which the entangling surface diverges from its boundary position and represents the leading term in the state-dependent part of the embedding functions. The state-independent terms of lower order in $z$ are all proportional to $k^i$. In vacuum, we also have $V^i \propto k^i$, and so for non-vacuum states $k \cdot \bar{X}_{SD} = \frac{1}{d} V \cdot k z^d + o(z^d)$.

---

\[19\] It is also worth noting that EWN together with $C \subseteq E$ can also be used to construct an argument. Suppose we start with a flat cut of a null plane, for which $e(A)$ is also a flat cut of a null plane in the vacuum (the bifurcation surface for the boost Killing horizon). We then deform this cut on the boundary to an arbitrary cut of the null plane in its future. In the bulk, EWN states that $e(A)$ would have to move in a space-like or null fashion, but if it moves in a space-like way, then $C \subseteq E$ is violated.

\[20\] The issue of gauge invariance for this proof should not be overlooked. On their own, each term in (5.38) is not gauge invariant under a general diffeomorphism. The sum of the two, on the other hand, does not transform under coordinate change:

$$g_{\mu \nu} \rightarrow g_{\mu \nu} + \nabla_\mu \xi_\nu$$

Plugging this into the formula for $k \cdot \eta$ shows that $\delta (k \cdot \eta) = -(k \cdot \xi)$, which is precisely the same as the change in position of the extremal surface $\delta (k \cdot \bar{X}_{SD}) = -(k \cdot \xi)$.  

Figure 5.5: This picture shows the various vectors defined in the proof. It depicts a cross-section of the extremal surface at constant \( z \). \( e(A)_{\text{vac}} \) denotes the extremal surface in the vacuum. For flat cuts of a null plane on the boundary, they agree. For wiggly cuts, they will differ by some multiple of \( k^i \).

Equation (5.34) tells us that \( \bar{X}_{SD} \) is proportional to boundary variations of the CFT entropy. Thus, equation (5.39) together with (5.34) tells us the simple result that

\[
k \cdot \bar{X}_{SD} = -\frac{4G_N}{d\sqrt{\lambda}} S_A \tilde{z}^{d-2}.
\]  

(5.40)

Now we explore the \( \eta \) deformation, where \( \eta \) is the vector denoting the shift in the position of the causal horizon. This discussion follows much of the formalism found in [63]. At a specific value of \((z, y)\), the null generator of the causal surface, \( k' \), is related to the vacuum vector \( k \) by

\[
k' = k + \delta k = k + k^a \nabla_a \eta
\]  

(5.41)

In the perturbed metric, \( k' \) must be null to leading order in \( \eta = \mathcal{O}(z^d) \). Imposing this condition we find that

\[
k^b \nabla_b (\eta \cdot k) = -\frac{1}{2} \delta g_{ab} k^a k^b
\]  

(5.42)

Here \( \delta g_{ab} \) is simply the difference between the excited state metric and the vacuum metric, which can be treated as a perturbation since we are in the near-boundary limit. This equation can be integrated back along the original null geodesic, with the boundary condition imposed that \( \eta(\infty) = 0 \). Thus, we find the simple relation

\[
(k \cdot \eta)(\lambda) = \frac{1}{2} \int_\lambda^\infty \delta g_{kk} d\tilde{\lambda}.
\]  

(5.43)

The holographic dictionary tells us how to relate \( \delta g_{kk} \) to boundary quantities. Namely, to leading order in \( z \), the expression above can be recast in terms of the CFT stress tensor

\[
k \cdot \eta = \frac{1}{2} \int_\lambda^\infty \frac{16\pi G_N}{d} z^{d-2} T_{kk} d\tilde{\lambda}.
\]  

(5.44)
Plugging all of this back in to (5.38), we finally arrive at the basic inequality

$$\int_{\lambda}^{\infty} T_{kk} d\tilde{\lambda} + \frac{\hbar}{2\pi\sqrt{h}} S'_A \geq 0.$$  \hspace{1cm} (5.45)$$

Note that all the factors of \( G_N \) have dropped out and we have obtained a purely field-theoretic QHANEC.

**Loop corrections** Here we will briefly comment on how bulk loop corrections affect the argument. Quantum effects do not just require that we add \( S_{\text{out}} \) to \( A \); higher derivative terms suppressed by the Planck-scale will be generated in the gravitational action which will modify the gravitational entropy functional. With Planck-scale suppressed higher derivative corrections, derivatives of the boundary entropy of a region have the form

$$S' = \frac{A'}{4G\hbar} + Q' + S'_{\text{out}}$$  \hspace{1cm} (5.46)$$

where \( Q' \) are the corrections which start at \( O((G\hbar)^0) \). The key point is that \( Q' \) is always one order behind \( A' \) in the \( G\hbar \) perturbation theory. As \( G\hbar \to 0 \), \( Q' \) can only possibly be relevant in situations where \( A' = 0 \) at \( O((G\hbar)^0) \). In this case, \( V' \sim k^4 \), and the bulk quantum extremal surface in the vacuum state is a cut of a bulk Killing horizon. But then \( Q' \) must be at least \( O(G\hbar) \), since \( Q' = 0 \) on a Killing horizon for any higher derivative theory. Thus we find Eq. (5.34) is unchanged at the leading nontrivial order in \( G\hbar \).

Higher derivative terms in the bulk action will also modify the definition of the boundary stress tensor. The appearance of the stress tensor in the QNEC and QHANEC proofs comes from the fact that it appears at \( O(z^2) \) in the near-boundary expansion of the bulk metric [115]. Higher derivative terms will modify the coefficient of \( T_{ij} \) in this expansion, and therefore in the QNEC and QHANEC. (They will not affect the structure of lower-order terms in the asymptotic metric expansion because there aren’t any tensors of appropriate weight besides the flat metric \( \eta_{ij} \) [115]). But the new coefficient will differ from the one in Einstein gravity by the addition of terms containing the higher derivative couplings, which are \( 1/N \)-suppressed relative to the Einstein gravity term, and will thus only contribute to the sub-leading parts of the QNEC and QHANEC. Thus the validity of the inequalities at small \( G\hbar \) is unaffected.

**Boundary Causality Condition implies the AANEC**

The proof of this statement was first described in [112]. We direct interested readers to that paper for more detail. Here we will sketch the proof and note some similarities to the previous two subsections.

As discussed above, the BCC states that no bulk null curve can connect boundary points that are not connected by a boundary causal curve. In the same way that we took a boundary limit of \( \mathcal{C} \subseteq \mathcal{E} \) to prove the QHANEC, the strategy here is to look at nearly null time-like
curves that hug the boundary. These curves will come asymptotically close to beating the boundary null geodesic and so in some sense derive the most stringent condition on the geometry.

Expanding the near boundary metric in powers of $z$, we use holographic renormalization to identify pieces of the metric as the stress tensor

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{dz^2 + \eta_{ij}dx^i dx^j + z^d\gamma_{ij}(z,x^i)dx^i dx^j}{z^2}$$

where $\gamma_{ij}(0,x^i) = \frac{16\pi G_N}{d}\langle T_{ij}\rangle$. Using null coordinates on the boundary, we can parameterize the example bulk curve by $u \mapsto (u,V(u),Z(u),y^i = 0)$. One constructs a nearly null, time-like curve that starts and ends on the boundary and imposes time delay. If $Z(-L) = Z(L) = 0$, then the BCC enforces that $V(L) - V(-L) \geq 0$. For the curve used in [112], the $L \to \infty$ limit turns this inequality directly into the boundary AANEC.

### 5.5 Discussion

We have identified two constraints on the bulk geometry, entanglement wedge nesting (EWN) and the $C \subseteq E$, coming directly from the consistency of subregion duality and entanglement wedge reconstruction. The former implies the latter, and the latter implies the boundary causality condition (BCC). Additionally, EWN can be understood as a consequence of the quantum focussing conjecture, and $C \subseteq E$ follows from the generalized second law. Both statements in turn have implications for the strongly-coupled large-$N$ theory living on the boundary: the QNEC and QHANEC, respectively. In this section, we list possible generalizations and extensions to this work.

**Unsuppressed higher derivative corrections**  There is no guarantee that higher derivative terms with un-suppressed coefficients are consistent with our conclusions. In fact, in [39] it was observed that Gauss-Bonnet gravity in AdS with an intermediate-scale coupling violates the BCC, and this fact was used to place constraints on the theory. We have seen that the geometrical conditions EWN and $C \subseteq E$ are fundamental to the proper functioning of the bulk/boundary duality. If it turns out that a higher derivative theory invalidates some of our conclusions, it seems more likely that this would be point to a particular pathology of that theory rather than an inconsistency of our results. It would be interesting if EWN and $C \subseteq E$ could be used to place constraints on higher derivative couplings, in the spirit of [39]. We leave this interesting possibility to future work.

**A further constraint from subregion duality**  Entanglement wedge reconstruction implies an additional property that we have not mentioned. Given two boundary regions $A$ and $B$ that are spacelike separated, $E(A)$ is spacelike separated from $E(B)$. This property is actually equivalent to EWN for pure states, but is a separate statement for mixed states.
In the latter case, it would be interesting to explore the logical relationships of this property to the constraints in Fig. 5.1.

**Beyond AdS** In this paper we have only discussed holography in asymptotically AdS spacetimes. While the QFC, QNEC, and GSL make no reference to asymptotically AdS spacetimes, EWN and \( C \subseteq \mathcal{E} \) currently only have meaning in this context. One could imagine however that a holographic correspondence with subregion duality makes sense in more general spacetimes — perhaps formulated in terms of a “theory” living on a holographic screen \([25, 28, 29]\). In this case, we expect analogues of EWN and \( C \subseteq \mathcal{E} \). For some initial steps in this direction, see \([155]\).

**Quantum generalizations of other bulk facts from generalized entropy** A key lesson of this paper is that classical results in AdS/CFT relying on the null energy condition (NEC) can often be made semiclassical by appealing to powerful properties of the generalized entropy: the quantum focussing conjecture and the generalized second law. We expect this to be more general than the semiclassical proofs of EWN and \( C \subseteq \mathcal{E} \) presented here. Indeed, Wall has shown that the generalized second law implies semiclassical generalizations of many celebrated results in classical general relativity, including the singularity theorem \([180]\). It would be illuminating to see how general this pattern is, both in and out of AdS/CFT. As an example, it is known that strong subadditivity of holographic entanglement entropy can be violated in spacetimes which don’t obey the NEC \([37]\). It seems likely that the QFC can be used to derive strong subadditivity in cases where the NEC is violated due to quantum effects in the bulk.

**Gravitational inequalities from field theory inequalities** We have seen that the bulk QFC and GSL, which are semi-classical quantum gravity inequalities, imply their non-gravitational limits on the boundary, the QNEC and QHANEC. But we can regard the bulk as an effective field theory of perturbative quantum gravity coupled to matter, and can consider the QNEC and QHANEC for the bulk matter sector. At least when including linearized backreaction of fields quantized on top of a Killing horizon, the QHANEC implies the GSL \([174]\), and the QNEC implies the QFC \([34]\). In some sense, this “completes” the logical relations of Fig. 5.1.

**Support for the quantum extremal surfaces conjecture** The logical structure uncovered in this paper relies heavily on the conjecture that the entanglement wedge should be defined in terms of the surface which extremizes the generalized entropy to one side \([64]\) (as opposed to the area). Perhaps similar arguments could be used to prove this conjecture, or at least find an explicit example where extremizing the area is inconsistent with subregion duality, as in \([83]\).
Connections to Recent Proofs of the AANEC  Recent proofs of the AANEC have illuminated the origin of this statement within field theory [71, 90]. In one proof, the engine of the inequality came from microcausality and reflection positivity. In the other, the proof relied on monotonocity of relative entropy for half spaces. A natural next question would be how these two proofs are related, if at all. Our paper seems to offer at least a partial answer for holographic CFTs. Both the monotonicity of relative entropy and microcausality - in our case the QHANEC and BCC, respectively - are implied by the same thing in the bulk: \( C \subseteq \mathcal{E} \). In 5.2, we gave a motivation for this geometric constraint from subregion duality. It would be interesting to see how the statement of \( C \subseteq \mathcal{E} \) in a purely field theoretic language is connected to both the QHANEC and causality.
Chapter 6

Local Modular Hamiltonians from the Quantum Null Energy Condition

6.1 Introduction and Summary

The reduced density operator \( \rho \) for a region in quantum field theory encodes all of the information about observables localized to that region. Given any \( \rho \), one can define the modular Hamiltonian \( K \) by

\[
\rho = e^{-K}.
\]

(6.1)

Knowledge of this operator is equivalent to knowledge of \( \rho \), but the modular Hamiltonian frequently appears in calculations involving entanglement entropy. In general, i.e. for arbitrary states reduced to arbitrary regions, \( K \) is a complicated non-local operator. However, in certain cases it is known to simplify.

The most basic example where \( K \) simplifies is the vacuum state of a QFT in Rindler space, i.e. the half-space \( t = 0, x \geq 0 \). The Bisognano–Wichmann theorem [20] states that in this case the modular Hamiltonian is

\[
\Delta K = \frac{2\pi}{\hbar} \int d^{d-2} y \int_0^\infty x T_{tt} \, dx
\]

(6.2)

where \( \Delta K \equiv K - \langle K \rangle_{\text{vac}} \) defines the vacuum-subtracted modular Hamiltonian, and \( y \) are \( d - 2 \) coordinates parametrizing the transverse directions. The vacuum subtraction generally removes regulator-dependent UV-divergences in \( K \). Other cases where the modular Hamiltonian is known to simplify to an integral of local operators are obtained via conformal transformation of Eq. (6.2), including spherical regions in CFTs [48], regions in a thermal state of 1+1 CFTs [43], and null slabs [32, 32].

Using conservation of the energy-momentum tensor, one can easily re-express the Rindler modular Hamiltonian in Eq. (6.2) as an integral over the future Rindler horizon \( u \equiv t - x = 0 \).
which bounds the future of the Rindler wedge:

$$\Delta K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_{V(y)}^{\infty} v T_{vv} \, dv,$$  \hspace{1cm} (6.3)

where $v \equiv t + x$. It is important to note that standard derivations of (6.2) or (6.3), e.g. [20, 48], do not apply when the entangling surface is defined by a non-constant cut of the Rindler horizon (see Fig. 6.1). One of the primary goals of this paper is to provide such a derivation.

For a large class of quantum field theories satisfying a precise condition specified momentarily, we will show that the vacuum modular Hamiltonian for the region $\mathcal{R}[V(y)]$ above an arbitrary cut $v = V(y)$ of a null plane is given by

$$\Delta K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_{V(y)}^{\infty} (v - V(y)) T_{vv} \, dv$$ \hspace{1cm} (6.4)

This equation has been previously derived by Wall for free field theories [174] building on [36, 159], and to linear order in the deformation away from $V(y) = \text{const}$ in general QFTs by Faulkner et al. [71]. In CFTs, conformal transformations of Eq. (6.4) yield versions of the modular Hamiltonian for non-constant cuts of the causal diamond of a sphere.

The condition leading to Eq. (6.4) is that the theory should satisfy the quantum null energy condition (QNEC) [34, 33, 115, 4] — an inequality between the stress tensor and the von Neumann entropy of a region — and saturate the QNEC in the vacuum for regions defined by cuts of a null plane. We will review the statement of the QNEC in Sec. 6.2.

The QNEC has been proven for free and superrenormalizable [33], as well as holographic [115, 4] quantum field theories. We take this as reasonable evidence that the QNEC is a true
fact about relativistic quantum field theories in general, and for the purposes of this paper take it as an assumption. In Sec. 6.2 we will show how saturation of the QNEC in a given state leads to an operator equality relating certain derivatives of the modular Hamiltonian of that state to the energy-momentum tensor. Applied to the case outlined above, this operator equality will be integrated to give Eq. (6.4).

Given the argument in Sec. 6.2, the only remaining question is whether the QNEC is in fact saturated in the vacuum state for entangling surfaces which are cuts of a null plane. This has been shown for free theories in [33]. In Sec. 6.3, we prove that this is the case for holographic theories to all orders in $1/N$. We emphasize that Eq. (6.4) holds purely as a consequence of the validity of the QNEC and the saturation in the vacuum for $\mathcal{R}$, two facts which are potentially true in quantum field theories much more generally than free and holographic theories.

Finally, in Sec. 6.4 we will conclude with a discussion of possible extensions to curved backgrounds and more general regions, connections between the relative entropy and the QNEC, and relations to other work.

### 6.2 Main Argument

#### Review of QNEC

The von Neumann entropy of a region in quantum field theory can be regarded as a functional of the entangling surface. We will primarily be interested in regions to one side of a cut of a null plane in flat space, for which the entangling surface can be specified by a function $V(y)$ which indicates the $v$-coordinate of the cut as a function of the transverse coordinates, collectively denoted $y$. See Fig. 6.1 for the basic setup. Each cut $V(y)$ defines a half-space, namely the region to one side of the cut. We will pick the side towards the future of the null plane. For the purposes of this section we are free to consider the more general situation where the entangling surface is only locally given by a cut of a null plane. Thus the von Neumann entropy can be considered as a functional of a profile $V(y)$ which defines the shape of the entangling surface, at least locally.

Suppose we define a one-parameter family of cuts $V(y; \lambda) \equiv V(y; 0) + \lambda \dot{V}(y)$, with $\dot{V}(y) > 0$ to ensure that $\mathcal{R}(\lambda_1) \subset \mathcal{R}(\lambda_2)$ if $\lambda_1 > \lambda_2$. If $S(\lambda)$ is the entropy of region $\mathcal{R}(\lambda)$, then the QNEC in integrated form states that

$$\int d^{d-2}y \langle T_{vv}(y) \rangle \dot{V}(y)^2 \geq \frac{\hbar}{2\pi} d^2S \frac{d\lambda}{d\lambda^2}. \quad (6.5)$$

In general there would be a $\sqrt{\hbar}$ induced metric factor weighting the integral, but here and in the rest of the paper we will assume that the $y$ coordinates have been chosen such that $\sqrt{\hbar} = 1$.

By taking advantage of the arbitrariness of $\dot{V}(y)$ we can derive from this the local form of the QNEC. If we take a limit where $\dot{V}(y')^2 \to \delta(y - y')$, then the l.h.s. reduces to $\langle T_{vv} \rangle$. 
We define \( S''(y) \) as the limit of \( \frac{d^2 S}{d\lambda^2} \) in the same situation:

\[
\frac{d^2 S}{d\lambda^2} \rightarrow S''(y) \quad \text{when} \quad \dot{V}(y')^2 \rightarrow \delta(y - y').
\] (6.6)

Taking the limit of the nonlocal QNEC then gives the local one:

\[
\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} S''.
\] (6.7)

The local QNEC together with strong subadditivity can likewise be used to go backward and derive the nonlocal QNEC [34, 33, 115]. The details of that argument are not important here. In the next section we will discuss the consequences of the saturation of the QNEC, and will have to distinguish whether we mean saturation of the nonlocal inequality Eq. (6.5) or the local inequality Eq. (6.7), the latter condition being weaker.

### The QNEC under state perturbations

In this section we consider how the QNEC behaves under small deformations of the state. We begin with a reference state \( \sigma \) and consider the deformed state \( \rho = \sigma + \delta \rho \), with \( \delta \rho \) traceless but otherwise arbitrary.

Consider a one-parameter family of regions \( \mathcal{R}(\lambda) \) as in the previous section. Define \( \overline{\mathcal{R}}(\lambda) \) to be the complement of \( \mathcal{R}(\lambda) \) within a Cauchy surface. The reduced density operator for any given region \( \mathcal{R}(\lambda) \) given by

\[
\rho(\lambda) = \sigma(\lambda) + \delta \rho(\lambda) = \text{Tr}_{\overline{\mathcal{R}}(\lambda)} \sigma + \text{Tr}_{\mathcal{R}(\lambda)} \delta \rho.
\] (6.8)

By the First Law of entanglement entropy, the entropy of \( \rho(\lambda) \) is given by

\[
S(\rho(\lambda)) = S(\sigma(\lambda)) - \text{Tr}_{\overline{\mathcal{R}}(\lambda)} \delta \rho(\lambda) \log \sigma(\lambda) + o(\delta \rho^2).
\] (6.9)

The second term can be written in a more useful way by defining the modular Hamiltonian \( K_\sigma(\lambda) \) as

\[
K_\sigma(\lambda) \equiv -\text{Tr}_{\overline{\mathcal{R}}(\lambda)} \log \sigma(\lambda).
\] (6.10)

Defining \( K_\sigma(\lambda) \) this way makes it a global operator, which makes taking derivatives with respect to \( \lambda \) formally simpler. Using this definition, we can write Eq. (6.9) as

\[
S(\rho(\lambda)) = S(\sigma(\lambda)) + \text{Tr} \delta \rho K_\sigma(\lambda) + o(\delta \rho^2).
\] (6.11)

Now in the second term the trace is over the global Hilbert space, and the \( \lambda \)-dependence has been isolated to the operator \( K_\sigma(\lambda) \). Taking two derivatives, and simplifying the notation slightly, we find

\[
\frac{d^2 S}{d\lambda^2}(\rho) = \frac{d^2 S}{d\lambda^2}(\sigma) + \text{Tr} \delta \rho \frac{d^2 K_\sigma}{d\lambda^2} + o(\delta \rho^2).
\] (6.12)
Suppose that the nonlocal QNEC, Eq. (6.5), is saturated in the state $\sigma$ for all profiles $\dot{V}(y)$. Then, using Eq. (6.12), the nonlocal QNEC for the state $\rho$ can be written as

$$\int d^{d-2}y \left( \text{Tr} \delta \rho T_{vv} \right) \dot{V}^2 \geq \frac{\hbar}{2\pi} \text{Tr} \delta \rho \frac{d^2 K_\sigma}{d\lambda^2} + o(\delta \rho^2). \tag{6.13}$$

The operator $\delta \rho$ was arbitrary, and in particular could be replaced by $-\delta \rho$. Then the only way that Eq. 6.13 can hold is if we have the operator equality

$$\frac{d^2 K_\sigma}{d\lambda^2} = C + \frac{2\pi \hbar}{\pi} \int d^{d-2}y T_{vv} \dot{V}^2. \tag{6.14}$$

Here $C$ is a number that we cannot fix using this method that is present because of the tracelessness of $\delta \rho$.

Eq. (6.14) can be integrated to derive the full modular Hamiltonian $K_\sigma$ if we have appropriate boundary conditions. Up until now we have only made use of local properties of the entangling surface, but in order to provide boundary conditions for the integration of Eq. (6.14) we will assume that the entangling surface is globally given by a cut of a null plane, and that $V(y;\lambda = 0) = 0$. We will also make $\sigma$ the vacuum state. In that situation it is known that the QNEC is saturated for free theories, and in the next section we will show that this is also true for holographic theories at all orders in the large-$N$ expansion.

Our first boundary condition is at $\lambda = \infty$.\footnote{It is not always possible to consider the $\lambda \to \infty$ limit of a null perturbation to an entangling surface because parts of the entangling surface may become timelike related to each other at some finite value of $\lambda$, at which point the surface is no longer the boundary of a region on a Cauchy surface. However, when the entangling surface is globally equal to a cut of a null plane this is not an issue.} Since we expect that $K_\sigma(\lambda)$ should have a finite expectation value in any state as $\lambda \to \infty$, it must be that $dK_\sigma/d\lambda \to 0$ as $\lambda \to \infty$.

Then integrating Eq. (6.14) gives

$$\frac{dK_\sigma}{d\lambda} = -\frac{2\pi \hbar}{\pi} \int d^{d-2}y \int_0^{\infty} dv T_{vv} \dot{V}. \tag{6.15}$$

Note that this equation implies that the vacuum expectation value $\langle K_\sigma(\lambda) \rangle_{\text{vac}}$ is actually $\lambda$-independent, which makes vacuum subtraction easy.

Our second boundary condition is Eq. (6.3), valid at $\lambda = 0$ when $V(y;\lambda) = 0$. Integrating once more and making use of this boundary condition, we find

$$\Delta K_\sigma(\lambda) = \frac{2\pi \hbar}{\pi} \int d^{d-2}y \int_0^{\infty} \left( v - V(y;\lambda) \right) T_{vv} dv \tag{6.16}$$

which is Eq. (6.4). Note that the l.h.s. of this equation is now the vacuum-subtracted modular Hamiltonian.
Before moving on, we will briefly comment on the situation where the local QNEC, Eq. (6.7), is saturated but the nonlocal QNEC, Eq. (6.5), is not. Then, analogously to $S''$ in Eq. (6.6), one may define a local second derivative of $K_\sigma$:

$$\frac{d^2 K_\sigma}{d\lambda^2} \rightarrow K''_\sigma(y) \quad \text{when} \quad \dot{V}(y')^2 \rightarrow \delta(y - y').$$

(6.17)

Very similar manipulations then show that saturation of the local QNEC implies the equality

$$K''_\sigma = \frac{2\pi}{\hbar} T_{vv}.$$  

(6.18)

This equation is weaker than Eq. (6.14), which is meant to be true for arbitrary profiles of $\dot{V}(y)$, but it may have a greater regime of validity. We will comment on this further in Sec. 6.4.

### 6.3 Holographic Calculation

In the previous section we argued that the form of the modular Hamiltonian could be deduced from saturation of the QNEC. In this section we will use the holographic entanglement entropy formula [153, 152, 101, 69] to show that the QNEC is saturated in vacuum for entangling surfaces defined by arbitrary cuts $v = V(y)$ of the null plane $u = 0$ in holographic theories. Our argument applies to any holographic theory defined by a relevant deformation to a holographic CFT, and will be at all orders in the large-$N$ expansion. To reach arbitrary order in $1/N$ we will assume that the all-orders prescription for von Neumann entropy is given by the quantum extremal surface proposal of Engelhardt and Wall [64]. This is the same context in which the holographic proof of the QNEC was extended to all orders in $1/N$ [4].

As before, the entangling surface in the field theory is given by the set of points $\partial \mathcal{R} = \{(u, v, y) : v = V(y), u = 0\}$ with null coordinates $u = t - x$ and $v = t + x$, and the region $\mathcal{R}$ is chosen to lie in the $u < 0$ portion of spacetime. Here $y$ represents $d-2$ transverse coordinates. The bulk quantum extremal surface anchored to this entangling surface is parameterized by the functions $\bar{V}(y, z)$ and $\bar{U}(y, z)$. It was shown in [115, 4] that if we let the profile $V(y)$ depend on a deformation parameter $\lambda$, then the second derivative of the entropy is given by

$$\frac{d^2 S}{d\lambda^2} = -\frac{d}{4G\hbar} \int d^{d-2}y \frac{d\bar{U}(d)}{d\lambda},$$

(6.19)

to all orders in $1/N$, where $\bar{U}(d)(y)$ is the coefficient of $z^d$ in the small-$z$ expansion of $\bar{U}(z, y)$. We will show that $\bar{U} = 0$ identically for any profile $V(y)$, which then implies that $d^2S/d\lambda^2 = 0$, which is the statement of QNEC saturation in the vacuum.

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2It is crucial that we demonstrate saturation beyond leading order in large-$N$. The argument in the previous section used exact saturation, and an error that is na"ively subleading when evaluated in certain states may become very large in others.
CHAPTER 6. LOCAL MODULAR HAMILTONIANS FROM THE QUANTUM NULL ENERGY CONDITION

One way to show that \( \bar{U} \) vanishes is to demonstrate that \( \bar{U} = 0 \) solves the quantum extremal surface equations of motion in the bulk geometry dual to the vacuum state of the boundary theory. The quantum extremal surface is defined by having the sum of the area plus the bulk entropy on one side be stationary with respect to first-order variations of its position. One can show that \( \bar{U} = 0 \) is a solution to the equations of motion if any only if

\[
\frac{\delta S_{\text{bulk}}}{\delta \bar{V}(y,z)} = 0
\]

in the vacuum everywhere along the extremal surface. This would follow from null quantization if the bulk fields were free [33], but that would only allow us to prove the result at order-one in the \( 1/N \) expansion.

For an all-orders argument, we opt for a more indirect approach using subregion duality, or entanglement wedge reconstruction [53, 96, 55, 88]. A version of this argument first appeared in [4], and we elaborate on it here.

Entanglement wedge reconstruction requires two important consistency conditions in the form of constraints on the bulk geometry which must hold at all orders in \( 1/N \): The first constraint, *entanglement wedge nesting* (EWN), states that if one boundary region is contained inside the domain of dependence of another, then the quantum extremal surface associated to the first boundary region must be contained within the entanglement wedge of the second boundary region [53, 177]. The second constraint, \( \mathcal{C} \subseteq \mathcal{E} \), demands that the causal wedge of a boundary region be contained inside the entanglement wedge of that region [53, 96, 177, 64, 100]. Equivalently, it says that no part of the quantum extremal surface of a given boundary region can be timelike-related to the (boundary) domain of dependence of that boundary region. It was shown in [4] that \( \mathcal{C} \subseteq \mathcal{E} \) follows from EWN, and EWN itself is simply the statement that a boundary region should contain all of the information about any of its subregions. We will now explain the consequences of these two constraints for \( \bar{U}(y,z) \).

Without loss of generality, suppose the region \( \mathcal{R} \) is defined by a coordinate profile which is positive, \( V(y) > 0 \). Consider a second region \( \mathcal{R}_0 \) which has an entangling surface at \( v = u = 0 \) and whose domain of dependence (i.e., Rindler space) contains \( \mathcal{R} \). The quantum extremal surface associated to \( \mathcal{R}_0 \) is given by \( \bar{U}_0 = \bar{V}_0 = 0 \). This essentially follows from symmetry. The entanglement wedge of \( \mathcal{R}_0 \) is then a bulk extension of the boundary Rindler space, namely the set of bulk points satisfying \( u \leq 0 \) and \( v \geq 0 \). Then EWN implies that \( \bar{U} \leq 0 \) and \( \bar{V} \geq 0 \).

The only additional constraint we need from \( \mathcal{C} \subseteq \mathcal{E} \) is the requirement that the quantum extremal surface for \( \mathcal{R} \) not be in the past of the domain of dependence of \( \mathcal{R} \). From the definition of \( \mathcal{R} \), it is clear that a bulk point is in the past of the domain of dependence of \( \mathcal{R} \) if and only if it is in the past of the region \( u < 0 \) on the boundary, which is the same as the

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3The entanglement wedge of a boundary region is the set of bulk points which are spacelike- or null-related to that region’s quantum extremal surface on the same side of the quantum extremal surface as the boundary region itself.

4One might worry that the quantum extremal surface equations display spontaneous symmetry breaking in the vacuum, but this can be ruled out using \( \mathcal{C} \subseteq \mathcal{E} \) with an argument similar to the one we present here.
region \( u < 0 \) in the bulk. Therefore it must be that \( \hat{U} \geq 0 \). Combined with the constraint from EWN above, we then conclude that the only possibility is \( \hat{U} = 0 \). This completes the proof that the QNEC is saturated to all orders in \( 1/N \).

### 6.4 Discussion

We conclude by discussing the generality of our analysis, some implications and future directions, and connections with previous work.

#### Generalizations and Future Directions

**General Killing horizons** Though we restricted to cuts of Rindler horizons in flat space for simplicity, all of our results continue to hold for cuts of bifurcate Killing horizons for QFTs defined in arbitrary spacetimes, assuming the QNEC is true and saturated in the vacuum in this context. In particular, Eq. (6.4) holds with \( v \) a coordinate along the horizon. For holographic theories, entanglement wedge nesting (EWN) and the entanglement wedge being outside of the causal wedge (\( \mathcal{C} \subseteq \mathcal{E} \)) continue to prove saturation of the QNEC. To see this, note that a Killing horizon on the boundary implies a corresponding Killing horizon in the bulk. Now take the reference region \( \mathcal{R}_0 \) satisfying \( V(y) = U(y) = 0 \) to be the boundary bifurcation surface. By symmetry, the associated quantum extremal surface lies on the bifurcation surface of the bulk Killing horizon. Then the quantum extremal surface of the region \( \mathcal{R} \) defined by \( V(y) \geq 0 \) must lie in the entanglement wedge of \( \mathcal{R}_0 \) — inside the bulk horizon — by entanglement wedge nesting, but must also lie on or outside of the bulk horizon by \( \mathcal{C} \subseteq \mathcal{E} \). Thus it lies on the bulk horizon, \( \hat{U} = 0 \), and the QNEC remains saturated by Eq. (6.19).

**Future work** In this work, we have only established the form of \( K_{\mathcal{R}} \) for regions \( \mathcal{R} \) bounded by arbitrary cuts of a null plane. A natural next direction would be to understand if and how we can extend Eq. (6.18) to more general entangling surfaces. As discussed above, the QNEC was shown to hold for locally flat entangling surfaces in holographic, free and super-renormalizable field theories [33, 115, 4]. Thus, if we could prove saturation, i.e. that \( S_{vac}'' = 0 \) at all orders in \( 1/N \), then we would establish (6.18) for all regions with a locally flat boundary.

One technique to probe this question is to perturb the entangling surface away from a flat cut and compute the contributions to the QNEC order-by-order in a perturbation parameter \( \epsilon \). Preliminary calculations [124] have revealed that for holographic theories at leading order in large \( N \), \( S_{vac}'' = 0 \) at all orders in \( \epsilon \).

Another interesting problem is to show that in a general QFT vacuum, null derivatives of entanglement entropy across arbitrary cuts of null planes vanish. That, along with a general proof of QNEC will establish (18) as a consequence. We will leave this to future work.
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The QNEC as $S(\rho\|\sigma)^{\prime\prime} \geq 0$

There is a connection between the QNEC and relative entropy, first pointed out in [4], that we elaborate on here. The relative entropy $S(\rho\|\sigma)$ between two states $\rho$ and $\sigma$ is defined as

$$S(\rho\|\sigma) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \sigma$$

(6.21)

and provides a measure of distinguishability between the two states [140]. Substituting the definition of $K$, Eq. (6.1), into Eq. (6.21) provides a useful alternate presentation:

$$S(\rho\|\sigma) = \langle K_\sigma \rangle_\rho - S(\rho).$$

(6.22)

If Eq. (6.4) is valid, then taking two derivatives with respect to a deformation parameter, as in the main text, shows that the nonlocal QNEC, Eq. (6.5), is equivalent to

$$\partial_\lambda^2 S(\rho(\lambda)\|\sigma(\lambda)) \geq 0.$$

(6.23)

For comparison, monotonicity of relative entropy for the types of regions and deformations we have been discussing can be written as

$$\partial_\lambda S(\rho(\lambda)\|\sigma(\lambda)) \leq 0.$$  

(6.24)

Eq. (6.23) is a sort of “convexity” of relative entropy.\(^5\) Unlike monotonicity of relative entropy, which says that the first derivative is non-positive, there is no general information-theoretic reason for the second derivative to be non-negative. In the event that Eq. (6.18) holds but not Eq. (6.4), we would still have

$$S(\rho\|\sigma)^{\prime\prime} \geq 0.$$  

(6.25)

where the $^{\prime\prime}$ notation denotes a local deformation as in Sec. 6.2.

It would be extremely interesting to characterize what about quantum field theory and null planes makes (6.23) true. We can model the null deformation as a non-unitary time evolution in the space of states, with the vacuum state serving as an equilibrium state for this evolution. Then an arbitrary finite-energy state will relax toward the equilibrium state, with the relative entropy $S(\rho\|\sigma)$ characterizing the free energy as a function of time. Monotonicity of relative entropy is then nothing more than the statement that free energy decreases, i.e. the second law of thermodynamics. The second derivative statement gives more information about the approach to equilibrium. If that approach is of the form of exponential decay, then all successive derivatives would alternate in sign. However, for null deformations in quantum field theory we do not expect to have a general bound on the behavior of derivatives of the energy-momentum tensor, meaning that the third derivative of the free energy should not have a definite sign.\(^6\) Perhaps there is some way of characterizing the approach to equilibrium we have here, which is in some sense smoother than the most general possibility but not so constrained as to force exponential behavior.

\(^{5}\)This is distinct from the well-known convexity of relative entropy, which says that $S(t\rho_1 + (1-t)\rho_2\|\sigma) \leq tS(\rho_1\|\sigma) + (1-t)S(\rho_2\|\sigma)$.

\(^{6}\)We thank Aron Wall for a discussion of this point.
Relation to previous work

Faulkner, Leigh, Parrikar and Wang [71] have discussed results very similar to the ones presented here. They demonstrated that for first-order null deformations $\delta V(y)$ to a flat cut of a null plane, the perturbation to the modular Hamiltonian takes the form

$$
\langle K_R \rangle_\psi - \langle K_{R_0} \rangle_\psi = -\frac{2\pi}{\hbar} \int d^{d-2}y \int_{\mathcal{V}(y)} dvT_{vv}(y) \delta V(y)
$$

This is precisely the form expected from our equation (6.4). Faulkner et al. went on to suggest that the natural generalization of the modular Hamiltonian to finite deformations away from a flat cut takes the form of Eq. (6.4). In the context of holography they showed that this conclusion applied both on the boundary and in the bulk is consistent with JLMS [109]. In the present paper, we have shown that Eq. (6.4) holds for theories which obey the QNEC, and for which the QNEC is saturated in the vacuum. A non-perturbative, field theoretic proof of these assumptions remains a primary goal of future work.
Chapter 7

Violating the Quantum Focusing Conjecture and Quantum Covariant Entropy Bound in \( d \geq 5 \) dimensions

7.1 Introduction

The gravitational focusing theorem plays a key role in the modern understanding of General Relativity. This key result states (see e.g. [172]) that the expansion of null congruences cannot increase toward the future in any solution to Einstein-Hilbert gravity sourced by matter satisfying the Null Energy Condition (NEC). It leads to the second law of black hole thermodynamics [91], singularity theorems [146, 93], the chronology protection theorem [92], topological censorship [79], and other fundamental results. It also guarantees essential properties of holographic entanglement entropy [177, 96] in the context of gauge/gravity duality.

However, the null energy condition is known to be violated by quantum effects [65]. This then raises the question of whether quantum corrections might enable fundamentally new and perhaps pathological gravitational phenomena. Indeed, it was recently established that traversable wormholes can be constructed in this way [83]. On the other hand, the conjectured Generalized Second Law of thermodynamics (GSL) would both limit the utility of traversable wormholes and prohibit even more troubling exotic physics [180].

Motivated in part by the GSL, and also in part by the covariant entropy conjecture [23], it was suggested in [34] that a generalization of the focusing theorem might continue to hold at the quantum level. Known as the Quantum Focussing Conjecture (QFC), it would imply both the GSL (for any causal horizon) and a form [34] of the covariant entropy bound of [23] related to the version discussed by Strominger and Thompson [163].

The QFC is formulated by noting that the expansion \( \theta \) of any null congruence can be expressed as a first functional derivative of the area of cuts of the congruence, and that Einstein-Hilbert gravity associates a Bekenstein-Hawking entropy \( S_{BH} = A/4G \) with many
surfaces of area $A$. In particular, given a region $R$ with boundary $\Sigma = \partial R$ in some Cauchy surface, and also given a null congruence $N$ orthogonal to $\Sigma$, we have

$$\theta[\Sigma, y] = \frac{4G}{\sqrt{\tilde{h}}} \frac{\delta S_{BH}}{\delta \Sigma(y)},$$

(7.1)

where $y$ labels the space of null generators, $\delta \Sigma(y)$ is an infinitesimal displacement of the surface along the null generator $y$, and $\tilde{h}$ denotes the determinant of the transverse metric in the $y$-coordinate system on the null congruence $N$. For semi-classical gravity (and in particular where the metric itself may be treated classically), ref. [34] then defines the generalized expansion $\Theta[\sigma, y]$ by replacing $S_{BH} = A/4G$ in (7.1) with the generalized entropy functional

$$S_{gen} = S_{grav} + S_{out}.$$  

(7.2)

Here $S_{grav}$ is an appropriate gravitational entropy functional (say, from [54, 134, 57, 175], which coincides with that of [107] for the case studied here) and $S_{out}$ is a von Neumann entropy for quantum fields outside the null congruence. Finally, the statement of the QFC is simply that $\Theta$ is semi-classically non-decreasing as we push the surface $\Sigma$ toward the future or, in other words, that a corresponding second derivative of $S_{gen}$ is negative or zero:

$$\frac{1}{\sqrt{\tilde{h}(y)}} \frac{\delta}{\delta \Sigma(y_2)} \delta \Sigma(y_1) \Theta[\Sigma; y_1] \leq 0.$$  

(7.3)

While (7.3) is divergent for $y_1 = y_2$, and in particular the contribution of the Einstein-Hilbert term to (7.3) is $\dot{\Theta} \delta(y_1 - y_2)$ where $\dot{\Theta} = k^a \nabla_a \Theta$, the quantity (7.3) remains meaningful when treated as a distribution.

As evidence for the QFC, one may recall [34] that in Einstein-Hilbert gravity, taking a weakly-gravitating ($G \to 0$) limit implies quantum fields satisfy a so-called Quantum Null Energy Condition (QNEC) generalizing the classical NEC, and that this QNEC has now been established in a variety of contexts [33, 115]. In such cases, an associated QFC follows immediately at first order in the coupling $G$ of such theories to Einstein-Hilbert gravity.

However, we argue here that for $d \geq 5$ spacetime dimensions the QFC generally fails. To do so, we recall that integrating out massive fields typically induces a Gauss-Bonnet term in the gravitational effective action; see e.g. [160]. Classical Einstein-Hilbert-Gauss-Bonnet gravity is analyzed in section 7.2, and is shown to violate the QFC at weak curvature for $d \geq 5$. The form of this violation shows that similar issues arise at the quantum level, and also in the presence of arbitrary higher derivative terms controlled by a single length scale so long as the coefficient of the Gauss-Bonnet term is non-zero. The QFC is thus

---

1 $S_{out}$ presumably includes an appropriate set of boundary terms for gauge fields as in e.g. [58, 59, 85, 6, 60, 61].

2 Causality violations implying pathologies for non-stringy theories with large Gauss-Bonnet couplings were found in [39]. By contrast, we emphasize that the QFC violation found in this paper is present for the less restrictive class of theories containing even a small effective field theory Gauss-Bonnet term.
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violated in generic $d \geq 5$ theories of semi-classical gravity coupled to massive quantum fields, and presumably in the presence of massless quantum fields as well. Our example also leads in section 7.3 to violations of the generalized covariant entropy bound (also called the quantum Bousso bound) conjectured in [34].

We close in section 7.4 with further discussion emphasizing future directions and the possibility that a reformulated QFC and quantum Bousso bound may nevertheless hold.

7.2 Violating the QFC in Gauss-Bonnet Gravity

Consider the the Einstein-Hilbert-Gauss-Bonnet action

$$I = \frac{1}{16\pi G} \int d^d x \sqrt{-g} R + \gamma \int d^d x \sqrt{-g} \left( R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \right).$$

(7.4)

As noted above, we will first treat this theory classically and identify violations of the associated QFC (7.3). We will then note that explicit quantum corrections are sub-leading in a long-wavelength expansion so our classical violation extends directly to the quantum level.

We work in the weak curvature limit, taking the Weyl tensor to be first order in some small quantity $\epsilon$:

$$C_{abcd} = O(\epsilon).$$

(7.5)

In this limit, iteratively solving the equation of motion yields

$$R_{ab} = \frac{16\pi G \gamma}{d-2} C_{cdef} C^{cdef} g_{ab} - 32\pi G \gamma C_{acde} C_b^{cde} + O(\epsilon^3).$$

(7.6)

Note that since the right-hand side is non-zero only due to contributions to the equations of motion from the variation of the Gauss-Bonnet term, the Gauss-Bonnet theorem requires it to vanish for $d = 4$. It also vanishes for $d < 4$ where the Weyl tensor is identically zero.

Now consider a null hypersurface $N$ generated by a hypersurface-orthogonal null normal vector field $k^a$. For simplicity, we choose both the expansion $\theta$ and the shear $\sigma_{ab}$ of $N$ to vanish at some point $p$, or equivalently that the extrinsic curvature along $k$ vanishes there for any cut $\Sigma$ of $N$ through $p$; i.e.,

$$K_{ab}^{(k)}|_p := (\tilde{h}_a^c \tilde{h}_b^d \nabla_c k_d)|_p = 0,$$

(7.7)

where $\tilde{h}_a^c$ is the projector onto $\Sigma$. As in e.g. [41], we will use the notation $K_{ab}^{(X)} := \tilde{h}_a^c \tilde{h}_b^d \nabla_c X_d$ below for any vector field $X_d$ orthogonal to $\Sigma$. Note that (7.7) does not restrict the spacetime at $p$ in any way; given any $p$ in any spacetime, we may choose $\Sigma$ and then define the orthogonal null congruence $N$ so that the above conditions are satisfied. We use indices $a$, $b$, $c$, $d$, . . . to denote coordinates in spacetime and indices $\alpha$, $\beta$, $\gamma$, $\delta$, . . . to denote coordinates on $\Sigma$.

---

3This conjecture is closely related to the Strominger-Thompson proposal [163].
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It is convenient to also introduce an auxiliary null vector field $l^a$ orthogonal to $\Sigma$ and satisfying $g_{ab}k^a l^b = -1$. The spacetime metric can then be written

$$g_{ab} = \tilde{h}_{ab} - k_a l_b - l_a k_b,$$  \hspace{1cm} (7.8)

where the transverse part $\tilde{h}_{ab} = \tilde{h}_{a}^{\ c} g_{cb}$ is the induced metric on $\Sigma$. We will reserve $k$ and $l$ “indices” to denote contractions with $k^a$ and $l^a$, as in e.g. $A_{kl} := A_{ab} k^a l^b$. Substituting (7.8) into equation (7.6) and noticing that $C_{klk\alpha} = -C_{k\beta\alpha}$ for all $d$, the Raychaudhuri equation

$$\dot{\theta} = -\frac{\theta^2}{d-2} - \sigma^{ab} \sigma_{ab} - R_{ab} k^a k^b$$

for hypersurface-orthogonal null congruences satisfying equation (7.7) yields

$$\dot{\theta} |_p = -R_{ab} k^a l^b = 32\pi G_{\gamma} C_{a cde} C_{b}^{cde} k^a k^b + O(\epsilon^3)$$

$$= 32\pi G_{\gamma} \left( C_{k\alpha\beta\gamma} C_{k}^{\alpha\beta\gamma} - 2C_{k\beta\alpha} C_{k}^{\alpha\beta} - 4C_{k\kappa\beta} C_{k}^{\alpha\beta} + O(\epsilon^3) \right).$$  \hspace{1cm} (7.9)

As noted above, (7.9) vanishes for $d = 4$. One may see this explicitly by using the $d = 4$ identity $C_{k\alpha\beta\gamma} C_{k}^{\alpha\beta\gamma} = 2C_{k\beta\alpha} C_{k}^{\alpha\beta}$ from [52] so that the first two terms cancel in (7.9). To deal with the final term we again use the $d = 4$ results from [52] to write $C_{k\alpha\beta\gamma} = -\frac{1}{4} A h_{\alpha\beta} + \frac{1}{2} B \epsilon_{\alpha\beta}$ where $\epsilon_{\alpha\beta}$ is the area element of $\Sigma$ and $A$ and $B$ are independent scalars; in particular, there is no traceless symmetric term. The final term in (7.9) then vanishes since $C_{k\alpha\kappa} = 0 = C_{k\alpha\kappa} \epsilon^{\alpha\beta}$ identically for all $d$.

To study the QFC, recall [54, 40] that the entropy functional associated with the Gauss-Bonnet term is

$$S_{\text{GB}} = -8\pi \gamma \int_{\Sigma} d^{d-2} y \sqrt{\tilde{h}} R,$$  \hspace{1cm} (7.10)

where $R$ is the scalar curvature of the induced metric $\tilde{h}_{a\beta}$. Let us introduce a deformation vector field $X^a = f h^a$ on $N$, where $f$ is a scalar function of the null generators $y$. Taking $f = \delta(y - y_p)$, when $\Sigma$ is is deformed along $X^a$ the first derivative of entropy (7.10) is

$$\delta_X S_{\text{GB}} = -8\pi \gamma \int_{\Sigma} d^{d-2} y \sqrt{\tilde{h}} \left( \tilde{R}^{ab} - \frac{1}{2} \tilde{h}^{ab} \right) \delta_X \tilde{h}_{ab}$$

$$= -16\pi \gamma \int_{\Sigma} d^{d-2} y \sqrt{\tilde{h}} \left( \tilde{R}^{ab} - \frac{1}{2} \tilde{h}^{ab} \right) K^{(X)}_{ab}$$

$$= -16\pi \gamma \sqrt{\tilde{h}} \left( \tilde{R}^{ab} - \frac{1}{2} \tilde{h}^{ab} \right) K^{(k)}_{ab}. $$  \hspace{1cm} (7.11)

Here, to obtain the second line, we used $\delta_X \tilde{h}_{ab} = 2 K^{(X)}_{ab}$ (i.e. equation (3.10) of [41]).

We now introduce another vector field $Z = \delta(y - y_Z) k^a$. Recalling that $K^{(k)}_{ab}$ vanishes at $p$, we find the second derivative

$$\delta_Z \left( \frac{1}{\sqrt{\tilde{h}}} \delta_X S_{\text{GB}} \right) = -16\pi \gamma \left( \tilde{R}^{ab} - \frac{1}{2} \tilde{h}^{ab} \right) \left( \delta_Z K^{(k)}_{ab} \right) |_p.$$  \hspace{1cm} (7.12)
Since $K_{ab}^{(k)}|_p = 0$ and $Z^a = \delta(y - y_Z)k^a$, the derivative of $K_{ab}^{(k)}$ at $p$ takes the simple form [41]

$$
(\delta_Z K_{ab}^{(k)})|_p = (-\tilde{h}_a^c \tilde{h}_b^d Z^e k^f R_{ecfd})|_p \tag{7.13}
$$

and (7.12) becomes $\delta_Z \left( \frac{1}{\sqrt{h}} \delta X S_{GB} \right) = \delta(y_p - y_Z)S''_{GB}$ for

$$
S''_{GB} = 16\pi\gamma \left( \tilde{R}^{ab} - \frac{1}{2} \tilde{R}\tilde{h}^{ab} \right) R_{kakb}. \tag{7.14}
$$

Since we treat the theory classically, we save for the end of this section consideration of any explicit $S_{out}$ term in equation (7.2) associated with the entropy of gravitons and thus find

$$
\frac{\delta}{\delta \Sigma(y_Z)} \Theta [\Sigma; y_p] = \sqrt{\tilde{h}}Q\delta(y_p - y_Z) \tag{7.15}
$$

for

$$
Q = \dot{\theta} + 4G S''_{GB}. \tag{7.16}
$$

Since $K_{ab}^{(k)}|_p = 0$, the Gauss equation (i.e. equation (2.14) of [41]) at $p$ is simply

$$
(\tilde{R}_{abcd})|_p = (h_a^e h_b^f h_c^g h_d^h R_{efgh})|_p, \tag{7.17}
$$

and expression (7.14) becomes

$$
S''_{GB} = 16\pi\gamma \left( C_{cedf} \tilde{h}^{cd} \tilde{h}^{ae} \tilde{h}^{bf} - \frac{1}{2} C_{cedf} \tilde{h}^{cd} \tilde{h}^{ef} \tilde{h}^{ab} \right) R_{kakb}. \tag{7.18}
$$

In the weak curvature limit, we may use (7.5) and (7.6) to further write

$$
S''_{GB} = 16\pi\gamma \left( C_{cedf} \tilde{h}^{cd} \tilde{h}^{ae} \tilde{h}^{bf} - \frac{1}{2} C_{cedf} \tilde{h}^{cd} \tilde{h}^{ef} \tilde{h}^{ab} \right) C_{kakb} + O(\epsilon^3) \tag{7.19}
$$

where in the last step we have used $\tilde{h}^{ab} C_{kakb} = C_{kkkl} + C_{klkk}$ which vanishes since the Weyl tensor is anti-symmetric in pairs of indices ($C_{abcd} = -C_{bacd} = -C_{abdc}$). Combining (7.9) and (7.19) with the definition (7.16) yields

$$
Q = 32\pi G\gamma \left( C_{k\alpha\beta\gamma} C_{k}^{\alpha\beta\gamma} - 2 C_{k\alpha\beta\gamma} C_{k}^{\alpha\beta\gamma} \right) + O(\epsilon^3). \tag{7.20}
$$

As with (7.9), expression (7.20) vanishes for $d = 4$. To show that it generally does not vanish for $d = 5$, we use further results from [52] to write it in terms of independent components of the Weyl tensor; the Weyl tensor at a point is constrained by its symmetries,
tracelessness, and the algebraic Bianchi identity. The block $C_{\kappa\alpha\beta\gamma}$, which has boost weight $-1$, can be written in terms of 8 independent components as

$$C_{\kappa\alpha\beta\gamma} = \tilde{h}_{\alpha\beta}v_{\gamma} - \tilde{h}_{\alpha\gamma}v_{\beta} + \epsilon_{\beta\gamma}^{\delta}n_{\delta\alpha}, \text{ for } d = 5,$$

(7.21)

where $\epsilon_{\alpha\beta\gamma}$ is the area element of $\Sigma$, $v_{\gamma}$ is a vector containing 3 independent components and $n_{\delta\alpha}$ is a traceless symmetric matrix containing 5 independent components. Thus,

$$Q = 64\pi G\gamma (n_{\alpha\beta}n^{\alpha\beta} - 2v_{\gamma}v_{\gamma}) + O(\epsilon^3), \text{ for } d = 5.$$  

(7.22)

Furthermore, for $d > 5$ we may again use [52] to take the block $C_{\kappa\alpha\beta\gamma}$ to be of the form (7.21), although (7.21) is no longer the most general form for $C_{\kappa\alpha\beta\gamma}$ and of course the number of components of each object above increases with the spacetime dimension $d$.

It is clear from (7.22) that (7.20) is generally non-zero for $d \geq 5$. Furthermore, while the QFC requires $Q$ to be non-positive, for $\gamma > 0$ it can be made positive by setting $v_{\gamma} = 0$ and taking $n_{\alpha\beta} \neq 0$, and for $\gamma < 0$ we can make $Q$ positive by taking $n_{\alpha\beta} = 0$ with $v_{\gamma} \neq 0$.

Violations of the QFC thus occur for either sign of the Gauss-Bonnet coupling $\gamma$ and the QFC generally fails for classical $d \geq 5$ Einstein-Hilbert-Gauss-Bonnet gravity. We may immediately extend this result to the quantum level by noting that graviton contributions to the $S_{out}$ term of equation (7.2) are of order $G$ while our violation above is of order $G\gamma$. The key point here is that $\gamma$ has dimensions $(\text{Length})^{-(d-4)}$ so that the $G\gamma$ term is more important at large length scales than the $G$ term in $S_{out}$. In other words, the classical contributions to (7.2) will dominate in the long-distance limit.

Let us now consider more general (perhaps, effective) theories of gravity with higher derivative terms. First, it is trivial to add a cosmological constant $\Lambda$ to the action (7.4). Noticing that $C_{\kappa\alpha\kappa} = 0$ identically for all $d$, one finds no change to equation (7.19). Next, recall that at the four-derivative level, up to total derivatives there are only two further independent terms that we may add to the action, and we may choose to write both in terms of the square of the Ricci tensor (so that they do not depend on the Weyl tensor). Thus Ricci-flat metrics continue to solve the theory with $\gamma = 0$, and there continue to be solutions of the form (7.6) in the presence of such terms, and in such cases we again find (7.20) (up to additional corrections that are also of order $\epsilon^2$ but involve additional derivatives and so remains smaller in the long-distance limit). Finally, so long as they are controlled by a common length scale, in a long-distance expansion any terms in the action with more than four derivatives can be ignored relative to those already discussed so that (7.6) continues to hold in that regime.

The key point, however, is the associated implication for generic quantum theories of massive fields when coupled to semi-classical gravity. Since integrating out massive fields gives an effective action of the above type, so long as the resulting Gauss-Bonnet coefficient

\footnote{The final Gauss-Bonnet coefficient is of course formally the sum of the Gauss-Bonnet coefficient in the gravitational action and the coefficient induced by integrating out the matter. For $d \geq 5$ the latter is generally divergent, so the former must be as well if the effective action is to be finite. In this sense, as usual, there is generally no meaning to attempting to couple the massive field theory to Einstein-Hilbert gravity alone.}
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is non-zero the theory will violate the associated QFC.

7.3 Violating the Generalized Covariant Entropy Bound

Bousso’s original covariant entropy bound [23] involved the concept of “entropy flux through a non-expanding null surface” and conjectured this to be bounded by $(\frac{1}{4G} \times$) the area of the largest cut. There has been much discussion of how this concept might be properly defined, with one seemingly-natural choice involving entropy defined directly on the null surface. This version was proven for free and interacting theories in the $G \to 0$ limit from the monotonicity property of the relative entropy [32, 31]. Alternatively, Strominger and Thompson [163] suggested focusing on the case where any cut of the null surface $N$ is closed and bounds a spacelike surface. One may then discuss the von Neumann entropy $S_{vN}$ of the region enclosed, and replace the “flux of entropy across $N$” with the change in $S_{vN}$ between the initial and final surfaces.

As noted in [34], this choice gives rise to a putative (generalized) covariant entropy bound which is intrinsically finite and does not require renormalization. The conjecture of [34] states that if some set of null generators has non-positive quantum expansion ($\Theta \leq 0$) on some cut $C_{\text{initial}}$ of $N$, then any cut $C_{\text{final}}$ obtained by moving $C_{\text{initial}}$ to the future along these generators will have smaller or equal generalized entropy $S_{\text{gen}}$ so long as no caustic lies between $C_{\text{initial}}$ and $C_{\text{final}}$. The non-increase of $S_{\text{gen}}$ is equivalent to the claim

$$\Delta S \leq \Delta A/4G,$$

(7.23)

which is a generalized covariant entropy bound of the form first discussed in [72]. Note, however, that the condition $\Theta|_{C_{\text{initial}}} \leq 0$ under which this was conjectured in [34] differs from the assumption used in [23, 72] which requires the classical expansion $\theta$ to be non-positive on all intermediate cuts. Furthermore, equation (7.23) follows directly from the QFC in cases where the latter is valid [34].

However, it turns out the QFC violation constructed above is also a counterexample to the generalized covariant entropy bound (i.e. the quantum Bousso bound) of [34]. The key point is that the Gauss-Bonnet contribution (7.11) to the quantum expansion vanishes at $p$ since $K^{(X)}_{ab} = 0$. But since $\theta|_p = 0$ as well, the full quantum expansion $\Theta$ also vanishes at $p$.

From here we need only note that we can then achieve $\Theta \leq 0$ near $p$ on $C_{\text{initial}}$ by taking the classical expansion $\theta$ sufficiently negative near $p$; i.e., by simply choosing $C_{\text{initial}}$ to have large enough extrinsic curvature of the appropriate sign. We then find that later cuts $C_{\text{final}}$ differing from $C_{\text{initial}}$ only very near $p$ and by small affine parameter distance along the QFC-violating generators must have larger generalized entropy $S_{\text{gen}}$, violating the conjecture of [34]. Indeed, in the appropriate limit the increase of $S_{\text{gen}}$ is determined by $(\mathcal{L}_k \Theta)|_p > 0$. 

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7.4 Discussion

Using an explicit calculation for classical Einstein-Hilbert-Gauss-Bonnet gravity, we argued that the QFC of [34] is violated in generic $d \geq 5$ theories of gravity coupled to massive quantum fields. The key point is that integrating out the massive fields generically induces a Gauss-Bonnet term which, at least for a certain class of solutions, dominates in the long-distance limit. There we may use the explicit Einstein-Hilbert-Gauss-Bonnet calculation of section 7.2. We expect similar violations to continue to arise when massless quantum fields are included as well. Our construction also provides a counterexample to the generalized covariant entropy bound (i.e. the quantum Bousso bound) conjectured in [34]. It remains an open question whether the QFC and covariant entropy bound could hold for $d \leq 4$, and it would be interesting to investigate the affect of Ricci-squared terms in this context.

As mentioned in the introduction, the QFC is closely related to the Quantum Null Energy Condition (QNEC). Indeed, when a matter theory satisfying the Quantum Null Energy Condition is coupled to Einstein-Hilbert gravity, the QFC will hold at least to first order in the gravitational coupling $G$. The reader may thus ask whether our results are in tension with the QNEC proofs in [33] and [115]. The answer is no, as those results prove the QNEC only for congruences $N$ through $p$ that form bifurcate Killing horizons at $G = 0$. And on a bifurcate Killing horizon components of the Weyl tensor with non-zero boost weight must vanish. This would then force $C_{k\alpha\beta\gamma} = 0$ and thus $Q = 0$ in (7.20), reproducing the expected result that the QFC hold at first order in $G\gamma$ for such cases.$^5$

Conversely, taking the limit $G\gamma \to 0$ of our results shows that for $d \geq 5$ the renormalized QNEC must generally fail$^6$ for surfaces $\Sigma$ defining null congruences $N$ that are only locally stationary at $p$; i.e., which satisfy $\theta = \sigma_{\alpha\beta} = R_{\alpha\beta k^a k^b} = 0$ in the background spacetime. However, one may ask if the QNEC can hold at locally stationary points of null congruences for $d < 5$ or where further conditions are satisfied. The forthcoming work [80] will provide results of this kind, including a proof for $d \leq 3$ holographic theories at locally stationary points.

It is natural to ask if our QFC violation also provides a perturbative counterexample to the GSL. While Einstein-Hilbert-Gauss-Bonnet gravity is known to violate the GSL at the non-perturbative level [107, 127, 157], these are of lesser interest as higher derivative theories of gravity are expected [39] to approximate UV-complete theories only when treated perturbatively as an effective field theory valid at lengths longer than some cutoff scale $\ell_c$. And indeed, as in section 7.3, one can certainly find cases where the generalized entropy inside the horizon increases and thus that outside decreases. But the GSL is naturally conjectured to hold at most for causal horizons (see e.g. [108], [174]), and determining whether

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$^5$Indeed, a result of [117] shows that the QFC holds for any Lovelock theory of gravity (a class which includes the Einstein-Hilbert-Gauss-Bonnet gravity) when evaluated at first order in $G$ about a Killing horizon. This result was then generalized in [156] and extended to arbitrary higher-derivative theories of gravity in [175].

$^6$As will be discussed in more detail in [80], the QNEC may still hold in some sense for appropriate bare quantities. But finite renormalized quantities cannot satisfy a QNEC-like bound.
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a given null $N$ is a causal horizon requires understanding the very far future. Analyzing the constraints on $N$, thus requires going well beyond the local approximations used here, and thus beyond the scope of this work, though see [175, 16, 18] for further work on the GSL for higher derivative gravity and more thorough reviews.

Finally, one may ask if some version of the QFC or quantum Bousso bound might yet be salvaged for general $d \geq 5$ theories. In particular, we recall again that higher derivative gravity should be treated as an effective field theory with a cutoff $\ell_c$. But the QFC, and in particular our construction of a counterexample, requires the choice of a null congruence $N$ that is taken to be arbitrarily well localized in the transverse directions. Furthermore, since the Gauss-Bonnet term should be treated as perturbatively small, correspondingly small changes in $N$ can make $\theta, \sigma_{ab}$ non zero at $p$ so that $\dot{\theta} = -\frac{\theta^2}{d-2} - \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b$ becomes sufficiently negative at $p$ that $Q < 0$ for the new surface. In other words, perturbatively close to any compact QFC-violating null congruence $N$ lies a QFC-respecting null congruence $N'$. If this can be interpreted as a distinction finer than the cutoff scale $\ell_c$, there is room for the formulation of an effective QFC valid only at larger scales.\footnote{We thank Aron Wall for this suggestion.} But such an interpretation is not immediately clear as the above mentioned deformation from $N$ to $N'$ involves adding extrinsic curvature of a particular sign; it is not just a transverse smearing of the surface. And while it is attractive from many perspectives to conjecture that a QFC-like inequality may hold in an appropriately cutoff sense, both the form that this effective QFC might take and how in practice it would be used to restrict possible pathologies of NEC-violating spacetimes remain open questions for future investigation.

Note added in v2. After the appearance of our paper on the arXiv, it was pointed out in [123] that the violation described above is removed by restricting the QFC to apply only to variations of the entropy defined by surfaces that are smooth on the scale set by $G\gamma$, and which is presumably associated with the cut-off that defines the effective theory. This emphasizes the importance of studying the effect of $R_{ab}R^{ab}$ terms in the action, which might contribute a different class of terms to the QFC.
Chapter 8

The Quantum Null Energy Condition in Curved Space

8.1 Introduction and Summary

Energy conditions are indispensable in understanding classical and quantum gravity. The weakest but most commonly used of the standard energy conditions is the null energy condition (NEC), which states that $T_{kk} \equiv T_{ab} k^a k^b \geq 0$ where $T_{ab}$ is the stress tensor of the matter coupled to gravity and $k^a$ is any null vector. It is sufficiently weak to be satisfied by familiar classical field theories, yet strong enough to prove the second law of black hole thermodynamics [91], singularity theorems [146, 93], the chronology protection theorem [92], topological censorship [79], and other fundamental results. It also guarantees essential properties of holographic entanglement entropy [177, 96] in the context of gauge/gravity duality.

On the other hand, it has long been known that the NEC is violated even in free quantum field theories [65]. Several quantum replacements for the NEC have been suggested — such as the averaged null energy condition (ANEC) [173, 87, 119, 120, 112] and “quantum inequalities” [74, 75, 76, 125] — which involve integrating $\langle T_{kk} \rangle$ over a region of spacetime, and others [133, 132, 131, 130]. In this paper we study the quantum null energy condition (QNEC) [34, 33, 115, 4],

$$\langle T_{kk} \rangle \geq \frac{1}{2\pi} S'' ,$$

which places a bound on the renormalized $\langle T_{kk} \rangle$ at a point $p$ in terms of a particular second derivative of the renormalized von Neumann entropy of a region touching $p$ with respect to deformations of the region at $p$.\footnote{The same inequality has been investigated [82] with the “causal holographic information” of [100, 78, 102, 111, 35] playing the role of $S$ instead of the von Neumann entropy. Another variant was studied in the hydrodynamic approximation in [72].} In (8.1) we have set $\hbar = 1$. Although shown in (8.1), we will often omit the expectation value brackets below.
The conjecture of [34] states that (8.1) holds when $k^a$ generates a locally stationary horizon through $p$; i.e., it generates a hypersurface-orthogonal null congruence with vanishing shear $\sigma_{ab}$ at $p$ and expansion $\theta$ at $p$ vanishing to second order along the generator ($\sigma_{ab}|_p = \theta|_p = \dot{\theta}|_p = 0$). Below, we restrict to backgrounds satisfying the null convergence condition $R_{ab}k^ak^b \geq 0$, so the Raychaudhuri equation

$$\dot{\theta} = -\frac{\theta^2}{d-2} - \sigma^{ab}\sigma_{ab} - R_{ab}k^ak^b \quad (8.2)$$

then requires $R_{ab}k^ak^b = 0$ at $p$.

This conjecture was motivated in [34] by taking a non-gravitating ($G \to 0$) limit of a “quantum focussing conjecture” (QFC), which was in turn motivated by the generalized second law (GSL) of thermodynamics [14] and the proposed covariant entropy bound [23]. Such conjectures suffice to preserve the most fundamental of the above results even in the presence of quantum corrections [180, 34]. For example, although quantum corrections allow the formation of traversable wormholes, the GSL severely limits their utility [83].

The QNEC has been proven for deformations along bifurcate Killing horizons in free bosonic theories [33] using the techniques of null quantization, and it was also shown to hold for holographic theories formulated in flat space in [115]. In the holographic case, the Ryu-Takayanagi-Hubeny-Rangamani formula [101] was used to translate the QNEC (applied in the boundary theory) at leading order in $1/N$ into a statement about how boundary-anchored extremal surfaces in AdS move when the anchoring region is deformed. The relevant condition is that when the boundary region is deformed within its domain of dependence, the corresponding extremal surface should move in a spacelike way, at least near the boundary. This condition — called “entanglement wedge nesting” in [4] — is automatically true assuming entanglement wedge reconstruction [53, 96], and can also be proven directly from the NEC applied in the bulk [177]. It was shown in [4] that the QNEC continues to hold at all orders in $1/N$, assuming the entanglement wedge nesting property and the quantum extremal surfaces prescription of [64] (building on [69]).

And as we note in section 8.3 below, the Koeller-Leichenauer holographic argument [115] admits a straightforward extension to arbitrary backgrounds. However, the quantities to which the resulting inequality applies are naturally divergent. One might expect that the inequality takes the form of a QNEC for the “bare” quantities that have not been fully renormalized. Though even this remains to be shown, it is nevertheless of central interest to understand when local counter-terms contribute to (8.1) — or, more specifically, when they contribute to the difference between the left- and right-hand sides. In such cases, a QNEC for renormalized quantities would also depend on the choice of renormalization scheme, as such choices induce finite shifts in naively-divergent couplings. We therefore refer to this phenomenon as scheme-dependence.

A result of [175] shows the QNEC to be scheme-independent when the null congruence $N$ lies on a bifurcate Killing horizon (see [117] and [156] for precursors in special cases), but the more general setting is studied in section 8.2 below. For $d \leq 3$ we find that this
extends\textsuperscript{2} to locally stationary null congruences $N$, and for $d = 4, 5$ it holds when some additional derivatives of the shear $\sigma_{ab}$ and expansion $\theta$ also vanish at $p$. For $d \geq 6$ we show that scheme-independence generally fails even for weakly isolated horizons [8] satisfying the dominant energy condition. The qualitative difference between the above cases is that in $d \leq 5$, finite counter-terms can only depend algebraically on the Riemann tensor by dimensional analysis, while in $d \geq 6$ derivatives of the Riemann tensor become allowed. In Sec. 8.2 we show explicitly that the QNEC fails to be invariant under the addition of the simplest possible such counter-term, $\int d^6 x \sqrt{-g} (\nabla_a R)(\nabla^a R)$.

In all cases where we find the QNEC to be scheme-independent, we show in section 8.3 that a renormalized QNEC can be proven for the universal sector of holographic theories using the method of [115]. This in particular establishes the QNEC for holographic theories on arbitrary backgrounds when the null congruence $N$ lies on a bifurcate Killing horizon.

We close with some final discussion in section 8.4. In particular, we note that a corollary to our work is a general proof of the QFC and GSL for holographic quantum field theories in $d \leq 3$ at leading order in both $1/N$ and the coupling $G$ to gravity\textsuperscript{3}, and for these theories on weakly-isolated horizons in $d \leq 5$. This is the first proof of the GSL in the semi-classical regime which does not require the quantum fields to be perturbations to a Killing horizon.

### 8.2 Scheme-(in)dependence of the QNEC

As discussed above, a crucial question is whether local counter-terms affect (8.1). To answer this question, it is useful to be more precise about how the various terms in (8.1) are to be computed. We briefly review such recipes in section 8.2 and then consider the effect of local counter-terms in 8.2.

\textsuperscript{2} A semantic subtlety is that [34] did not spell out in detail the set of backgrounds in which the QNEC should hold. In particular, although the focussing theorem applies only to spacetimes satisfying the null convergence condition $R_{ab}k^a k^b \geq 0$ for null vectors $k^a$, the utility of this theorem in Einstein-Hilbert gravity (where $R_{ab}k^a k^b = 8\pi T_{ab}k^a k^b$) stems from the fact that reasonable matter theories satisfy the null energy condition (NEC) $T_{ab}k^a k^b \geq 0$. Thus the focussing theorem holds on solutions to reasonable theories. The derivation [34] of the QNEC from the QFC suggests the former to hold on backgrounds that solve reasonable theories of gravity and indeed the discussion in [34] assumed that the QNEC was to be studied on an Einstein space. In contrast, the idea that the QNEC is an intrinsically field-theoretic property (having nothing to do with coupling to gravity) suggests that – like the classical NEC for reasonable matter theories – it should in fact hold on any background spacetime. We will focus on the latter perspective for several reasons: First, the results of [33, 115] hold on any bifurcate Killing horizon without other restrictions on the background. Second, as explained in footnote 5 below, the results of [81] imply for $d \geq 5$ that the QNEC is scheme-dependent on general null congruences even for standard scalar field theories on Ricci-flat backgrounds. Third, the result below that the QNEC holds on any null congruence $N$ in an arbitrary $d \leq 3$ background.

\textsuperscript{3} I.e., this is the “boundary” Newton constant, not the bulk Newton constant of the gravitational dual.
CHAPTER 8. THE QUANTUM NULL ENERGY CONDITION IN CURVED SPACE

Preliminaries

We consider states $|\Psi\rangle$ that are pure on a sufficiently enlarged spacetime (with metric $g_{ab}$) and which are defined by a path integral with arbitrary operator insertions $O[\Phi]$ and sources (included in the action $I$):

$$|\Psi\rangle = \int_{\tau=-\infty}^{\tau=0} [D\Phi]O[\Phi]\, e^{-I[\Phi,g_{ab}]}$$  (8.3)

We use Euclidean notation for familiarity, though the integration contour may also include Lorentzian or complex pieces of spacetime (as in e.g. the discussion of entropy in [56]).

The partition function $Z = \text{Tr} [|\Psi\rangle\langle\Psi|]$ for such a state is computed by sewing together two copies of (8.3) to form the path integral

$$Z[g_{ab}] = \int [D\Phi]O[\Phi]\, e^{-I[\Phi,g_{ab}]}.$$  (8.4)

As usual, (8.4) is a functional of the background geometry $g_{ab}$. We take the definitions of $[D\Phi]$ and $I[\Phi,g_{ab}]$ to include appropriate renormalizations to make $Z[g_{ab}]$ well-defined. The renormalized effective action $W$ is defined by

$$Z[g_{ab}] = e^{-W[g_{ab}]}.$$  (8.5)

Both $T_{kk}$ and $S''$ are to be computed from $W[g_{ab}]$. The expectation value of the renormalized gravitational stress tensor in $|\Psi\rangle$ is defined by

$$\langle T_{ab} \rangle \equiv -\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{ab}}.$$  (8.6)

Given a region $\mathcal{R}$ with boundary $\Sigma = \partial \mathcal{R}$, we take the renormalized $S$ to be computed from $W[g_{ab}]$ via the replica trick, as the response of $W[g_{ab}]$ to a conical singularity at $\Sigma$ (henceforth called the entangling surface):

$$S = -\text{Tr} \rho \log \rho = (1 - \partial_n) \log \text{Tr} \rho^n.$$  (8.7)

The density matrix can be written in terms of the path integral, $\rho^n = \frac{Z[g_{(n)ab}]}{Z[g_{(1)ab}]}$, where $g_{(n)ab}$ denotes the geometry with $n$ replicas of the original geometry, glued together at the entangling surface. Thus $S$ can be expressed in terms of $W[g_{(n)ab}]$ as

$$S = W_{(1)} - \partial_n W_{(n)} \big|_{n=1},$$  (8.8)

with $W_{(n)} \equiv W[g_{(n)ab}]$.

It then remains to compute $S''$. Given a null congruence $N$ orthogonal to $\Sigma$ (by convention taken outgoing relative to the region $\mathcal{R}$), we may vary $\Sigma$ by displacing it along the null
generators of $N$. This defines an associated deformation of $\mathcal{R}$, and thus a change of $S$. The quantity $S''$ is well-defined at $p$ when the second such variation takes the form

$$\frac{\delta}{\delta \Sigma (y_1)} \left[ \frac{1}{\sqrt{h}} \frac{\delta}{\delta \Sigma (y_p)} S \right] = \sqrt{h} \ S''(y_p) \delta(y_1 - y_p) + f(y_p, y_1)$$  \hspace{1cm} (8.9)

for smooth functions $S''(y_p), f(y_p, y_1)$. Here $y$ labels the generators of the null congruence $N$, $y_p$ is the null generator through $p$, and $h$ denotes the determinant of the metric on $\Sigma$ in the $y$-coordinate system.\(^4\)

### The QNEC and local counter-terms

The renormalized effective action $W$ depends on the choice of renormalization scheme, though any two schemes will differ only by adding finite local counter-terms; i.e., by the addition to $W$ of integrals of marginal or relevant operators built from background curvatures of $g_{ab}$ and/or matter fields. Such terms might in principle affect either side of (8.1). Below, we calculate the net effect on the QNEC quantity

$$Q := T^{ab} k^a k^b - \frac{1}{2\pi} S''$$  \hspace{1cm} (8.10)

at points $p$ where $k^c$ generates a locally stationary null congruence in a background satisfying the null convergence condition $R_{ab} k^a k^b \geq 0$. In particular, as noted in the introduction we must have

$$(R_{ab} k^a k^b)|_p = 0$$  \hspace{1cm} (8.11)

at all such points.

We consider a theory that approaches a (unitary) conformal fixed point in the UV. The possible terms thus depend on the spacetime dimension $d$. We will assume in all cases that there are no scalar operators saturating the unitarity bound $\Delta = \frac{d-2}{2}$, so the addition to $W$ of kinetic terms like $\int d^d x \sqrt{-g} \phi^2$ are not allowed. For simplicity, for $d = 2$ we also neglect conserved currents as in this case they would require special treatment. Apart from this one case, in the absence of non-metric sources combining covariance with unitarity bounds forbids the appearance of terms in $W$ involving CFT operators with spin $j \geq 1$. For later use in section 8.3 we note that, using an argument like that in footnote 5 below, a result of \[175\] thus shows that the QNEC is scheme-independent on any bifurcate Killing horizon.

$d \leq 3$

For $d \leq 3$, the terms one may add to $W$ are only

$$\int d^d x \sqrt{-g} \phi_1, \int d^d x \sqrt{-g} R \phi_2,$$  \hspace{1cm} (8.12)

\(^4\)In general, one might expect even more singular terms (involving e.g. derivatives of delta-functions) to appear in (8.9). In such cases a QNEC of the form (8.1) cannot hold.
for scalar operators $\phi_1, \phi_2$ of dimensions $\Delta_1, \Delta_2$ with $\Delta_1 \leq d$ and $\Delta_2 \leq d - 2$. Note that this includes the case $\phi_2 = 1$. The contribution of the first term to $T_{ab}$ is proportional to $g_{ab}$ and thus vanishes when contracted with $k^a k^b$ for null $k^c$. Its explicit contribution to $S$ also vanishes, so it does not affect the QNEC.

For terms of the second form in (8.12) one finds

$$\Delta S = 4\pi \int_{C_B} d^{d-2} y \sqrt{h} \phi_2,$$

$$\Delta T_{ab} = 2 \left( \nabla_a \nabla_b - g_{ab} \nabla^2 \right) \phi_2 - 2 \left( R_{ab} - \frac{1}{2} g_{ab} R \right) \phi_2.$$  \hspace{1cm} (8.14)

Using (8.11) and the fact that $k^c$ is null, direct computation then gives

$$\Delta Q = -2\phi_2 \left( \dot{\theta} + R_{ab} k^a k^b \right) - 4\theta \dot{\phi}_2 = 2\phi_2 \left( \frac{\theta^2}{d-2} + \sigma_{ab} \sigma^{ab} \right) - 4\theta \dot{\phi}_2,$$  \hspace{1cm} (8.15)

where the final step uses (8.2). Both terms vanish on a locally stationary horizon, and in fact $\sigma_{ab}$ vanishes identically for $d = 3$. So under the above conditions the QNEC is scheme-independent for $d \leq 3$. In fact, we see that it is really only necessary to impose $\theta = 0$.

$d = 4, 5$

Increasing $d$ leads to additional terms. The allowed terms for $d = 4, 5$ are those in (8.12) together with

$$\int d^d x \sqrt{-g} R_{ab} R^{ab}, \quad \int d^d x \sqrt{-g} R_{abcd} R^{abcd}.$$  \hspace{1cm} (8.16)

Since scalar operators $\phi$ have dimensions larger than $\frac{d-2}{2} \geq 1$, for $d = 4, 5$ they cannot be inserted into the terms (8.16). Note that terms like $\int d^d x \sqrt{-g} R^2$ can be written as the second term in (8.12) by taking $\phi_2$ to involve $R$, so such terms were already considered above.

The contributions of (8.16) to the QNEC quantity $Q$ are complicated and do not appear to vanish at the desired points $p$. Indeed, for $d \geq 5$ the results of [81] show that the particular combination of the terms in (8.16) with $\int d^d x \sqrt{-g} R^2$ that defines the Gauss-Bonnet term contributes $\Delta Q \neq 0$ even on Ricci-flat backgrounds\(^5\), though the Gauss-Bonnet contribution to $Q$ vanishes for $d = 4$.

The fact that terms in (8.16) are four-derivative counter-terms suggests that their contributions to $\Delta T_{ab} k^a k^b$ and $\Delta S''$ contain fourth derivatives of the horizon generator $k^a$. It is thus natural to ask if we can force $\Delta Q = 0$ for (8.16) by setting the first, second, and third

---

\(^5\) Ref. [81] considered a perturbative computation of $Q_{\text{QFC}} := \dot{\theta} + 4 G S''_{GB}$, where $\dot{\theta}$ is affine derivative of the expansion of $N$. The computation was done at first order in the Gauss-Bonnet coupling $\gamma$ about a Ricci-flat background. From the Raychaudhuri equation (8.2), the first order change in $\dot{\theta}$ is precisely $-\Delta T_{ab} k^a k^b$ where $\Delta T_{ab}$ is the Gauss-Bonnet term’s contribution to $T_{ab}$. Thus $Q_{\text{QFC}} = -4GQ$ with $Q$ defined by (8.10).
derivatives of the expansion $\theta$ and shear $\sigma_{\alpha\beta}$ to zero, where here and from now on, we use indices $\alpha, \beta, \ldots$ to indicate coordinates on $\Sigma$. It turns out that this is the case. For $d = 4, 5$ we impose the conditions

$$
\theta|_p = (\mathcal{D}_a \theta)|_p = (\mathcal{D}_b \mathcal{D}_a \theta)|_p = (\mathcal{D}_c \mathcal{D}_b \mathcal{D}_a \theta)|_p = 0,
$$

$$
\sigma_{\alpha\beta}|_p = (\mathcal{D}_a \sigma_{\alpha\beta})|_p = (\mathcal{D}_b \mathcal{D}_a \sigma_{\alpha\beta})|_p = (\mathcal{D}_c \mathcal{D}_b \mathcal{D}_a \sigma_{\alpha\beta})|_p = 0,
$$

(8.17)

where $\mathcal{D}_a$ is the covariant derivative along the congruence $N$. We will show below that (when combined with a positive energy condition) these requirements suffice to show $\Delta Q = 0$, though the question remains open which conditions are precisely necessary.

A final condition we impose is that the background solve the Einstein equations with a source respecting the dominant energy condition (DEC) up to a term proportional to the metric. This is equivalent to requiring $R_{ab}$ to be of the form

$$
R_{ab} = R_{ab}^{(DEC)} + \alpha g_{ab},
$$

(8.18)

for some scalar field $\alpha$, where for any future-pointing causal (either timelike or null) vector field $\nu^a$, the vector field $-R_b^{(DEC)} \nu^b$ must also be both future-pointing and causal. A short argument (see appendix A.5) using (8.17) then shows that on the null generator through $p$ we have

$$
R_{ab} k^b = f k_a + O(\lambda^3),
$$

(8.19)

for some scalar function $f$ and that

$$
R_{abcd} k^b k^d = \zeta k_a k_c + O(\lambda^3),
$$

(8.20)

for some scalar function $\zeta$. Since (8.17) implies that equation (A.42) holds at point $p$, the contracted Bianchi identity implies that equation (A.58) holds at point $p$, namely

$$
(k^a \partial_a f)|_p = \left(\frac{1}{2} k^a \partial_a R\right)|_p = (-k^a \partial_a \zeta)|_p.
$$

(8.21)

The expressions for $\Delta S$ for the counter-terms (8.16) can be found in [54] and the expressions for $\Delta T_{ab}$ for these counter-terms can be calculated by using the definition (8.6). These expressions are simplified greatly by using conditions (8.17) and (8.18). After such simplifications it is straightforward (see appendix A.6) to compute $\Delta S''$ and, as shown in table 8.1, the above conditions suffice to force $\Delta Q = 0$ for both terms (8.16) in all dimensions. In table 8.1 we have used the notation $\partial_k := k^a \partial_a$, which we continue to use elsewhere below.

As a final comment, we note that a careful analysis of the calculation shows that although for $d = 5$ we require the full list of conditions (8.17) show $\Delta Q = 0$, for $d = 4$ it suffices to use only a subset of the conditions. The reason is that for $d = 4$ we may choose to study the Gauss-Bonnet term $R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2$ (instead of $R_{abcd} R^{abcd}$). This term is topological and so contributes to neither $T_{ab}$ nor $S''$, and to guarantee $\Delta Q = 0$ for the only remaining counter-term $R_{ab} R^{ab}$ we need only (8.18) and conditions on the first line of (8.17).

---

6This condition is motivated by the discussion of weakly isolated horizons proposed in [8].
Table 8.1: Scheme-independence of QNEC for all four-derivative counter-terms $\Delta L$ from (8.16) when (8.17) and (8.18) hold.

<table>
<thead>
<tr>
<th>$\Delta L$</th>
<th>$\Delta T_{ab}k^ak^b$</th>
<th>$\Delta S$</th>
<th>$\Delta Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>$4\partial_k^2 R$</td>
<td>$8\pi R$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{ab}R_{ab}$</td>
<td>$4\partial_k^2 f$</td>
<td>$8\pi f$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{abcd}R_{abcd}$</td>
<td>$-8\partial_k^2 \zeta$</td>
<td>$-16\pi \zeta$</td>
<td>0</td>
</tr>
</tbody>
</table>

$d \geq 6$

In six dimensions, six-derivative counter-terms become allowed. Based on the results above, one might hope to maintain scheme-independence of the QNEC in this case by requiring even more derivatives of the extrinsic curvature to vanish. However, we now show that for $d \geq 6$ the contribution $\Delta Q$ is generally non-zero even on weakly isolated horizons (where $\theta, \sigma_{\alpha\beta}$ vanish identically on $N$ [8]) in backgrounds satisfying (8.18).

Since all derivatives of $\theta, \sigma_{\alpha\beta}$ vanish, the results (8.19), (8.20), and (8.21) now exactly hold on a finite neighborhood of point $p$ on the horizon. Furthermore, we show in appendix A.5 that on weakly isolated horizons the Riemann tensor $R_{abck}$ can be written as $R_{abck} = k_c\tilde{A}_{ab} + k_{[a}\tilde{B}_{b]c}$, where $\tilde{A}_{ab}$ is antisymmetric and satisfies $k^a\tilde{A}_{ab} \propto k_b$ and $\tilde{B}_{ab}$ satisfies $k^a\tilde{B}_{ab} \propto k_b$ and $k^b\tilde{B}_{ab} \propto k_a$. This allows one to write down additional relations also listed in appendix A.5. Together, they allow one to show the QNEC to be unchanged by adding six-derivative counter-terms built from polynomial contractions of the Riemann tensor. The computations are presented in appendix A.6 and the results are summarized in table 8.2.

However, we will shortly see that scheme-independence of the QNEC can fail even on weakly isolated horizons for counter-terms that contain derivatives of the Riemann tensor. There are four such counter-terms, $(\nabla_a R)(\nabla^a R)$, $(\nabla_a R_{bc})(\nabla^a R^{bc})$, $(\nabla_e R_{abcd})(\nabla^e R^{abcd})$, and $(\nabla_a R_{bc})(\nabla^b R^{ac})$. Neglecting total derivatives of the action, these counter-terms are not linearly independent and one can write the last two previous counter-terms in terms of the other ten [143].

We will show this failure for the term $(\nabla_a R)(\nabla^a R)$. From [143] we have

$$\Delta T_{kk} = k^a k^b \Delta T_{ab}$$

$$= -4k^a k^b \nabla_a \nabla_b (\nabla_e \nabla^e R) - 2(k^a \nabla_a R)(k^b \nabla_b R)$$

$$= -4\partial_k^2 (\nabla_a \nabla^a R) - 2(\partial_k R)^2. \quad (8.22)$$

The formula used to calculate the change in the entropy is given in [134], where it was written as

$$\Delta S = \Delta S_{\text{G-Wald:1984rg}} + \Delta S_{\text{Anomaly}}. \quad (8.23)$$
Table 8.2: Scheme-independence of QNEC for the six-derivative counter-terms $\Delta L$ built from polynomial contractions of the Riemann tensor when the null congruence $N$ is a weakly isolated horizon. However, as shown in the main text, scheme-independence can fail for counter-terms involving derivatives of the Riemann tensor.

<table>
<thead>
<tr>
<th>$\Delta L$</th>
<th>$\Delta T_{ab}^{k^a_k^b}$</th>
<th>$\Delta S$</th>
<th>$\Delta Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^3$</td>
<td>$6\partial^2_k(R^2)$</td>
<td>$12\pi R^2$</td>
<td>0</td>
</tr>
<tr>
<td>$RR^{ab}_{ab}$</td>
<td>$2\partial^2_k(R^{ab}_{ab}) + 4\partial^2_k(Rf)$</td>
<td>$4\pi R^{ab}_{ab} + 8\pi Rf$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{ab}_{bc}R^c_a$</td>
<td>$6\partial^2_k(f^2)$</td>
<td>$12\pi f^2$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{abcd}<em>{ac}R</em>{bd}$</td>
<td>$2\partial^2_k(f^2) - 4\partial^2_k(R_{akel}R^{ac})$</td>
<td>$4\pi f^2 - 8\pi R_{akel}R^{ac}$</td>
<td>0</td>
</tr>
<tr>
<td>$RR^{abcd}_{abcd}$</td>
<td>$2\partial^2_k(R^{abcd}_{abcd}) - 8\partial^2_k(R\zeta)$</td>
<td>$4\pi R^{abcd}_{abcd} - 16\pi R\zeta$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{abcd}_{cdbe}R^e_a$</td>
<td>$-8\partial^2_k(\zeta f) - 2\partial^2_k(R_{ikel}R^{cdbl})$</td>
<td>$-16\pi \zeta f - 4\pi R^{abcd}<em>{cdbe}R</em>{cdbl}$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{ab}<em>{cd}R^{ce}</em>{bf}R^{def}_{ae}$</td>
<td>$-6\partial^2_k(R_{ikel}^{f}R_{jfk}^{e}) + 6\partial^2_k(\zeta^2)$</td>
<td>$-12\pi R_{ikel}^{f}R_{jfk}^{e} + 12\pi \zeta^2$</td>
<td>0</td>
</tr>
<tr>
<td>$R^{abcd}<em>{cdef}R^{e^f}</em>{ab}$</td>
<td>$-12\partial^2_k(R_{ik}^{ef}R_{e,flk})$</td>
<td>$-24\pi R_{ik}^{ef}R_{e,flk}$</td>
<td>0</td>
</tr>
</tbody>
</table>

On weakly isolated horizons, the “generalized Wald:1984rg entropy” term in (8.23) reduces to the ordinary Wald:1984rg entropy and is given by

$$
\Delta S_{G-Wald:1984rg} = \Delta S_{Wald:1984rg} = -2\pi \times (-2\nabla_a \nabla^a R) \times (-2) = -8\pi \nabla_a \nabla^a R. \quad (8.24)
$$

On weakly isolated horizons, by applying equation (A.12) of [134], one finds that the “entropy anomaly” term in (8.23) vanishes for counter-term $(\nabla_a R)(\nabla^a R)$. Thus,

$$
\Delta S_{\text{Anomaly}} = 0. \quad (8.25)
$$

Finally, we obtain

$$
\Delta Q = \Delta T_{kk} - \frac{1}{2\pi} \Delta S'' = -2(\partial_k R)^2. \quad (8.26)
$$

This quantity does not vanish generally, although it vanishes on Ricci-flat background.

It is not hard to find an explicit geometry in which this counter-term spoils the scheme-independence of the QNEC. Consider the spacetime metric

$$
ds^2 = -du dv - dv du - cu^2 v^2 du^2 + \sum_\alpha (dy_\alpha)^2, \quad (8.27)
$$

where $c$ is a positive constant which is not assumed to be small. In this spacetime, there is a non-expanding null surface $v = 0$. Its Ricci tensor is

$$
R_{ab} = c^2 u^4 v^2 (du)_a (du)_b + cu^2 (du)_a (dv)_b + cu^2 (dv)_a (du)_b = -cu^2 g_{\perp ab}, \quad (8.28)
$$
where $g_{\perp}^{a\, b}$ is the projector onto the $u-v$ plane. Since this plane is timelike, for any future-pointing causal vector $v^b$ the vector $-R^a_{\ b}v^b = cu^2 g_{\perp}^{a\, b}v^b$ is again future-pointing and causal and the Ricci tensor (8.28) satisfies (8.18) with $\alpha = 0$. Thus the plane $v = 0$ is a weakly isolated horizon. But from (8.28) we find scalar curvature $R = -2cu^2$, hence
\[ \Delta Q = -32c^2u^2 \neq 0 \] (8.29)
and the QNEC fails to be scheme-independent.

### 8.3 Holographic Proofs of the QNEC

The QNEC was proven to hold in [115] for leading-order holographic field theories on flat spacetimes. We review this derivation in section 8.3 below and show that the argument admits a straightforward generalization to arbitrary curved backgrounds; i.e., to the case where the boundary of the asymptotically locally AdS bulk is arbitrary. However, the resulting inequality is generally divergent, and we expect it to yield a finite renormalized QNEC only in the contexts where the QNEC is scheme-independent. For the scheme-independent cases described in section 8.2, we will indeed be able to derive such a finite renormalized QNEC below.

Since all the proofs in this section are provided in the context of AdS/CFT correspondence, we change our index notation to make it more suitable for this context. In this section, we use indices $\mu, \nu, \ldots$ to indicate the $d+1$ coordinates on the bulk spacetime and use indices $i, j, \ldots$ to indicate the $d$ coordinates on the boundary spacetime.

#### Outline of holographic proofs

The central idea of [115] is to reformulate both $T_{kk}$ and $S''$ in terms of quantities in the dual bulk asymptotically AdS spacetime, and to use a fact about extremal surfaces known as “entanglement wedge nesting” (EWN) [53, 177, 4] to provide the desired inequality. To begin, consider two regions $A, B$ on the asymptotically AdS boundary. Entanglement wedge nesting states that if these boundary regions are nested in the sense that $D(B) \subseteq D(A)$ then their extremal surfaces $e(A), e(B)$ must also be nested, i.e. everywhere spacelike related. Here $D(A)$ is the domain of dependence of $A$ in the asymptotically AdS boundary.

Now consider a family of boundary regions $A(\lambda)$ with entangling surfaces $\partial A(\lambda)$, which differ by localized deformations along a single null generator $k^i(y)$ of a null hypersurface shot out from the initial surface $A(0)$. The derivatives of the entropy used in the QNEC are then derivatives with respect to this particular $\lambda$, which we can take to be an affine parameter for $k^i$. Consider the codimension-1 bulk surface $M$ foliated by the (smallest for each $A(\lambda)$) extremal surfaces $e(\lambda) \equiv e(A(\lambda))$,
\[ M := \bigcup_{\lambda} e(\lambda) \] (8.30)
EWN for $A(\lambda)$ implies that $M$ is a spacelike surface.
The surface $\mathcal{M}$ can be parametrized by $\lambda$ and the $d-1$ coordinates $g^\alpha = \{\varepsilon, y^\alpha\}$ associated with each $\varepsilon(\lambda)$. The coordinate basis for the tangent space of $\mathcal{M}$ then consists of $\partial_\lambda X^\mu$ and the $d-1$ coordinate basis tangent vectors of the extremal surfaces, $\partial_\alpha X^\mu$. By EWN, each of these vectors has positive norm. The norm of $\partial_\lambda X^\mu$ can be expanded in $z$ near the boundary, and will involve the near-boundary expansion of both the metric and the extremal surface embedding functions. EWN implies in particular that as $z \to 0$ the most dominant term in this expansion is positive. In [115] it was shown that for locally-stationary surfaces (satisfying $\theta|_p = \sigma_{ab}|_p = 0$ at a point $p$) in flat space one has\footnote{Any vector tangent to $\mathcal{M}$ has positive norm. The original proof of [115] used $s^\mu \equiv t^\mu_\nu \partial_\lambda X^\nu$, where $t^\mu_\nu$ projects onto the 2-dimensional subspace orthogonal to $e(\lambda)$, instead of $\partial_\lambda X^\mu$ itself. Both work equally well to derive the QNEC in $d \geq 3$. We restrict to $d \geq 3$ and use $\partial_\lambda X^\mu$ for simplicity. Note that for $d = 2$ by a change of conformal frame one may always choose to work on a flat background.}:

$$0 \leq G_{\mu\nu} \partial_\lambda X^\mu \partial_\lambda X^\nu = \frac{16\pi G}{d} z^{d-2} \left( T_{kk} - \frac{1}{2\pi} S'' \right), \tag{8.31}$$

where the quantities on the right-hand side are have been renormalized.\footnote{In flat space, the local stationarity condition makes the renormalization trivial; $T_{kk}$ and $S''$ are finite to begin with [115]. This is not guaranteed if the boundary spacetime is curved.} Thus for these surfaces in flat space, EWN implies the renormalized QNEC.

As indicated in equation (8.31), the renormalized quantities in the QNEC appear in the EWN inequality at $O(z^{d-2})$. So in order to derive the renormalized QNEC, the terms in the $z$-expansion of $(\partial_\lambda X^\mu)^2$ must vanish at all lower orders. This condition provides restrictions on the surfaces and space-times. In flat space, it is sufficient to have local stationarity, i.e. $\theta|_p = 0, \sigma_{ab}|_p = 0$ at a point $p$ [115]. More generally the condition may be more complicated.

To compute (8.31) explicitly, we set $\ell_{\text{AdS}} = 1$ and use Fefferman-Graham-style coordinates to introduce the near-boundary expansion of the bulk metric $G_{\mu\nu}$:

$$G_{zz}(x^\mu) = \frac{1}{z^2}, \quad G_{zi} = 0,$$

$$G_{ij}(x^\mu) \equiv \frac{1}{z^2} g_{ij} = \frac{1}{z^2} \left( g(0)_{ij} + g(2)_{ij} + \ldots + g(d)_{ij} + \frac{16\pi G}{d} z^d T_{ij} \right), \tag{8.32}$$

and the embedding functions of the extremal surfaces

$$X^z(y^\alpha) = z,$$

$$X^i(y^\alpha) = X^i(0) + X^i(2) + \ldots + X^i(d) = \frac{4G}{d} z^d g(0)^{ij} S'_i + X^i(d), \tag{8.33}$$

where

$$S'_i \equiv \frac{1}{\sqrt{h(0)}} \frac{\delta S_{\text{ren}}}{\delta X^i(0)} \tag{8.34}$$
is the renormalized entropy directional derivative per unit area and subscripts denote powers of \( z \), e.g. \( X^i_{(m)} \) is \( O(z^m) \), while \( g_{(d)ij} \) is a log term of order \( z^d \log z \). As a result, \( \partial_{\lambda} X^d \) involves \( S'' = \partial_{\lambda} (k^i S'_i) \). The terms \( \bar{g}_{(d)ij} \) and \( \bar{X}^i_{(d)} \) refer to the geometric parts of the \( O(z^{d-2}) \) and \( O(z^d) \) parts of the metric and embedding functions, respectively. For more details, see [103, 115, 158] for the extremal surface expansion, and e.g. [89, 103] for the metric.

The key point is that \( S' \equiv k^i_{(0)} S'_i \) and \( T_{ij} k^i_{(0)} k^j_{(0)} \) appear at \( O(z^{d-2}) \). This is why plugging these expansions into \( (\partial_{\lambda} X^d)^2 \) gives the QNEC at \( O(z^{d-2}) \), as in (8.31). That the CFT stress tensor appears at \( O(z^{d-2}) \) in the near-boundary expansion for the metric is well known. We now derive the appearance of \( S'_i \) in equation (8.33).

In Einstein-Hilbert gravity, the HRT prescription [101] is

\[
S(\lambda) = \frac{A(e(\lambda))}{4G} = \frac{1}{4G} \int dz d^{d-2} y \sqrt{H[X]},
\]

where \( H[X] \) is the determinant of the induced metric on \( e(\lambda) \) written as a functional of \( X \). Varying the on-shell area functional with respect to \( X^i \) gives a boundary term evaluated at a cutoff surface \( z = \text{const} \). This produces the regulated entropy variation

\[
\frac{1}{\sqrt{h}} \frac{\delta S_{\text{reg}}}{\delta X^i} = - \frac{1}{4G} z^{1-d} \frac{g_{ij} \partial_z X^j}{\sqrt{1 + g_{nm} \partial_z X^n \partial_z X^m}} \bigg|_{z=\text{const}},
\]

where \( h \) is the determinant of the induced metric. Everything in equation (8.36) (including \( \delta X^i \)) is to be expanded in \( z \) and evaluated at a cutoff surface at \( z = \text{const} \). In general, there will be terms that diverge as \( z \to 0 \).

The entropy can be renormalized using in a manner similar to that used for the on-shell action (see e.g. [89]). Indeed, the two are intimately related [166]. Adding local, geometrical, covariant counter-terms gives the renormalized entropy via

\[
S_{\text{ren}} = \lim_{z \to 0} (S_{\text{reg}} + S_{\text{ct}}).
\]

With an arbitrary choice of renormalization scheme, \( S_{\text{ct}} \) can contain finite counter-terms in addition to those required to cancel divergences. Expanding (8.36) and removing the divergences, we have the finite renormalized entropy variation given in general by

\[
\frac{1}{\sqrt{h_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta X^i_{(0)}} = - \frac{d}{4G z^d g_{(0)ij} X^j_{(d)}} + \ldots
\]

where \( h_{(0)} \) is the determinant of the metric induced on \( \Sigma \) by \( g_{(0)ij} \), \( X^j_{(d)} \) denotes the \( O(z^d) \) part of \( X^j \), and the “…” denotes finite, local, geometric terms, which include both finite contributions to equation (8.36) from products of lower-order terms in the embedding functions, as well as possible finite counter-terms from \( S_{\text{ct}} \). Re-arranging for \( X^i_{(d)} \) gives

\[
X^i_{(d)} = - \frac{4G}{d} z^d g_{ij} \frac{1}{\sqrt{h_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta X^j_{(0)}} + X^i_{(d)}
\]
In this expression, $\bar{X}^i_{(d)}$ contains the contribution from the “…” of equation (8.38). Plugging (8.39) into the expansion of $X^i$ yields equation (8.33), as promised.

The geometric restrictions on the surface and geometry which guarantee that the lower-order terms in expression (8.31) vanish can in principle be determined by solving the relevant equations, though we will not carry out an exhaustive analysis here.

We show in section 8.3 below that the above argument leads to a finite renormalized scheme-independent QNEC for $d \leq 3$ at points $p$ where the expansion $\theta$ vanishes for the chosen null congruence $N$. We then show in 8.3 and 8.3 below that for $d = 4, 5$ it leads to a finite renormalized scheme-independent QNEC at points $p$ where the chosen null congruence $N$ satisfies the conditions on the first line of (8.17) on backgrounds satisfying (8.18). From the results of section 8.2 and the fact [89] that – for Einstein-Hilbert gravity in the bulk – holographic renormalization requires only counter-terms that can be built from the Ricci tensor, it is no surprise that we do not require the full list of conditions (8.17). Finally, in section 8.3 we provide a finite renormalized scheme-independent QNEC for holographic theories on Killing horizons in arbitrary backgrounds.

**Proof of the $d \leq 3$ holographic QNEC**

For $d = 3$ the asymptotic metric expansion (8.32) and the asymptotic embedding function expansion (8.33) take the form

$$g_{ij}(x, z) = g_{(0)ij} + g_{(2)ij} + g_{(3)ij} + \ldots, \quad (8.40)$$

$$X^i(x, z) = X^i_{(0)} + X^i_{(2)} + X^i_{(3)} + \ldots. \quad (8.41)$$

The causal property of extremal surfaces then implies

$$0 \leq g_{ij}(\partial_\lambda X^i)(\partial_\lambda X^j) = g_{(0)ij}k^ik^j + g_{(2)ij}k^ik^j + 2g_{(0)ij}(\nabla_\lambda X^i_{(2)})k^j + g_{(3)ij}k^ik^j + 2g_{(0)ij}(\nabla_\lambda X^i_{(3)})k^j. \quad (8.42)$$

One can easily check that the Einstein equations and extremal surface equation at second order in $z$ give

$$g_{(2)ij} = \frac{z^2}{d-2} \left(R_{ij} - \frac{1}{2(d-1)} R g_{(0)ij} \right), \quad X^i_2 = \frac{1}{2(d-2)} z^2 K^i, \quad (8.43)$$

in terms of the (traced) extrinsic curvature $K^i := g^j_{(0)i} K^i_{jk}$ of the (boundary) codimension-2 surface $\partial A$ with conventions given by (A.44). Since $k^i$ is null, $\dot{\theta} = \nabla_\lambda(g_{(0)ij} K^j k^i)$ and that $k^i$ satisfies the geodesic equation $\nabla_\lambda k^i := k^j \nabla_j k^i = 0$, the terms on the second line of (8.42) combine to give

$$\frac{z^2}{d-2} \left(R_{ij}k^ik^j + \dot{\theta} \right) = -\frac{z^2}{d-2} \left(\frac{\theta^2}{d-2} + \sigma^{ij}\sigma_{ij} \right), \quad (8.44)$$
where in the last step we have used (8.2). Both terms vanish on a locally stationary horizon, and in fact \( \sigma_{ij} \) vanishes identically for \( d = 3 \). The terms on the third line of (8.42) then give the renormalized QNEC (8.31).

The \( d = 2 \) argument is identical in form. Though terms at order \( z^2 \) are not divergent for \( z = 2 \), there are then divergent terms at order \( z^2 \log z \) which are structurally the same as the \( z^2 \) terms for \( d = 3 \).

**Proof of the \( d = 4 \) holographic QNEC**

For the case of four dimensional boundary, the asymptotic metric expansion (8.32) and the asymptotic embedding function expansion (8.33) take the form

\[
g_{ij}(x, z) = g_{(0)ij} + g_{(2)ij} + g_{(4)ij} + \ldots, \tag{8.45}
\]

\[
X^i(x, z) = X^i_{(0)} + X^i_{(2)} + X^i_{(4)} + \ldots. \tag{8.46}
\]

The causal property of extremal surfaces provides us the inequality in QNEC. We have

\[
0 \leq g_{ij}(\partial_\lambda X^i)(\partial_\lambda X^j) = g_{(0)ij}k^i_ik^j + 2g_{(0)ij}(\nabla_\lambda X^i_{(2)})k^j + 2g_{(0)ij}(\nabla_\lambda X^i_{(4)})k^j. \tag{8.47}
\]

Terms on the second line of (8.47) vanish just as in section 8.3. From [89, 128] we have

\[
g_{(4)ij}k^i_ik^j = -\frac{z^4 \log z}{24} \partial_\lambda^2 R, \tag{8.48}
\]

\[
g_{(4)ij}k^i_ik^j = z^4 \left( 4\pi G T_{kk} + \frac{1}{32} \partial_\lambda^2 R \right), \tag{8.49}
\]

which may be used to calculate the third line of equation (8.47). Here and below we freely use equation (A.42) which follows from the first line of (8.17). We will first show that the log terms cancel each other, and then show that the \( O(z^4) \) terms produce the QNEC.

We introduce the standard notation

\[
A_{\text{ren}} = A_{\text{reg}} + A_{\text{ct}}. \tag{8.50}
\]

The entropy counter-terms \( A_{\text{ct}} \) generically contain a finite part which we must extract. This comes from the requirement that the counter-terms are covariant functionals of the geometric quantities on the cutoff surface. The counter-term \( A_{\text{ct}, O(\log z)} \) which cancels the log divergence has an explicit \( \log z \), and consequently has no finite part. The finite part \( A_{\text{ct, finite}} \) of the counter-terms comes from the counter-term which cancels the leading area-law divergence. This is [166]

\[
A_{\text{ct, } A} = -\frac{1}{d-2} \int_{z=\epsilon} d^{d-2}y \sqrt{\gamma} = -\frac{1}{2} \int_{z=\epsilon} d^{d-2}y \sqrt{\gamma(0)} \frac{1}{z^2} \left[ 1 + \frac{1}{2} g^{||ij}g_{(2)ij} + \ldots \right], \tag{8.51}
\]
where $\sqrt{\gamma}$ is the induced metric on the intersection of the HRT surface with the cutoff surface, and $g_{\parallel}^{ij}$ is the part of the boundary metric parallel to the entangling surface. Using the first equality of (8.21), the finite part of the counter-term can be written

$$A_{ct, \text{finite}} = -\frac{1}{4} \int d^{d-2} y \sqrt{\gamma^{(0)}} (g^{ij} - g_{\parallel}^{ij}) \left( -\frac{1}{2} R_{ij} + \frac{1}{12} R g_{ij} \right)$$

(8.52)

Thus we find

$$A'_{ct, \text{finite}} = -\frac{1}{24} \partial_k R.$$  

(8.53)

From Eq. (8.36) we have

$$A'_{\text{reg}} = -\frac{1}{z^{d-1}} g_{ij} (\partial_z X^i) (\partial_\lambda X^j).$$

(8.54)

Expanding the above equation and using equation (8.43), we find that the non-trivial contributions are at the same order as in (8.47), which are

$$A'_{\text{reg}, \mathcal{O}(\log z)} = -\frac{1}{2} (\log z) \partial_k \left( f - \frac{R}{3} \right) = -\frac{1}{12} (\log z) \partial_k R$$

(8.56)

From this, we can infer that

$$g_{(4l)ij} k^i k^j |_{z^3 \log z} = 12 (\log z) \partial_k R,$$

(8.57)

$$g_{(0)ij} X^i_{(4l)} k^j |_{z^3} = \frac{1}{48} (\log z) \partial_k R,$$

(8.58)

$$g_{(0)ij} (\nabla_\lambda X^i_{(4l)}) k^j |_{z^3} = \frac{1}{48} (\log z) \partial_k R,$$

(8.59)

$$g_{(0)ij} (\nabla_\lambda X^i_{(4l)}) k^j |_{z^3} = \frac{1}{48} (\log z) \partial_k R.$$  

(8.60)

As expected, the $\mathcal{O}(z^4 \log z)$ terms in equation (8.47) cancel:

$$g_{(4l)ij} k^i k^j + 2 g_{(0)ij} (\nabla_\lambda X^i_{(4l)}) k^j = 0.$$  

(8.61)

And since the rate of change of the renormalized area is

$$A'_{\text{ren}} = A'_{\text{reg, finite}} + A'_{ct, \text{finite}} = -\frac{1}{z^3} \left[ \frac{1}{48} z^3 \partial_k R + g_{(0)ij} (\partial_z X^i_{(4l)}) k^j \right] - \frac{1}{24} \partial_k R,$$

(8.62)
we obtain

\[ g_{(0)ij}(\nabla_z X^i (4))k^j = z^3 \left( -A'_{\text{ren}} - \frac{1}{16} \partial_k R \right) , \quad (8.63) \]

\[ g_{(0)ij} X^i (4)k^j = \frac{A'_{\text{ren}}}{4} + \frac{1}{64} \partial_k R , \quad (8.64) \]

\[ g_{(0)ij}(\nabla_\lambda X^i (4))k^j = z^4 \left( -A''_{\text{ren}} - \frac{1}{64} \partial_k R \right) . \quad (8.65) \]

We now have all we need to evaluate the rest of the final line of equation (8.47) and derive the QNEC. Plugging (8.65) and equation (8.49) into equation (8.47), we obtain the QNEC

\[ 0 \leq T_{kk} - \frac{A''_{\text{ren}}}{8\pi G} . \quad (8.66) \]

**Proof of the \( d = 5 \) holographic QNEC**

The \( d = 5 \) case is similar. We find

\[ 0 \leq g_{ij}(\partial_{\lambda} X^i)(\partial_{\lambda} X^j) \]

\[ = g_{(0)ij}k^i k^j + g_{(2)ij}k^i k^j + 2g_{(0)ij}(\nabla_\lambda X^i (2))k^j \]

\[ + g_{(4)ij}k^i k^j + 2g_{(0)ij}(\nabla_\lambda X^i (4))k^j + g_{(5)ij}k^i k^j + 2g_{(0)ij}(\nabla_\lambda X^i (5))k^j , \quad (8.67) \]

with terms on the second line vanishing just as in section 8.3. As before, we will freely use equation (A.42) which follows from (8.17). For \( d = 5 \) we find [89, 128]

\[ g_{(4)ij}k^i k^j = \frac{z^4}{32} \partial^2_k R , \quad (8.68) \]

\[ g_{(5)ij}k^i k^j = z^5 \frac{16\pi G}{5} T_{kk} . \quad (8.69) \]

We again consider (8.50). Because the boundary dimension is odd, the counter-term

\[ A_{\text{ct}, A} = -\frac{1}{d-2} \int_{z=\epsilon} d^{d-2} y \sqrt{\gamma} = -\frac{1}{3} \int_{z=\epsilon} d^{d-2} y \sqrt{\gamma(0)} \frac{1}{z^3} \left[ 1 + \frac{1}{2} g^{|i|j} g_{(2)ij} + \ldots \right] . \quad (8.70) \]

contributes no finite part to the renormalized entropy;

\[ A_{\text{ct}, \text{finite}} = 0 . \quad (8.71) \]

From Eq. (8.36) we have

\[ A'_{\text{reg}} = -\frac{1}{z^{d-1}} g_{ij}(\partial_z X^i)(\partial_\lambda X^j) , \quad (8.72) \]
Expanding the above equation, we find that the non-trivial contributions are at the same order as above, which are

\[ A'_{\text{reg}} = -\frac{1}{z^4} g(0)_{ij} (\nabla_z X^i_{(4)}) k^j - \frac{1}{z^4} g(0)_{ij} (\nabla_z X^i_{(5)}) k^j. \]  

(8.73)

The first term on right-hand side is a divergence which must be canceled by the counter-term and the second term on right-hand side is the finite part. We introduce the following notations:

\[ A'_{\text{reg}, O(z^{-1})} = -\frac{1}{z^4} g(0)_{ij} (\nabla_z X^i_{(4)}) k^j, \]  

(8.74)

\[ A'_{\text{reg}, \text{finite}} = -\frac{1}{z^4} g(0)_{ij} (\nabla_z X^i_{(5)}) k^j. \]  

(8.75)

Using (8.19), the \( O(z^{-1}) \) divergent part of the regulated area can be computed as [136]

\[ A_{\text{reg}, O(z^{-1})} = \int d^{d-2} y \sqrt{\gamma (0)} \left( \frac{1}{2z} \left( \frac{1}{3} R_{ij} g^{kij} - \frac{5}{24} R \right) \right) \]  

\[ = \int d^{d-2} y \sqrt{\gamma (0)} \left( \frac{2}{3} f - \frac{5}{24} R \right). \]  

(8.76)

Together with the first equality in (8.21), this further implies

\[ A'_{\text{reg}, O(z^{-1})} = \frac{1}{16z} \partial_k R. \]  

(8.77)

Therefore we have

\[ g(0)_{ij} (\nabla_z X^i_{(4)}) k^j = -\frac{z^3}{16} \partial_k R, \]  

(8.78)

\[ g(0)_{ij} X^i_{(4)} k^j = -\frac{z^4}{64} \partial_k R, \]  

(8.79)

\[ g(0)_{ij} (\nabla_\lambda X^i_{(4)}) k^j = -\frac{z^4}{64} \partial^2_k R. \]  

(8.80)

Plugging equation (8.80) and (8.68) into (8.67), we find as expected that the \( O(z^4) \) terms cancel:

\[ g(4)_{ij} k^i k^j + 2g(0)_{ij} (\nabla_\lambda X^i_{(4)}) k^j = 0. \]  

(8.81)

Combining the above results yields

\[ A'_{\text{ren}} = A'_{\text{reg, finite}} + A'_{\text{ct, finite}} = -\frac{1}{z^4} g(0)_{ij} (\nabla_z X^i_{(5)}) k^j + 0, \]  

(8.82)

which implies

\[ g(0)_{ij} X^i_{(5)} k^j = -\frac{z^5}{5} A'_{\text{ren}}. \]  

(8.83)

Using (8.67), (8.69), (8.81), (8.83), we thus obtain the renormalized QNEC

\[ 0 \leq T_{kk} = \frac{1}{8\pi G} A''_{\text{ren}}. \]  

(8.84)
Killing horizons

As noted in section 8.2, a result of [175] implies the QNEC to be scheme-independent on any bifurcate Killing horizon. So for any \( d \) we would expect the holographic argument to yield a finite renormalized QNEC in this case as well.

In particular, we consider a boundary metric \( g_{(0)ij} \) with a bifurcate Killing horizon \( H_{(0)} \) generated by the Killing vector \( \xi^i_{(0)} \), i.e.

\[
\mathcal{L}_{\xi} g_{(0)ij} = 0 \tag{8.85}
\]

We evaluate the QNEC for deformations generated by \( k^i \xi^j_{(0)} \) acting on a cut \( \partial A \) of \( H_{(0)} \).

The critical fact is that, as explained above, the possible (finite or divergent) corrections to (8.31) are of the form \( Z_{ij} k^i k^j \) for some smooth (covariant) geometric tensor \( Z_{ij} \) built from the spacetime metric \( g_{(0)ij} \), the extrinsic curvature \( K^i_{jk} \), the projector \( h_{(0)ij} \) onto \( \partial A \), and their derivatives. Since \( \xi^i_{(0)} \) generates a symmetry, the quantity \( Z_{ij} \xi^i_{(0)} \xi^j_{(0)} \) is some constant \( C \) along the flow generated by \( \xi^i_{(0)} \); and thus along each generator. But \( \xi^i_{(0)} = f k^i \) for some scalar function \( f \) that vanishes on the bifurcation surface. The fact that \( Z_{ij} k^i k^j = f^{-2} C \) must be smooth and thus finite at the bifurcation surface then forces \( C = 0 \), so all possible corrections to (8.31) in fact vanish. Note that this argument relies only on the general form of the Fefferman-Graham expansion and not on the detailed equations of motion. In particular, it continues to hold in the presence of bulk higher-derivative corrections.

8.4 Discussion

We have investigated the QNEC in curved space by analyzing the scheme-independence of the QNEC and its validity in holographic field theories. For \( d \leq 3 \), for arbitrary background metric we found that the QNEC (8.1) is naturally finite and independent of renormalization scheme for points \( p \) and null congruences \( N \) for which the expansion \( \theta \) vanishes at \( p \). It is interesting that this condition is weaker than the local stationarity assumption (\( \theta|_p = \dot{\theta}|_p = 0, \sigma_{ab}|_p = 0 \)) under which the QNEC was previously proposed to hold, and it is in particular weaker than the conditions under which it can be derived from the quantum focusing conjecture [34]. But for \( d = 4, 5 \) we require local stationarity as well as the vanishing of additional derivatives as in (8.17), as well as a dominant energy condition (8.18). Under the above conditions, we also showed the universal sector of leading-order holographic theories to satisfy a finite renormalized QNEC.

The success of this derivation for \( d \leq 3 \) (using only \( \theta = 0 \)) suggests that the QNEC may hold for general field theories in contexts where it cannot be derived from the quantum focusing conjecture (QFC). If so, it would be incorrect to think of the QFC as being more fundamental than the QNEC; the QNEC seems to have a life of its own.

For \( d \geq 6 \) we argued these properties to generally fail even for weakly isolated horizons (where all derivatives of \( \theta, \sigma_{ab} \) vanish) satisfying the dominant energy condition, though they do hold on Killing horizons. The issue in \( d = 6 \) is that finite counter-terms in the
effective action can contain derivatives of the Riemann tensor, and that these terms change the definition of the entropy and stress tensor in such a way that the combination entering the QNEC is not invariant.

Our $d = 5$ argument for scheme-independence required the conditions (8.17) and (8.18), while for $d = 4$ we need only (8.18) and the first line of (8.17). It certainly appears that local stationarity is not itself sufficient for the four-derivative counter-terms in (8.16), this has been shown (using [81] and in footnote 5) only for $d \geq 5$ in the case of the Gauss-Bonnet term. For $d = 4$ the Gauss-Bonnet term gives $\Delta Q = 0$. Terms involving only the scalar curvature (i.e., the $R^2$ term) are easily handled by changing conformal frame to write the theory as Einstein-Hilbert gravity coupled to a scalar field [181]. So if the QNEC is invariant under the remaining $R_{ab}R^{ab}$ term, one would expect a useful QNEC (and thus perhaps also a useful quantum focussing condition (QFC)) to hold in $d = 4$ as well.

As explained in [34], the QNEC implies the perturbative semi-classical generalized second law (GSL) of thermodynamics at first non-trivial order in the gravitational coupling $G$. A consequence of our work is a thus proof of the (first-order) semi-classical GSL on causal horizons satisfying the conditions above, and in particular on general causal horizons for $d \leq 3$. Even at this order, this is the first GSL proof valid when the null congruence $N$ does not reduce to a Killing horizon in the background.

Now, as described in footnote 5, it was recently shown in [81] that for $d \geq 5$ the QNEC generally fails to be scheme-independent when the change of renormalization scheme entails the addition of a Gauss-Bonnet term to the action, and furthermore that the associated change $\Delta Q$ can have either sign. It then follows that (for theories that require a Gauss-Bonnet counter-term) a renormalized QNEC cannot hold in general renormalization schemes as, if one finds a finite $Q \geq 0$ with some scheme, we may always change the scheme to add a Gauss-Bonnet term so that $Q_{modified} = Q + \Delta Q_{GB} < 0$.

In a rather different direction, it was recently noted [4, 116] that on Killing horizons the quantum null energy condition is related to a property of the relative entropy $S(\rho||\sigma)$ between an arbitrary state $\rho$ and the vacuum state $\sigma$:

$$0 \leq T_{kk} - \frac{\hbar}{2\pi} S'' = S(\rho||\sigma)''$$  \hspace{1cm} (8.86)

In this equation, the derivatives of the relative entropy are the same type of local derivatives with respect to null deformations of the region that appear in $S''$. (This “concavity” property is about the second derivative, while the well-known monotonicity of relative entropy bounds the first derivative, $S(\rho||\sigma)' \leq 0$. ) While we have seen that the QNEC is not always well defined or true in curved space, the relative entropy is known to be scheme-independent. It would thus be interesting to understand if an inequality of the form $S(\rho||\sigma)'' \geq 0$ for appropriate $\sigma$ might hold more generally, perhaps even in cases where the QNEC fails. One might also investigate whether, without introducing any smearing, this could lead to a new conjecture for theories with dynamical gravity that could replace the quantum focussing condition [34] and which might hold even when the original QFC is violated [81].
Finally, we comment on the relation of the QNEC to the observation of [123] that at least the QFC violation of [81] can be avoided by taking the QFC to apply only to suitably smooth variations of the generalized entropy. One might then ask if such a “smeared QFC” could lead to a suitably smeared version of the QNEC that would hold even when the original QNEC fails. However, the original QFC reduces to the QNEC only at locally stationary points where $\theta|_p = \dot{\theta}|_p = \sigma_{ab}|_p = 0$. And the point of the averaging in [123] is precisely that, when $d > 3$ and $k^a h^b h^c h^d R_{abcd} \neq 0$ (where $k^a$ and $h^a b$ are respectively the null normal vector and the projector onto the chosen cut of the null congruence $N$), the locally stationary condition can hold only a set of measure zero. Dimensional analysis then shows that smearing out the QFC on scales long compared to the cutoff leads manifestly non-positive QFC contributions from violations of local stationarity to swamp those from failures of the QNEC. In other words, any QNEC-like inequality is irrelevant to the smeared QFC of [123] unless

$$d \leq 3 \text{ or } k^a h^b h^c h^d R_{abcd} = 0,$$

(8.87)

so that only under one of these conditions could any QNEC be derived from this smeared QFC. We therefore suspect that these are the most general conditions under which any QNEC could possibly hold, and the analysis of section 8.2 suggests that further conditions are likely required at least for $d \geq 6$. Indeed, as shown in appendix A.5, the condition (8.87) follows from (8.17), and (8.18), and then from [81] one sees that it suffices to avoid the $d \geq 5$ QNEC violation associated with the Gauss-Bonnet term. However, it remains to further investigate the effect of the $R_{ab} R^{ab}$ counter-term for both $d = 4, 5$ because (8.87) does not appear to guarantee the existence of a null congruence $N$ satisfying the sufficient conditions which we used in this paper. Similar comments must also apply to the proposed “quantum dominant energy condition” of [176], which reduces to the QNEC when considering pairs of variations that act in the same null direction.

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Appendix A

Appendices

A.1 Correlation Functions

Scalar Field

The chiral scalar operator $\partial \Phi(z)$ is a conformal primary of dimension $(h, \bar{h}) = (1, 0)$. Its two point function on the Euclidean plane is fixed by conformal symmetry up to an overall constant. We will take the following normalization:

$$\langle \partial \Phi(z) \partial \Phi(w) \rangle = -\frac{1}{(z-w)^2}. \quad (A.1)$$

The two point function on the $n$-sheeted replicated manifold is obtained by application of the conformal transformation $z \rightarrow z^n$:

$$\langle \partial \Phi(z) \partial \Phi(w) \rangle_n = -\frac{1}{n^2zw} \frac{(zw)^{1/n}}{(z^{1/n} - w^{1/n})^2}. \quad (A.2)$$

The second-derivative of this two point function under translations of the holomorphic coordinate, evaluated at $\lambda = 0$, is defined by

$$\langle \partial \Phi(z - \lambda) \partial \Phi(w - \lambda) \rangle_n'' = \langle \partial^3 \Phi(z) \partial \Phi(w) \rangle_n + \langle \partial \Phi(z) \partial^3 \Phi(w) \rangle_n + 2 \langle \partial^2 \Phi(z) \partial^2 \Phi(w) \rangle_n. \quad (A.3)$$

One can show that this combination of correlation functions can be written as

$$\frac{1}{n(zw)^2} \sum_{|q|<1} \text{sign}(q) q(q^2 - 1) \left( \frac{w}{z} \right)^q, \quad (A.4)$$

where $q$ is an integer divided by $n$. Notice that this implies that the sum vanishes for $n = 1$, as required by translation invariance.
Our convention for the only nonzero component of the stress tensor for the holomorphic sector of the theory is
\[ T(z) = -2\pi T_{zz}(z) = -\frac{1}{2} : \partial \Phi(z) \partial \Phi(z) : , \] (A.5)
where : AB : denotes the normal-ordered product. Thus using Wick’s theorem we have
\[ \langle \partial \Phi(z) \partial \Phi(w) T(0) \rangle = \frac{-1}{(zw)^2} . \] (A.6)

**Auxiliary System**
In this appendix we will evaluate the \( \theta \)-ordered correlation functions of the auxiliary system,
\[ \langle E_{ij}(\theta) E_{i'j'}(\theta') \rangle_n = \text{Tr} \left[ e^{-2\pi n K_{\text{aux}}} T [ E_{ij}(\theta) E_{i'j'}(\theta') ] \right] \right] . \] (A.7)

First, consider the case \( \theta > \theta' \):
\[ \text{Tr} \left[ e^{-2\pi n K_{\text{aux}}} E_{ij}(\theta) E_{i'j'}(\theta') \right] = e^{-2\pi n K_i} e^{(\theta - \theta') \alpha_{ij}} \delta_{ij'} \delta_{ji'} , \] (A.8)
where \( \alpha_{ij} \equiv K_i - K_j \) is the difference in two of the eigenvalues of \( K_{\text{aux}} \). For \( \theta < \theta' \), we have the opposite ordering inside the expectation value, which gives
\[ \text{Tr} \left[ e^{-2\pi n K_{\text{aux}}} E_{i'j'}(\theta') E_{ij}(\theta) \right] = e^{-2\pi n K_i} e^{(\theta - \theta' + 2\pi n) \alpha_{ij}} \delta_{ij'} \delta_{ji'} \] (A.9)

We will find it convenient to use the following complex exponential representation of \( e^{(\theta - \theta') \alpha_{ij}} \), valid for \( \theta - \theta' \in (0, 2\pi n) \):
\[ e^{(\theta - \theta') \alpha_{ij}} = \frac{1}{\pi n} \sum_p e^{-ip(\theta - \theta')} \frac{\sinh n\pi \alpha_{ij} \alpha_{ij} \alpha_{ij}}{ip + \alpha_{ij}} e^{n\pi \alpha_{ij}} . \] (A.10)

Here \( p \) is being summed over all rational numbers which are integers divided by \( n \). This can be substituted directly into (A.8). For the expectation value when \( \theta < \theta' \) given by (A.9), we can take \( \theta - \theta' + 2\pi n \) as our Fourier series variable instead of \( \theta - \theta' \), which also lies in \( (0, 2\pi n) \) in this case. This means we can substitute this into (A.10), giving the same complex exponential representation:
\[ e^{(\theta - \theta' + 2\pi n) \alpha_{ij}} = \frac{1}{\pi n} \sum_p e^{-ip(\theta - \theta')} \frac{\sinh n\pi \alpha_{ij} \alpha_{ij} \alpha_{ij}}{ip + \alpha_{ij}} e^{n\pi \alpha_{ij}} . \] (A.11)

Collecting these results, the \( \theta \)-ordered correlation function in the auxiliary system is simply
\[ \langle E_{ij}(\theta) E_{i'j'}(\theta') \rangle_n = \delta_{ij'} \delta_{ji'} e^{-2\pi n K_i} \frac{1}{\pi n \tilde{Z}_{\text{aux}}^n} \sum_p e^{-ip(\theta - \theta')} \frac{\sinh n\pi \alpha_{ij} \alpha_{ij} \alpha_{ij}}{ip + \alpha_{ij}} e^{n\pi \alpha_{ij}} , \] (A.12)

where
\[ \tilde{Z}_{\text{aux}}^n \equiv \text{Tr} \left[ e^{-2\pi n K_{\text{aux}}} \right] . \] (A.13)

Note that \( \tilde{Z}_{\text{aux}}^1 = 1 \).
A.2 Details of the Asymptotic Expansions

In this appendix we will provide a few more details about the asymptotic expansions appearing in section 3.3. Consider an Einstein-scalar field system where the scalar field $\Phi$ has mass $m^2 = \Delta(\Delta - d)$, and $\Delta$ is the dimension of the relevant boundary operator $O$. It is useful to also define $\alpha = d - \Delta$. Let us assume first that $\Delta \geq \frac{d}{2}$, so that $\alpha < \Delta$. This is the case for the standard quantization of the scalar field. Near $z = 0$, the leading part of the field is then $\Phi \sim \phi_0 z^\alpha$, where $\phi_0$ is a constant which is proportional to the coupling constant of the relevant operator. Then the Einstein equations have a solution of the form given by (3.10),

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + \left[ f(z) \eta_{ij} + \frac{16\pi G_N}{dL^{d-1}} z^d t_{ij} \right] dx^i dx^j + o(z^d) \right),$$  \hspace{1cm} (A.14)

where $f(z)$ is state-independent and has an expansion

$$f(z) = 1 + \sum_{m=2}^{m \alpha \leq d} f_{(m\alpha)} z^{m\alpha}. \hspace{1cm} (A.15)$$

Here $f_{(m\alpha)}$ is proportional to $\phi_0^m$. The minimal value $m = 2$ corresponds to the fact that, in Einstein gravity, the metric couples quadratically to $\Phi$.

It is important for this proof that all terms in the expansion of the metric and embedding functions of lower order than $z^d$ are proportional to $\eta_{ij}$ and $k^i$, respectively. For the metric, we can see this immediately from (A.14) for operators with $\Delta \geq \frac{d}{2}$. One has to be more careful in the case where $\frac{d}{2} > \alpha \geq \frac{(d - 2)}{2}$. The lower bound here represents the unitarity bound. Treatment of this case requires the alternative quantization, which means that the roles of $\alpha$ and $\Delta$ are switched [113]. In particular, it means that when we solve Einstein's equations there will be terms of order less than $z^d$ which are state-dependent:

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + \left[ \eta_{ij} + \sum_{m=2}^{2n+m\Delta \leq d} g^{(2n+m\Delta)}_{ij} z^{2n+m\Delta} \right] + \frac{16\pi G_N}{dL^{d-1}} z^d t_{ij} \right] dx^i dx^j + o(z^d) \right).$$  \hspace{1cm} (A.16)

Here the $g^{(2n+m\alpha)}_{ij}$ are built out of the expectation value of the relevant operator, $\langle O \rangle$, rather than its coupling constant. However, all is not lost. Because $\Delta > (d - 2)/2$, only the coefficients with $n = 0$ actually appear in this sum because the others are $o(z^d)$. But $g^{(m\Delta)}_{ij}$ depends only on $\langle O \rangle^m$ and not any of its derivatives (this follows from a scaling argument [103]). So $g^{(m\Delta)}_{ij} \propto \eta_{ij}$, which is what we need for the argument in the main text.

We also have to make sure that derivatives of $\langle O \rangle$ do not contaminate the expansion of the embedding functions $\tilde{X}^i$. From the equation of motion, we see that the lowest order at which $\partial \langle O \rangle$ enters the expansion of $\tilde{X}^i$ is $z^{2+2\Delta}$, but $2 + 2\Delta > d$ for $\Delta > (d - 2)/2$.

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1We are excluding operators which saturate the unitarity bound, $\Delta = (d - 2)/2$, because those are expected to be free scalars not coupled to the rest of the system.
A.3 KRS Bound

Ultimately the equivalence principle is a hypothesis, supported by a certain amount of evidence. Indeed, Strominger [162] (building on earlier work of Ashtekar [7] and others) has argued that the vacuum of asymptotically flat spacetimes, Minkowski space, is infinitely degenerate, i.e., that it corresponds to an infinite number of distinct quantum states labeled by the quantity $C_{AB}$ in Eq. (4.39).

If these states could be distinguished by any observation, empty space would contain an infinite amount of information. This would constitute a violation of the equivalence principle in its usual, classical sense: in a basis where Minkowski-like states are labeled by $C_{AB}$, they can be naturally identified with a choice of coordinates, so coordinates would be measurable. Kapec, Raclariu, and Strominger [110] (KRS) recently proposed an entropy bound that contains an extra term (denoted $X^{KRS}$ below), designed to account for this possibility.

A precise definition of the relevant entropy was not yet given in Ref. [110]. More importantly, no measurement protocol has been suggested for extracting the information contained in empty space. Such a measurement would rule out the equivalence principle experimentally. Conversely, absent experimental evidence to the contrary, we would argue that the equivalence principle should be retained: we should not consider Minkowski space written with different coordinate parameters $C_{AB}$ to be physically distinct spacetimes.

In order to facilitate further study, we will summarize our understanding of the differences between our bounds and the KRS bound. We will offer a geometric interpretation of the extra term $X^{KRS}$. We will explain why it appears in their derivation of an asymptotic bound but not in ours. We will also describe how the presence of this term conflicts with the equivalence principle.

KRS considered the asymptotic limit of bulk null hypersurfaces with approximately spherical cross-sections. This simplifies the approach to $I^+$ in spherical Bondi coordinates, compared to our use of planar light-sheets. Unlike the planar null surfaces $H(u_p)$ used above, however, existing bulk entropy bounds become divergent and hence trivial in the asymptotic limit of spherical null surfaces. Hence they cannot be used as a starting point if one wishes to work with spherical cross-sections. A new subtraction method was proposed to cancel the divergence [110]. The KRS bound is

$$S_0^{KRS}[\hat{\sigma}_2] \leq \Delta K[\hat{\sigma}_2] + X^{KRS}[\hat{\sigma}_2; C_{AB}(\infty)]$$  \hspace{1cm} (A.17)

where $u_2(\Omega)$ defines the position of the cut.\(^2\) A full definition of $S_0^{KRS}$ was left to future work, but we will argue below that the choice is tightly constrained by coordinate invariance.

Let us first consider the r.h.s. of Eq. (A.17). The first term is given by

$$\Delta K[\hat{\sigma}_2] \equiv \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} d^2\Omega \ du \ u - u_2(\Omega) \ \hat{T}.$$  \hspace{1cm} (A.18)

\(^2\)In the notation of Ref. [110], our $\hat{\sigma}_2$ is their $\Sigma$; our $\Delta K$ is $-A_F^2/4G\hbar$; and our $X^{KRS}$ is $(-A_0^2 + A_F^2)/4G\hbar$.\]
This is precisely the r.h.s. of our integrated Boundary GSL, Eq. (4.19):

\[ \hat{S}_C[\hat{\sigma}_2] \leq \Delta K[\hat{\sigma}_2] . \]  

(A.19)

However the r.h.s. of the KRS bound contains the extra term

\[ X^{KRS} \equiv -\frac{1}{8G\hbar} \int d^2\Omega \, D_A u_2(\Omega) \, D_B \bar{C}^{AB} \]  

(A.20)

\[ = \frac{1}{8G\hbar} \int d^2\Omega \, u_2(\Omega) \, D_A D_B \bar{C}^{AB} \]  

(A.21)

where

\[ \bar{C}^{AB} \equiv C^{AB}(\infty) - C_0^{AB} . \]  

(A.22)

Here \( C_0^{AB} \) refers to a fiducial choice of Bondi coordinates (or of a “late-time vacuum” in the sense of Ref. [162]) at \( u \to \infty \), whereas \( C^{AB} \) refers to the “actual” Bondi coordinates (or “late-time vacuum”) that will be attained as \( u \to \infty \). Because \( D_A D_B \bar{C}^{AB} \) is a total derivative, its average on the cut \( \hat{\sigma}_2 \) vanishes, so unless it vanishes identically, it will have indefinite sign on the sphere. It also follows that \( X^{KRS} = 0 \) if \( u_2 = \text{const} \), so the extra term only contributes if the cut has nontrivial angular dependence in the chosen coordinates.

In the bulk, \( D_A D_B \bar{C}^{AB} \) arises geometrically from a nonvanishing expansion of the null hypersurfaces at late times, which remains after the KRS regularization. Namely, the null expansion orthogonal to a surface of constant \( u, r \) in Minkowski space in the metric of Eq. (4.39) is

\[ \theta[C_{AB}] = -\frac{1}{r} - \frac{1}{2r^2} D^A D^B C_{AB} , \]  

(A.23)

so the difference between two choices \( C^{AB}, C_0^{AB} \) yields

\[ \bar{\theta}(\Omega) = -\frac{1}{2r^2} D^A D^B \bar{C}_{AB} . \]  

(A.24)

Substituting this result in Eq. (A.21), the term \( X^{KRS} \) can thus be understood as an extra area difference accumulated due to a nonzero regulated expansion \( \bar{\theta} \) of the KRS null surface at late times.

The extra term \( X^{KRS} \) was motivated in Ref. [110] by covariance of their geometric construction under BMS transformations, so it is worth explaining its absence in Eq. (A.19) and our other bounds. KRS consider a coordinate sphere at fixed \( u, r \) at late times, and construct a null hypersurface orthogonal to it. BMS transformations act nontrivially by deforming the geometry of this coordinate sphere and changing its null expansion as a function of angle. This change propagates along the entire null hypersurface and leads to an extra area difference \( X^{KRS} \) as described above.

A bulk BMS supertranslation of a given late-time cross-section of the null plane \( H(u_p) \) would yield a similar term. The null surface \( \tilde{H}(u_p) \) orthogonal to the new cross-section would be neither a light-sheet nor a causal horizon, because the expansion at late times has
the wrong sign on some generators. From this perspective, the KRS conjecture involves a modification of the nonexpansion condition of the covariant entropy bound, such that the permitted range of the (regulated) expansion depends on the late-time Bondi frame.

However, with our definition of $H(u_p)$, BMS supertranslations do not act in this way. We defined $H(u_p)$ not in terms of a given bulk cross-section, but as the boundary of the past of a point $p$ on $I^+$ [24]. BMS supertranslations can only move this point along the null geodesic generator on which it lies. The boundary of the past of any point on $I^+$ has vanishing late-time expansion and is a causal horizon. Thus, supertranslations map the set of all $H(u_p)$ to itself. Therefore they have no effect when the limit as $u_p \to \infty$ is taken, and they leave no imprint in our asymptotic entropy bounds.

We now turn to the l.h.s. of Eq. (A.17). The indefinite sign of $D_A D_B C^{AB}(\infty)$ on the sphere constrains possible definitions of $S_0^{KRS}$. It implies that $S_0^{KRS}[\sigma_2]$ cannot be unique in Minkowski space, for any nonconstant cut $\sigma_2$. In particular, it is not possible for $S_0^{KRS}$ to always vanish for arbitrary subregions of the boundary of Minkowski space regardless of the choice of coordinates.

To see this, choose asymptotic coordinates such that $\bar{C}_{AB} = \beta \tilde{C}_{AB}$ where $\tilde{C}_{AB}$ is nonvanishing and satisfies Eq. (4.41), and $\beta$ is a constant. By Eq. (A.20), $X^{KRS}$ is linear in $\beta$ so it can be made negative and arbitrarily large in magnitude by an appropriate choice of $\beta$. This would violate the KRS bound so $S_0^{KRS}[\sigma_2]$ must depend on $\bar{C}_{AB}$.

The above considerations also imply that $S_0^{KRS}$ cannot be bounded from below by the Shannon entropy—not even approximately—in the case where classical Bondi news is present.

Indeed, KRS advocate that $S_0^{KRS}$ should not be unique in Minkowski space. Rather it should contain a “soft term” that depends on $C^{AB}$ in some way, so that the KRS bound is satisfied independently of the choice of the “reference vacuum” $C^{AB}_0$.

Here we note that the only definition consistent with the equivalence principle is

$$ S_0^{KRS} \equiv \hat{S}_C + X^{KRS}, \quad (A.25) $$

where $S_C$ has no dependence on $C_{AB}$. With this choice, the $X^{KRS}$ terms would cancel, and thus, all dependence on $C_{AB}$ would drop out. Then Eq. (A.17) would reduce to Eq. (A.19). With any inequivalent definition, the physical content of Eq. (A.17) would depend on a coordinate choice.

This is because $X^{KRS}$ depends on the quantity $\bar{C}_{AB}$ defined in Eq. (A.22). We have argued in Sec. 4.3 that $C_{AB}(\infty)$ can be changed by changing the coordinate choice. Therefore, neither $C_{AB}(\infty)$ nor its difference from a fiducial value, $\tilde{C}_{AB}$, can be observable, if the equivalence principle is valid. [Note that the fiducial value $C^{AB}_0$ need not correspond to the value of $C_{AB}$ at any cut on $I^+$. If it did, $\tilde{C}_{AB}$ could be measured, and it would originate with physical radiation whose information content satisfies Eq. (A.19).]

In particular, if $S_0^{KRS}$ could be constructed entirely from observable quantities, then Eq. (A.17) could be used to constrain $\bar{C}_{AB}$, thus making it accessible to observation. This would be a problem: $\bar{C}_{AB}$ must remain unobservable by the equivalence principle, because it corresponds to a coordinate choice in Minkowski space.
In closing, we stress again that by the equivalence principle we mean the statement that empty Riemann-flat space contains no classical information. In Sec. 4.4 we showed that the bounds of Sec. 4.2 are consistent with this principle. In this appendix we have argued that the KRS bound is not consistent with it, except for a particular choice of definition of entropy under which it would reduce to Eq. (4.19). We make no claims about the compatibility of the KRS bound with any other formulation of the equivalence principle.

A.4 Single Graviton Wavepacket

In this appendix, we study the implications of asymptotic bounds in a quantum setting; we will find that in some cases they restrict the entropy more strongly than the equivalence principle did for classical waves.

We consider a classical probabilistic ensemble of single graviton wave packets, of characteristic wavelength $\lambda$ in the $u$-direction. Like the classical gravitational wave of Sec. 4.4, the wave packets shall be roughly centered on $u = 0$, and delocalized on the sphere. This is a global quantum state, defined on all of $I^+$. Any such state is orthogonal to the vacuum. Here we shall take the global state $\rho_g$ to be a mixed state with global von Neumann entropy of order unity:

$$\hat{S}_g = -\text{tr} \rho_g \log \rho_g \sim O(1).$$

For example, $\rho_g$ could be an incoherent superposition of the graviton wavepacket in two different polarization states. Alice could encode a message about the weather in the choice of polarization, and Bob could decode this message if he is able to measure the polarization.

In the region occupied by the wave packet, we have

$$N_{AB}N^{AB} \sim O\left(\frac{l_P^2}{\lambda^2}\right),$$
$$\bar{T} \sim O\left(\frac{\hbar}{\lambda^2}\right),$$
$$N_{AB} \sim O\left(\frac{l_p}{\lambda}\right),$$

where expectation value brackets are left implicit. The gravitational memory created by the wavepacket is

$$\Delta C_{AB}^\infty = \int_{-\infty}^{\infty} N_{AB} \, du \approx \int_{-\lambda}^{\lambda} N_{AB} \, du \sim O(l_P),$$

where

$$l_p \equiv \sqrt{G\hbar}$$

is the Planck length. Note that the memory is independent of $\lambda$ and so remains finite as $\lambda$ is taken large.
Boundary Quantum Bousso Bound

The Boundary QBB, Eq. (4.16), bounds the entropy on finite portions of $I^+$. This is particularly relevant to actual experiments. There are no experiments that started infinitely long ago and will complete an infinite time from now. When we measure something, we do it in finite time.

Hence, we will consider an experiment of finite duration of order $T$. It will be convenient to center this time interval near $u = 0$. Thus, we consider an observer who has access to the subregion

$$-T \lesssim u \lesssim T$$  \hspace{1cm} (A.32)

of $I^+$ (or to the subregion of the asymptotic region defined by the same range, in Bondi coordinates). It will not be important whether the cuts $\hat{\sigma}_1, \hat{\sigma}_2$ are at constant $u$.

All observables that can be measured by this observer can be computed from the reduced density operator

$$\rho_T \equiv \text{tr}_T \rho_g .$$  \hspace{1cm} (A.33)

We must also consider the global vacuum state, restricted to the observation interval:

$$\chi_T = \text{tr}_T |0\rangle \langle 0| .$$  \hspace{1cm} (A.34)

In this notation, the vacuum-subtracted entropy, Eq. (4.17), is written as

$$\hat{S}_C = -\text{tr}_T \rho_T \log \rho_T + \text{tr}_T \chi_T \log \chi_T .$$  \hspace{1cm} (A.35)

The subscript $T$ (or $\mathcal{T}$) on the trace indicates that the trace is taken over the Hilbert space factor associated with the observation interval (or its complement).
Short Observation Regime  We begin by considering the case where $\lambda \gg T$. In this regime, the observer has access to a region occupied by the graviton wavepacket, but much smaller than the wavepacket (Fig. A.1). The Boundary QBB implies

$$\hat{S}_C \lesssim O(\hat{T}T^2/\hbar) \sim O(T^2/\lambda^2)$$  \hspace{1cm} (A.36)

so the upper bound vanishes quadratically with $T/\lambda$.

To understand this result, it is instructive to return to the bulk and consider the case of a scalar field wavepacket passing through $H(u_p)$. In this setting, the entropy can be computed explicitly; and the bound has been proven [32, 31]. A beautiful explanation of the vanishing of the information content was given by Casini [45], building on pioneering work of Marolf, Minic, and Ross [129].

To an observer with access to a finite or semi-infinite region, the vacuum (restricted to this region) is a noisy state. For example, in the simplest case of a semi-infinite region (Rindler space), the restricted vacuum is a thermal state. Further restrictions only make the fluctuations larger. This means that the global vacuum restricted to the interval $(-T, T)$ is a state in which thermal-like excitations with energy up to order $\hbar/T$ are unsuppressed. This energy is larger, by a huge factor $\lambda^2/T^2$, than the total energy of the graviton in this region. This is the physical origin of Eq. (A.36): because of thermal noise, states with and without the graviton wavepacket cannot be distinguished by an observer with access to a small subregion of the wavepacket. In short, the vacuum-subtracted entropy is a physical quantity that correctly captures how much information can be gained by a given observer.

We can also shift the observation interval so that it fails to overlap with the graviton. This is analogous to a case of classical Bondi news studied in Sec. 4.4, and it gives the same result: In this case it does not matter how long or short the observation is; if it does not overlap with the news, then upper bound vanishes.

Long Observation Regime  Next, let us consider the case where the observer has access to a region that includes the whole wavepacket: $T \gg \lambda$ (Fig. A.2). In the long-observation regime, the experiment begins well before the graviton starts arriving, and ends well after. From Eq. (A.28) we see that the energy density $\hat{T}$ scales as $\lambda^{-2}$. The Boundary QBB, Eq. (4.16), evaluates to

$$\hat{S}_C \lesssim O(\hat{T}T\lambda/\hbar) \sim O\left(\frac{T}{\lambda}\right)$$  \hspace{1cm} (A.37)

as the integral has support only only on the central interval of size $O(\lambda)$ where $\hat{g} \sim O(T)$. Since we have $T \gg \lambda$, Eq. (A.37) is consistent with the ability of the observer to extract information from the graviton.

We may specify a “soft limit” of the long-observation regime as follows: Let $T = \alpha\lambda$, with $\alpha \gg 1$ fixed. Then we take $\lambda$ to become as large as we like, while the experiment always lasts longer than the wavepacket. We note that the upper bound remains fixed in this limit, at $O(\alpha) \gg 1$. We can tighten the upper bound to $O(1)$ while remaining marginally within the long-observation regime by taking $\alpha \sim 1$.  

Figure A.2: A long observation (green shaded rectangle) can distinguish the reduced graviton state from the reduced vacuum. The graviton carries information to this observer.

We can gain further intuition by returning to the bulk and considering the same graviton as it crosses a planar light-sheet $H(u_p)$. It induces focussing as $\frac{d\theta}{dw} \sim O(G\hbar/(A\lambda^2))$, were $A$ is the transverse area on which the wavepacket has support. Integrating twice along the light-rays and once transversally, we see that the area loss between the two ends of the wave packet is of order the Planck area for $\alpha \sim 1$. Thus, a single quantum induces loss of about a Planck area in planar light-sheets, independently of wavelength [26]. Hence the bound on its entropy is of order unity.

We will not try to compute the entropy of the graviton directly, but we expect it to be of order unity. To see this, let us again consider instead a scalar field wavepacket, for which the QBB has been proven [32, 31]. We understand the presence of nonzero entropy: the experiment can access the whole wavepacket, and the excitation can be distinguished reasonably well from the thermal noise that pollutes any finite-duration measurements [129, 45]. Thus as $\alpha \sim 1$, the bound becomes approximately saturated at the order-of-magnitude level.

**Boundary Generalized Second Law**

Finally, let us consider an observer with access to a semi-infinite region above some cut $\hat{\sigma}_2$ of $I^+$. The bound that applies to this case is the integrated Boundary GSL, Eq. (4.19):

$$\hat{S}_C[\hat{\sigma}_2] \leq \Delta K[\hat{\sigma}_2] \ ,$$

(A.38)

where

$$\Delta K[\hat{\sigma}_2] \equiv \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} d^2\Omega \, du \left[ u - u_2(\Omega) \right] \hat{T}$$

(A.39)
is the modular Hamiltonian.

We stressed earlier that all real experiments are finite. Nevertheless, the above bound is a useful approximation for long but finite observations: first specify the global state, which must obey fall-off conditions [51] on the news. Then restrict to an interval \((u_2, T)\) such that \(T\) lies far inside the future region with essentially no news, and consider the QBB for this interval. Since the slope of \(\hat{g}\) is unity near the lower end of the interval, and since \(\hat{S}_C\) will no longer depend on \(T\) in this regime, the Boundary QBB reduces to Eq. (A.38).

![Figure A.3: A graviton conveys \(O(1)\) information as long as it has appreciable support in the region of observation.](image)

Let us apply Eq. (A.38) to a graviton wavepacket with support in the region \((-\lambda, \lambda)\). First suppose that the cut \(\hat{\sigma}_2\) lies, say, around the center of the wavepacket, as depicted in Fig. A.3. By the previous paragraph, the results will be the same as for the QBB in the regime \(\alpha \sim 1\): the asymptotic geometry can be distinguished from Minkowski space, and the upper bound will be of order unity. On the other hand, if we shift the wavepacket so as to lie entirely prior to \(\hat{\sigma}_2\), then the upper bound vanishes.

We can also consider the differential version of the Boundary GSL, which can be written as

\[
-\frac{1}{\delta \Omega} \frac{d}{du} \hat{S}_C[\hat{\sigma}_2; \Omega] \leq \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} du \hat{T}.
\]  

(A.40)

This vanishes if the news has no support in the region above the cut \(\hat{\sigma}_2\). Thus, for the case of news that arrives entirely prior to \(\hat{\sigma}\), the upper bounds on the entropy, and on its variation under deformations of \(\hat{\sigma}\), both vanish. This is the same behavior we encountered for the classical case in Sec. 4.4.

In the case where a graviton wavepacket lies partially or entirely above the cut (Fig. A.3), we see that the derivative of \(\hat{S}_C\) is bounded by the energy of the wavepacket. This is
nonzero for any finite $\lambda$. We note that the upper bound depends on the energy, not on the gravitational memory created by the wavepacket. Therefore, the upper bound on the derivative of $\hat{S}_C$ vanishes in the soft limit as $\lambda$ becomes large, even though $\Delta C_{AB}$ remains fixed in this limit. Thus, the differential Boundary GSL implies that the variation in entropy of the region above a cut $\hat{\sigma}$, under fixed length deformations of $\hat{\sigma}$, is insensitive to the addition of gravitons of much greater wavelength.

**A.5 Non-Expanding Horizons and Weakly Isolated Horizons**

In this appendix, we study properties of non-expanding horizons and weakly isolated horizons. First, we provide geometric identities for non-expanding null surfaces. We use $k^a$ to indicate the generator of a null surface which satisfies the null condition $k^a k^a = 0$ and the geodesic equation $k^a \nabla_b k^a = 0$. We introduce an auxiliary null vector field $l^a$ satisfying $l^a l^a = 0$, $k^a l^a = -1$, and $L^a l^a = 0$. The transverse metric of the null surface is given by $h_{ab} = g_{ab} + k^a l^b + l^a k^b$. To make the notation more precise, we use the sign “$\doteq$” to denote “equal on the horizon” in this appendix and in appendix A.6.

A non-expanding null surface is defined by

$$h^c k^d \nabla_c k_d \doteq 0. \quad \text{(A.41)}$$

Substituting the definition of $h_{ab}$ into the above equation, we obtain

$$\nabla_a k_b \doteq L_a k_b + k_a R_b + B k_a k_b, \quad \text{(A.42)}$$

where $L_a \equiv -l^d \nabla_d k_a$, $R_b \equiv -l^c \nabla_c k_b$, and $B \equiv -l^d l^c \nabla_d k_c$. $L^a$, $R^a$ and $B$ satisfy relations $L^a k^a = 0$, $R^a k^a = 0$, and $L^a l^a = B = R^a l^a$. Furthermore, there is

$$\nabla_a k^a \doteq 0. \quad \text{(A.43)}$$

The extrinsic curvature, defined by

$$K^c_{ab} \doteq -h^c (h^d l^e \nabla_d h_e^c), \quad \text{(A.44)}$$

of a non-expanding null surface can always be written as

$$K^c_{ab} \doteq k_c A_{ab}, \quad \text{(A.45)}$$

where $A_{ab} \equiv -h^c (h^d l^e \nabla_d l_e^a) + L_a l_b + B k_a l_b$.

On a non-expanding null surface, the Riemann tensor contracting with a $k^a$ can be written in terms of $L^a$, $R^a$, and $B$ as

$$R_{abck} \equiv 2 \nabla_{[a} \nabla_{b]} k_c$$

$$\doteq 2k_c \nabla_{[a} L_{b]} + 2R_c k_{[a} R_{b]} + 2k_{[b} \nabla_{a]} R_c + 2k_c k_{[b} \nabla_{a]} B + 2B k_c k_{[a} R_{b]} + 2B k_{[b} L_{a]} k_c. \quad \text{(A.46)}$$
From the above equation, one immediately obtains

\[ R_{akck} = -L_bk_ck_aR^b_c - k^bL_aR^c - k^bL_cR^a - k^bL_c k_a \nabla_b B, \quad (A.47) \]
\[ R_{bk} = k^a \nabla_a L_b + k_b \nabla_a R^a + k_b L^a R_a + k_b \partial_k B, \quad (A.48) \]
\[ R_{kk} = 0. \quad (A.49) \]

The above results are for non-expanding horizons in general. By further imposing the dominant energy condition (8.18), one observes special properties of weakly isolated horizons. First, by noticing that equation (A.49) implies that vector \( R_{akb} \) must be a vector tangent to the horizon and equation (8.18) implies that \( R_{akb} \) can be written as a term proportional to \( k^a \) plus a causal vector, one concludes that for weakly isolated horizons there is

\[ R_{akb} \hat{=} f k^a. \quad (A.50) \]

Second, we argue that on weakly isolated horizons the Weyl tensor is Petrov type II. To prove this, we only need to show that for weakly isolated horizons there is

\[ C_{abcd} k^a h^b h^c h^d f \hat{=} 0, \quad (A.51) \]
\[ C_{abcd} k^a h^b h^c h^d g \hat{=} 0. \quad (A.52) \]

Equation (A.50) and equation

\[ R_{abcd} = C_{abcd} + \frac{2}{d-2} (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) - \frac{2}{(d-1)(d-2)} R g_{a[c} g_{d]b} \quad (A.53) \]

together imply that

\[ C_{abcd} k^a k^c h^b h^d f \hat{=} R_{abcd} k^a k^c h^b h^d f, \quad (A.54) \]
\[ C_{abcd} k^a h^b h^c h^d g \hat{=} R_{abcd} k^a h^b h^c h^d g. \quad (A.55) \]

Equation (A.46) implies that the right-hand side of equation (A.55) vanishes and equation (A.47) implies that the right-hand side of equation (A.54) vanishes. Therefore, equations (A.51) and (A.52) do hold and the Weyl tensor is indeed Petrov type II on weakly isolated horizons. This leads us to conclude that on weakly isolated horizons we have

\[ C_{abcd} k^b k^d \hat{=} \hat{\zeta} k_a k_c, \quad (A.56) \]

which, together with (A.53) and (A.50), further leads to

\[ R_{abcd} k^b k^d \hat{=} \zeta k_a k_c. \quad (A.57) \]

The contracted Bianchi identity \( \nabla_a R_{bcd} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0 \) provides a relationship among \( \zeta, f \), and the spacetime scalar curvature \( R \):

\[ \frac{1}{2} \partial_k R \hat{=} \partial_k f \hat{=} - \partial_k \zeta. \quad (A.58) \]
These features guide one to conclude $k^a \nabla_a L_b \propto k_b$ and $k^b \nabla_b R_c \propto k_c$. Therefore, on weakly isolated horizons, the Riemann tensor $R_{abck}$ can be written as $R_{abck} \hat{=} k_c \tilde{A}_{ab} + k_b [\tilde{B}_{ab} c]$, where $\tilde{A}_{ab}$ is antisymmetric and satisfies $k^a \tilde{A}_{ab} \propto k_b$ and $\tilde{B}_{ab}$ satisfies $k^a \tilde{B}_{ab} \propto k_b$ and $k^b \tilde{B}_{ab} \propto k_a$. This guarantees more proportion relations, including $R_{ac} R_{abck} \propto k_b$, $R_{bcd} k R_{cdbe} \propto k_e$, $R_{fkec} R_{eafk} \propto k_a k_c$, and $R_{abck} R_{abdk} \propto k_c k_d$.

These relations are crucial to the scheme-independence of QNEC for $d = 4, 5$. However, in those cases we do not require them to hold on the entire horizon, but only to the appropriate order in $\lambda$ about the point $p$. As a result, it suffices to impose only (8.17).

### A.6 Scheme Independence of the QNEC

In this appendix, we derive the scheme independence of the QNEC for four-derivative counter-terms and six-derivative counter-terms built from arbitrary polynomial contractions of the Riemann tensor. Equations (A.42), (A.50), and (A.57) are our three inputs. As we have mentioned at the end of appendix A.5, with these three assumptions the Riemann tensor $R_{abck}$ can be written in the form $R_{abck} \hat{=} k_c \tilde{A}_{ab} + k_b [\tilde{B}_{ab} c]$, where $\tilde{A}_{ab}$ is antisymmetric and satisfies $k^a \tilde{A}_{ab} \propto k_b$ and $\tilde{B}_{ab}$ satisfies $k^a \tilde{B}_{ab} \propto k_b$ and $k^b \tilde{B}_{ab} \propto k_a$. Moreover, the contracted Bianchi identity implies that equation (A.58) holds on the horizon. We introduce notations $g^\perp_{ab} \equiv -k_a l_b - l_a k_b$ and $\epsilon_{ab} \equiv -k_a l_b + l_a k_b$, where the meaning of $l^a$ has been explained at the beginning of appendix A.5.

In principle, to calculate the gravitational entropy associated with the type of counter-terms considered in this appendix, we need to use Dong’s entropy formula [54]. However, it is easy to see that the $S'_{ct}$ from Dong’s entropy formula reduces to that computed from Wald:1984rg’s formula when (8.17) and (8.18) hold. Therefore, in this appendix, we compute the gravitational entropy from the formula

$$S'_{ct} = \partial_k \left( -2 \pi \epsilon_{ab} \epsilon_{cd} \frac{\partial L_{ct}}{\partial R_{abcd}} \right).$$

(A.59)

Furthermore, the $kk$-component of the stress tensor associated with these counter-terms is given by the standard functional derivative formula

$$T_{kk, ct} = k^b k^d - \frac{2}{\sqrt{-g}} \frac{\delta I_{ct}}{\delta g^{bd}}.$$

(A.60)

The change of the QNEC associated with these counter-terms is thus given by

$$Q_{ct} = T_{kk, ct} - \frac{1}{2 \pi} S''_{ct}.$$

(A.61)

### Four-derivative counter-terms

We now consider the three possible four-derivative counter-terms,

$$I_1 = \int d^4 x \sqrt{-g} R^{abcd} R_{abcd}, \quad I_2 = \int d^4 x \sqrt{-g} R^{ab} R_{ab}, \quad \text{and} \quad I_3 = \int d^4 x \sqrt{-g} R^2.$$  

(A.62)
For the counter-term $I_3$, the entropy is

$$S'_{ct3} = \partial_k (8\pi R),$$

so that

$$\frac{1}{2\pi} S''_{ct3} = 4\partial^2_k R.$$ (A.64)

To compute the stress tensor term we write

$$\delta I_3 = \int d^4x \sqrt{-g} 2R \left( \nabla^d \nabla^b \delta g_{bd} - g^{bd} \nabla^c \nabla_c \delta g_{bd} \right),$$ (A.65)

$$T_{kk, ct3} = k^b k^d \frac{-2}{\sqrt{-g}} \frac{\delta I_3}{\delta g^{bd}} = 4k^b k^d \nabla_b \nabla_d R = 4\partial^2_k R,$$ (A.66)

so the QNEC remains unchanged

$$\Delta Q_{ct3} = T_{kk, ct3} - \frac{1}{2\pi} S''_{ct3} = 0.$$ (A.67)

For $I_2$, we find

$$S'_{ct2} = \partial_k \left( 4\pi g^{1ab} R_{ab} \right) = \partial_k \left( -8\pi R_{kl} \right) = \partial_k \left( 8\pi f \right),$$ (A.68)

$$\frac{1}{2\pi} S''_{ct2} = 4\partial^2_k f,$$ (A.69)

and

$$\delta I_2 = \int d^4x \sqrt{-g} 2R^{ab} \left( -\frac{1}{2} g^{cd} \nabla_a \nabla_b \delta g_{cd} - \frac{1}{2} g^{cd} \nabla_c \nabla_d \delta g_{ab} + g^{cd} \nabla_c \nabla_b \delta g_{ad} \right),$$ (A.70)

$$T_{kk, ct2} = k^a k^b \frac{-2}{\sqrt{-g}} \frac{\delta I_2}{\delta g^{ab}} = -2k^a k^b \nabla_c \nabla^c R_{ab} + 4k^a k^b \nabla_c \nabla_b R^c_a = 4\partial^2_k f.$$ (A.71)

So again we find

$$\Delta Q_{ct2} = T_{kk, ct2} - \frac{1}{2\pi} S''_{ct2} = 0.$$ (A.72)

For $I_1$, we have

$$S'_{ct1} = \partial_k \left( 8\pi g^{1ac} g^{1bd} R_{abcd} \right) = \partial_k \left( -16\pi R_{klk} \right) = \partial_k \left( -16\pi \zeta \right),$$ (A.73)

$$\frac{1}{2\pi} S''_{ct1} = -8\partial^2_k \zeta,$$ (A.74)

and

$$\delta I_1 = \int d^4x \sqrt{-g} 2R^{abcd} \left( -2\nabla_a \nabla_c \delta g_{bd} \right),$$ (A.75)

$$T_{kk, ct1} = k^b k^d \frac{-2}{\sqrt{-g}} \frac{\delta I_1}{\delta g^{bd}} = -8k^b k^d \nabla^a \nabla^a R_{abcd} = -8\partial^2_k \zeta,$$ (A.76)

and again

$$\Delta Q_{ct1} = T_{kk, ct1} - \frac{1}{2\pi} S''_{ct1} = 0.$$ (A.77)

These results are summarized in table 8.1.
Six-derivative counter-terms

We now consider six-derivative counter-terms built from arbitrary polynomial contractions of the Riemann tensor. These terms are listed in [143]. They are

\[ I_1 = \int d^dx \sqrt{-g} R^{abcd} R_{cdef} R_{ef}^{ab}, \]
\[ I_2 = \int d^dx \sqrt{-g} R^{ab} R_{cd}^{ef} R^{ce}_{bf} R^{df}_{ae}, \]
\[ I_3 = \int d^dx \sqrt{-g} R^{abcd} R_{edbe} R_{c}^{e} a, \]
\[ I_4 = \int d^dx \sqrt{-g} R^{ab} R_{cdeb} R_{abcd}, \]
\[ I_5 = \int d^dx \sqrt{-g} R^{abcd} R_{ac} R_{bd}, \]
\[ I_6 = \int d^dx \sqrt{-g} R^{ab} R_{bc} R_{c}^{a}, \]
\[ I_7 = \int d^dx \sqrt{-g} R^{ab} R_{ab}, \]
\[ I_8 = \int d^dx \sqrt{-g} R^{3}. \]

For \( I_8 \), we find

\[ S'_{ct8} = \partial_k (12\pi R^2), \]  
\[ \frac{1}{2\pi} S''_{ct8} = 6\partial_k^2 R^2, \]

and

\[ \delta I_8 = \int d^dx \sqrt{-g} 3R^2 \left( \nabla^d \nabla^b \delta g_{bd} - g^{bd} \nabla^c \nabla_c \delta g_{bd} \right), \]

\[ T_{kk, ct8} = k^b k^d \frac{-2}{\sqrt{-g}} \partial_k^3 I_3 = k^b k^d (6) \nabla_b \nabla_d R^2 = 6\partial_k^2 R^2, \]

and thus

\[ \Delta Q_{ct8} = T_{kk, ct8} - \frac{1}{2\pi} S''_{ct8} = 0. \]

For \( I_7 \), we find

\[ S'_{ct7} = \partial_k \left[ -2\pi \left( -2R^{ab} R_{ab} - 4f R \right) \right], \]
\[ \frac{1}{2\pi} S''_{ct7} = 2\partial_k^2 (R^{ab} R_{ab}) + 4\partial_k^2 (f R), \]

and

\[ \delta I_7 = \int d^dx \sqrt{-g} R^{ab} R_{ab} \left( \nabla^d \nabla^c \delta g_{cd} - g^{cd} \nabla^e \nabla_e \delta g_{cd} \right) \]
\[ + \int d^dx \sqrt{-g} 2R \nabla^{ab} \left( -\frac{1}{2} g^{cd} \nabla_a \nabla_b \delta g_{cd} - \frac{1}{2} g^{cd} \nabla_c \nabla_d \delta g_{ab} + g^{cd} \nabla_c \nabla_b \delta g_{ad} \right), \]

\[ T_{kk, ct7} = k^e k^d \frac{-2}{\sqrt{-g}} \partial_k^4 I_7 \]
\[ = 2k^e k^d \nabla_c \nabla_d \left( R^{ab} R_{ab} \right) - 2k^e k^d \nabla_c \nabla^e \left( RR_{cd} \right) + 4k^e k^d \nabla_b \nabla_d \left( RR_{c}^{b} \right) \]
\[ = 2\partial_k^2 (R^{ab} R_{ab}) + 4\partial_k^2 (f R), \]
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and thus
\[ \Delta Q_{ct7} = T_{kk, ct7} - \frac{1}{2\pi} S'_{ct7} = 0. \]  
(A.88)

For \( I_6 \), we have
\[ S'_{ct6} = \partial_k \left( 6\pi R_{bc} R^c_a g^{ab} \right) = -\partial_k \left( 12\pi R_{kc} R^c_i \right) = \partial_k \left( 12\pi f^2 \right), \]  
(A.89)
\[ \frac{1}{2\pi} S''_{ct6} = 6\partial^2_k \left( f^2 \right), \]  
(A.90)

and
\[ \delta I_6 = \int d^d x \sqrt{-g} 3 R^{bc} R^a_c \left( -\frac{1}{2} g^{cd} \nabla_a \nabla_b \delta g_{cd} - \frac{1}{2} \nabla^c \nabla_c \delta g_{ab} + \nabla^d \nabla_b \delta g_{ad} \right), \]  
(A.91)
\[ T_{kk, ct6} = k^a k^b - \frac{2}{\sqrt{-g}} \frac{\delta I_6}{\delta g^{ab}} = -3 k^a k^b \nabla_c \nabla^c (R_{bc} R^c_a) + 6 k^a k^b \nabla_c \nabla_b (R^c_e R_{ea}) \]  
(A.92)
\[ = 6\partial^2_k \left( f^2 \right), \]  
(A.93)

and so
\[ \Delta Q_{ct6} = T_{kk, ct6} - \frac{1}{2\pi} S''_{ct6} = 0. \]  
(A.94)

To deal with counter-term \( I_5 \), not first that \( R_{kb} \hat{=} \hat{f} k_b \) implies \( R^a_c R_{abck} \hat{=} \hat{f} k_b \) and that contracting both sides with \( l^b \) then gives the function \( \hat{f} \hat{=} - R^a_c R_{alc} \). Thus we find
\[ R^a_c R_{abck} \hat{=} - k_b R^a_c R_{alc}. \]  
(A.95)

We may now compute
\[ S'_{ct5} = \partial_k \left( 2\pi g^{1/2} R^a_c R^{bd} + 4\pi g^{1/2} R^{abcd} R_{ac} \right) = \partial_k \left( 4\pi f^2 - 8\pi R^a_c R_{alc} \right), \]  
(A.96)
\[ \frac{1}{2\pi} S''_{ct5} = 2\partial^2_k \left( f^2 \right) - 4\partial^2_k \left( R^a_c R_{alc} \right), \]  
(A.97)

and
\[ \delta I_5 = \int d^d x \sqrt{-g} R^a_c R^{bd} \left( -2 \nabla_a \nabla_c \delta g_{bd} \right) \]  
(A.98)
\[ + \int d^d x \sqrt{-g} R^{abcd} R_{ac} \left( -\frac{1}{2} g^{ef} \nabla_b \nabla_a \delta g_{ef} - \frac{1}{2} \nabla^e \nabla_e \delta g_{bd} + \nabla^f \nabla_a \delta g_{bf} \right), \]  
(A.99)
\[ T_{kk, ct5} = k^c k^d - \frac{2}{\sqrt{-g}} \frac{\delta I_5}{\delta g^{bd}} = -4 k^b k^d \nabla^e \nabla^a R_{a[c} R_{b|d]} - 2 k^b k^d \nabla^e \nabla_e (R_{abcd} R^a_c) + 4 k^b k^f \nabla^d \nabla_f (R_{abcd} R^a_c) \]  
(A.99)
\[ = 2 k^b k^d \nabla^e \nabla^a R_{ad} R_{bc} + 4 k^b k^f \nabla^d \nabla_f (R_{abcd} R^a_c) \]  
(A.99)
\[ = 2\partial^2_k \left( f^2 \right) - 4\partial^2_k \left( R_{alc} R^a_c \right), \]  
(A.99)
which together imply
\[ \Delta Q_{ct5} = T_{kk, ct5} - \frac{1}{2\pi} S''_{ct5} = 0. \] (A.99)

For \( I_4 \), we find
\[ S'_{ct4} = \partial_k \left[ -2\pi \left( -2R^{abcd} R_{abcd} + 8\zeta R \right) \right], \]  (A.100)
\[ \frac{1}{2\pi} S''_{ct4} = 2\partial_k^2 \left( R^{abcd} R_{abcd} \right) - 8\hat{\partial}_k^2 (\zeta R), \]  (A.101)
and
\[ \delta I_4 = \int d^d x \sqrt{-g} R^{abcd} R_{abcd} \left( \nabla^c \nabla^e \delta g_{ef} - g^{ef} \nabla^a \nabla_g \delta g_{ef} \right) \] 
\[ + \int d^d x \sqrt{-g} 2R^{abcd} \left( -2\nabla_a \nabla_c \delta g_{bd} \right), \]  (A.102)
\[ T_{kk, ct4} = k^b k^d \frac{-2}{\sqrt{-g}} \delta I_4 \] 
\[ = 2k^c k^d \nabla_c \nabla_f \left( R^{abcd} R_{abcd} \right) - 8k^b k^d \nabla_c \nabla^a \left( RR_{abcd} \right) \] 
\[ = 2\partial_k^2 \left( R^{abcd} R_{abcd} \right) - 8\hat{\partial}_k^2 (R \zeta), \]  (A.103)
so that
\[ \Delta Q_{ct4} = T_{kk, ct4} - \frac{1}{2\pi} S''_{ct4} = 0. \] (A.104)

For counter-term \( I_3 \), note that \( R_{akck} = \zeta k_a k_c \) implies \( R_k^{bcd} R_{cdbe} = \tilde{\zeta} k_e \). Contracting both sides with \( l^e \) then gives the function \( \tilde{\zeta} = -R_k^{bcd} R_{cdbl} \). Thus we find
\[ R_k^{bcd} R_{cdbe} = -k_e R_k^{bcd} R_{cdlb}. \] (A.105)

We may now compute
\[ S'_{ct3} = \partial_k \left( -8\pi R^{cdbe} R_e^a g_{[a|c]}^+ g_{b|d}^+ + 2\pi R^{abcd} R_{cde} g_{e[a]}^+ \right) \] 
\[ = \partial_k \left( -16\pi R_{klke}^k R_e^a - 4\pi R_k^{bcd} R_{cdlb} \right) \] 
\[ = \partial_k \left( -16\pi \zeta f - 4\pi R_k^{bcd} R_{cdlb} \right), \]  (A.106)
\[ \frac{1}{2\pi} S''_{ct3} = -8\hat{\partial}_k^2 (\zeta f) - 2\hat{\partial}_k^2 \left( R_k^{bcd} R_{cdlb} \right), \]  (A.107)
and
\[ \delta I_3 = \int d^d x \sqrt{-g} 2R^{cdbe} R_e^a \left( -2\nabla_a \nabla_c \delta g_{bd} \right) \] 
\[ + \int d^d x \sqrt{-g} R^{abcd} R_{cde} \left( -\frac{1}{2} g^{fg} \nabla_a \nabla_c \delta g_{gf} - \frac{1}{2} \nabla_f \nabla^f \delta g_{ea} + \nabla^f \nabla_e \delta g_{af} \right), \]  (A.108)
\[ T_{k k, c t 3} = k^b k^d \frac{-2}{\sqrt{-g}} \frac{\delta I_3}{\delta g^{bd}} = -8 k^b k^d \nabla_c \nabla_a (R^{c de b} R^{e a}) - k^e k^d \nabla^f \nabla_f (R^{a b c d e} R^{c de b}) \]

\[ = -8 k^b k^d \nabla^f \nabla_f (R^{a b c d e} R^{c de b}) \]

Putting these together yields

\[ \Delta Q_{c t 3} = T_{k k, c t 3} - \frac{1}{2 \pi} S''_{c t 3} = 0. \] (A.109)

For counter-term \( I_2 \), notice that \( R_{a k c c} = \zeta A_k A_c \) implies \( R^e_{a f k} R^{e f}_{c c} = \zeta A_k A_c \). Contracting both sides with \( l^a l^c \) gives the function \( \bar{\zeta} = R^e_{a f k} R^{e f}_{c c} \). One then finds

\[ R^e_{a f k} R^{e f}_{c c} = \zeta A_k A_c R^e_{a f k} R^{e f}_{c c}. \] (A.110)

With this in hand, we calculate

\[ S''_{c t 2} = \partial_k \left( -12 \pi g^{l \perp a} g^{l \perp b} R^{e c b f} R^{d e f}_{a e} \right) \]

\[ = \partial_k \left[ -12 \pi \left( R^{f}_{k e l} R^{f}_{l f k} - R^{f}_{l e l} R^{f}_{f k k} \right) \right] \]

\[ = \partial_k \left[ -12 \pi \left( R^{f}_{k e l} R^{f}_{l f k} - \zeta^2 \right) \right], \]

\[ = \frac{1}{2 \pi} S''_{c t 2} = -6 \partial_k^2 \left( R^{f}_{k e l} R^{f}_{l f k} - \zeta^2 \right), \] (A.112)

and

\[ \delta I_2 = \int d^d x \sqrt{-g} 3 R^{e f g h}_{e f g h} R^{a b c}_{a b c} \delta g^{a b} (-2 \nabla_a \nabla_c \delta g^{a b}), \] (A.113)

\[ T_{k k, c t 2} = k^b k^d \frac{-2}{\sqrt{-g}} \frac{\delta I_2}{\delta g^{bd}} \]

\[ = -12 k^b k^d \nabla_c \nabla_a \left( R^{e f g h}_{e f g h} R^{a b c}_{a b c} \right) \]

\[ = -6 \partial_k^2 \zeta + 6 \partial_k^2 \zeta^2 \]

\[ = -6 \partial_k^2 \zeta + 6 \partial_k^2 \zeta^2 \] (A.115)

The result is then

\[ \Delta Q_{c t 2} = T_{k k, c t 2} - \frac{1}{2 \pi} S''_{c t 2} = 0. \] (A.116)
For the final counter-term $I_1$, notice that $R_{a k c k} = \zeta_{k a k c}$ implies $R_{a b}^{ab} R_{a b d k} = \zeta_{c k d}$. Contracting both sides with $l^c l^d$ gives the function $\zeta = R_{l k}^{ab} R_{a b d k}$. Thus we find

$$R_{a b}^{ab} R_{a b d k} = \zeta_{c k d} R_{l k}^{ab} R_{a b d k}. \quad \text{(A.117)}$$

This gives

$$S'_{c t 1} = \partial_k \left(-12\pi R^{c d e f} R_{e f}^{a b} g_{[a c]}^{\perp} g_{b d}^{\perp} \right)$$

$$= \partial_k \left(-24\pi R_{l k}^{e f} R_{e f l k} \right), \quad \text{(A.118)}$$

$$\frac{1}{2\pi} S''_{c t 1} = -12\partial_k^2 \left(R_{l k}^{e f} R_{e f l k} \right), \quad \text{(A.119)}$$

and

$$\delta I_1 = \int d^d x \sqrt{-g} 3 R^{c d e f} R_{e f}^{a b} \left(-2\nabla_a \nabla_c \delta g_{b d} \right), \quad \text{(A.120)}$$

$$T_{k k, c t 1} = k^b k^d \frac{-2}{\sqrt{-g}} \frac{\delta I_1}{\delta g^{b d}}$$

$$= -12 k^b k^d \nabla_c \nabla_a \left( R^{c d e f} R_{e f}^{a b} \right)$$

$$= -12 \nabla_c \nabla^a \left(R_{c k}^{e f} R_{e f a k} \right)$$

$$= -12 \partial_k^2 \left(R_{l k}^{e f} R_{e f l k} \right). \quad \text{(A.121)}$$

The result is then once again that

$$\Delta Q_{c t 1} = T_{k k, c t 1} - \frac{1}{2\pi} S''_{c t 1} = 0. \quad \text{(A.122)}$$

The above results are summarized in table 8.2.
Bibliography


