Independence Relations in Theories with the Tree Property

by

Gwyneth Fae Harrison-Shermoen

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Committee in charge:

Professor Thomas Scanlon, Chair
Professor Leo Harrington
Professor John Steel
Associate Professor Antonio Montalbán
Professor Martin Olsson

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Abstract

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This thesis investigates theories with the tree property and, in particular, notions of independence in such theories. We discuss the example of the two-sorted theory of an infinite dimensional vector space over an algebraically closed field and with a bilinear form (which we refer to as $T_\infty$), examined by Granger in [9]. Granger notes that there is a well-behaved notion of independence, which he calls $\Gamma$-non-forking, in this theory, and that it can be viewed as the limit of the non-forking independence in the theories of its finite dimensional subspaces, which are $\omega$-stable. He defines a notion of an “approximating sequence” of substructures, and shows that $\Gamma$-non-forking in a model of $T_\infty$ corresponds to “eventual” non-forking in an approximating sequence. We generalize his notion of approximation by substructures, and in the case of a theory whose large models can be approximated in this way, define a relation $\downarrow_{\lim}$, which is the “limit” of the non-forking independence in the theories of the approximating substructures. We show that if the approximating substructures have simple theories, $\downarrow_{\lim}$ satisfies invariance, monotonicity, base monotonicity, transitivity, normality, extension, finite character, and symmetry. Under certain additional assumptions, $\downarrow_{\lim}$ also satisfies anti-reflexivity and the Independence Theorem over algebraically closed sets.

We also consider the two-sorted theory of infinitely many cross-cutting equivalence relations, $T^*_\text{feq}$. We give a proof, explaining in detail the argument of Shelah and Usvyatsov for Theorem 2.1 in [23], that $T^*_\text{feq}$ does not have SOP$_2$ (equivalently, TP$_1$). The argument makes use of a theorem of Kim and Kim, from [14], along with several other lemmas involving tree indiscernibility.
To my parents.
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Chapter 1

Introduction

Classification theory is a far-reaching project within model theory, originating with the work of Morley in the 1960s and Shelah in the 1970s. Its goal is, roughly, to categorize first-order theories based on how much variation there is among their models and, in the other direction, to recover structural information about the models of a theory from facts about the theory. One measure of the “amount of variation among models” is the number of types consistent with the theory.

Definition 1.0.1. A first-order theory, $T$, is stable if there is some infinite cardinal $\kappa$ such that for all models $\mathcal{M} \models T$, subsets $A$ of $M$ with $|A| \leq \kappa$, and $n < \omega$, the number of complete $n$-types in $\mathcal{M}$ with parameters from $A$ is no more than $\kappa$, i.e.

$$|S^n_\kappa(A)| \leq \kappa.$$ 

A rich theory has been developed around the notion of stability, and as a result, stable theories are relatively well understood. There are many mathematical structures with unstable theories, however, and so there has been a concerted effort to generalize various results of stability theory in several different directions (for example, to simple theories, theories without the independence property, and rosy theories).

One of the aspects of stable theories that pops up in many unstable theories is the existence of a well-behaved independence relation, similar to linear independence in vector spaces. (In fact, the abstract notion of independence generalizes both linear independence in vector spaces and algebraic independence in fields, as well as many other geometric notions of independence.) Essentially, given sets $A$, $B$, and $C$ of elements in whatever structure we are working in, $A$ is “independent” from $B$ over $C$ if $B \cup C$ has no more “information” about $A$ than $C$ does. (In the case of a vector space over a fixed field, “$A$ is independent from $B$ over $C$” boils down to $\text{span}(A \cup C) \cap \text{span}(B \cup C) = \text{span}(C)$.) Independence relations give us a way to determine what behavior is “generic,” even in theories with many models.

In particular, the relation of non-forking independence satisfies, in stable theories, many of the properties that are desirable for an independence relation. In the late 1990s, Byunghan Kim showed ([12], [13]) that non-forking independence satisfies almost all of the same
properties in simple theories, and, in fact, that a theory is simple if and only if non-forking independence is a symmetric relation. He and Anand Pillay showed in [16] that if a theory has an abstract independence relation satisfying a certain set of axioms, then that theory is simple and the independence relation is non-forking independence. In more recent years, model theorists have discovered various theories that are not simple but still have a nice notion of independence. Since these theories are not simple, the independence relation(s) in question cannot satisfy all the axioms satisfied by non-forking independence in a simple theory, but they may satisfy almost all of them. One example of such a theory is the theory of $\omega$-free pseudo algebraically closed fields, studied by Chatzidakis in [4], [3], and [2]. (Chatzidakis actually identifies several distinct notions of independence, satisfying different subsets of the properties of non-forking independence in a simple theory.) Another example is the two-sorted theory of an infinite dimensional vector space over an algebraically closed field and with a bilinear form, studied by Granger in his thesis [9].

In the case of the infinite dimensional vector spaces with a bilinear form, the independence relation that Granger singles out (which he refers to as “$\Gamma$-non-forking”) can be viewed as the limit of the non-forking independence in the finite dimensional subspaces, whose theories are stable. There is a sense in which the finite dimensional subspaces collectively “approximate” the infinite dimensional vector space. In chapter 3, I generalize this notion of approximation by substructures, and define an independence relation that is the limit of the non-forking independence in the theories of the approximating substructures.

**Definition 1.0.2.** Given a theory $T$, a model, $M$, of $T$, and a directed system, $H$, of substructures $N$ of $M$, we define the following notion of independence: $A \downarrow_{C}^{\lim} B$ ($A$ is limit independent from $B$ over $C$) if for each finite subset $A_0$ of $A$, there is $N_{A_0} \in H$ such that $A_0 \subseteq N_{A_0}$ and for all $N_{A_0} \subseteq N \in H$, $A_0 \downarrow_{\text{acl}^N(M)(C) \cap N}^{N} \text{acl}^N(B \cap N)$, where $\downarrow^N$ is non-forking independence as computed in $N$, and acl is the model-theoretic notion of algebraic closure. We say that $H$ approximates $M$ if the following conditions hold:

- $H$ covers $M$.
- $H$ is closed under automorphism.
- Any sentence that is true in $M$ is “eventually true” in $H$.
- Every formula either “eventually forks” in $H$ or “eventually does not fork” in $H$.
- If $A \not\downarrow_{C}^{\lim} B$, this dependence is witnessed by the “eventual forking” of some formula.
- (optional) Algebraically closed sets in $M$ are “eventually algebraically closed” in $H$.
- (optional) For each $N \in H$, $a, b \in N$, and $A \subseteq M$, if $\text{tp}^M(a/A) = \text{tp}^M(b/A)$, then $\text{tp}^N(a/A \cap N) = \text{tp}^N(b/A \cap N)$. 
I then show that if the theories of the approximating substructures are simple, “limit independence” satisfies many of the properties satisfied by non-forking independence in simple theories.

**Theorem 1.0.3.** Given a $\kappa$-saturated, strongly $\kappa$-homogeneous model $M$ of $T$ approximated by $H$, where each $N \in H$ has a simple theory, $\lim^{\kappa}$ has the following properties: automorphism invariance, transitivity, finite character, symmetry, existence, and extension. If $H$ satisfies the optional conditions and in the theories of its elements, non-forking independence satisfies the Independence Theorem over algebraically closed sets, then $\lim^{\kappa}$ satisfies anti-reflexivity and the Independence Theorem over algebraically closed sets.

This thesis is structured as follows: in the remainder of this chapter, we present our notational conventions and necessary background from stability and simplicity theory, as well as some definitions and results of a combinatorial nature on “tree indiscernibility.” Chapter 2 examines the two-sorted theory of infinitely many cross-cutting equivalence relations ($T^*_{feq}$). Following Shelah and Usvyatsov [23], we give a proof that $T^*_{feq}$ does not have the tree property of the first kind (making heavy use of the tree indiscernibility material from this chapter). In Chapter 3, we introduce the example of the infinite dimensional bilinear spaces, define notions of “approximation” and “limit independence,” and prove Theorem 1.0.3. We then look at the relationship between Granger’s $\Gamma$-non-forking ($\lim^{\Gamma}$) and limit independence, and see to what extent $\lim^{\kappa}$ generalizes $\Gamma$.

### 1.1 Notation and conventions

We assume knowledge of basic model theory, at the level of chapters 1 through 5 of [17]. Given a complete theory $T$, we work in a large, $\kappa$-saturated, strongly $\kappa$-homogeneous (for some large regular cardinal $\kappa$) model $\mathfrak{C}$ of $T$. We consider only models $M$ of $T$ of cardinality less than $\kappa$ and such that $M \prec \mathfrak{C}$. (This is without loss of generality, as every model of $T$ of cardinality less than $\kappa$ can be elementarily embedded into $\mathfrak{C}$.) We consider only sets $A \subset \mathfrak{C}$ of cardinality less than $\kappa$. Hence, we may write acl($A$) or $\models \varphi(a)$ without specifying a model.

Usage of lowercase and uppercase Greek and Latin letters varies somewhat with context, but the most common (and default) uses are as follows. Uppercase $A$, $B$, $C$, and $D$ denote sets, while $\mathcal{M}$ and $\mathcal{N}$ denote models with universes $M$ and $N$, respectively. $\mathcal{L}$ is a first-order (usually countable) language, and $T$ is a first-order theory. $R$ is a relation symbol, $E$ an equivalence relation, and $P$ a unary predicate. $I$ and $J$ are index sets. $K$ is a field ($L$ and $F$ may be, as well). Concerning (lowercase) Greek letters, $\lambda$ and $\kappa$ are infinite cardinals, $\varphi$ and $\psi$ (and occasionally $\theta$ and $\chi$) are formulae, and $\eta, \nu$ are finite sequences of natural numbers (i.e. initial segments of tree branches). The lowercase letters $a, b, c, d$ are elements (or tuples) of the ambient structure; $f, g$, and $h$ are functions; $i, j, k, l$ (or $\ell$), $m$, and $n$ are natural numbers; $p$ and $q$ are types. The letters $r, s$, and $t$ are occasionally types, but are
more often natural numbers, and \( u, v, w, x, y, z \) are variables. In languages with multiple sorts, we may use uppercase letters \( X, Y, \) and \( Z \) as variables, too.

The symbols \( x, a, \) and so on, may denote tuples of variables and parameters - in general, we do not distinguish a tuple of length greater than 1 by use of an overline. We shall warn the reader of exceptions as they arise. We denote the length of the tuple \( a \) by \( \lg(a) \). (That is, if \( a = (a_0, \ldots, a_{n-1}) \), \( \lg(a) = n \).) We abuse notation by writing \( a \in A \) (where \( a \) is a tuple of length \( n \)) rather than \( a \in A^n \).

We often denote a union by concatenation, e.g. we may write \( BC \) for \( B \cup C \). We use \( \acl(A) \) to denote the model-theoretic algebraic closure of \( A \), and \( K^\text{alg} \) to denote the field-theoretic algebraic closure of \( K \). An automorphism over \( A \) or \( A \text{-automorphism} \) is an automorphism of \( \mathcal{C} \) (or perhaps some smaller model, where specified) fixing \( A \) pointwise. \( B \equiv_A C \) means that \( \text{tp}(B/A) = \text{tp}(C/A) \) and hence, by the strong \( \kappa \)-homogeneity of \( \mathcal{C} \), that there is a \( A \text{-automorphism} \) of \( \mathcal{C} \) sending \( B \) to \( C \). If \( \Delta \subseteq \mathcal{L}, B \equiv_\Delta C \) means that \( B \) and \( C \) have the same \( \Delta \)-types. We may use \( (\exists^{=1} x)\varphi(x) \) to abbreviate
\[
\exists x (\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x)),
\]
(and similarly with \( (\exists^n x)\varphi(x) \), for \( n > 1 \).

We work quite a bit with trees in Section 1.5 and Chapter 2. The set of functions \( f : \omega \rightarrow \omega, \omega^\omega \), can be viewed as an infinitely branching tree of height \( \omega \). Hence, \( \beta \in \omega^\omega \) is a branch from such a tree, and \( \alpha \in \downarrow \omega \omega \) is an initial segment of such a branch, that is, \( \beta \upharpoonright k \) for some \( \beta \in \omega^\omega, k < \omega \). We also work with finitely-branching trees \( \omega^q \) for \( q < \omega \), especially \( \omega^2 \). In all cases, we denote the tree order by \( \preceq \): that is, \( \eta \preceq \nu \) if there are \( k \leq \ell \in \omega \) and \( \beta \in \omega^\omega \) (respectively, \( \beta \in \omega^2 \)) such that \( \eta = \beta \upharpoonright k \) and \( \nu = \beta \upharpoonright \ell \).

### 1.2 Tree properties, and the SOP hierarchy

In the effort to push the results of stability theory beyond the boundaries of stable theories, Shelah has identified a number of properties that an unstable theory might have (or lack). First of all, a theory is unstable if and only if it has at least one of the independence property (that is, there are a formula \( \varphi(x; y) \) and sequences \( \langle a_i : i < \omega \rangle, \langle c_\sigma : \sigma \in 2^\omega \rangle \) such that \( \models \varphi(a_i; c_\sigma) \) if and only if \( \sigma(i) = 0 \)) or the strict order property \([19]\) (that is, there are a formula \( \varphi(x, y) \) and a sequence \( \langle b_i : i < \omega \rangle \) such that \( \models \forall x (\varphi(x; b_i) \rightarrow \varphi(x; b_j)) \) if and only if \( i < j \)). In this thesis, we shall focus primarily on unstable theories without the strict order property. In this section, we introduce a number of properties weaker than the strict order property. All definitions are due to Shelah.

**Definition 1.2.1.** A set of formulae is \( k \)-inconsistent if every subset of size \( k \) is inconsistent. That is, a set of formulae \( \{ \psi_i(x) : i \in I \} \) (possibly with parameters) is \( k \)-inconsistent if for any \( i_0, \ldots, i_{k-1} \in I \), \( \models \neg \exists x (\bigwedge \psi_{i_j}(x)) \).

**Definition 1.2.2 ([20], Definition 0.1).** 1. A theory \( T \) has the tree property if there are a formula \( \varphi(x; y), k < \omega, \) and sequences \( a_\eta \in \mathcal{C} (\eta \in <\omega \omega) \) such that:
• for any $\eta \in <\omega \omega$, $\{\varphi(x, a_{\eta-l}) : l < \omega\}$ is $k$-inconsistent, but
• for every $\beta \in \omega \omega$, $\{\varphi(x, a_{\beta|n}) : n < \omega\}$ is consistent.

2. A theory $T$ is simple if it does not have the tree property.

Remark 1.2.3. The semicolon in $\varphi(x; y)$ above is significant: it separates the object variables $(x)$ from the parameter variables $(y)$. Instances of $\varphi$ are formulae $\varphi(x; b)$, in which the parameter variables have been replaced by parameters.

The tree property is a robust dividing line among unstable theories: the class of simple theories is exactly the class of theories in which non-forking independence is symmetric, as we shall see in Section 1.4. Simple theories can also be characterized by having an independence relation (to be defined in Section 1.3) satisfying a certain type amalgamation property (analogous to stationarity of types in stable theories). The theories of the random graph and of pseudo-finite fields are two examples of simple unstable theories.

In [19], Shelah proves that if a theory has the tree property, then it has one of two “extreme” tree properties. The tree property specifies that instances along a branch are consistent, while instances at nodes that are siblings are inconsistent. It leaves open the consistency of instances at incomparable, non-sibling nodes. It is therefore natural to consider tree properties that do make a decision on the consistency of these instances.

Definition 1.2.4. 1. A theory $T$ has the $k$-tree property of the first kind ($k$-TP$_1$) if there are a formula $\varphi(x; y)$ and tuples $\{a_\alpha : \alpha \in <\omega \omega\}$ such that:

- for $\alpha_0, \ldots, \alpha_{k-1} \in <\omega \omega$ pairwise incomparable, $\{\varphi(x, a_{\alpha_i}) : 1 \leq i \leq k\}$ is inconsistent, but
- for $\beta \in \omega \omega$, $\{\varphi(x, a_{\beta|n}) : n \in \omega\}$ is consistent.

2. A theory has TP$_1$ if it has 2-TP$_1$.

3. A theory is called NTP$_1$ or subtle if it does not have TP$_1$.

The two-sorted theory of infinitely many cross-cutting equivalence relations, $T_{\text{eq}}^*$ (to be discussed extensively in Chapter 2), is NTP$_1$ [23] but not simple. Recently, Chernikov has identified a sufficient condition for a theory to be NTP$_1$, and used it to show that both the two-sorted theory of infinite dimensional vector spaces over algebraically closed fields with a bilinear form (described in Section 3.1) and the theory of $\omega$-free PAC fields of characteristic 0 are NTP$_1$ [5]. This makes use of results from [3], [2], [9], and [6]. Each of these theories is not simple, and so by Theorem 1.2.7 below, has the tree property of the second kind.

Definition 1.2.5. A theory $T$ has the tree property of the second kind (TP$_2$) if there are a formula $\varphi(x; y)$, tuples $\{a_\alpha : \alpha \in <\omega \omega\}$, and $k < \omega$ such that:

- for any $\alpha \in <\omega \omega$, $\{\varphi(x; a_{\alpha-i}) : i < \omega\}$ is $k$-inconsistent, but
• for any \( n \) and any \( \alpha_0, \ldots, \alpha_{n-1} \in <\omega \omega \), no two of which are siblings, \( \{ \varphi(x; a_\alpha) : i < n \} \) is consistent.

A theory without TP\(_2\) is called NTP\(_2\).

Remark 1.2.6. TP\(_2\) is equivalent to the following condition (which is more commonly given as the definition of TP\(_2\)): there are a formula \( \varphi(x; y) \) and tuples \( \{ a_{i,j} : i,j < \omega \} \) such that

• for any \( f : \omega \to \omega \), \( \{ \varphi(x; a_{i,f(i)}) : i < \omega \} \) is consistent, but

• for all \( i < \omega \), \( \{ \varphi(x; a_{i,j}) : j < \omega \} \) is \( k \)-inconsistent for some \( k \).

Various theories of valued fields - for example, any theory of an ultraproduct of \( p \)-adics - are NTP\(_2\) but not simple [6].

Theorem 1.2.7 ([19], Theorem III.7.11). If a theory \( T \) has the tree property, then either \( T \) has TP\(_1\) or \( T \) has TP\(_2\).

Definition 1.2.8 ([22], Definitions 2.1, 2.2, and 2.5). 1. A theory \( T \) has the strict order property if some formula \( \varphi(x,y) \) (where \( x \) and \( y \) are tuples of the same length) defines, in some model \( M \models T \), a partial order with infinite chains.

2. A (complete) theory \( T \) has the strong order property (SOP) if there is some sequence \( \varphi = \langle \varphi_n(x^n,y^n) : n < \omega \rangle \) of formulae such that for every \( \lambda \),
   a) \( \lg(x^n) = \lg(y^n) \) are finite, and \( x^n \) (respectively \( y^n \)) is an initial segment of \( x^{n+1} \) (respectively \( y^{n+1} \));
   b) \( T \cup \{ \varphi_{n+1}(x^{n+1},y^{n+1}) \} \vdash \varphi_n(x^n,y^n) \);
   c) for \( m \leq n \), \( \neg(\exists x_0, \ldots, x_{n+1})[\bigwedge \{ \varphi_n(x^n,k,n) : k = \ell + 1 \bmod m \}] \) belongs to \( T \);
   d) there is a model \( M \) of \( T \) and \( a^n_\alpha \in M \) (of length \( y^n \), for \( n < \omega, \alpha < \lambda \)) such that
      \( a^n_\alpha = a^n_{\alpha+1} \bmod \lg(y^n) \) and \( M \models \varphi_n[a^n_\alpha, a^n_\beta] \) for \( n < \omega \) and \( \alpha < \beta < \lambda \).

3. A theory \( T \) has the \( n \)-stronger order property (SOP\(_n\)) if there are a formula \( \varphi(x,y) \) (where \( x \) and \( y \) are tuples of the same length), a model \( M \) of \( T \), and tuples \( \{ a_k \in M^{\lg(x)} : k < \omega \} \) such that
   • \( M \models \varphi(a_k, a_m) \) for \( k < m < \omega \), and
   • \( M \models \neg \exists x_0, \ldots, x_{n-1}(\bigwedge \{ \varphi(x_l,x_k) : l, k < n \text{ and } k = l + 1 \bmod n \}) \).

Definition 1.2.9 ([7], Definition 2.2). 1. \( T \) has SOP\(_2\) if there are a formula \( \varphi(x,y) \) and tuples \( a_\eta \) for \( \eta \in <\omega 2 \) such that
   • for every \( \rho \in <\omega 2 \), the set \( \{ \varphi(x, a_{\rho[n]} : n < \omega \} \) is consistent, while
   • if \( \eta, \nu \in <\omega 2 \) are incomparable, \( \{ \varphi(x,a_\eta), \varphi(x,a_\nu) \} \) is inconsistent.
2. $T$ has SOP$_1$ if there are a formula $\varphi(x, y)$ and tuples $a_\eta$ for $\eta \in \omega^2$ such that
   - for $\rho \in \omega^2$ the set $\{\varphi(x, a_\rho(n)) : n < \omega\}$ is consistent, but
   - if $\nu \prec \langle 0 \rangle \subseteq \eta \in \omega^2$, then $\{\varphi(x, a_\nu), \varphi(x, a_{\nu \langle 1 \rangle})\}$ is inconsistent.

**Fact 1.2.10.** A theory $T$ has TP$_1$ if and only if it has SOP$_2$. (See, e.g., [14].)

The various notions defined in this section are related as follows: the strict order property implies the strong order property (SOP), and in turn,

$$\text{SOP} \Rightarrow \ldots \Rightarrow \text{SOP}_{n+1} \Rightarrow \text{SOP}_n \Rightarrow \ldots \Rightarrow \text{SOP}_3 \Rightarrow \text{SOP}_2(\Leftrightarrow \text{TP}_1) \Rightarrow \text{SOP}_1 \Rightarrow \text{TP}$$

It is unknown whether the implications SOP$_3 \Rightarrow$ SOP$_2$ and SOP$_2 \Rightarrow$ SOP$_1$ are strict. All of the other implications (i.e. strict order property $\Rightarrow$ SOP, SOP $\Rightarrow$ SOP$_n$, SOP$_{n+1} \Rightarrow$ SOP$_n$ for $n \geq 3$, and SOP$_1 \Rightarrow$ TP) are strict. See [22] for more details.

### 1.3 Independence

As mentioned above, notions of independence provide us with a measure of genericity. There is some disagreement about which properties should be required of a ternary relation in order for it to be called an independence relation. Below we give Adler’s definition from [1]. In general, we shall use the phrase notion of independence as a catch-all term for a relation satisfying a “reasonable” number of the axioms below.

**Definition 1.3.1.** An independence relation is a ternary relation (which we denote by $\downarrow$) on small sets satisfying the following axioms:

- **invariance:** If $A \downarrow_C B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \downarrow_C B'$.
- **monotonicity:** If $A \downarrow_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \downarrow_C B'$.
- **base monotonicity:** Suppose $D \subseteq C \subseteq B$. If $B \downarrow D A$ and $C \downarrow D A$, then $B \downarrow_D A$.
- **transitivity:** Suppose $D \subseteq C \subseteq B$. If $B \downarrow_C A$ and $C \downarrow_D A$, then $B \downarrow_D A$.
- **normality:** $A \downarrow_C B$ implies $AC \downarrow_C B$.
- **extension:** If $A \downarrow_C B$ and $B' \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \downarrow_C B'$.
- **finite character:** If $A_0 \downarrow_C B$ for all finite $A_0 \subseteq A$, then $A \downarrow_C B$.
- **local character:** For every $A$ there is a cardinal $\kappa(A)$ such that for any set $B$ there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \downarrow_C B$.

An independence relation is strict if it also satisfies anti-reflexivity:

$$a \downarrow_B a \text{ implies } a \in acl(B).$$
Recalling the slogan from the beginning of this chapter - that $A$ should be independent from $B$ over $C$ just in case $B \cup C$ has no more “information” about $A$ than $C$ has on its own - many of these axioms should seem intuitive.

**Remark 1.3.2.** Some additional properties one might ask a notion of independence to satisfy are:

- **existence:** For any $A$, $B$, and $C$ there is $A' \equiv_C A$ such that $A' \downarrow_C B$.

- **symmetry:** $A \downarrow_C B$ if and only if $B \downarrow_C A$.

- **strong finite character:** If $A \downarrow_C B$, then there are finite tuples $a \in A$, $b \in B$, and $c \in C$, and a formula $\varphi(x, y, z)$ without parameters such that
  
  - $\models \varphi(a, b, c)$, and
  - $a' \downarrow_C b$ for all $a'$ such that $\models \varphi(a', b, c)$.

- **stationarity:** If $a \downarrow_C B$ and $B \subseteq B'$, there is $a' \equiv_{BC} a$ such that $a' \downarrow_C B'$, and if $d \downarrow_C B'$ and $d \equiv_{BC} a$, then $d \equiv_{BC} a'$. (A relation with the property of “stationarity over algebraically closed sets” is one that satisfies the property of stationarity when $C$ is algebraically closed.)

- **independence theorem over a model:** If $a \equiv_M b$, $a \downarrow_M A$, $b \downarrow_M B$ (for $M$ a model and $A$, $B \supseteq M$), and $A \downarrow_M B$, then there is $c$ such that $c \equiv_A a$, $c \equiv_B b$, and $c \downarrow_M AB$. (One should read this as a weakening of the previous property. Roughly, if $\downarrow$ satisfies stationarity, independent extensions of types are unique. If $\downarrow$ satisfies the independence theorem (over a model), independent extensions (of types over models) can be amalgamated.)

In the next section, we shall see that the syntactic relations of non-forking and non-dividing are notions of independence (and, in certain theories, satisfy all of the properties mentioned here). For now, we give a few concrete examples from [1].

**Example 1.3.3.** 1. The trivial relation holding of all triples is an independence relation (as defined above), but not a strict independence relation.

2. In the theory of an equivalence relation with infinitely many infinite classes (and no finite classes), the relation given by

$$A \downarrow_C B \text{ if and only if } A \cap B \subseteq C$$

is a strict independence relation.

3. The relation given by

$$A \downarrow^a_C B \text{ if and only if } \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$$
satisfies base monotonicity if and only if whenever \( A, B, C \) are algebraically closed sets such that \( B \supseteq C, B \cap \text{acl}(AC) = \text{acl}((B \cap A)C) \). In this case, it is a strict independence relation.

4. In the theory of the random graph, the relation given by
\[
A \downarrow_{C} B \text{ if and only if } A \cap B \subseteq C \text{ and there is no edge from } A \setminus C \text{ to } B \setminus C
\]
satisfies all the axioms of a strict independence relation except for local character.

1.4 Forking

In this section, we discuss the concepts of forking and dividing, and some of the properties they satisfy, paying particular attention to the behaviour of forking and dividing in simple theories. We give proofs of several easy facts, as this may provide the reader with a better sense of dividing and forking, which can be difficult to grasp. The reader should assume that these facts have already been proven many times, by many people, even where no specific reference is given.

**Definition 1.4.1** ([19], Chapter III, Definitions 1.3-1.4).
- A formula \( \varphi(x;b) \) *divides over* \( A \) if there are \( \{b_i : i < \omega\} \) and \( k < \omega \) such that
  - \( \text{tp}(b_i/A) = \text{tp}(b/A) \) for all \( i < \omega \), and
  - \( \{\varphi(x;b_i) : i < \omega\} \) is \( k \)-inconsistent.
- A type \( p(x) \) *divides over* \( A \) if there is a formula \( \varphi(x;b) \) such that \( p(x) \vdash \varphi(x;b) \) and \( \varphi(x;b) \) divides over \( A \).
- A type \( p(x) \) *forks over* \( A \) if there are formulae \( \varphi_0(x;a_0), \ldots, \varphi_{n-1}(x;a_{n-1}) \) such that
  1. \( p(x) \vdash \bigvee_{i<n} \varphi_i(x;a_i) \), and
  2. \( \varphi_i(x;a_i) \) divides over \( A \) for each \( i \).

**Definition 1.4.2.** A sequence \( (a_i : i < \alpha) \) (where for all \( i, j, \log(a_i) = \log(a_j) \)) is *indiscernible over* \( A \) or *\( A \)-indiscernible* if for every \( n < \omega \) and every \( i_0 < \ldots < i_{n-1}, j_0 < \ldots < j_{n-1} \),
\[
\text{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/A) = \text{tp}(a_{j_0}, \ldots, a_{j_{n-1}}/A).
\]

**Remark 1.4.3** ([19], Lemma III.1.1(3)). In the first part of Definition 1.4.1, we could have taken \( \{b_i : i < \omega\} \) to be an indiscernible sequence. In that case, the second condition is equivalent to
\[
\{\varphi(x;b_i) : i < \omega\} \text{ is inconsistent.}
\]
Both non-dividing and non-forking can be viewed as notions of independence. We write $a \downarrow^d B$ for “$\text{tp}(a/BC)$ does not divide over $C$” and $a \downarrow^f C B$ for “$\text{tp}(a/BC)$ does not fork over $C$.”

**Fact 1.4.4.** For a set $C$, parameters $b \notin C$ and $c \in C$, formulae $\psi(x,y,z), \varphi(x,w) \in \mathcal{L}$, and $n < \omega$, if $\psi(x,b,c)$ divides over $C$, then the following formula does, as well:

$$(\exists^n x)\varphi(x,w) \land \exists x(\varphi(x,w) \land \psi(x,b,c)).$$

*Proof.* By the definition of dividing and Remark 1.4.3, we have an infinite, $C$-indiscernible sequence $\langle b_i c_i : i \in I \rangle = \langle b_i c_i : i \in I \rangle$, with $b_0 c_0 = bc$ (since $c \in C$, $c_i = c$ for all $i \in I$) and such that

$$\{\psi(x,b_i,c) : i \in I\}$$

is inconsistent. We show that

$$\{(\exists^n x)\varphi(x,y) \land \exists x(\varphi(x,y) \land \psi(x,b_i,c)) : i \in I\}$$

is inconsistent. If not, let $d$ be a realization. That is, for each $i \in I$,

$$\models (\exists^n x)\varphi(x,d) \land \exists x(\varphi(x,d) \land \psi(x,b_i,c))$$

Suppose $\{a_0, \ldots, a_{n-1}\}$ is the set defined by $\varphi(x,d)$. Then for each $i \in I$, there is $j < n$, such that

$$\models \psi(a_j, b_i, c).$$

It follows that for at least one $j < n$, there is an infinite subset $I' \subseteq I$ such that

$$\models \psi(a_j, b_i, c)$$

for all $i \in I'$.

However, there is some $k < \omega$ such that $\{\psi(x,b_i,c) : i \in I\}$ is $k$-inconsistent, so in particular, $\{\psi(x,b_i,c) : i \in I'\}$ is $k$-inconsistent. Contradiction. 

**Corollary 1.4.5.** If $A \downarrow^d C B$, then $\text{acl}(A) \downarrow^d C B$.

*Proof.* We prove the contrapositive. If $\text{acl}(A) \not\downarrow^d C B$, then there is some formula

$$\psi(x,b,c) \in \text{tp}(\text{acl}(A)/BC)$$

(where $\psi(x,y,z) \in \mathcal{L}$, $b \in B$, and $c \in C$) that divides over $C$. Let $a' \in \text{acl}(A)$ realize $\psi(x,b,c)$. Since $a' \in \text{acl}(A)$, there are $n < \omega$, $\varphi(x,w) \in \mathcal{L}$, and $a \in A$ such that

$$\varphi(x,a) \land (\exists^n x)\varphi(x,a) \in \text{tp}(a'/A).$$

It follows that

$$(\exists^n x)\varphi(x,w) \land \exists x(\varphi(x,w) \land \psi(x,b,c)) \in \text{tp}(a/BC).$$

By fact 1.4.4 this formula divides over $C$, so $A \downarrow^d C B$. 

\[\square\]
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Fact 1.4.6. Dividing over $A$ and forking over $A$ are preserved under $A$-automorphism.

Proof. This is clear from the definitions. \hfill \square

Fact 1.4.7. If $\psi(x,d)$ forks over $C$, and $d_0, \ldots, d_{n-1}$ are $C$-conjugates of $d$, then $\bigvee_{i<n} \psi(x,d_i)$ forks over $C$.

Proof. This follows from the definition and Fact 1.4.6. \hfill \square

Fact 1.4.8. If $d \in \text{acl}(B)$, $\varphi(y,b)$ isolates $\text{tp}(d/B)$ (for $\varphi(y,z) \in \mathcal{L}$ and $b \in B$), and $\psi(x,d)$ forks over $C \subseteq B$, then $\exists y(\varphi(y,b) \land \psi(x,y))$ forks over $C$.

Proof. By assumption, there are $d_0 = d, d_1, \ldots, d_{n-1}$ such that $\{d_i : i < n\} = \varphi(C,b)$, and so

$$\exists y(\varphi(y,b) \land \psi(x,y)) \vdash \bigvee_{i} \psi(x,d_i).$$

By the fact that for $i, j < n$, $\text{tp}(d_i/B) = \text{tp}(d_j/B)$, strong $\kappa$-homogeneity of $C$, and Fact 1.4.7, $\exists y(\varphi(y,b) \land \psi(x,y))$ forks over $C$. \hfill \square

Corollary 1.4.9 ([19], Lemma III.6.3(3)). If $A \downarrow^f_C B$, then $A \downarrow^f_C \text{acl}(B)$.

Fact 1.4.10. An infinite, $A$-indiscernible sequence is $\text{acl}(A)$-indiscernible.

Fact 1.4.11. $A \downarrow_C B$ if and only if $A \downarrow_{\text{acl}(C)} B$ (for $\downarrow = \downarrow^d$ or $\downarrow = \downarrow^f$).

Proposition 1.4.12 ([12], Proposition 2.1). If $T$ is simple, $\varphi(x,a)$ divides over $A$ if and only if $\varphi(x,a)$ forks over $A$.

Proposition 1.4.13 (Properties of forking independence). Let $T$ be a simple theory. The independence relation given by $A \downarrow^f_C B$ if $\text{tp}(A/BC)$ forks over $C$ satisfies the axioms of existence, anti-reflexivity, monotonicity, base monotonicity, finite character, local character, extension, symmetry, and transitivity.

Proof. Anti-reflexivity, monotonicity, base monotonicity, and finite character follow from the definition of forking (and are true in any theory). Existence follows from the equivalence of forking and dividing in a simple theory. Local character is proved in [20], Claim 4.2. Extension is proved in [19], chapter III, Theorem 1.4. Symmetry and transitivity are proved in [12], Theorems 2.5 and 2.6, respectively. \hfill \square

Remark 1.4.14. If $T$ is simple, we shall write $\downarrow$ for forking (equivalently, dividing) independence.

Remark 1.4.15. In a simple theory, Corollary 1.4.5 can be strengthened to the following: if $A \downarrow_C B$, then $\text{acl}(A) \downarrow_C \text{acl}(B)$. This is by the equivalence of forking and dividing and symmetry of $\downarrow$. 

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The next two theorems give us ways to identify simple theories by looking at the behaviour of either non-forking independence or an abstract independence relation.

**Theorem 1.4.16** ([13], Theorem 2.4). *Let $T$ be arbitrary. The following are all equivalent.*

1. $T$ is simple.
2. Forking (Dividing) satisfies local character.
3. Forking (Dividing) satisfies symmetry.
4. Forking (Dividing) satisfies transitivity.

**Remark 1.4.17.** The equivalence of $T$’s simplicity and local character of forking was proved by Shelah in [20] (Claim 4.2 and Theorem 4.3).

**Theorem 1.4.18** (Characterization of Simplicity: [16], Theorem 4.2). *Let $T$ be an arbitrary theory. Then $T$ is simple if and only if $T$ has a notion of independence $\downarrow^\circ$ which satisfies the Independence Theorem over a model. Moreover, if $\downarrow^\circ$ is such, then for all $a,B,C$, $a \downarrow^\circ_C B$ if and only if $tp(a/BC)$ does not fork over $C$ (namely, $\downarrow^\circ$-non-forking coincides with non-forking).*

**Remark 1.4.19.** In [16], the authors take ‘notion of independence’ to mean ternary relation satisfying invariance, monotonicity, base monotonicity, local character, finite character, extension, symmetry, and transitivity.

An analogous result for stable theories was proved by Harnik and Harrington in 1984. Rephrasing to match the terminology of the previous theorem, the result states:

**Theorem 1.4.20** ([10], Theorem 5.8). *Given a theory $T$ and a notion of independence $\downarrow^\circ$, if $\downarrow^\circ$ satisfies invariance, monotonicity, base monotonicity, transitivity, extension, local character, and stationarity over algebraically closed sets, then $T$ is stable and $\downarrow^\circ$ is non-forking independence.*

### 1.5 Tree indiscernibility

In general, it is not easy to prove directly that a theory does not have a tree property (of any kind). The problem becomes somewhat more tractable if we may assume certain facts about the tree. In particular, it is helpful to assume that the tree in question has some level of *indiscernibility*. There are several kinds of tree indiscernibility, but the idea behind all of them is that given two configurations of nodes on the tree that “look alike,” the parameters decorating the nodes of those two configurations should have the same type. The differences among the various notions of tree indiscernibility come from the different interpretations of what it means for two tree configurations to “look alike.” (For comparison, in the case of an indiscernible *sequence*, two sets of elements from the sequence “look alike” if they have the same order type.) For an in-depth discussion of “generalized indiscernibles,” see [18].
Remark 1.5.1 (Notation). 1. In this section, we will make use of overlines to denote tuples of nodes, and tuples of the tuples decorating those nodes. For example, \( \eta \) denotes, for some \( n < \omega \), a sequence \((\eta_0, \ldots, \eta_{n-1})\), and \( \overline{a_\eta} \) denotes \((a_{\eta_0}, \ldots, a_{\eta_{n-1}})\). Note that each \( a_{\eta_i} \) may itself be a tuple, but we shall follow the same convention we use elsewhere, and not use an overline in this case. For \( \eta = (\eta_0, \ldots, \eta_{n-1}) \) and \( i < n \), we write \( \eta_i \in \eta \).

2. For a single node \( \eta \), we will use \(|\eta|\) to refer to the height of the node. That is, if \( \eta \) is a node in a \( q \)-branching tree, \( |\eta| = k \) if \( \eta \in \langle \eta \rangle^k \).

The following definition is taken from [14], although an equivalent version of parts 1 through 3 appeared in [7] several years earlier. As the authors of [14] note, the terminology of part 4 of this definition is due to Scow [18].

**Definition 1.5.2.**

1. A tuple \( \eta \in \langle \omega \rangle^q \) is \( \cap \)-closed if for each \( \eta_i, \eta_j \in \eta \), \( \eta_i \cap \eta_j \in \eta \), too.

2. Given tuples \( \eta, \nu \in \langle \omega \rangle^q \), \( \eta \approx_1 \nu \) if:
   - \( \eta \) and \( \nu \) are \( \cap \)-closed tuples of the same arity, and
   - \( \forall i, j < \lg(\eta) \) and \( \forall t < q \),
     - \( \eta_i \leq \eta_j \) iff \( \nu_i \leq \nu_j \), and
     - \( \eta_i^\cap(t) \leq \eta_j \) iff \( \nu_i^\cap(t) \leq \nu_j \).

We shall read \( \eta \approx_1 \nu \) as “\( \eta \) is 1-similar to \( \nu \).”

3. We say \( \langle a_\eta | \eta \in \langle \omega \rangle^q \rangle \) is 1-fti (or 1-fully tree indiscernible) if for all \( \eta, \nu \in \langle \omega \rangle^q \), \( \eta \approx_1 \nu \) implies \( \overline{a_\eta} \equiv \overline{a_\nu} \).

4. We say a sequence \( \langle a_\eta | \eta \in \langle \omega \rangle^q \rangle \) is 1-modeled by a sequence \( \langle b_\eta | \eta \in \langle \omega \rangle^q \rangle \) if, for any \( d < \omega \), finite set \( \Delta(x_0, \ldots, x_{d-1}) \) of \( L \)-formulae, and \( \cap \)-closed tuple \( \eta = (\eta_0, \ldots, \eta_{d-1}) \in \langle \omega \rangle^q \), there exists \( \nu \in \langle \omega \rangle^q \) such that \( \eta \approx_1 \nu \) and \( \overline{b_\eta} \equiv_\Delta \overline{a_\nu} \).

**Remark 1.5.3.** The concept of 1-modeling allows us to “transform” one tree into another tree that is 1-fti (see Kim and Kim’s Theorem 1.5.6, below). Importantly, this transformation preserves the property of being SOP \(_2\) (Remark 1.5.7, below), so if we are given an SOP \(_2\) tree, we can find a 1-fti SOP \(_2\) tree.

Before stating the results that will be important in Chapter 2, we give a few examples of the properties defined above.

**Example 1.5.4.** Let \( \eta = (\langle 1 \rangle, \langle 101 \rangle, \langle 1001 \rangle) \). The tuple \( \eta \) is not \( \cap \)-closed, because \( \langle 101 \rangle \cap \langle 1001 \rangle = \langle 10 \rangle \notin \eta \).
Example 1.5.5. Let

\[ \eta = (\langle \rangle, \langle 0 \rangle, \langle 001 \rangle, \langle 01 \rangle, \langle 111 \rangle) \]
\[ \nu = (\langle 1 \rangle, \langle 101 \rangle, \langle 1010 \rangle, \langle 10110 \rangle, \langle 110 \rangle) \]

Then \( \eta \approx_1 \nu \).

Theorem 1.5.6 ([14], Proposition 2.3). Any sequence \( \langle \alpha_\eta | \eta \in \omega^\omega \rangle \) can be 1-modeled (in a sufficiently saturated model) by some 1-fti sequence \( \langle b_\eta | \eta \in \omega^\omega \rangle \).

Remark 1.5.7 (See also [15], Remark 5.2, for the \( \omega \)-branching case). If \( \langle a_\eta | \eta \in \omega^2 \rangle \) witnesses SOP\(_2\) for a formula \( \psi(x,y) \) and \( \langle b_\eta | \eta \in \omega^2 \rangle \) 1-models \( \langle a_\eta | \eta \in \omega^2 \rangle \) in some model \( M \), then \( \langle b_\eta | \eta \in \omega^2 \rangle \) is also an SOP\(_2\) tree for \( \psi(x,y) \).

Proof. 1. First we show that for any \( \beta \in \omega^2 \), \( \{ \psi(x,b_\beta|m) | m \in \omega \} \) is consistent. Fix such a \( \beta \), and consider a finite subset of \( \{ \psi(x,b_\beta|m) | m \in \omega \} \). It is contained in a set of the form \( \{ \psi(x,b_\beta|m) | m < n \} \) for some \( n \in \omega \), so it suffices to show that sets of this form are consistent. Choose \( n \in \omega \), and let \( \eta = \langle \beta \upharpoonright 0, \ldots, \beta \upharpoonright n-1 \rangle \). This is an \( \cap \)-closed tuple. Consider \( \Delta(y) := \{ \exists x ( \bigwedge_{i<n \psi(x,y_i))} \} \). By hypothesis, we can find a \( \nu \in \omega^2 \) such that \( \eta \approx_1 \nu \) and \( \bar{\eta} \equiv_{\Delta} \bar{\nu} \). Since \( \nu \approx_1 \eta \), we have \( \nu_i \leq \nu_j \) for all \( i \leq j \). That is, \( \nu \) lies
on a branch. Since \( \langle \bar{\pi}_\eta | \eta \in \langle \omega \rangle \rangle \) is an SOP\(_2\) tree for \( \psi \), \( \{ \psi(x, a_\nu) | i < n \} \) is consistent, and thus \( \mathcal{M} \models \exists x( \bigwedge_{i < n} \psi(x, a_\nu) ) \). It follows that \( \mathcal{M} \models \exists x( \bigwedge_{i < n} \psi(x, b_\eta) ) \), and thus the set \( \{ \psi(x, b_\beta | m) | m < n \} \) is consistent, as desired.

2. Now suppose that \( \alpha, \gamma \in \langle \omega \rangle \) are incomparable - we must show that \( \{ \psi(x, b_\alpha), \psi(x, b_\gamma) \} \) is inconsistent. Let \( \bar{\eta} := \langle \alpha, \gamma, \alpha \land \gamma \rangle \). This is clearly \( \cap \)-closed. Let

\[
\Delta(y) := \{ \exists x(\psi(x, y_0) \land \psi(x, y_1)) \}.
\]

We can find \( \bar{\tau} \in \langle \omega \rangle \) such that \( \bar{\tau} \simeq_1 \bar{\eta} \) and \( \bar{b}_\tau \equiv_\Delta \bar{a}_\tau \). Since \( \alpha \land \gamma \) is the most recent common ancestor of \( \alpha \) and \( \gamma \), it must be that one of \( \alpha \), \( \gamma \) is a descendant of \( (\alpha \land \gamma) \land (\langle 0 \rangle) \), while the other is a descendant of \( (\alpha \land \gamma) \land (\langle 1 \rangle) \). Without loss of generality, assume \( (\alpha \land \gamma) \land (\langle 0 \rangle) \subseteq \alpha \). It follows that \( \nu_2 \langle 0 \rangle \subseteq \nu_0 \), while \( \nu_2 \langle 1 \rangle \subseteq \nu_1 \), and thus, that \( \nu_0 \) and \( \nu_1 \) are incomparable. Since \( \langle a_\alpha | \eta \in \langle \omega \rangle \rangle \) is an SOP\(_2\) tree for \( \psi \), it follows that \( \{ \psi(x, a_\alpha), \psi(x, a_\nu) \} \) is inconsistent, and thus that \( \mathcal{M} \models \neg \exists x(\psi(x, a_\nu) \land \psi(x, a_\nu)) \). Since \( \bar{b}_\tau \equiv_\Delta \bar{a}_\tau \), \( \mathcal{M} \models \neg \exists x(\psi(x, b_\alpha) \land \psi(x, b_\gamma)) \), and \( \{ \psi(x, b_\alpha), \psi(x, b_\gamma) \} \) is inconsistent, as desired.

\[\square\]

Remark 1.5.8. 1-modeling does not necessarily preserve TP, however. We omit a formal argument, and simply observe that in a tree 1-modeled on a tree with TP, nodes that are siblings might be modeled on nodes that are not siblings in the original tree. Since TP only guarantees inconsistency among instances of the formula at sibling nodes, the new tree might not satisfy the inconsistency requirement.

The above facts tell us that if a theory \( T \) has TP\(_1\) (and hence, SOP\(_2\)), then there are a formula \( \psi(x; y) \) and a sequence \( \langle a_\alpha | \alpha \in \langle \omega \rangle \rangle \) which is a 1-fti tree witnessing SOP\(_2\) for \( \psi(x; y) \). We would also like to be able to narrow down the set of formulae which we must show do not have SOP\(_2\) in order to show that no formula has SOP\(_2\). To that end, we show that we can “eliminate the disjunction” - that is, if a disjunction of formulae has SOP\(_2\), then one of the disjuncts has SOP\(_2\).

Lemma 1.5.9. If \( \langle a_\alpha : \alpha \in \langle \omega \rangle \rangle \) is a 1-fti tree witnessing SOP\(_2\) for \( \psi_0 \lor \psi_1 \), then we can find a tree witnessing SOP\(_2\) for either \( \psi_0 \) or \( \psi_1 \).

Proof. We may assume that both \( \psi_0(x, a_\alpha) \) and \( \psi_1(x, a_\beta) \) are consistent - otherwise \( \langle a_\alpha : \alpha \in \langle \omega \rangle \rangle \) is already a tree for one or the other. (This uses the fact that the tree is 1-fti: if, say, \( \psi_0(x, a_\alpha) \) is inconsistent, then so is \( \psi_0(x, a_\alpha) \) for every \( \alpha \in \langle \omega \rangle \). It follows that consistency of the instances of \( \psi_0 \lor \psi_1 \) along a branch implies consistency of the corresponding instances of \( \psi_1 \). Inconsistency of incomparable instances of the disjunction automatically implies inconsistency of incomparable instances of \( \psi_1 \), so the original tree witnesses SOP\(_2\) for \( \psi_1 \).) Consider the leftmost branch (i.e. \( \{ \langle \rangle, \langle 0 \rangle, \langle 00 \rangle, \langle 000 \rangle, \ldots \} \)). Let \( b \) realize

\[\{(\psi_0 \lor \psi_1)(x; a_\langle 0^n \rangle) : n < \omega\},\]
where \( \langle 0^0 \rangle = \langle \rangle, \langle 0^1 \rangle = \langle 0 \rangle, \langle 0^2 \rangle = \langle 00 \rangle \), etc. Let \( f \) be a function from \( \omega \) to \( \{0, 1\} \) defined as follows:

\[
f(j) = \begin{cases} 
0 & \text{if } \models \psi_0(b; a_{\langle 0^j \rangle}) \\
1 & \text{otherwise}
\end{cases}
\]

The function \( f \) must take at least one of \( \{0, 1\} \) as its value for arbitrarily large \( j \). We may assume \( f(j) = 0 \) for arbitrarily large \( j \). Now, for any \( m \in \omega \), consider \( \overline{\eta} := \{\langle \rangle, \langle 0 \rangle, \ldots, \langle 0^m \rangle\} \). By our assumption, we can find \( j_0, \ldots, j_M \in \omega \) with \( j_0 < \ldots < j_M \) and such that

\[
\{\psi_0(x, a_{\langle 0^{j_i} \rangle}) : i \leq m\}
\]

is consistent. Let \( \overline{\nu} := \{\langle 0^{j_0} \rangle, \langle 0^{j_1} \rangle, \ldots, \langle 0^{j_m} \rangle\} \). Clearly, \( \overline{\eta} \approx_1 \overline{\nu} \). Since our tree is 1-fti, it follows that \( \{\psi_0(x, a_{\langle 0^{j_i} \rangle}) : i \leq m\} \) is consistent. By compactness, \( \{\psi_0(x, a_{\langle 0^{j_i} \rangle}) : i \in \omega\} \) is consistent.

We now define a function \( g : \prec \omega^2 \rightarrow \prec \omega^2 \) by induction, as follows:

\[
g(\langle \rangle) := \langle \rangle \\
g(\alpha \downarrow \langle 0 \rangle) := g(\alpha) \downarrow \langle 00 \rangle \\
g(\alpha \downarrow \langle 1 \rangle) := g(\alpha) \downarrow \langle 01 \rangle
\]

This is our pruning function: it picks out the nodes from the original tree that will form our new tree. We illustrate the first few steps of the function below. Think of the tree on the left as the tree we are constructing (the candidate for an SOP\(_2\) tree for \( \psi_0(x; y) \)), and the tree on the right as the original tree. The black dots are the nodes in the image of \( g \), and these are the nodes that form our new tree. The circles are nodes not in the image of \( g \).

Notice that we are picking nodes from every other level of the original tree, and each node that appears in the image of \( g \) lies above the left child of the previous node appearing in the image. This construction gives us a tree all of whose branches “look like” (are 1-similar to) the leftmost branch of the original tree. We shall show that the new tree witnesses that \( \psi_0(x; y) \) has SOP\(_2\).

For \( \alpha \in \prec \omega^2 \), we let \( b_\alpha := a_{g(\alpha)} \). We claim that \( \langle b_\alpha : \alpha \in \prec \omega^2 \rangle \) is a tree witnessing SOP\(_2\) for \( \psi_0 \).
Claim 1.5.10. Given \( \eta, \nu \in <^\omega 2, \eta \leq \nu \) if and only if \( g(\eta) \leq g(\nu) \). (In particular, if \( \eta \) and \( \nu \) are incomparable, so are \( g(\eta) \) and \( g(\nu) \).)

Proof of claim. \( \rightarrow \): Suppose \( \eta \leq \nu \). We show that \( g(\eta) \leq g(\nu) \) by induction on the distance between \( \eta \) and \( \nu \).

- Distance = 0; i.e. \( \eta = \nu \). Then \( g(\eta) = g(\nu) \), and so \( g(\eta) \leq g(\nu) \).

- Distance = \( n + 1 \); i.e. \( \nu = \alpha \langle i \rangle \) for some \( i \in \{0, 1\} \) and \( \alpha \geq \eta \), with the distance between \( \eta \) and \( \alpha \) equal to \( n \). By the induction hypothesis, \( g(\eta) \leq g(\alpha) \). By the definition of \( g \),

\[
\begin{align*}
g(\nu) &= g(\alpha \langle i \rangle) = g(\alpha) \langle 0i \rangle \geq g(\alpha) \geq g(\eta),
\end{align*}
\]

as desired.

\( \leftarrow \): We prove the contrapositive. Suppose \( \not\models \nu \). There are two cases.

- \( \nu \not\leq \eta \). Then there is \( i \in \{0, 1\} \) such that \( \nu^{-}\langle i \rangle \leq \eta \). By the forward direction of this claim, \( g(\nu^{-}\langle i \rangle) \leq g(\eta) \), hence \( g(\nu^{-}\langle 0i \rangle) \leq g(\eta) \). It follows that \( g(\eta) \not\models g(\nu) \), as desired.

- \( \nu \) and \( \eta \) are incomparable. We may assume without loss of generality that:

\[
\begin{align*}
(\nu \cap \eta)^{-}\langle 0 \rangle &\leq \nu; \\
(\nu \cap \eta)^{-}\langle 1 \rangle &\leq \eta.
\end{align*}
\]

Again by the forward direction of the claim, we have:

\[
\begin{align*}
g(\nu) &\geq g((\nu \cap \eta)^{-}\langle 0 \rangle) = g(\nu \cap \eta)^{-}\langle 00 \rangle = (g(\nu \cap \eta)^{-}\langle 0 \rangle)^{-}\langle 0 \rangle; \\
g(\eta) &\geq g((\nu \cap \eta)^{-}\langle 1 \rangle) = g(\nu \cap \eta)^{-}\langle 01 \rangle = (g(\nu \cap \eta)^{-}\langle 0 \rangle)^{-}\langle 1 \rangle.
\end{align*}
\]

Hence \( g(\eta) \) and \( g(\nu) \) are incomparable, and \( g(\eta) \not\models g(\nu) \), as desired.

\[\square\]

Remark 1.5.11. In fact, if \( \eta \not\leq \nu \), then \( g(\eta)^{-}\langle 0 \rangle \leq g(\nu) \).

Proof of remark. If \( \eta \not\leq \nu \), then \( \eta^{-}\langle i \rangle \leq \nu \), for \( i = 0 \) or \( i = 1 \). By Claim 1.5.10 and the definition of \( g \),

\[
\begin{align*}
g(\eta^{-}\langle i \rangle) &= g(\eta)^{-}\langle 0i \rangle = (g(\eta)^{-}\langle 0 \rangle)^{-}\langle i \rangle \leq \nu
\end{align*}
\]

\[\square\]

Claim 1.5.12. If \( \eta, \nu \in <^\omega 2 \) are incomparable, then \( \psi_0(x, b_\eta) \land \psi_0(x, b_\nu) \) is inconsistent.

Proof of claim. Recall that \( b_\eta = a_{g(\eta)} \) and \( b_\nu = a_{g(\nu)} \). By the above, \( g(\eta) \) and \( g(\nu) \) are incomparable, so \( \psi_0(x, a_{g(\eta)}) \land \psi_0(x, a_{g(\nu)}) \) - i.e. \( \psi_0(x, b_\eta) \land \psi_0(x, b_\nu) \) - is inconsistent, since \( (\psi_0 \lor \psi_1(x, a_{g(\eta)})) \land (\psi_0 \lor \psi_1(x, a_{g(\nu)})) \) is.

\[\square\]
Claim 1.5.13. If $\beta \in \omega^2$, \{\$\psi_0(x, b_{\beta i}) : i \in \omega$\} is consistent.

Proof of claim. It suffices to show that for any $n \in \omega$, \{\$\psi_0(x, b_{\beta i}) : i \leq n$\} is consistent. By Remark 1.5.11, if $i < j$, then $g(\beta \upharpoonright i)^\beta \not\models g(\beta \upharpoonright j)$. Let $\eta := (g(\beta \upharpoonright i) : i \leq n)$. We just observed that $\eta$ lies on a branch, and so it is $\cap$-closed. Now let $\nu := (\langle 0^i \rangle : i \leq n)$. This is also $\cap$-closed. Furthermore, for $i, j \leq n$,

\[
\langle 0^i \rangle \sqsubseteq \langle 0^j \rangle \text{ iff } \langle 0^i \rangle = \langle 0^j \rangle \text{ or } \langle 0^i \rangle \prec \langle 0^j \rangle
\]

if $i < j$

\[
\text{iff } g(\beta \upharpoonright i) = g(\beta \upharpoonright j) \text{ or } g(\beta \upharpoonright i)^\beta \not\models g(\beta \upharpoonright j)
\]

It follows that $\eta \models_1 \nu$. Since \{\$\psi_0(x, a_{\beta i}) : i \leq n$\} is consistent and $\langle a_\alpha : \alpha \in \omega^2$\} is a 1-fiti tree, \{\$\psi_0(x, a_{\beta(i)}) : i \leq n$\} is also consistent, as desired.

Claims 1.5.12 and 1.5.13 tell us that $\langle b_\alpha : \alpha \in \omega^2$\} is a tree witnessing SOP$_2$ for $\psi_0$, so the proof of the lemma is finished.

Corollary 1.5.14. If $T$ is $\aleph_0$-categorical and $T$ has SOP$_2$, then there is a complete formula (that is, a formula isolating a complete type) $\varphi(x; y)$ with SOP$_2$.

Proof. Suppose $\psi(x; y)$ has SOP$_2$, and $\psi(x; y)$ is incomplete. By $\aleph_0$-categoricity of $T$, there are finitely many complete types in the variables of $\psi$. Of these, let $p_0, \ldots, p_{n-1}$ list the ones that contain $\psi$. Again by $\aleph_0$-categoricity, each $p_i$ is isolated by some formula - say, $\psi_i$. Clearly, $\psi_i \vdash \psi$ for each $i$. Further, since any realization of $\psi$ realizes one of $p_0, \ldots, p_{n-1}$, $\psi \equiv \bigvee_{i<n} \psi_i$. By Lemma 1.5.9 and induction on $n$, $\bigvee_{i<n} \psi_i$ has SOP$_2$ if and only if one of the $\psi_i$ does.

We now make an observation about 1-fiti trees: if the parameters at two different nodes in such a tree share any elements, then those elements are common to the parameters at all nodes in the tree.

Lemma 1.5.15. Suppose $\langle b_\eta : \eta \in \omega^2$\} is a 1-fiti tree, where the $b_\eta$ are tuples of length $n$. If for some $\eta, \nu \in \omega^2$ and some $i \leq n$, $b^i_\eta = b^i_\nu$, then for all $\rho \in \omega^2$, $b^i_\rho = b^i_\eta$.

Proof. We begin by considering the case where $\eta$ and $\nu$ are siblings, and have the same $i^{th}$ coordinate as each other and as their parent.

Claim 1.5.16. If for some $\rho \in \omega^2$, $b^i_\rho = b^i_{\rho^\beta\eta} = b^i_{\rho^\beta\eta^\beta\nu}$, then for all $\xi \in \omega^2$, $b^i_\xi = b^i_\rho$.

Proof of claim. We split the proof into two cases, depending on whether or not $\rho$ is the root of the tree.
• Case 1: $\rho = \langle \rangle$. For all $\xi \in ^{<\omega}2$, either
\[
\langle \rangle \xi \approx_1 \langle \rangle (0)
\]
or
\[
\langle \rangle \xi \approx_1 \langle \rangle (1)
\]
Since both $b_{\langle \rangle}^i = b_{\langle \rangle (0)}^i$ and $b_{\langle \rangle}^i = b_{\langle \rangle (1)}^i$, tree-indiscernibility implies that $b_{\langle \rangle}^i = b_{\xi}^i$.

• Case 2: $\rho \neq \langle \rangle$. Then either $\langle 0 \rangle \leq \rho$ or $\langle 1 \rangle \leq \rho$, and hence either
\[
\langle \rho \rangle \approx_1 \rho \rho^\prec (0)
\]
or
\[
\langle \rho \rangle \approx_1 \rho \rho^\prec (1)
\]
Since both $b_{\rho}^i = b_{\rho^\prec (0)}^i$ and $b_{\rho}^i = b_{\rho^\prec (1)}^i$, by tree-indiscernibility. Then, since
\[
(\langle \rangle, \langle 0 \rangle, \langle 1 \rangle) \approx_1 (\rho, \rho^\prec (0), \rho^\prec (1)),
\]
tree-indiscernibility implies that $b_{\langle 0 \rangle}^i = b_{\langle 1 \rangle}^i = b_{\xi}^i$, and we are now in case 1 (giving the result that for all $\xi \in ^{<\omega}2$, $b_{\xi}^i = b_{\langle \rangle}^i$).

We split the proof of the lemma into two cases, depending on whether or not $\eta$ and $\nu$ are comparable. In the following, refer to $b_{\eta}^i = b_{\nu}^i$ as $b^i$.

• Case 1. Suppose $\eta$ and $\nu$ lie on the same branch. Without loss of generality, suppose $\eta^\prec (0) \leq \nu$. Then for any $\nu' \ni \eta^\prec (0)$, $\eta \nu \approx_1 \eta \nu'$, and so $b_{\nu'}^i = b_{\eta}^i = b^i$. In particular, $b_{\eta^\prec (0)}^i = b_{\eta^\prec (00)}^i = b_{\eta^\prec (01)}^i = b^i$. By Claim 1.5.16, we are done (take $\rho$ to be $\eta^\prec (0)$).

• Case 2. Suppose $\eta$ and $\nu$ are incomparable. Without loss of generality, suppose $(\eta \cap \nu)^\prec (0) \leq \eta$ and $(\eta \cap \nu)^\prec (1) \leq \nu$. Then for all $\eta' \ni (\eta \cap \nu)^\prec (0)$, $b_{\eta'}^i = b_{\eta}^i = b^i$ (since $(\nu, \eta, \eta \cap \nu) \approx_1 (\eta, \eta', \eta \cap \nu)$), and for all $\nu' \ni (\eta \cap \nu)^\prec (1)$, $b_{\nu'}^i = b_{\eta}^i = b^i$ (since $(\eta, \nu, \eta \cap \nu) \approx_1 (\eta, \nu', \eta \cap \nu)$). In particular,
\[
\begin{align*}
b^i &= b_{(\eta \cap \nu)^\prec (0)}^i \\
    &= b_{(\eta \cap \nu)^\prec (00)}^i \\
    &= b_{(\eta \cap \nu)^\prec (01)}^i
\end{align*}
\]
We apply Claim 1.5.16 again, this time taking $\rho$ to be $(\eta \cap \nu)^\prec (0)$.

\qed
**Corollary 1.5.17.** Suppose \( \langle b_\eta : \eta \in <_2^\omega \rangle \) is a 1-fti tree, where the \( b_\eta \) are tuples of length \( n \). If for some \( \eta, \nu \in <_2^\omega \) \( c \in b_\eta \cap b_\nu \) (that is, \( c \) is a coordinate in each tuple, though not necessarily the same coordinate), then for all \( \rho \in <_2^\omega \), \( c \in b_\rho \).

**Proof.** Take \( i \neq j < n \) such that \( c = b^i_\eta = b^j_\nu \). (If \( i = j \), the statement reduces to Lemma 1.5.15.) We split the proof into two cases, depending on whether or not \( \eta \) and \( \nu \) are comparable.

- **Case 1:** Suppose \( \eta \) and \( \nu \) are comparable. Without loss of generality, we may assume that \( \eta \vdash \langle 0 \rangle \preceq \nu \). Then

  \[
  (\eta, \nu) \approx_1 (\eta \vdash \langle 0 \rangle, \eta \vdash \langle 00 \rangle) \approx_1 (\eta, \eta \vdash \langle 00 \rangle)
  \]

  Since \( b^i_\eta = b^j_\nu \), we have

  \[
  b^i_{\eta \vdash \langle 0 \rangle} = b^j_{\eta \vdash \langle 00 \rangle}
  \]

  \[
  b^i_\eta = b^j_{\eta \vdash \langle 00 \rangle}
  \]

  Hence \( b^i_\eta = b^i_{\eta \vdash \langle 0 \rangle} \), and by Lemma 1.5.15, \( c = b^i_\eta = b^i_\rho \) for all \( \rho \in <_2^\omega \).

- **Case 2:** Suppose \( \eta \) and \( \nu \) are incomparable. Without loss of generality, we may assume that \( (\eta \cap \nu) \vdash \langle 0 \rangle \preceq \eta \) and \( (\eta \cap \nu) \vdash \langle 1 \rangle \preceq \nu \). It follows that

  \[
  (\eta, \nu, \eta \cap \nu) \approx_1 ((\eta \cap \nu) \vdash \langle 0 \rangle, \nu, \eta \cap \nu).
  \]

  Since \( b^i_\eta = b^j_\nu \), \( b^i_{(\eta \cap \nu) \vdash \langle 0 \rangle} = b^j_\nu \), and we have \( c = b^i_{(\eta \cap \nu) \vdash \langle 0 \rangle} = b^j_\eta \). By Lemma 1.5.15, \( c = b^j_\rho \) for all \( \rho \in <_2^\omega \).

\[\square\]

**Remark 1.5.18.** If \( \langle b_\eta : \eta \in <_2^\omega \rangle \) is a 1-fti tree (where the \( b_\eta \) are tuples of length \( n \)) and \( c \) is an element of the common intersection of \( \langle b_\eta : \eta \in <_2^\omega \rangle \) (that is, \( c \in b_\rho \) for all \( \rho \in <_2^\omega \)), then for any \( \eta, \nu \in <_2^\omega \), \( \text{tp}(b_\eta/c) = \text{tp}(b_\nu/c) \).

**Proof.** By the proof of Corollary 1.5.17, there is some \( i < n \) such that \( c = b^i_\rho \), all \( \rho \in <_2^\omega \). Since \( \langle b_\eta : \eta \in <_2^\omega \rangle \) is 1-fti, \( \text{tp}(b_\eta/c) = \text{tp}(b_\eta/b^i_\eta) = \text{tp}(b_\nu/b^i_\nu) = \text{tp}(b_\nu/c) \) for all \( \eta, \nu \in <_2^\omega \).

\[\square\]

We can generalize Corollary 1.5.17 a bit, so that it applies to the intersections of the definable closures of tuples at distinct nodes on a 1-fti tree. We begin by showing that we can add elements of the definable closure (in a reasonable way) to a 1-fti tree, and the resulting tree will also be 1-fti.
Lemma 1.5.19. If \( \langle b_\eta : \eta \in \langle \omega \rangle \rangle \) is a 1-fti tree (where the \( b_\eta \) are tuples of length \( n \)), \( \varphi(x, y) \) is a formula such that \( M \models (\exists x) \varphi(x, b_0) \), and for each \( \eta \in \langle \omega \rangle \) \( c_\eta \) is the unique realization of \( \varphi(x, b_\eta) \), then the tree \( \langle b_\eta c_\eta : \eta \in \langle \omega \rangle \rangle \) is 1-fti.

Proof. First note that the statement of the lemma makes sense: if \( M \models (\exists x) \varphi(x; b_0) \), then \( M \models (\exists x) \varphi(x; b_\eta) \) for all \( \eta \in \langle \omega \rangle \), so the \( c_\eta \)'s exist. We must show that if \( \eta \) and \( \nu \) are tuples from 2 such that \( \eta \approx 1 \nu \), then \( b_\eta c_\eta \equiv b_\nu c_\nu \). Let \( \eta \approx 1 \nu \) with \( m = \lg(\eta) = \lg(\nu) \), and let \( \psi \) be any formula (to which we add dummy variables, if necessary). Suppose that \( M \models \psi(\bar{\eta}, \bar{x}) \). Then

\[
M \models \exists x (\psi(\bar{\eta}, \bar{x}) \land \bigwedge_{i < m} \varphi(x_i, b_\eta)).
\]

Since \( \langle b_\eta : \eta \in \langle \omega \rangle \rangle \) is 1-fti,

\[
M \models \exists x (\psi(\bar{\eta}, \bar{x}) \land \bigwedge_{i < m} \varphi(x_i, b_\eta)).
\]

Since \( c_\nu \) is the only realization of \( \varphi(x_i, b_\nu) \), \( c_\nu \) is the only possible realization of \( \psi(\bar{\eta}, \bar{x}) \land \bigwedge_{i < m} \varphi(x_i, b_\eta)) \). Since we know this formula has a realization, \( M \models \psi(\bar{\eta}, \bar{x}) \). Since \( \psi \) was arbitrary, \( \bar{\eta} \bar{\eta} \equiv \bar{\eta} \bar{\nu} \), and \( \langle b_\eta c_\eta : \eta \in \langle \omega \rangle \rangle \) is 1-fti. \( \square \)

Corollary 1.5.20. If \( \langle b_\eta : \eta \in \langle \omega \rangle \rangle \) is a 1-fti tree and for some \( \eta, \nu \) (\( \eta \neq \nu \))

\[
c \in \text{dcl}(b_\eta) \cap \text{dcl}(b_\nu),
\]

then \( c \in \text{dcl}(b_\rho) \) for all \( \rho \in \langle \omega \rangle \).

Proof. Suppose \( \varphi_1 \) witnesses that \( c \in \text{dcl}(b_\eta) \) and \( \varphi_2 \) witnesses that \( c \in \text{dcl}(b_\nu) \). For each \( \rho \in \langle \omega \rangle \), let \( c_\rho \) be the unique realization of \( \varphi_1(x, b_\rho) \), and let \( d_\rho \) be the unique realization of \( \varphi_2(x, b_\rho) \). (So \( c_\eta = c = d_\nu \).) By Lemma 1.5.19, the tree \( \langle b_\rho c_\rho d_\rho : \rho \in \langle \omega \rangle \rangle \) is 1-fti. Since \( c \in b_\eta c_\nu d_\eta \cap b_\nu c_\nu d_\nu \), \( c \in b_\rho c_\rho d_\rho \) - and hence, \( c \in \text{dcl}(b_\rho) \) - for all \( \rho \in \langle \omega \rangle \), by Corollary 1.5.17. (In fact, by the proof of Corollary 1.5.17, \( c = c_\rho = d_\rho \) for all \( \rho \in \langle \omega \rangle \).) \( \square \)
Chapter 2

Parameterized Equivalence Relations

In this chapter, we consider the two-sorted theory of infinitely many cross-cutting parameterized equivalence relations, which is known to have the tree property, but not the tree property of the first kind. This example was introduced by Shelah in [21]. In [1] and [14], Adler and Kim and Kim (respectively) discuss possible notions of independence for $T^*_\text{feq}$. In [23], Shelah and Usvyatsov show that $T^*_\text{feq}$ does not have SOP$_2$. We give a version of this proof, making use of the results on indiscernible trees presented in the first chapter.

2.1 The Theory

We take $\mathcal{L}$ to be a language whose signature has two sorts: one, $P$, for points, and another, $E$, for equivalence relations, and one three-place relation, $R$, on $P \times P \times E$. Given an $E$-element $r$, and two $P$-elements, $a$ and $b$, we shall also write $r(a, b)$ for $R(a, b, r)$. We shall use lowercase letters $x, y, z$ to denote variables of sort $P$, uppercase letters $X, Y, Z$ to denote variables of sort $E$, lowercase $a, b, c, d$ to denote parameters of sort $P$, and lowercase $r, s, t$ to denote parameters of sort $E$. The universal $\mathcal{L}$-theory $T_0$ says that for each $r \in E$, $r(\cdot, \cdot)$ is an equivalence relation on $P$:

$$T_0 := \{\forall X, y(X(y, y)), \forall X, y, z(X(y, z) \rightarrow X(z, y)), \forall X, y, z, w((X(y, z) \land X(z, w)) \rightarrow X(y, w))\}$$

Fact 2.1.1. The class $\mathcal{K}$ of finite models of $T_0$ has the Hereditary Property, the Joint Embedding Property, and the Amalgamation Property, and hence (see, e.g., [11], Theorem 6.1.2) has a Fraïssé limit, $\mathcal{F}$.

---

1The stated result in [23], Theorem 2.1 is, in fact, that $T^*_\text{feq}$ does not have SOP$_1$. The proof, however, relies on a faulty result from [7] (Claims 2.11 and 2.14). Those Claims were fixed in [8], a few years after [23] was published. The issue is, in short, that one cannot assume as great a degree of indiscernibility for SOP$_1$ trees as was initially thought. However, given that one can assume that SOP$_2$ trees are highly indiscernible, Shelah and Usvyatsov’s argument for Theorem 2.1 shows that $T^*_\text{feq}$ does not have SOP$_2$. It is that argument that we give here, filling in the details that were less obvious to us than they would have been to the authors of [23]. It is still conjectured that $T^*_\text{feq}$ does not have SOP$_1$. 
Proof. We recall the definition of each property before proving that $K$ has it.

1. **Hereditary Property**: If $A \in K$ and $B$ is a finitely generated substructure of $A$, then $B \in K$.

(Note that as $K$ consists of finite structures in a relational language, “finitely generated” just means “finite” here.) By universality of $T_0$, any substructure of a (finite) model of $T_0$ is itself a (finite) model of $T_0$.

2. **Joint Embedding Property**: If $A, B \in K$, then there is $C \in K$ such that $A$ and $B$ are embeddable in $C$.

We shall construct $C$. Suppose $A = \{a_0, \ldots, a_{n-1}, r_0, \ldots, r_{m-1}\}$, and $B = \{b_0, \ldots, b_{n'-1}, s_0, \ldots, s_{m'-1}\}$.

Take the universe of $C$ to be the disjoint union of the universes of $A$ and $B$, that is, $C = \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n'-1}, r_0, \ldots, r_{m-1}, s_0, \ldots, s_{m'-1}\}$ where the $a_i$ and $b_i$, and the $r_i$ and $s_i$ are all distinct. (We may replace $B$ by an isomorphic copy, if necessary, to achieve this.) We now define $R(\cdot, \cdot, \cdot)$ on $C$.

- For $i < m$, $j, k < n$, and $j', k' < n'$,
  
  $C \models r_i(a_j, a_k)$ if and only if $A \models r_i(a_j, a_k)$;
  
  $C \models \neg r_i(a_j, b_{j'}) \wedge \neg r_i(b_{j'}, a_j)$;
  
  $C \models r_i(b_{j'}, b_{k'})$.

- For $i' < m'$, $j, k < n$, and $j', k' < n'$,

  $C \models s_{i'}(b_{j'}, b_{k'})$ if and only if $B \models s_{i'}(b_{j'}, b_{k'})$;
  
  $C \models \neg s_{i'}(b_{j'}, a_j) \wedge \neg s_{i'}(a_j, b_{j'})$;
  
  $C \models s_{i'}(a_j, a_k)$.

In $C$, each $r_i$ has all of the classes it has in $A$, plus an additional class for the $b_i$’s. Similarly, each $s_i$ has all of the classes it has in $B$, plus an additional class for the $a_i$’s. It is clear that each of the $r_i$’s and $s_j$’s is an equivalence relation on the points of $C$, and that the inclusion maps from $A$ into $C$ and from $B$ into $C$ are embeddings.

3. **Amalgamation Property**: If $A, B, C \in K$ and $e : A \to B$, $f : A \to C$ are embeddings, then there are $D \in K$ and embeddings $g : B \to D$, $h : C \to D$ such that $ge = hf$. 


CHAPTER 2. PARAMETERIZED EQUIVALENCE RELATIONS

We construct $\mathcal{D}$. Let the universe of $\mathcal{D}$ be the disjoint union of $B \setminus e(A)$, $C \setminus f(A)$, and $A$, where

$$B \setminus e(A) = \{b_0, \ldots, b_{n_B-1}, s_0, \ldots, s_{m_B-1}\},$$
$$C \setminus f(A) = \{c_0, \ldots, c_{n_C-1}, t_0, \ldots, t_{m_C-1}\},$$
$$A = \{a_0, \ldots, a_{n_A-1}, r_0, \ldots, r_{m_A-1}\}.$$

Our definition of $R^D$ is similar to that of $R^C$ in the proof of JEP: each $s_i$ has the same classes as in $\mathcal{B}$, plus an additional class for the elements of $C \setminus f(A)$, and each $t_i$ has the same classes as in $\mathcal{C}$, plus an additional class for the elements of $B \setminus e(A)$. We must be more careful when extending the definition of the $r_i$’s, however, in order to make sure they satisfy transitivity. Thus, in our formal definition of $R^D$, we consider three separate cases, depending on whether the equivalence relation element is from $A$, $B \setminus e(A)$, or $C \setminus f(A)$.

- For all $i < m_B$, $j_A, k_A < n_A$, $j_B, k_B < n_B$, and $j_C, k_C < n_C$,

  $$\mathcal{D} \models s_i(a_{j_A}, a_{k_A}) \text{ if and only if } \mathcal{B} \models s_i(e(a_{j_A}), e(a_{j_A}));$$
  $$\mathcal{D} \models s_i(b_{j_B}, b_{k_B}) \text{ if and only if } \mathcal{B} \models s_i(b_{j_B}, b_{k_B});$$
  $$\mathcal{D} \models s_i(c_{j_C}, c_{k_C});$$

  $$\mathcal{D} \models \begin{cases} 
  s_i(a_{j_A}, b_{j_B}) \land s_i(b_{j_B}, a_{j_A}) & \text{if } \mathcal{B} \models s_i(e(a_{j_A}), b_{j_B}), \\
  \neg s_i(a_{j_A}, b_{j_B}) \land \neg s_i(b_{j_B}, a_{j_A}) & \text{otherwise};
  \end{cases}$$

  $$\mathcal{D} \models \neg s_i(a_{j_A}, c_{j_C}) \land \neg s_i(c_{j_C}, a_{j_A}) \land \neg s_i(b_{j_B}, c_{j_C}) \land \neg s_i(c_{j_C}, b_{j_B}).$$

We can see that for any $s_i$, elements of $\mathcal{B}$ are $s_i$-related in $\mathcal{D}$ if and only if they are $s_i$-related in $\mathcal{B}$, while elements of $\mathcal{C} \setminus f(A)$ are all $s_i$-related to one another, and are not $s_i$-related to any element of $\mathcal{B}$.

- For all $i < m_C$, $j_A, k_A < n_A$, $j_B, k_B < n_B$, and $j_C, k_C < n_C$,

  $$\mathcal{D} \models t_i(a_{j_A}, a_{k_A}) \text{ if and only if } \mathcal{C} \models t_i(f(a_{j_A}), f(a_{k_A}));$$
  $$\mathcal{D} \models t_i(c_{j_C}, c_{k_C}) \text{ if and only if } \mathcal{C} \models t_i(c_{j_C}, c_{k_C});$$
  $$\mathcal{D} \models t_i(b_{j_B}, b_{k_B});$$

  $$\mathcal{D} \models \begin{cases} 
  t_i(a_{j_A}, c_{j_C}) \land t_i(c_{j_C}, a_{j_A}) & \text{if } \mathcal{C} \models t_i(f(a_{j_A}), c_{j_C}), \\
  \neg t_i(a_{j_A}, c_{j_C}) \land \neg t_i(c_{j_C}, a_{j_A}) & \text{otherwise};
  \end{cases}$$

  $$\mathcal{D} \models \neg t_i(a_{j_A}, b_{j_B}) \land \neg t_i(b_{j_B}, a_{j_A}) \land \neg t_i(c_{j_C}, b_{j_B}) \land \neg t_i(b_{j_B}, c_{j_C}).$$

As in the definition of $s_i$ on $\mathcal{D}$, elements of $\mathcal{C}$ are $t_i$-related in $\mathcal{D}$ if and only if they are $t_i$-related in $\mathcal{C}$, while the elements of $B \setminus e(A)$ form a separate $t_i$-class.
For all $i < m_A, j_A, k_A < n_A, j_B, k_B < n_B$, and $j_C, k_C < n_C$,
\[ D \models r_i(a_{j_A}, a_{k_A}) \text{ if and only if } A \models r_i(a_{j_A}, a_{k_A}) \]
\[ \text{if and only if } B \models e(r_i)(e(a_{j_A}), e(a_{k_A})) \]
\[ \text{if and only if } C \models f(r_i)(f(a_{j_A}), f(a_{k_A})) \]
\[ D \models r_i(b_{j_B}, b_{k_B}) \text{ if and only if } B \models e(r_i)(b_{j_B}, b_{k_B}) \]
\[ D \models r_i(c_{j_C}, c_{k_C}) \text{ if and only if } C \models f(r_i)(c_{j_C}, c_{k_C}) \]
\[ D \models \begin{cases} r_i(a_{j_A}, b_{j_B}) \land r_i(b_{j_B}, a_{j_A}) & \text{if } B \models e(r_i)(e(a_{j_A}), b_{j_B}), \\ -r_i(a_{j_A}, b_{j_B}) \land -r_i(b_{j_B}, a_{j_A}) & \text{otherwise}; \end{cases} \]
\[ D \models \begin{cases} r_i(a_{j_A}, c_{j_C}) \land r_i(c_{j_C}, a_{j_A}) & \text{if } C \models f(r_i)(f(a_{j_A}), c_{j_C}), \\ -r_i(a_{j_A}, c_{j_C}) \land -r_i(c_{j_C}, a_{j_A}) & \text{otherwise}; \end{cases} \]
\[ D \models r_i(b_{j_B}, c_{j_C}) \land r_i(c_{j_C}, b_{j_B}) \text{ if there is } 1 \leq \ell \leq n_A \text{ such that} \]
\[ B \models e(r_i)(b_{j_B}, e(a_{\ell})) \text{ and} \]
\[ C \models f(r_i)(f(a_{\ell}), c_{j_C}), \]
\[ D \models -r_i(b_{j_B}, c_{j_C}) \land -r_i(c_{j_C}, b_{j_B}) \text{ otherwise,} \]

Once again, elements of $B$ are $r_i$-related in $D$ if and only if they are $e(r_i)$-related in $B$, and elements of $C$ are $r_i$-related in $D$ if and only if they are $f(r_i)$-related in $C$. We extend $r_i$ to $(B \setminus e(A)) 	imes (C \setminus f(A))$ so that $b_j$ and $c_k$ are $r_i$ related just in case there is some element of $A$ that “connects” them. This is well defined: either $b_j$ and $c_k$ are $e(r_i)$-related (respective, $f(r_i)$-related) to (images of) all the same elements of $A$, or to (images of) none of the same elements of $A$, since $e$ and $f$ preserve the $r_i$-classes of $A$.

It should be clear that each $r_i, s_j$, and $t_k$ is an equivalence relation, so $D \models T_0$. We define $g : B \to D$ and $h : C \to D$ in the obvious way: $g \upharpoonright (B \setminus e(A))$ and $h \upharpoonright (C \setminus f(A))$ are the identity map, while $g \upharpoonright e(A) = e^{-1}$ and $h \upharpoonright f(A) = f^{-1}$. The maps $g$ and $h$ are embeddings, and $ge = hf$.

Let $T^*_{\leq}$ be the theory of $\mathcal{F}$. As the theory of the Fraïssé limit of the finite models of $T_0$, $T^*_{\leq}$ is $\aleph_0$-categorical and has elimination of quantifiers (see [11], Theorem 6.4.1), and it is
the model completion of $T_0$. For convenience, we spell out the intermediate theory $T_{\text{feq}}$:

$$T_{\text{feq}} := T_0 \cup \left\{ \forall X \exists y_0, \ldots, y_{n-1} \left( \bigwedge_{i \neq j} \neg X(y_i, y_j) \right) : n < \omega \right\}$$

$$\cup \left\{ \exists X_0, \ldots, X_{n-1} \left( \bigwedge_{i \neq j} X_i \neq X_j \right) : n < \omega \right\}$$

$$\cup \left\{ \forall X_0, \ldots, X_{n-1} \forall y_0, \ldots, y_{n-1} \exists y \left( \bigwedge_{i \neq j} X_i \neq X_j \rightarrow \bigwedge_{i < n} X_i(y, y_i) \right) : n < \omega \right\}$$

$T_{\text{feq}}$ says that each equivalence relation has infinitely many classes, that there are infinitely many $E$-sort elements, and that given any $n$ distinct classes $r_0, \ldots, r_{n-1}$, and $n$ (not necessarily distinct) points $a_0, \ldots, a_{n-1}$, we can find a point that is $r_i$-equivalent to $a_i$ for $i < n$. We shall refer to this as the “cross-cutting axiom.”

**Fact 2.1.2.** The theory $T^*_{\text{feq}}$ is an extension of $T_{\text{feq}}$.

**Proof.** Suppose $\mathcal{M} \models T^*_{\text{feq}}$. We show $\mathcal{M} \models T_{\text{feq}}$.

1. For each $n < \omega$, $\mathcal{M} \models \forall X \exists y_0, \ldots, y_{n-1} \left( \bigwedge_{i \neq j} \neg X(y_i, y_j) \right)$.

Suppose $r \in E^M$, and $r$ has $m$ classes for some $m < n$. We build a model $\mathcal{N}$ of $T_0$ extending $\mathcal{M}$: let the universe of $\mathcal{N}$ be $M \cup \{a_0, \ldots, a_{n-m-1}\}$ (where $a_0, \ldots, a_{n-m-1}$ are new elements of sort $P$). For $s \in E^M \setminus \{r\}$, let the new elements form a single, new $s$-class. (This is not actually relevant, but we must make some choice with regard to the $s$-classes of the new elements.) Let each new element form its own $r$-class. More precisely, we define $R^N$ by:

$$\mathcal{N} \models s(b, c) \text{ if and only if } \mathcal{M} \models s(b, c) \text{ for all } s \in E^M, b, c \in P^M;$$

$$\mathcal{N} \models \neg s(b, a_i) \land \neg s(a_i, b) \text{ for any } s \in E^M, b \in P^M \text{ and } i < n - m;$$

$$\mathcal{N} \models \bigwedge_{i,j < n-m} s(a_i, a_j) \text{ for all } s \in E^M \setminus \{r\};$$

$$\mathcal{N} \models \bigwedge_{i \neq j} \neg r(a_i, a_j).$$

Under this definition of $R^N$, $\mathcal{N} \models T_0$, $\mathcal{N}$ extends $\mathcal{M}$, and $\mathcal{N} \models \exists y_0, \ldots, y_{n-1} \left( \bigwedge_{i \neq j} \neg r(y_i, y_j) \right)$.

As $\mathcal{M}$ is existentially closed for models of $T_0$, $\mathcal{M}$ must already contain at least $n$ $r$-classes. Since $r$ was arbitrary, we have that

$$\mathcal{M} \models \forall X \exists y_0, \ldots, y_{n-1} \left( \bigwedge_{i \neq j} \neg X(y_i, y_j) \right),$$

$$\mathcal{M} \models \exists X_0, \ldots, X_{n-1} \left( \bigwedge_{i \neq j} X_i \neq X_j \right),$$

$$\mathcal{M} \models \forall X_0, \ldots, X_{n-1} \forall y_0, \ldots, y_{n-1} \exists y \left( \bigwedge_{i \neq j} X_i \neq X_j \rightarrow \bigwedge_{i < n} X_i(y, y_i) \right).$$
as desired.

2. For each \( n < \omega \), \( \mathcal{M} \models \exists X_0, \ldots, X_{n-1} \left( \bigwedge_{i \neq j} X_i \neq X_j \right) \).

Again, we build a model \( \mathcal{N} \) of \( T_0 \) extending \( \mathcal{M} \). Let the universe of \( \mathcal{N} \) be \( \mathcal{M} \cup \{ r_0, \ldots, r_{n-1} \} \), and define \( R^\mathcal{N} \) as follows:

\[
\mathcal{N} \models s(a, b) \text{ if and only if } \mathcal{M} \models s(a, b) \text{ for all } s \in E^\mathcal{M}, a, b \in P^\mathcal{M}; \\
\mathcal{N} \models r_i(a, b) \text{ for any } i < n, \text{ any } a, b \in P^\mathcal{M}.
\]

That is, \( \mathcal{N} \) adds \( n \) equivalence relations to \( \mathcal{M} \), each of which has only one class. It is clear that \( \mathcal{M} \) embeds into \( \mathcal{N} \), \( \mathcal{N} \models T_0 \), and \( \mathcal{N} \models \exists X_0, \ldots, X_{n-1} \left( \bigwedge_{i \neq j} X_i \neq X_j \right) \). Since \( \mathcal{M} \) is existentially closed for models of \( T_0 \), it satisfies the desired axiom.

3. For each \( n < \omega \), \( \mathcal{M} \models \forall X_0, \ldots, X_{n-1} \forall y_0, \ldots, y_{n-1} \exists y \left( \bigwedge_{i \neq j} X_i \neq X_j \rightarrow \bigwedge_{i < n} X_i(y, y_i) \right) \).

Suppose \( r_0, \ldots, r_{n-1} \) are distinct elements of \( E^\mathcal{M} \), and \( a_0, \ldots, a_{n-1} \) are (not necessarily distinct) elements of \( P^\mathcal{M} \). Again, we build a model \( \mathcal{N} \) of \( T_0 \) extending \( \mathcal{M} \). Let the universe of \( \mathcal{N} \) be the union of \( \mathcal{M} \) and a single new element, \( b \). For each \( s \in E^\mathcal{M} \setminus \{ r_0, \ldots, r_{n-1} \} \), let \( b \) be in a new \( s \)-class of its own. For each \( r_i \), put \( b \) in the same \( r_i \) class as \( a_i \) (and all of the other elements in \( a_i \)'s \( r_i \)-class, according to \( \mathcal{M} \)). More precisely, we define \( R^\mathcal{N} \) as follows:

\[
\mathcal{N} \models s(c, d) \text{ if and only if } \mathcal{M} \models s(c, d) \text{ for all } s \in E^\mathcal{M}, c, d \in P^\mathcal{M}; \\
\mathcal{N} \models \neg s(c, b) \land \neg s(b, c) \text{ for all } s \in E^\mathcal{M}, c \in P^\mathcal{M}; \\
\mathcal{N} \models \bigwedge_{i < n} r_i(b, a_i) \land r_i(a_i, b); \\
\mathcal{N} \models \begin{cases} 
 r_i(b, c) \land r_i(c, b) & \text{if } \mathcal{M} \models r_i(a_i, c) \\
 \neg r_i(b, c) \land \neg r_i(c, b) & \text{otherwise}
\end{cases}
\text{ for any } c \in P^\mathcal{M}.
\]

With this definition of \( R^\mathcal{N} \), it is clear that \( \mathcal{N} \models T_0 \), \( \mathcal{N} \) is an extension of \( \mathcal{M} \), and \( \mathcal{N} \models \exists y \left( \bigwedge_{i < n} r_i(y, a_i) \right) \). Again, we note that since \( \mathcal{M} \) is existentially closed for models of \( T_0 \), this must already be true in \( \mathcal{M} \), that is,

\[
\mathcal{M} \models \exists y \left( \bigwedge_{i < n} r_i(y, a_i) \right).
\]
Since $r_0, \ldots, r_{n-1}$ were arbitrary distinct elements of $E^M$ and $a_0, \ldots, a_{n-1}$ were arbitrary elements of $P^M$, $M$ satisfies the desired axiom.

\[ \square \]

**Fact 2.1.3.** If $T$ extends $T_{\text{eq}}$, the formula $\varphi(x; y, Z) := Z(x, y)$ has the tree property of the second kind, and in particular, $T$ is not simple.

**Proof.** Work in any sufficiently saturated model $M$ of $T_{\text{eq}}$. We begin by showing that the following type, in variables $X_i$ for $i \in \omega$, $z_{(i,j)}$ for $i, j \in \omega$, and $y_\alpha$ for $\alpha \in \omega$ is consistent:

$$
\pi(X_i, z_{(j,k)}, y_\alpha : i, j, k \in \omega, \alpha \in \omega) = \bigcup_{\alpha \in \omega} \{X_i(y_\alpha, z_{(i,\alpha(i))}) : i < \omega\}
$$

$$
\cup \{X_i \neq X_j : i < j < \omega\}
$$

$$
\cup \bigcup_{i < \omega} \{\neg X_i(z_{(i,j)}, z_{(i,k)}) : j < k < \omega\}.
$$

Given a finite subset of this type, there are $n, m, l \in \omega$ such that the subset is contained in a type of the following form, in variables $X_i$ for $i < n$, $y_{\alpha_0}, \ldots, y_{\alpha_l}$ for $\alpha_i \in \omega$, and $z_{(i,j)}$ for $i < n$ and $j < m$, where $m > \max\{\alpha_j(i) : j < l, i < n\}$:

$$
\pi_0(X_i, z_{(j,k)}, y_{\alpha_r} : i, j < n, k < m, r < l) = \bigcup_{r < l} \{X_i(y_{\alpha_r}, z_{(i,\alpha_r(i))}) : i < n\}
$$

$$
\cup \{X_i \neq X_j : i < j < n\}
$$

$$
\cup \bigcup_{i < n} \{\neg X_i(z_{(i,j)}, z_{(i,k)}) : j < k < m\}.
$$

To realize $\pi_0$, we begin by choosing any $n$ distinct equivalence relation elements $s_0, \ldots, s_{n-1}$ from $M$. We can do this because $E^M$ is infinite. Then, for each $i$, we choose $m$ elements $a_{(i,0)}, \ldots, a_{(i,m-1)}$ of $P^M$ from distinct $s_i$-classes. We can do this because each element of $E^M$ has infinitely many classes. Lastly, for each $r < l$, we find $b_{\alpha_r}$ satisfying

$$
\bigwedge_{i < n} s_i(y, a_{(i,\alpha_r(i))}).
$$

Such $b_{\alpha_r}$ exist by the cross-cutting axiom of $T_{\text{eq}}$. By the choice of these elements,

$$
M \models \pi_0(s_i, a_{(j,k)}, b_{\alpha_r} : i, j < n, k < m, r < l).
$$

By compactness, $\pi$ is consistent and (by saturation) realized in $M$. Let

$$
(s_i, a_{(i,j)}, b_\alpha : i, j < \omega, \alpha \in \omega)
$$

realize $\pi$ in $M$, and for each $i$, let $t_{(i,j)} = s_i$. To recap, we have:

$$
M \models t_{(i,\alpha(i))}(b_\alpha, a_{(i,\alpha(i))})
$$

(2.1)
for each \( \alpha \in \omega \) and \( i < \omega \), and

\[
\mathcal{M} \models \neg \exists x (t(i,j)(x, a_{(i,j)}) \land t(i,k)(x, a_{(i,k)}))
\]

(2.2)

for any \( i < \omega \) and any \( j \neq k < \omega \). (This last comes from the fact that \( t(i,j) = t(i,k) \) and \( a_{(i,j)} \) and \( a_{(i,k)} \) are in different \( t(i,j) \)-classes.) We claim that \( (a_{(i,j)})_{i,j} : i,j < \omega \) witnesses that \( \varphi(x; y, Z) \) has TP

• For each \( i < \omega \),

\[
\{ \varphi(x; a_{(i,j)}, t(i,j)) : j < \omega \} = \{ t(i,j)(x, a_{(i,j)}) : j < \omega \}
\]

is inconsistent, by 2.2, above.

• For each \( \alpha \in \omega \),

\[
\{ \varphi(x; a_{(i,\alpha(i))}, t(i,\alpha(i))) : i < \omega \} = \{ t(i,\alpha(i))(x, a_{(i,\alpha(i))}) : i < \omega \}
\]

is consistent, realized by \( b_\alpha \) (by 2.1, above).

\[\square\]

2.2 Demonstration of subtlety

In this section, we show that \( T_{\text{eq}}^{\ast} \) does not have the tree property of the first kind (or, in the terminology of Kim and Kim, that it is \textit{subtle}).

Remark 2.2.1. By Fact 1.2.10 and Corollary 1.5.14, it suffices to show that no “complete” formula (that is, a formula that isolates some complete type in its variables) has SOP

Remark 2.2.2. Any isolating formula \( \psi(x_0, \ldots, x_{m-1}, Y_0, \ldots, Y_{n-1}; z_0, \ldots, z_{r-1}, W_0, \ldots, W_{s-1}) \)
is (equivalent to a formula) of the form

\[
\psi(xY; zW) := \bigwedge_{(i,j,k) \in I_1} Y_k(x_i, x_j) \land \bigwedge_{(i,j,k) \notin I_1} \neg Y_k(x_i, x_j) \land \bigwedge_{(i,j,k) \in I_2} Y_k(x_i, z_j) \\
\land \bigwedge_{(i,j,k) \notin I_2} \neg Y_k(x_i, z_j) \land \bigwedge_{(i,j,k) \in I_3} Y_k(z_i, z_j) \land \bigwedge_{(i,j,k) \notin I_3} \neg Y_k(z_i, z_j) \\
\land \bigwedge_{(i,j,k) \in J_1} W_k(x_i, x_j) \land \bigwedge_{(i,j,k) \notin J_1} \neg W_k(x_i, x_j) \land \bigwedge_{(i,j,k) \in J_2} Y_k(x_i, z_j) \\
\land \bigwedge_{(i,j,k) \notin J_2} \neg Y_k(x_i, z_j) \land \bigwedge_{(i,j,k) \in J_3} W_k(z_i, z_j) \land \bigwedge_{(i,j,k) \notin J_3} \neg W_k(z_i, z_j) \\
\land \bigwedge_{(i,j) \in K_1} x_i = x_j \land \bigwedge_{(i,j) \notin K_1} x_i \neq x_j \land \bigwedge_{(i,j) \in K_2} Y_i = Y_j \land \bigwedge_{(i,j) \notin K_2} Y_i \neq Y_j \\
\land \bigwedge_{(i,j) \in K_3} z_i = z_j \land \bigwedge_{(i,j) \notin K_3} z_i \neq z_j \land \bigwedge_{(i,j) \in K_4} W_i = W_j \land \bigwedge_{(i,j) \notin K_4} W_i \neq W_j \\
\land \bigwedge_{(i,j) \in L_1} x_i = z_j \land \bigwedge_{(i,j) \notin L_1} x_i \neq z_j \land \bigwedge_{(i,j) \in L_2} Y_i = W_j \land \bigwedge_{(i,j) \notin L_2} Y_i \neq W_j
\]

for some \( I_1 \subseteq m \times m \times n, I_2 \subseteq m \times r \times n, I_3 \subseteq r \times r \times n, J_1 \subseteq m \times m \times s, J_2 \subseteq m \times r \times s, \)

\( J_3 \subseteq r \times r \times s, K_1 \subseteq m \times m, K_2 \subseteq n \times n, K_3 \subseteq r \times r, K_4 \subseteq s \times s, L_1 \subseteq m \times r, \) and

\( L_2 \subseteq n \times s. \) However, we may eliminate the equalities: any equalities among variables of the same ‘type’ (where by ‘type’ we mean both sort and either object or parameter - so \( x_i \) is the same type as \( x_j \), but not the same type as \( z_j \), for example) simply make one of the variables redundant. That is, since the formula is consistent in the first place, we may replace it by a formula without the redundant variables and the conjuncts containing them. Similarly, suppose there is some equality between an object variable and a parameter variable, say, \( x_i = z_j \). Then the realization of \( x_i \) is determined by the parameter at \( z_j \). Since the branches of the tree are consistent, the \( j^{th} \) point parameter must be the same along any branch, and hence (by Lemma 1.5.15), throughout the tree. If \( \psi(xY; zW) \) has \( \text{SOP}_2 \), so does the formula \( \psi' \) obtained by removing \( x_i \) and any conjuncts containing it. (\( \text{SOP}_2 \) for \( \psi' \) is witnessed by the same parameters as for \( \psi \).)
Proposition 2.2.3 (Adapted from [23], Theorem 2.1). No formula of the form

\[ \psi(xY; zW) := \bigwedge_{(i,j,k) \in I_1} Y_k(x_i, x_j) \wedge \bigwedge_{(i,j,k) \notin I_1} \neg Y_k(x_i, x_j) \wedge \bigwedge_{(i,j,k) \in I_2} Y_k(x_i, z_j) \wedge \bigwedge_{(i,j,k) \notin I_2} \neg Y_k(x_i, z_j) \]

(\text{where } I_1 \subseteq \lg(x) \times \lg(x) \times \lg(Y), I_2 \subseteq \lg(x) \times \lg(z) \times \lg(Y), \text{ etc.}) has SOP\_2, and hence, \( T_{\text{eq}}^* \) is NTP\_1.

Proof. We work in a sufficiently saturated model \( \mathcal{M} \) of \( T_{\text{eq}}^* \). (Recall that in Theorem 1.5.6, we need some degree of saturation to obtain a tree that is 1-fti.) In what follows, we use superscripts to denote coordinates (e.g. \( a^i \) is the \( i^{th} \) coordinate of \( a \)), so as to avoid conflict with subscripts denoting nodes.

Suppose, for a contradiction, that a formula \( \psi(xY; zW) \) (of the form listed in the statement of the proposition) has SOP\_2. By Theorem 1.5.6, there is a 1-fti tree \( \langle a_\alpha r_\alpha : \alpha \in \omega^2 \rangle \) witnessing SOP\_2 for \( \psi(xY; zW) \) (so \( \lg(a_\alpha) = \lg(z) \) and \( \lg(r_\alpha) = \lg(W) \) for each \( \alpha \)). Note that by Corollary 1.5.17, since our tree is 1-fti, \( a_\eta r_\eta \cap a_\nu r_\nu \) is the same for any \( \eta \neq \nu \in \omega^2 \).

In what follows, let \( ct := a_0 r_0 \cap a_1 r_1 \) (where \( c \in \mathcal{P}^M \) and \( t \in \mathcal{E}^M \)). By Remark 1.5.18, since the tree is 1-fti, \( \text{tp}(a_\eta r_\eta / ct) = \text{tp}(a_\nu r_\nu / ct) \) for all \( \eta, \nu \in \omega^2 \). In fact, by the proof of Corollary 1.5.17, for each \( c^i \in c \) and \( t^i \in t \) there are \( k, l \) such that \( a^k_\eta = c^i \) and \( r^i_\eta = t^i \) for all \( \eta \).

Take \( b_{(0)} s_{(0)} \) and \( b_{(1)} s_{(1)} \) to be realizations of the instances of \( \psi \) at the nodes \( \langle 0 \rangle \) and \( \langle 1 \rangle \), respectively:

\[ b_{(0)} s_{(0)} \models \psi(xY; a_0 r_0); \]
\[ b_{(1)} s_{(1)} \models \psi(xY; a_1 r_1). \]

Let \( N_0, N_1, \) and \( B \) be the substructures of \( \mathcal{M} \) with universes

\[ N_0 = b_{(0)} \cup s_{(0)} \cup a_{(0)} \cup r_{(0)}; \]
\[ N_1 = b_{(1)} \cup s_{(1)} \cup a_{(1)} \cup r_{(1)}; \] and
\[ B = a_{(0)} \cup r_{(0)} \cup a_{(1)} \cup r_{(1)}, \]
We claim that we can define \( R \). Proof of claim. There are three cases to consider.

- \( q \in C \) is in \( N \) and \( q \) is not in \( B \), then \( f_C(d, e, q) \in R_0 \) if and only if \( (d, e, q) \in R^C \). Equivalently, if \( C_1 \) and \( C_2 \) are two distinct structures from the set \( \{ N_0, N_1, B \} \), and \( d, e \in f_{C_1}(C_1^P) \cap f_{C_2}(C_2^P) \) and \( q \in f_{C_1}(C_1^F) \cap f_{C_2}(C_2^F) \), then \( (d, e, q) \in f_{C_1}(R^C) \) if and only if \( (d, e, q) \in f_{C_2}(R^{C_2}) \).

Proof of claim. There are three cases to consider.

- \( (d, e, q) \in f_0(N_0) \cap f_1(N_1) \): There are several subcases, depending on whether each of \( d, e, \) and \( q \) is in \( B \) or not, but in all cases the result follows from the fact that \( N_0 \) and \( N_1 \) have the same diagram. For example, if there are \( i, j, \) and \( k \) such that \( d = b^i, e = b^j, \) and \( q = s^k, \) then
  \[
  (d, e, q) \in f(R_0) \iff N_0 \models s_{(0)}^k(b_{(0)}^i, b_{(0)}^j) \\
  \iff \psi(xY; zW) \models R(x^i, x^j, Y^k) \\
  \iff N_1 \models s_{(1)}^k(b_{(1)}^i, b_{(1)}^j) \\
  \iff (d, e, q) \in f(R_1). 
  \]

The arguments in the other subcases are the same, except that some or all of the relevant variables in the second line may be from \( z \) and \( W \), rather than \( x \) and \( Y \). (We are using the fact that elements of \( c \) and \( t \) come from the same coordinates in \( a_{(0)} \) as in \( a_{(1)} \).)
• \((d, e, q) \in f_0(N_0) \cap f_B(B)\): In this case, the result follows from the facts that 
f_0(N_0) \cap f_B(B) = N_0 \cap B\) and that \(N_0\) and \(B\) are both substructures of \(M\), so \(R^{N_0}\) and \(R^B\) agree on \(N_0 \cap B\). Suppose 
\(d = f(a_i^{(0)}) = a_i^{(0)}\), \(e = f(a_j^{(0)}) = a_j^{(0)}\), and 
\(q = f(r_k^{(0)}) = r_k^{(0)}\). Then we have:

\[(d, e, q) \in f_0(R^{N_0}) \iff N_0 \models r_k^{(0)}(a_i^{(0)}, a_j^{(0)})\]

\[\iff M \models r_k^{(0)}(a_i^{(0)}, a_j^{(0)})\]

\[\iff B \models r_k^{(0)}(a_i^{(0)}, a_j^{(0)})\]

\[\iff (d, e, q) \in f_B(R^B).
\]

• \((d, e, q) \in f_1(N_1) \cap f_B(B)\): The argument is the same as in the previous case.

We extend \(R_0\) to all of \(P^N \times P^N \times E^N\) as follows:

• \(R^N(b_i^{(0)}, a_j^{(0)}, r_k^{(1)})\) and \(R^N(a_i^{(0)}, b_i^{(0)}, r_k^{(1)})\) hold if and only if there is \(l\) such that:

\[N_1 \models R(b_i^{(1)}, a_j^{(1)}, r_k^{(1)})\] and

\[B \models R(a_i^{(1)}, b_i^{(1)}, r_k^{(1)}).
\]

• \(R^N(b_i^{(0)}, a_j^{(1)}, r_k^{(0)})\) and \(R^N(a_i^{(1)}, b_i^{(1)}, r_k^{(0)})\) hold if and only if there is an \(l\) such that:

\[N_0 \models R(b_i^{(0)}, a_j^{(0)}, r_k^{(0)})\] and

\[B \models R(a_i^{(0)}, b_i^{(0)}, r_k^{(0)}).
\]

• \(R^N(a_i^{(0)}, a_j^{(0)}, s_{lk})\) and \(R^N(a_j^{(0)}, a_i^{(0)}, s_{lk})\) hold if and only if there is an \(l\) such that:

\[N_0 \models R(a_i^{(0)}, b_i^{(0)}, s_{lk}(0))\] and

\[N_1 \models R(a_j^{(0)}, b_j^{(0)}, s_{lk}(1)).
\]

Claim 2.2.6. \(R^N\), as defined above, gives an equivalence relation for each \(i \in E^N\).

Proof of claim. The reflexivity of and symmetry of \(R^N\) come from the definition (and the fact that \(R^{N_0}, R^{N_1},\) and \(R^B\) are reflexive and symmetric). We must show that \(R^N(\cdot, \cdot, q)\) is transitive for all \(q \in E^N\).

1. \(q = r_k^{(1)} \in r_{(1)}\) \(R^N(u, v, r_k^{(1)})\) and \(R^N(v, w, r_k^{(1)}).

a) \( v \in a_{(1)} \): then \( R^N(u, w, r'_{(1)}) \) follows from one of: transitivity of \( R^N_i(\cdot, \cdot, r'_{(1)}) \) (if \( u, w \in f_1(N_1) \)), transitivity of \( R^S(\cdot, \cdot, r'_{(1)}) \) (if \( u, w \in f_0(B) = B \)), or the definition of \( R^N \) (if one of \( u, w \) comes from \( b' \) - i.e., \( f_1(N_1) \setminus (f_B(B) \cap f_1(N_1)) \) - and the other comes from \( a_{(0)} \) - i.e., \( f_B(B) \setminus (f_B(B) \cap f_1(N_1)) \)).

b) \( v = a^j_{(0)} \in a_{(0)} \).

- If \( u, w \in f_B(B) = B \), then \( R^N(u, w, r'_{(1)}) \) follows from transitivity of \( R^S(\cdot, \cdot, r'_{(1)}) \). Otherwise, one or both of \( u, w \) comes from \( b' \).
- Suppose \( u = b^i \), \( w = b^k \) (i.e., \( u, b \in N \setminus f_B(B) \)). Then by the definition of \( R^N, R^N(b^i, a^j_{(0)}, r'_{(1)}) \) and \( R^N(a^j_{(0)}, b^k, r'_{(1)}) \) imply that there are \( m \) and \( n \) such that:

\[
\begin{align*}
R^N_i(b^i_{(1)}, a^m_{(1)}, r'_{(1)}) & \quad \text{and} \quad R^S(a^j_{(0)}, a^m_{(1)}, r'_{(1)}) \\
R^S(a^j_{(0)}, a^n_{(1)}, r'_{(1)}) & \quad \text{and} \quad R^N_i(b^k_{(1)}, a^n_{(1)}, r'_{(1)})
\end{align*}
\]

By transitivity of \( R^S(\cdot, \cdot, r'_{(1)}) \), it follows that \( R^S(a^m_{(1)}, a^n_{(1)}, r'_{(1)}) \). By Claim 2.2.5, \( R^N_i(a^m_{(1)}, a^n_{(1)}, r'_{(1)}) \) holds, too. Two applications of the transitivity of \( R^N_i(\cdot, \cdot, r'_{(1)}) \) give us that \( R^N_i(b^i_{(1)}, b^k_{(1)}, r'_{(1)}) \), and thus that \( R^N(b^i, b^k, r'_{(1)}) \).

- Suppose \( u = b^i \) and \( w \in B \). We have \( a^m_{(1)} \) as above. Since \( R^S(a^j_{(0)}, w, r'_{(1)}) \), transitivity of \( R^S(\cdot, \cdot, r'_{(1)}) \) gives us that \( R^S(a^m_{(1)}, w, r'_{(1)}) \). Then \( R^N(b^i, w, r'_{(1)}) \) comes from either transitivity of \( R^N_i(\cdot, \cdot, r'_{(1)}) \) (if \( w \in a_{(1)} \), in which case we use, once again, the fact that \( R^S \) and \( R^N \) agree on elements of \( a_{(1)}r'_{(1)} \) or from the definition of \( R^N \) (if \( w \in a_{(0)} \)).

c) \( v = b^j \in b' \). The reasoning here will be the same as in the previous subcase.

2. \( q = r'_{(0)} \in r_{(0)} \). \( R^N(u, v, r'_{(1)}) \) and \( R^N(v, w, r'_{(0)}) \). The reasoning here will be the same as in the previous case.

3. \( q = s'^{i} \in s' \). \( R^N(u, v, s'^{i}) \) and \( R^N(v, w, s'^{i}) \).

a) \( v \in b' \). Then \( R^N(u, w, s'^{i}) \) follows from transitivity in \( R^{N_i} \) (respectively, \( R^{N_0} \)) if \( u \) and \( w \) are both in \( f_1(N_1) \) (respectively, \( f_0(N_0) \)). If \( u \in a_{(0)} \) and \( w \in a_{(1)} \) (or vice versa), \( R^N(u, w, s'^{i}) \) comes from the definition of \( R^N \).

b) \( v = a^j_{(0)} \in a_{(0)} \).

- If \( u, w \in f_0(N_0) \), then \( R^N(u, w, s'^{i}) \) comes from transitivity in \( R^{N_0} \). Otherwise, one or both of \( u, w \) comes from \( a_{(1)} \).
- Suppose \( u = a^i_{(1)} \), \( w = a^k_{(1)} \). Then, by definition of \( R^N, R^N(a^i_{(1)}, a^j_{(0)}, s'^{i}) \) and \( R^N(a^j_{(0)}, a^k_{(1)}, s'^{i}) \) imply that there are \( m \) and \( n \) such that:

\[
\begin{align*}
R^N_i(a^i_{(1)}, b^m_{(1)}, s'^{i}_{(1)}) & \quad \text{and} \quad R^{N_0}(a^j_{(0)}, b^m_{(0)}, s'^{i}_{(0)}) \\
R^{N_0}(a^j_{(0)}, b^n_{(0)}, s'^{i}_{(0)}) & \quad \text{and} \quad R^N_i(a^k_{(1)}, b^n_{(1)}, s'^{i}_{(1)})
\end{align*}
\]
By transitivity in $R_{N_0}$, this gives us that $R_{N_0}^N(b^{m}_0, b^{n}_0, s^l_0)$. Since $R_{N_0}$ and $R_{N_1}$ agree on their respective preimages of $b's'$, $R_{N_1}(b^{m}_0, b^{n}_0, s^l_0)$ holds. Then by transitivity (twice) in $R_{N_1}$, we have $R_{N_1}(a^i_1, a^k_1, s^l_1)$, and thus $R_{N}^N(a^i_1, a^k_1, s^l_1)$, as desired.

- Suppose $u = a^i_1$ and $w \in f_0(N_0)$. We have $b^m$ as above. Then, since $R_{N_0}(a^i_0, w, s^l_0)$, transitivity in $R_{N_0}$ gives us $R_{N_0}(w, b^m_0, s^l_0)$. Then, by definition of $R_{N}^N$ (if $w \in a(0)$) or by transitivity in $R_{N_1}$ (if $w \in b'$), it follows that $R_{N}^N(a^i_1, w, s^l)$.

c) $v = a^j_1 \in a(1)$. The reasoning here will be the same as in the previous subcase.

Claim 2.2.6 shows that $\mathcal{N} \models T_0$. We now observe that, by Claim 2.2.5 and the fact that the extension of the definition of $R_0$ to $P_0 \times P_0 \times E_0$ adds no new relations among images of elements from the same structure, $f_0$, $f_1$, and $f_B$ are embeddings, and $\mathcal{N}$ is the desired amalgam. This finishes the proof of Lemma 2.2.4.

Since $f_0 : \mathcal{N}_0 \hookrightarrow \mathcal{N}$ and $f_1 : \mathcal{N}_1 \hookrightarrow \mathcal{N}$, and since $\psi$ is quantifier free,

$$\mathcal{N} \models \psi(b's'; a(0)r_0(0)) \land \psi(b's'; a(1)r_1(1)).$$

As $\mathcal{N} \models T_0$ and $T_{feq}^*$ is the model completion of $T_0$, $\mathcal{N}$ embeds into a model $\mathcal{N}^*$ of $T_{feq}^*$. Again, since $\psi$ is quantifier free,

$$\mathcal{N}^* \models \psi(b's'; a(0)r_0(0)) \land \psi(b's'; a(1)r_1(1)).$$

Using, once again, the fact that $T_{feq}^*$ is the model completion of $T_0$, we note that $T_{feq}^* \cup \text{diag}(B)$ is a complete $\mathcal{L}_B$-theory. Since

$$(\mathcal{N}^*, a(0), r(0), a(1), r(1)) \models T_{feq}^* \cup \text{diag}(B),$$

and since $\exists x \exists Y(\psi(xy; a(0)r_0) \land \psi(xy; a(1)r_1))$ is a $\mathcal{L}_B$-sentence in $\text{Th}(\mathcal{N}^*, a(0), r(0), a(1), r(1))$, $T_{feq}^* \cup \text{diag}(B) \models \exists x \exists Y(\psi(xy; a(0)r_0) \land \psi(xy; a(1)r_1))$. We also know that

$$(\mathcal{M}, a(0), r(0), a(1), r(1)) \models T_{feq}^* \cup \text{diag}(B),$$

and so

$$\mathcal{M} \models \exists x \exists Y(\psi(xy; a(0)r_0) \land \psi(xy; a(1)r_1)),$$

contradicting the choice of $\langle a_\alpha r_\alpha : \alpha \in \omega \rangle$ as an SOP_2 tree for $\psi(xy; zW)$. 

**Remark 2.2.7.** At first glance, one might worry that the proof of Proposition 2.2.3 contradicts the fact that $T_{feq}^*$ has the tree property, since a tree witnessing TP must contain nodes $\langle 0 \rangle$ and $\langle 1 \rangle$ such that the instances of the formula in question corresponding to these nodes are inconsistent with each other. However, Proposition 2.2.3 relies on the fact that an SOP_2 (or TP_1) tree can be chosen to be 1-fti. The same is not always true for formulae with (only) the tree property, as we noted in Remark 1.5.8.
Chapter 3

Limit Independence

In this chapter, we begin by examining theories of infinite dimensional vector spaces with a non-degenerate bilinear form, studied by Nicolas Granger in his thesis [9]. The well-behaved non-forking independence in finite dimensional subspaces (which are $\omega$-stable of finite Morley rank) of a model of this theory can be lifted to another independence relation (not quite, but almost, as well-behaved) in the larger structure. We generalize this independence relation construction to any theory whose models can be “approximated” by certain substructures in the same way that Granger’s infinite dimensional vector space is approximated by finite dimensional subspaces.

3.1 Motivation

Here we present the two-sorted theory of an infinite dimensional vector space with a bilinear form, along with Granger’s independence relation for this theory. Most definitions and results in this section are from chapters 9, 10 and 12 of [9].

Granger uses a language with sorts $V$ (for the vector elements) and $K$ (for the field elements). The language contains the following functions and constants:

- $0_V$ vector additive identity
- $0_K$ field additive identity
- $1_K$ field multiplicative identity
- $+_V : V \times V \to V$ vector addition
- $+_K : K \times K \to K$ field addition
- $\circ_K : K \times K \to K$ field multiplication
- $\gamma : K \times V \to V$ scalar multiplication
- $[\cdot, \cdot] : V \times V \to K$ bilinear form

In the following, we will use the capital letters $X, Y, Z,$ and $W$ for vector variables, the lower case letters $x, y, z,$ and $w$ for field variables, the lower case letters $a, b, c,$ and $d$ for vector
constants, and lower case greek letters $\alpha, \beta, \gamma$, etc. for field constants. We will denote field and scalar multiplication simply by concatenation of the elements being multiplied, rather than using the symbols given above. We may leave out the subscripts of $'+_V'$ and $'+_K'$ (it should be clear from the context which one we intend) and will make use of $'\sum'$ (without any subscript).

We start with the two-sorted theory of a vector space with a bilinear form. Our base theory, $T_u$, states that:

- $K$ is a field,
- $V$ is a vector space over $K$, and
- $[,]$ is a bilinear form, that is:

\[
\forall X, Y, Z ([X + Y, Z] = [X, Z] + [Y, Z]) \\
\forall X, Y, Z ([X, Y + Z] = [X, Y] + [X, Z]) \\
\forall X, Y \forall z ([zX, Y] = z[X, Y] = [X, zY]).
\]

(The subscript "u" stands for "universal": although this theory is not universal, it could be made so by adding symbols for $-V$, $-K$, and $\div_K$. We include only universal axioms for the bilinear form. In particular, $T_u$ does not require the form to be non-degenerate.) We build up to a completion of $T_u$ as follows.

- Let $T$ be the two-sorted theory of a vector space with a non-degenerate bilinear form:

\[
T = T_u \cup \{ \forall X (\forall Y ([X, Y] = 0_K) \rightarrow X = 0_V) \}.
\]

- Given a field $\tilde{K}$, $T\tilde{K}$ adds to the above the complete theory of $\tilde{K}$ (for the sort $K$):

\[
T\tilde{K} := T \cup Th(\tilde{K}).
\]

- $A_T$ declares that the bilinear form is alternating, while $S_T$ declares that it is symmetric:

\[
A_T := T \cup \{ \forall X ([X, X] = 0_K) \}; \\
S_T := T \cup \{ \forall X, Y ([X, Y] = [Y, X]) \}.
\]

- Lastly, we specify the dimension. For each $n < \omega$, let $\Theta_n$ be the statement that there are at least $n$ linearly independent vectors:

\[
\Theta_n := \exists X_0, \ldots, X_{n-1} \forall y_0, \ldots, y_{n-1} \left( \sum_{i=0}^{n-1} y_i X_i = 0_V \rightarrow \bigwedge_{i=0}^{n-1} y_i = 0_K \right).
\]
A vector space with an alternating form must be of even dimension, so in that case, we will let the subscript $n$ indicate that the dimension is $2n$. If the form is symmetric, a subscript $n$ indicates that the dimension is $n$:

$$A_T^n := A_T \cup \{ \Theta_{2n} \land \neg \Theta_{2n+1} \}, \text{ and}$$

$$S_T^n := S_T \cup \{ \Theta_n \land \neg \Theta_{n+1} \}. \quad (1)$$

The theory of an infinite dimensional vector space with a bilinear form is given by

$$T_\infty := T \cup \{ \Theta_n : n < \omega \}. \quad (2)$$

If $\tilde{K}$ is finite and $F \in \{ S, A \}$, $F T^K_\infty$ is $\kappa_0$-categorical, super simple of SU-rank 1, and unstable. (The formula $[X, Y] = 0$ has the order property.) See [9], Corollary 6.2.4, Proposition 6.2.5, and Remark 6.2.6. From now on, we assume $\tilde{K}$ is infinite.

To obtain a quantifier elimination result when the field is infinite, we expand the language to include predicates for linear independence. For each $n \in \omega$, let

$$\theta_n(X_0, \ldots, X_{n-1}) := \forall y_0, \ldots, y_{n-1} \left( \sum_{i=0}^{n-1} y_i X_i = 0 \right) \rightarrow \left( \bigwedge_{i=0}^{n-1} y_i = 0 \right). \quad (3)$$

Let $L_{\theta}$ be the language obtained from $L$ by adding each $\theta_n$ as an atomic formula. (More precisely, $L_{\theta}$ is the definitional expansion of $L$ having new $n$-ary predicate symbols on the sort $V$ to be interpreted as $\theta_n$.) Let $L^K_{\theta}$ be the language obtained from $L_{\theta}$ by adding an elimination set for $\tilde{K}$.

**Proposition 3.1.1** ([9], Theorem 9.2.3 and Proposition 9.3.4). Given a field $\tilde{K}$, $F \in \{ S, A \}$, and $m \in \mathbb{N} \cup \{ \infty \}$, the theory $F T^K_m$ has elimination of quantifiers in the language $L^K_{\theta}$. In particular, if $\tilde{K}$ is an algebraically closed field, then $F T^K_m$ has elimination of quantifiers in the language $L_{\theta}$. Further, if $\tilde{K}$ is algebraically closed, $F T^K_m$ is model complete in the language $L$.

**Proposition 3.1.2** ([9], Corollary 9.2.9). Let $\tilde{K}$ be a field and $m \in \mathbb{N} \cup \{ \infty \}$.

1. The theory $A T^K_m$ is complete.

2. If $\tilde{K}$ is a field with square roots for each element then the theory $S T^K_m$ is complete.

Since we find it rather cumbersome to continually refer to a theory with three distinct decorations, we will generally abuse language and notation by speaking of “the theory $T_\infty$” (or “the theory $T_n$”). By this, we mean $F T^K_\infty$ ($F T^K_n$) for some field $\tilde{K}$ and $F \in \{ S, A \}$. We do not mean the incomplete theory which we earlier called $T_\infty$ ($T_n$). Unless otherwise specified, we shall assume that $\tilde{K}$ is algebraically closed.

**Proposition 3.1.3** ([9], Corollary 10.1.2). If $m$ is finite, $T_m$ is $\omega$-stable of finite Morley rank.
Fact 3.1.4. $T_\infty$ is not simple.

Proof. (See also [9], Proposition 7.4.1.) Consider the formula $\varphi(X;Y,y) := \Gamma[X,Y] = y$, and work in any model $\mathcal{M} = (V,K) \models T_\infty$. We construct a tree $\{a_\eta \gamma_\eta : \eta \in {}^{<\omega}\omega\}$ witnessing that this formula has the tree property.

- Let $a_{\eta^{-i}}$ be some vector that is linearly independent from $\{a_{\eta_j} : j \leq |\eta|\}$ (the same vector for all $i$). (This is possible because $V$ is infinite dimensional.)
- Take a countable set of field elements $\{\gamma_i : i < \omega\}$ distinct from one another and from $\{\gamma_{\eta_j} : j \leq |\eta|\}$. Let $\gamma_{\eta^{-i}} = \gamma_i$. (This is possible because $K$ is infinite.)

Now we check the tree property conditions:

- For each $\eta \in {}^{<\omega}\omega$, $\{[X,a_{\eta_j}] = \gamma_{\eta_j} : j \leq |\eta|\}$ is consistent, since the $a_{\eta_j}$’s are linearly independent. (See [9], Lemma 5.2.3.) By compactness, for any $\beta \in {}^{<\omega}\omega$,

  $$\{[X,a_{\beta_j}] = \gamma_{\beta_j} : j < \omega\}$$

  is consistent.

- For any $\eta \in {}^{<\omega}\omega$, $\{[X,a_{\eta^{-i}}] = \gamma_{\eta^{-i}} : i < \omega\}$ is inconsistent, since the $a_{\eta^{-i}}$’s are all the same, but the $\gamma_{\eta^{-i}}$’s are distinct.

Hence, the formula $[X,Y] = y$ (with $Yy$ as the parameter variables) has the tree property in $T_\infty$, and $T_\infty$ is not simple. Note that this argument holds whenever $\tilde{K}$ (the field such that $Th(\tilde{K}) \subset T_\infty$) is infinite - not just for algebraically closed fields.

We need a few more definitions before we can state Granger’s independence relation. Let $\mathcal{M} \models T_\infty$ be sufficiently saturated. If $A \subseteq M$, we call the set of vector elements of $A$ (i.e. $A \cap V^{\mathcal{M}}$) $A_V$, and the set of field elements of $A$ (i.e. $A \cap K^{\mathcal{M}}$) $A_K$. If $\text{char}(K^{\mathcal{M}}) = 0$, Granger takes $K_A$ to be the subfield of $K^{\mathcal{M}}$ generated by

- $A_K$,
- $\{[a,b] : a, b \in A_V\}$, and
- for each $n$, the sets $\{\alpha_1, \ldots, \alpha_n\}$ such that there are $b_1, \ldots, b_n, b \in A_V$ with $b_1, \ldots, b_n$ linearly independent and $b = \sum_{i=1}^n \alpha_i b_i$.

If $\text{char}(K^{\mathcal{M}}) > 0$, he takes $K_A$ to be the perfect closure of the above field.

If $A$ is a set and $L$ is a subfield of $K^{\mathcal{M}}$, we denote the $L$-linear span of the vector part of $A$ by $\text{span}_L(A_V)$. We denote the $K^{\mathcal{M}}$-linear span of $A_V$ by $\langle A \rangle$.

Remark 3.1.5 ([9], Proposition 9.5.1). For $m \in \mathbb{N} \cup \{\infty\}$, $\mathcal{M} = (V,K) \models T_m$, and $A \subseteq M$,

1. $\text{dcl}(A) = (\text{span}_{K_A}(A_V), K_A)$, and
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2. except in the case that \( m \) is finite and \( \dim(A) = m - 1 \), \( \text{acl}(A) = (\text{span}_{K^\text{alg}}(A_V), K^\text{alg}_A) \).

Remark 3.1.6. In the original statement of this Proposition, Granger specifies that \( \mathcal{M} \) should be “sufficiently saturated.” This is unnecessary, however: since each \( T_m \) is model complete with respect to \( \mathcal{L} \), \( \text{dcl}(A) \) and \( \text{acl}(A) \) are the same in any model of \( T_m \) containing \( A \).

Granger’s independence relation is as follows:

Definition 3.1.7 ([9], 12.2.1). Let \( r \in \omega, \mathcal{M} \models T_\infty, C \subseteq B \subseteq M \), and \( a \in M \). Say that \( \text{tp}(a/B) \) does not \( \Gamma \)-fork (\( \text{dnf} \)) over \( C \) if \( K_{Ca} \not\subseteq_{K_C} K_B \) (that is, \( K_{Ca} \) and \( K_B \) are algebraically independent over \( K_C \) in the field \( K^\mathcal{M} \)) and one of the following three conditions holds:

1. \( a \in K^\mathcal{M} \)
2. \( a \in \langle C \rangle \)
3. \( a \notin \langle B \rangle \) and whenever \( \{b_1, \ldots, b_n\} \subset B_V \) is linearly independent modulo \( \langle C \rangle \) then the set \( \{[a, b_1], \ldots, [a, b_n]\} \) is algebraically independent over \( K_B(K_{Ca}) \), the field generated over \( K_B \) by the elements of \( K_{Ca} \).

Extend the relation inductively to \( n \)-types as follows: \( \text{tp}(a_1, \ldots, a_n, a_{n+1}/B) \) \( \text{dnf} \) over \( C \) if and only if \( \text{tp}(a_1, \ldots, a_n/B) \) \( \text{dnf} \) over \( C \) and \( \text{tp}(a_{n+1}/Ba_1, \ldots, a_n) \) \( \text{dnf} \) over \( Ca_1, \ldots, a_n \).

This gist of this relation is this: dependence either comes from the field or from the existence of an algebraic relation among some of the pairings that was not forced by a linear relation among the vectors being paired. (Or it is trivial dependence: \( c \in \langle B \rangle \setminus \langle C \rangle \)).

Granger had already defined a very similar relation for finite dimensional vector spaces ([9], Definition 10.2.1), and showed that it was the same as non-forking independence ([9], Proposition 10.2.2) using Theorem 1.4.18, the characterization of simple theories via abstract independence. The finite dimensional relation is almost identical to the infinite dimensional relation, but contains one additional possible condition for \( c \):

4. \( a \in \langle B \rangle \setminus \langle C \rangle \) and \( \langle B \rangle = V \) and whenever \( \{b_1, \ldots, b_n\} \subset B_V \) is linearly independent modulo \( \langle C \rangle \) then the set \( \{[a, b_1], \ldots, [a, b_n]\} \) is algebraically independent over the field \( K_B(K_{Ca}) \).

Granger shows that his \( \Gamma \)-non-forking satisfies automorphism invariance, symmetry, transitivity, finite character, extension, and stationarity of types over models. In the finite dimensional case, \( \Gamma \)-non-forking also satisfies local character, but the same is not true in \( T_\infty \). (In fact, local character does hold in all countable models of \( T_\infty \), but can fail in uncountable models.)

\(^1\)Note that independence of field elements as computed in \( \mathcal{M} \) is the same as independence as computed in \( K^\mathcal{M} \), since \( K^\mathcal{M} \) is stably embedded.
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Remark 3.1.8. Since \( \Gamma \)-non-forking is symmetric for \( T_\infty \) and \( T_\infty \) is not simple, \( \Gamma \)-non-forking is not the same as non-forking in \( T_\infty \) (by Theorem 1.4.16). More concretely, the formula \( \varphi(X; b) := \Gamma[X, b] = 0_K \wedge X \neq 0_V^- \) (for some \( b \neq 0_V \)) divides over \( \emptyset \) in \( T_n \) for each \( n < \omega \) (and hence, by Proposition 3.1.11 below, \( \Gamma \)-forks over \( \emptyset \) in \( T_\infty \), but does not divide over \( \emptyset \) in \( T_\infty \). (This is discussed in [9], Example 12.3.3.)

Proof. 1. To see that \( \varphi(X; b) \) divides over \( \emptyset \) in \( T_n \) (for \( n < \omega \)), take a model \( \mathcal{N} \models T_n \) containing \( b \), and an indiscernible sequence \( (b_i : i < \omega) \) of elements of \( \mathcal{N} \) (or some elementary extension of \( \mathcal{N} \)) where \( b_0 = b \) and \( \{b_0, \ldots, b_{n-1}\} \) (and thus, any set of \( n \) distinct elements from the sequence) is linearly independent. Then \( \{\varphi(X; b_i) : i < \omega\} \) is \( n \)-inconsistent, by the non-degeneracy of the bilinear form: given distinct elements \( b_{i_0}, \ldots, b_{i_{n-1}} \) from the sequence, if \( [a, b_{j}] = 0_K \) for \( j < n \), then \( [a, d] = 0_K \) for all \( d \in V^\mathcal{N} \), since \( \{b_{i_0}, \ldots, b_{i_{n-1}}\} \) spans \( V^\mathcal{N} \). Since the form is non-degenerate, \( a = 0_V \), so \( \mathcal{N} \not\models \varphi(a; b) \).

Alternatively, take \( a \in \mathcal{M} \models T_\infty \) such that \( \mathcal{M} \models \varphi(a; b) \). Clearly \( a \notin K^\mathcal{M} \), \( a \notin \emptyset \), and \( \{[a, b]\} = \{0_K\} \) is not algebraically independent over \( K_\mathcal{b}(K_a) \), so tp\( (a/b) \) \( \Gamma \)-forks over \( \emptyset \).

2. If \( (b_i : i < \omega) \) is an indiscernible sequence in \( \mathcal{M} \models T_\infty \) with \( b_0 = b \), then \( \{\varphi(X; b_i) : i < \omega\} \) is consistent. (A finite subset just says that there is some nonzero vector that pairs with finitely many other vectors to give 0, and this presents no problem in an infinite dimensional space.) So \( \varphi(X; b) \) does not divide in \( T_\infty \).

After establishing that \( \Gamma \)-non-forking in \( T_\infty \) has many of the nice properties of an independence relation, Granger goes on to show that in a countable dimensional model of \( T_\infty \), it is the “limit” of \( \Gamma \)-non-forking (and hence, non-forking) in a sequence of finite dimensional subspaces.

Definition 3.1.9 ([9], Section 12.1). If \( \mathcal{M} \models T_\infty \), an approximating sequence for \( \mathcal{M} \) is a sequence of homogeneous substructures \( \mathcal{N}_r \subseteq \mathcal{M} \) such that \( \mathcal{M} = \bigcup_{r \in \omega} \mathcal{N}_r \), and for each \( r \), \( \mathcal{N}_r \models T_r \) and \( K^{\mathcal{N}_r} = K^\mathcal{M} \). (\( \mathcal{N} \) is a homogeneous substructure of \( \mathcal{M} \) if given tuples \( a, b \in N \) such that there is \( \sigma \in \text{Aut}(\mathcal{M}) \) sending \( a \) to \( b \), then there is \( \tau \in \text{Aut}(\mathcal{M}) \) sending \( a \) to \( b \) and fixing \( N \) setwise.)

Remark 3.1.10. By this definition, only countable dimensional models of \( T_\infty \) have approximating sequences, since a countable union of finite dimensional subspaces could not cover an uncountable dimensional model. Approximating a model of \( T_\infty \) by substructures will not be fruitful unless the substructures in question have a nicer theory than \( T_\infty \), so it would not be helpful to extend this definition by allowing a longer sequence of substructures, the tail end of which consists of infinite dimensional spaces.

Admittedly, Granger does not require that an approximating sequence of substructures be a chain of substructures (although he may be assuming it implicitly), so one could imagine
taking an uncountable sequence of finite dimensional subspaces that simply does not satisfy $N_r \subset N_s$ for $r < s$. This seems unnecessarily messy, however, and there are more reasonable ways to extend the definition. When we generalize this framework in the next section, we shall make use of directed systems rather than sequences, and this will allow us to include uncountable models in our discussion.

**Proposition 3.1.11** ([9], Proposition 12.2.3). Let $M \models T_\infty$, $C \subseteq B \subseteq M$, $p(x) \in S(B)$, and $(N_r : r \in \omega)$ be an approximating sequence for $M$. Then the following are equivalent.

1. $p(x)$ does not $\Gamma$-fork over $C$.

2. Given any formula $\varphi = \varphi(x,b) \in p(x)$, there is $R_\varphi \in \omega$ such that $\varphi(x,b)$ does not fork over $C \cap N_r$ in the structure $N_r$ for all $r \geq R_\varphi$.

**Remark 3.1.12.** The original Proposition contained a third statement: “For each finite $b \subseteq B$ there is $R_b \in \omega$ such that $p(x) \mid_{N_r \cap Cb}$ does not fork over $C \cap N_r$ in $N_r$ for all $r \geq R_b$.” This is not actually equivalent to the other two. The error in the proof lies in Granger’s assumption that $p(x) \mid_{N_r \cap Cb}$ has a realization in $N_r$ (that is, an element of $N_r$ that $N_r$ believes satisfies the type). If $p(x)$ is a complete type, however, then for every $r < \omega$, $p(x) \mid_{N_r \cap Cb}$ is not realized in $N_r$ because it contains the sentence $\Theta_{r+1}$. Thus, any complete type (trivially) forks in every $N_r$.

### 3.2 Definitions

We wish to generalize Granger’s “approximation” approach to dealing with the unsimple theory $T_\infty$. Given some theory $T$, we begin with a strongly $\kappa$-homogeneous, $\kappa$-saturated model $M \models T$, and find a directed system, $\mathcal{H}$, of substructures of $M$. In the case of $M = (K,V) \models T_\infty$, $\mathcal{H}$ is the collection of finite dimensional $K$-subspaces of $(K,V)$. Before imposing requirements on $\mathcal{H}$, let us develop some terminology to talk about behaviour in $\mathcal{H}$. Unless otherwise stated, any subset of $M$ we discuss is small (of cardinality less than $\kappa$).

**Definition 3.2.1.**

1. For $\psi(x) \in \mathcal{L}$, $a \in M$, we say that $\psi(a)$ is eventually true in (or true in the limit of) $\mathcal{H}$ if there is some $N_{a,\psi} \in \mathcal{H}$ such that $a \in N_{a,\psi}$ and for $N_{a,\psi} \subseteq N \in \mathcal{H}$, $N \models \psi(a)$. We define eventually false analogously.

2. Given a formula or finite partial type $\pi(x;y) \in \mathcal{L}$, $b \in M$, and $C \subseteq M$, we say that $\pi(x;b)$ eventually forks over $C$ if there is $N_{\pi,b,C} \in \mathcal{H}$ such that $b \in N_{\pi,b,C}$ and for all $N_{\pi,b,C} \subseteq N \in \mathcal{H}$, $\pi(x,b)$ forks over $C \cap N$ in $N$. We define eventual non-forking of a finite partial type analogously.

3. Given a (not necessarily complete) type $p(x)$ over $B$ and $C \subseteq B$, we say that $p(x)$ eventually forks over $C$ if it contains some finite subtype that eventually forks over $C$, and that it eventually does not fork over $C$ if every finite subtype eventually does not fork over $C$. 

CHAPTER 3. LIMIT INDEPENDENCE

4. For subsets $A, B,$ and $C$ of $M$, we say that $A \lVert_C^\lim B$ (A is limit independent or eventually independent from $B$ over $C$) if for each finite tuple $a \in A$, there is $N_a \in \mathcal{H}$ such that $a \in N_a$ and for all $N_a \subseteq N \in \mathcal{H}$,

$$a \lVert^{N}_{\text{acl}^M(C) \cap N} \text{acl}^M(B) \cap N$$

where $\lVert^N$ is non-forking independence as computed in $N$. That is,

$$\text{tp}^N(a / (\text{acl}^M(B) \cup \text{acl}^M(C)) \cap N)$$

does not fork over $\text{acl}^M(C) \cap N$ in $N$.

**Notation:**

1. In the following, we shall occasionally use the phrase “for large enough $N$” to abbreviate “there is $N' \in \mathcal{H}$ such that for all $N' \subseteq N \in \mathcal{H}$...” (For example, the definition of “$\pi(x, b)$ eventually forks over $C$” could be rephrased as: “for large enough $N'$, $\pi(x, b)$ forks over $C \cap N$ in $N$.”)

2. Unless otherwise stated, all structures ‘$\mathcal{N}$’ (possibly with decoration) come from $\mathcal{H}$.

3. We shall refer to $\text{acl}^M(A)$ as $\overline{A}$.

**Remark 3.2.2.**

1. When we say that a formula or type $\pi(x)$ “forks over $A$ in $\mathcal{N}$,” we mean that for some formulae $\varphi_0(x, b_0), \ldots, \varphi_{n-1}(x, b_{n-1})$ and some $k < \omega$, the following infinitary type is consistent with $\text{Th}(\mathcal{N})$:

$$\{\pi(x) \to \bigvee_{i<n} \varphi_i(x, b_i)\} \cup \bigcup_{i<n} \{\psi(x^j_i) \leftrightarrow \psi(b_i) : j < \omega, \psi \in L(A)\}$$

$$\cup \bigcup_{i<n} \{\neg \exists x \bigwedge_{j \in S} \varphi_i(x, x^j_i) : S \subset \omega, |S| = k\}.$$

Notice that under this definition of “forking in $\mathcal{N}$,” the elements of $\mathcal{H}$ need not be saturated.

2. Note that eventual non-forking is not the same as not eventually forking. In principle, a formula or type could both fork in arbitrarily large $\mathcal{N} \in \mathcal{H}$ and not fork in arbitrarily large $\mathcal{N} \in \mathcal{H}$.

3. If $p(x)$ is a complete type that eventually forks over $C$, then $p(x)$ contains a formula that eventually forks over $C$.

4. It is tempting to define eventual forking of an infinite type as the forking of restrictions of the type in large enough $\mathcal{N}$. This harks back to the problem mentioned in Remark 3.1.12: unless the elements of $\mathcal{H}$ have the same theory as $\mathcal{M}$ (which would make $\mathcal{H}$ rather unhelpful), for any $A$ and any complete type $p \in S^M(A)$, $p \lVert_N$ is false in each $\mathcal{N}$, and hence trivially forks in $\mathcal{N}$. We do not want every complete type to eventually fork, so we must instead define eventual forking of types finitarily.
5. In principle, $A \downarrow^\lim_C B$ is not equivalent to the eventual non-forking of any type from $\mathcal{M}$ (say, $\text{tp}^\mathcal{M}(A/BC)$), since limit independence involves the forking of types as computed in the approximating structures, while eventual non-forking of a type involves forking in the approximating structures of a fixed set of formulae. For example,

$$a \downarrow^\mathcal{N}_{C \cap N} \overline{B} \cap N$$

might be caused by the forking of some formula from $\text{tp}^\mathcal{N}(a/(\overline{B} \cup \overline{C}) \cap N)$ that does not appear in $\text{tp}^\mathcal{M}(a/\overline{B} \cup \overline{C})$.

**Definition 3.2.3.** Given $\mathcal{M} \models T$ and $\mathcal{H}$ a directed system of substructures $\mathcal{N}$ of $\mathcal{M}$, we say that $\mathcal{H}$ approximates $\mathcal{M}$ just in case conditions 1 through 5 hold. We also identify optional conditions 6 and 7.

1. **$\mathcal{H}$ covers $\mathcal{M}$:** $\bigcup \mathcal{H} = \mathcal{M}$.

2. **Automorphism Invariance:** $\mathcal{H}$ is closed under automorphism.

3. **Convergence of truth value:** For any $a \in \mathcal{M}$ and formula $\psi(x) \in \mathcal{L}$, if $\mathcal{M} \models \psi(a)$, then $\psi(a)$ is eventually true in $\mathcal{H}$. (It follows that if $\mathcal{M} \not\models \psi(a)$, then $\psi(a)$ is eventually false in $\mathcal{H}$: $\mathcal{M} \models \neg\psi(a)$, so for large enough $\mathcal{N}$, $\mathcal{N} \models \neg\psi(a)$, and hence $\mathcal{N} \not\models \psi(a)$.)

4. **Stabilization of (non)forking:** For every formula or finite partial type $\pi(x,y)$, tuple $b$, and set $C$ we have one of the following:
   - $\pi(x,b)$ eventually forks over $C$ in $\mathcal{H}$.
   - $\pi(x,b)$ eventually does not fork over $C$ in $\mathcal{H}$.

5. **Strong finite character of limit dependence:** If $A \downarrow^\lim_C B$, then there are $\psi(x,y,z)$ without parameters, $a \in A$, $b \in \overline{B}$, and $c \in \overline{C}$ such that $\mathcal{M} \models \psi(a,b,c)$, and $\psi(x,b,c)$ eventually forks over $\overline{C}$ in $\mathcal{H}$.

6. **Stabilization of algebraic closure:** If $\overline{A} = A$, there is $\mathcal{N}_{\text{alg}} \in \mathcal{H}$ such that for all $\mathcal{N} \supseteq \mathcal{N}_{\text{alg}}$, $A \cap N = \text{acl}^\mathcal{N}(A \cap N)$.

7. **Homogeneity:** For each $\mathcal{N} \in \mathcal{H}$, $a,b \in M$, and $A \subseteq M$, if $\text{tp}^\mathcal{M}(a/A) = \text{tp}^\mathcal{M}(b/A)$, then $\text{tp}^\mathcal{N}(a/A \cap N) = \text{tp}^\mathcal{N}(b/A \cap N)$.

**Remark 3.2.4.** 1. Condition 6 implies the following:

Given $a \in M$ and $B \subseteq M$, if there is $\mathcal{N}_{\text{alg}}$ such that for all $\mathcal{N} \supseteq \mathcal{N}_{\text{alg}}$, $a \in \text{acl}^\mathcal{N}(B \cap N)$, then $a \in \overline{B}$. (That is, if $a$ is eventually in the algebraic closure of $B$ in $\mathcal{H}$, then $a$ is in the algebraic closure of $B$ in $\mathcal{M}$.)
To see this, suppose $a \in \text{acl}^N(B \cap N)$ for $N \supseteq N_{\text{alg}}$. Then $a \in \text{acl}^N(\overline{B} \cap N)$ for $N \supseteq N_{\text{alg}}$, since $B \subseteq \overline{B}$. Since $\overline{B}$ is algebraically closed in $\mathcal{M}$, condition 6 implies that for large enough $N$, $\text{acl}^N(\overline{B} \cap N) = \overline{B} \cap N$. Hence, for large enough $N$, $a \in \overline{B} \cap N$. Thus $a \in \overline{B}$, as desired.

The converse of this statement follows from condition 3: if $a \in \overline{B}$, then for large enough $N$, $a \in \text{acl}^N(B \cap N)$, since the formula declaring $a$’s algebraicity over $B$ is eventually true in $\mathcal{H}$. (More precisely, if $\mathcal{M} \models \phi(a, b) \land (\exists n)x \phi(x, b)$, then by condition 3 there is $N_{\text{alg}}$ such that for all $N \supseteq N_{\text{alg}}$, $N \models \phi(a, b) \land (\exists n)x \phi(x, b)$, and hence $a \in \text{acl}^N(B \cap N)$.)

2. One might consider adding the following uniformity condition: “There is a collection of formulae $\{\phi_i(x, y, z) : i \in I\}$ such that for any $\mathcal{M} \models T$, there are parameters $\{a_i : i \in I\}$ such that $\mathcal{H} = \{\phi_i(x, \mathcal{M}, a_i) : i \in I\}$ satisfies the requirements of approximation.” If this holds, then approximation is really a property of the theory $T$, not of the particular model chosen. In the case of Granger’s $T_{\infty}$, the $\phi_i$ are of the form

$$\sum_{j \leq i} y_j Z_j = X,$$

and $\phi_i(X, \mathcal{M}, a_1, \ldots, a_i)$ gives rise to the $K^\mathcal{M}$-vector space generated by $\{a_1, \ldots, a_i\}$.

**Lemma 3.2.5.** Suppose $\mathcal{H}$ satisfies conditions 1 through 5. Given sets $A$, $B$, and $C$, $A \downarrow_C \text{lim} B$ if and only if $\text{tp}^\mathcal{M}(A/\overline{B} \cup C)$ eventually does not fork over $\overline{C}$.

**Proof.** $\leftarrow$: Suppose $A \nless_C \text{lim} B$. By condition 5 on $\mathcal{H}$, there are $\psi(x, y, z)$, $a \in A$, $b \in \overline{B}$, and $c \in \overline{C}$ such that $\mathcal{M} \models \psi(a, b, c)$ and $\psi(x, b, c)$ eventually forks over $\overline{C}$ in $\mathcal{H}$. Since $\psi(x, b, c) \in p = \text{tp}^\mathcal{M}(A/\overline{B} \cup C)$, $p$ eventually forks over $\overline{C}$ by definition.

$\rightarrow$: Suppose $p = \text{tp}^\mathcal{M}(A/\overline{B} \cup C)$ does not eventually not fork over $\overline{C}$. By condition 4 on $\mathcal{H}$, $p$ eventually forks over $\overline{C}$, and by definition, there is a finite subtype $\pi(x, b, c)$ of $p$ that eventually forks over $\overline{C}$. Let $a \in A$ realize $\pi(x)$ in $\mathcal{M}$. Since $\pi$ is finite, by condition 3 on $\mathcal{H}$ there is some $N_{\text{tv}} \in \mathcal{H}$ containing $a, b$, and $c$ such that for $N \supseteq N_{\text{tv}}$, $N \models \pi(a, b, c)$ (that is, $\pi(x, b, c) \subseteq \text{tp}^N(a/(\overline{B} \cup C) \cap N)$). Since $\pi$ eventually forks over $\overline{C}$, there is some $N_f$ such that for $N \supseteq N_f$, $\pi(x, b, c)$ forks over $\overline{C} \cap N$ in $N$. For $N \supseteq N_{\text{tv}} \cup N_f$, $\text{tp}^N(a/(\overline{B} \cup C) \cap N)$ forks over $\overline{C} \cap N$ in $N$. That is,

$$a \nless_N C_{\overline{C} \cap N} \overline{B} \cap N.$$

By definition, $A \nless_C \text{lim} B$. \hfill $\square$

**Remark 3.2.6.** In fact, Lemma 3.2.5 does not require condition 4 on $\mathcal{H}$, but the proof is simpler to state with that assumption.

We have defined $\downarrow_C \text{lim}$ without imposing the requirement that the set on the right side must contain the base set. We now show that this does not pose a problem; namely, that if $A \downarrow_C \text{lim} B$, we may enlarge the righthand side to contain $C$, while maintaining the independence. We shall refer to this condition as “normality on the right.”
Lemma 3.2.7. Suppose \( \mathcal{H} \) satisfies conditions 1 through 5. Given sets \( A, B, \) and \( C, \) if \( A \downarrow_C \lim \overline{B} \), then \( A \downarrow_C \lim BC. \)

Proof. Suppose \( A \downarrow_C \lim BC. \) By condition 5 on \( \mathcal{H}, \) there are \( \psi(x, y, z) \in L, a \in A, b \in \overline{BC}, \) and \( c \in \overline{C} \) such that \( M \models \psi(a, b, c) \) and \( \psi(x, b, c) \) eventually forks over \( \overline{C}. \) Take \( N_{tv}, N_f, \) and \( N_{alg} \) such that for all \( N \supseteq N_{tv}, M \models \psi(a, b, c) \), for all \( N \supseteq N_f, \psi(x, b, c) \) forks over \( \overline{C} \cap N \) in \( N, \) and for all \( N \supseteq N_{alg}, b \in \text{acl}^N(BC \cap N). \) For \( N \supseteq N_{tv} \cup N_f \cup N_{alg}, \psi(x, b, c) \in \text{tp}^N(a/\text{acl}^N(BC \cap N)) \) and forks over \( \overline{C} \cap N \) in \( N. \) That is,

\[
a \downarrow_{\overline{C} \cap N} a \text{cl}^N(BC \cap N).
\]

By Corollary 1.4.9,

\[
a \downarrow_{\overline{C} \cap N} BC \cap N.
\]

By monotonicity of non-forking,

\[
a \downarrow_{\overline{C} \cap N} (B \cup \overline{C}) \cap N,
\]

and by the definition of forking independence,

\[
a \downarrow_{\overline{C} \cap N} B \cap N.
\]

It follows by definition that \( A \downarrow_C \lim B. \)

3.3 Properties of the relation

We now show that in the event that the structures in \( \mathcal{H} \) have simple theories, the relation defined in the last section satisfies many of the properties that forking independence satisfies in a simple theory. In particular, \( \downarrow \lim \) is preserved under taking algebraic closures and has automorphism-invariance, monotonicity, base monotonicity, transitivity, normality, extension, finite character, (a weakened version of) strong finite character, and symmetry. If we impose some additional conditions on \( \mathcal{H}, \) we also get anti-reflexivity and an independence theorem over algebraically closed sets.

Lemma 3.3.1. Suppose \( \mathcal{H} \) satisfies conditions 1 through 3, and for each \( N \in \mathcal{H}, \) forking and dividing are equivalent in \( \text{Th}(N). \) For sets \( A, B, \) and \( C, A \downarrow_C \lim B \) if and only if \( \overline{A} \downarrow_C \lim \overline{B}. \)

Proof. We break the proof up into three parts.

1. \( A \downarrow_C \lim B \) if and only if \( \overline{A} \downarrow_C \lim B. \)

\[
\iff \text{: Suppose } A \downarrow_C \lim B. \text{ Then there is some finite } a \in A \text{ such that for arbitrarily large } N \in \mathcal{H}, a \downarrow_{\overline{C} \cap N} \overline{B} \cap N. \text{ But } a \in \overline{A}, \text{ too, so by definition } \overline{A} \downarrow_C \lim B.
\]
→: Suppose $A \downarrow_C^{\text{lim}} B$. Then there is some finite $a' \in A$ such that for arbitrarily large $N \in \mathcal{H}$, $a' \left[ \frac{N}{C \cap N} \right] B \cap N$. As noted in Remark 3.2.4, by condition 3 there is $N_{\text{alg}} \in \mathcal{H}$ such that for $N \supseteq N_{\text{alg}}$, $a' \in \text{acl}^N(A \cap N)$. Take $a \in A$ finite such that $a' \in \text{acl}^M(a)$, and for $N \supseteq N_{\text{alg}}$, $a \in \text{acl}^N(a)$. If $N \supseteq N_{\text{alg}}$ and $a' \left[ \frac{N}{C \cap N} \right] B \cap N$, then $\text{acl}^N(a) \left[ \frac{N}{C \cap N} \right] B \cap N$ by monotonicity of $\left[ \frac{N}{C \cap N} \right]$. Since forking and dividing are the same in $N$, it follows by Corollary 1.4.5 that $a \left[ \frac{N}{C \cap N} \right] B \cap N$. Since this is the case for arbitrarily large $N$, $A \downarrow_C^{\text{lim}} B$.

2. $A \downarrow_C^{\text{lim}} B$ if and only if $A \downarrow_C^{\text{lim}} B$.

\[
A \downarrow_C^{\text{lim}} B \iff a \in A, \ a \left[ \frac{N}{C \cap N} \right] B \cap N \text{ for arbitrarily large } N \\
\iff a \in A, \ a \left[ \frac{N}{C \cap N} \text{acl}(B) \right] \cap N \text{ for arbitrarily large } N \\
\iff A \downarrow_C^{\text{lim}} B.
\]

3. $A \downarrow_C^{\text{lim}} B$ if and only if $A \downarrow_C^{\text{lim}} B$. The proof is the same as in the previous case. That is, the statement follows from the fact that $\text{acl}^M(\text{acl}^N(C)) = \text{acl}^M(C)$.

Combining these three facts, we have:

\[
A \downarrow_C^{\text{lim}} B \text{ if and only if } \overline{A} \downarrow_C^{\text{lim}} B \\
\text{if and only if } \overline{A} \downarrow_C^{\text{lim}} B \\
\text{if and only if } \overline{A} \downarrow_C^{\text{lim}} B.
\]

\[\square\]

**Theorem 3.3.2.** Given a $\kappa$-saturated, strongly $\kappa$-homogeneous model $\mathcal{M}$ of $T$, approximated by $\mathcal{H}$ (that is, satisfying conditions 1 through 5, above), where each $N \in \mathcal{H}$ has a simple theory, $\downarrow^{\text{lim}}$ has the following properties:

- **invariance** If $A \downarrow_C^{\text{lim}} B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \downarrow_C^{\text{lim}} B'$.
- **monotonicity** If $A \downarrow_C^{\text{lim}} B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \downarrow_C^{\text{lim}} B'$.
- **base monotonicity** If $D \subseteq C \subseteq B$ and $A \downarrow_D^{\text{lim}} B$, then $A \downarrow_C^{\text{lim}} B$
- **transitivity** If $D \subseteq C \subseteq B$ then $A \downarrow_D^{\text{lim}} C$ and $A \downarrow_C^{\text{lim}} B$ if and only if $A \downarrow_D^{\text{lim}} B$. (The ‘if’ - known as partial transitivity - follows from monotonicity and base monotonicity.)
- **normality** If $A \downarrow_C^{\text{lim}} B$, then $AC \downarrow_C^{\text{lim}} B$. 


• **extension** If $A \downarrow_c^{\lim} B$ and $B \subseteq B'$, then there is $A' \equiv_{BC} A$ such that $A' \downarrow_c^{\lim} B'$.

• **finite character** If $A \uparrow_c^{\lim} B$, then there are finite tuples $a \in A$, $b \in B$ such that $a \uparrow_c^{\lim} b$.

• **strong finite character** If $A \uparrow_c^{\lim} B$, then there are finite tuples $a \in A$, $b \in B$, $c \in C$, and a formula $\psi(x,y,z)$ without parameters such that
  
  \[ \forall a, b, c \quad a \uparrow_c^{\lim} b \quad \text{for all} \quad a \models \psi(x,b,c). \]

• **symmetry** $A \downarrow_c^{\lim} B$ if and only if $B \downarrow_c^{\lim} A$.

**Proof.** Since $\text{Th}(N)$ is simple for each $N \in \mathcal{H}$, $\Downarrow^N_c$ is dividing independence in $N$. In the following, we will frequently refer to “$N_a$, as in the definition of $\Downarrow^N_c$.” Recall that if $A \downarrow_c^{\lim} B$, then for each finite $a \in A$ there is $N_a$ such that for $N \supseteq N_a$,

\[ a \downarrow_c^{N} \mathcal{N}_N \cap \mathcal{B} \cap N. \]

This is the $N_a$ to which we are referring.

**Invariance.** Suppose $A \downarrow_c^{\lim} B$ and $(A,B,C) \equiv (A', B', C')$. We would like to show that $A' \downarrow_{c'}^{\lim} B'$. Let $\sigma$ be an automorphism sending $A$ to $A'$, $B$ to $B'$, and $C$ to $C'$. For any $a \in A$, let $N_a$ be as in the definition of $\Downarrow_{c'}^{\lim}$. By condition 2 on $\mathcal{H}$, $\sigma(N) \in \mathcal{H}$ for any $N \in \mathcal{H}$, so we may let $N_{\sigma(a)} = \sigma(N_a)$. We claim that $N_{\sigma(a)}$ is as required.

Write $a'$ for $\sigma(a)$, and let $N' \supseteq N_{a'}$. We must show that

\[ a' \downarrow_{c'}^{N'_{\sigma(N)}} \mathcal{B} \cap N'. \]

If not, there are $b' \in \mathcal{B} \cap N'$, $c' \in \mathcal{C} \cap N'$ and $\psi(x,y,z)$ such that

\[ \psi(x,b',c') \in \text{tp}_{N'}((\mathcal{B} \cup \mathcal{C}) \cap N') \]

and $\psi(x,b',c')$ divides over $\mathcal{C} \cap N'$ in $N'$. Hence, the following type is consistent with $\text{Th}(N')$ (for some $k < \omega$):

\[ \{ \varphi(y_i, z_i) \leftrightarrow \varphi(b', c') : i < \omega, \varphi \in \mathcal{L}(\mathcal{C} \cap N') \cup \{ \exists x \bigwedge_{i \in S} \psi(x,y_i,z_i) : S \subseteq \omega, |S| = k \}. \]

It follows that the image under $\sigma^{-1}$ of this type is consistent with $\text{Th}(N)$:

\[ \{ \varphi(y_i, z_i) \leftrightarrow \varphi(b, c) : i < \omega, \varphi \in \mathcal{L}(\mathcal{C} \cap N) \cup \{ \exists x \bigwedge_{i \in S} \psi(x,y_i,z_i) : S \subseteq \omega, |S| = k \}. \]
(where $\sigma(b) = b', \sigma(c) = c'$, and $\sigma(N) = N'$). This implies that $\psi(x, b, c)$ $k$-divides over $B \cap C \cap N$ in $N$. Since $\psi(x, b', c') \in \text{tp}^{N'}((a'/(B \cup C') \cap N')$, $\psi(x, b, c) \in \text{tp}^{N}(a/(B \cup C) \cap N)$, and we have

$$a \nmid \frac{\text{tp}^{N}}{C \cap N} B \cap N.$$ 

Since $N' \supseteq N_a'$, $N \supseteq N_a$, and the above is a contradiction.

**Monotonicity.** Suppose $A_0 \subseteq A$, $B_0 \subseteq B$, and $A \nmid \limits_c^{\text{lim}} B$. If $a \in A_0$, then $a \in A$, and so there is $N_a$ such that for $N \supseteq N_a$,

$$a \nmid \frac{\text{tp}^{N}}{C \cap N} B \cap N.$$ 

By monotonicity of non-forking independence in $N$, and since $B_0 \cap N \subseteq B \cap N$,

$$a \nmid \frac{\text{tp}^{N}}{C \cap N} B_0 \cap N.$$ 

So the same $N_a$’s witness that $A_0 \nmid \limits_c^{\text{lim}} B_0$.

**Base monotonicity.** Suppose $D \subseteq C$, and $A \nmid \limits_c^{\text{lim}} B$. We must show that $A \nmid \limits_c^{\text{lim}} B$. If not, then by condition 5 on $H$, there are $a \in A$, $b \in B$, $c \in C$ and $\psi$ such that $M \models \psi(a, b, c)$ and $\psi(x, b, c)$ eventually forks over $\overline{C}$. That is, there is some $N_f \in H$ such that for $N \supseteq N_f$, $\psi(x, b, c)$ forks over $\overline{C} \cap N$ in $N$. Since $M \models \psi(a, b, c)$, there is some $N_{tv} \in H$ such that for $N \supseteq N_{tv}$, $N \models \psi(a, b, c)$. So for $N \supseteq N_f \cup N_{tv}$, $\text{tp}^{N}(a/bc)$ forks over $\overline{C} \cap N$ in $N$. It follows that $\text{tp}^{N}(a/(B \cup C) \cap N)$ does as well, that is,

$$a \nmid \frac{\text{tp}^{N}}{C \cap N} B \cap N.$$ 

By base monotonicity for $\nmid$,

$$a \nmid \frac{\text{tp}^{N}}{D \cap N} B \cap N.$$ 

For $N \supseteq N_a$ (where $N_a$ is as in the definition of $\nmid_c^{\text{lim}}$), this contradicts our assumption that $A \nmid \limits_c^{\text{lim}} B$.

**Transitivity.** Suppose $D \subseteq C \subseteq B$, $A \nmid \limits_c^{\text{lim}} C$, and $A \nmid \limits_c^{\text{lim}} B$. Then for each $a \in A$ there are $N_a^{DC}$ and $N_a^{CB}$ such that:

- for $N \supseteq N_a^{DC}$, $a \nmid \frac{\text{tp}^{N}}{C \cap N} B \cap N$;
- and for $N \supseteq N_a^{CB}$, $a \nmid \frac{\text{tp}^{N}}{C \cap N} B \cap N$.

Let $N_a^{DB}$ be some element of $H$ containing $N_a^{DC} \cup N_a^{CB}$. For $N \supseteq N_a^{DB}$ we have, by transitivity of $\nmid$,

$$a \nmid \frac{\text{tp}^{N}}{D \cap N} B \cap N.$$ 

It follows that $A \nmid \limits_D^{\text{lim}} B$, as desired.
Normality. Suppose \( A \downarrow^\lim_C B \). We must show that \( AC \downarrow^\lim_C B \). If not, then by condition 5 on \( \mathcal{H} \), there are \( \psi(xy;zw), a \in A, c \in C, b \in \overline{B}, \) and \( c' \in \overline{C} \) such that \( \mathcal{M} \models \psi(ac;bc') \) and \( \psi(xy;bc') \) eventually forks over \( \overline{C} \) in \( \mathcal{H} \). Take \( \mathcal{N}_f \) such that for \( \mathcal{N} \supseteq \mathcal{N}_f, \psi(xy;bc') \) forks over \( \overline{C} \cap N \) in \( \mathcal{N} \). Take \( \mathcal{N}_{tv} \) such that for \( \mathcal{N} \supseteq \mathcal{N}_{tv}, \mathcal{N} \models \psi(ac;bc') \). For \( \mathcal{N} \supseteq \mathcal{N}_f \cup \mathcal{N}_{tv} \), then, \( \text{tp}^{\mathcal{N}}(ac/(\overline{B} \cup \overline{C}) \cap N) \) forks over \( \overline{C} \cap N \) in \( \mathcal{N} \), that is,
\[
ac \vdash_{\overline{C} \cap N}^{\mathcal{N}} B \cap N.
\]
By monotonicity of \( \downarrow^N \), \( \{a\} \cup (\overline{C} \cap N) \vdash_{\overline{C} \cap N}^{\mathcal{N}} B \cap N \). By normality of \( \downarrow^N \), \( a \vdash_{\overline{C} \cap N}^{\mathcal{N}} B \cap N \). For \( \mathcal{N} \supseteq \mathcal{N}_0 \) (where \( \mathcal{N}_0 \) is as in the definition of \( \downarrow^\lim \)), this is a contradiction.

Extension. We first show that the extension property for types holds, that is:

**Claim 3.3.3.** Given a partial type \( \pi(x) \) over \( B \) that eventually does not fork over \( C \subseteq B \), there is a complete type \( p \in S(B) \) extending \( \pi \) such that \( p \) eventually does not fork over \( C \).

**Proof of Claim 3.3.3.**

**Subclaim.** If \( \pi(x) \) is a partial type that eventually does not fork over \( C \), and \( \psi(x,b) \) is a formula, then either \( \pi(x) \cup \{\psi(x,b)\} \) eventually does not fork over \( C \), or \( \pi(x) \cup \{\neg\psi(x,b)\} \) eventually does not fork over \( C \).

**Proof of subclaim.** Suppose both \( \pi(x) \cup \{\psi(x,b)\} \) and \( \pi(x) \cup \{\neg\psi(x,b)\} \) eventually fork over \( C \). Then by the definition of eventual forking for types, there are finite subtypes \( \pi_0 \) and \( \pi_1 \) of \( \pi \) such that \( \pi_0(x) \cup \{\psi(x,b)\} \) and \( \pi_1(x) \cup \{\neg\psi(x,b)\} \) eventually fork over \( C \). For \( i \in \{0,1\} \), let \( \mathcal{N}_i \) be such that for \( \mathcal{N}_i \subseteq \mathcal{N} \in \mathcal{H} \), \( \pi_i \cup \{\psi_i(x,b)\} \) forks over \( C \cap N \) in \( \mathcal{N} \) (where \( \psi_0 = \psi \) and \( \psi_1 = \neg\psi \)). For \( \mathcal{N}_0 \cup \mathcal{N}_1 \subseteq \mathcal{N} \in \mathcal{H} \), both of these finite types fork over \( C \cap N \) in \( \mathcal{N} \), and we have
\[
\pi_0(x) \cup \{\psi(x,b)\} \vdash_{\mathcal{N}} \bigvee_{i \in I_\mathcal{N}} \theta_i(x)
\]
and
\[
\pi_1(x) \cup \{\neg\psi(x,b)\} \vdash_{\mathcal{N}} \bigvee_{i \in I_\mathcal{N}} \eta_i(x)
\]
where each \( \theta_i \) and \( \eta_i \) divides over \( C \cap N \) in \( \mathcal{N} \) (and the particular formulae may depend on \( \mathcal{N} \)). It follows that
\[
\pi_0(x) \vdash_{\mathcal{N}} \psi(x,b) \rightarrow \bigvee_{i \in I_\mathcal{N}} \theta_i(x)
\]
\[
\pi_1(x) \vdash_{\mathcal{N}} \neg\psi(x,b) \rightarrow \bigvee_{i \in I_\mathcal{N}} \eta_i(x)
\]
Since \( \pi_0(x) \cup \pi_1(x) \vdash_{\mathcal{N}} \psi(x,b) \lor \neg\psi(x,b) \) (trivially), we have
\[
\pi_0(x) \cup \pi_1(x) \vdash_{\mathcal{N}} \bigvee_{i \in I_\mathcal{N}} \theta_i(x) \lor \bigvee_{i \in I_\mathcal{N}} \eta_i(x)
\]
But then \( \pi(x) \vdash_{\mathcal{N}} \bigvee_{i \in I_{\mathcal{N}}} \theta_i(x) \lor \bigvee_{i \in I_{\mathcal{N}}} \eta_i(x) \), and \( \pi(x) \) forks over \( C \cap \mathcal{N} \) in \( \mathcal{N} \). Since this happens in all large enough \( \mathcal{N} \) (with the witnessing disjunctions of dividing formulae possibly changing), \( \pi \) eventually forks over \( C \) in \( \mathcal{H} \), contrary to our hypothesis. \( \square \)

To finish the proof of the claim, we enumerate the formulae with parameters from \( B \): \( \{ \psi_i(x) : \psi_i(x) \in \mathcal{L}(B) \} \). We construct a consistent (in \( \mathcal{M} \)), eventually non-forking completion of \( \pi(x) \) by induction:

- \( \pi^0(x) = \pi(x) \) (This is consistent and eventually non-forking by hypothesis.)
- For limit cardinals \( \lambda < |B| \), let \( \pi^\lambda := \bigcup_{\beta < \lambda} \pi^\beta \). (Since consistency and eventual non-forking are local properties, and all the \( \pi^\beta \)'s are consistent and eventually non-forking, so is \( \pi^\lambda \).)
- Given a consistent (in \( \mathcal{M} \)) eventually non-forking type \( \pi^\alpha \), consider \( \psi_\alpha \): by the subclaim, at least one of \( \pi^\alpha \cup \{ \psi_\alpha \} \), \( \pi^\alpha \cup \{ \neg \psi_\alpha \} \) is consistent and eventually non-forking. (The consistency in \( \mathcal{M} \) follows from the eventual non-forking.) If the former is, let \( \pi^{\alpha+1} := \pi^\alpha \cup \{ \psi_\alpha \} \). Otherwise, let \( \pi^{\alpha+1} := \pi^\alpha \cup \{ \neg \psi_\alpha \} \).
- Let \( p(x) := \bigcup_{\beta < |B|} \pi^\alpha \). By construction, \( p \) is consistent, complete over \( B \), extends \( \pi(x) \), and eventually does not fork over \( C \).

By Lemma 3.2.5, \( A \downarrow^\lim_{C} B \) implies that \( p(X) = \text{tp}^{\mathcal{M}}(A/B \cup \overline{C}) \) eventually does not fork over \( \overline{C} \). Since \( B \subset B', \overline{B} \cup \overline{C} \subset \overline{B'} \cup \overline{C} \), and by Claim 3.3.3, there is an extension \( q(X) \in S^{\mathcal{M}}(\overline{B'} \cup \overline{C}) \) of \( p(X) \) that eventually does not fork over \( \overline{C} \).

Let \( A' \models q(X) \). Since \( q(X) = \text{tp}^{\mathcal{M}}(A'/\overline{B'} \cup \overline{C}) \) eventually does not fork over \( \overline{C} \), we apply Lemma 3.2.5 again to get

\[ A' \downarrow_{C}^\lim B' \]

Lastly, since \( q(X) \supset p(X) \), \( A' \equiv_{\overline{B}, \overline{C}} A \), and so \( A' \equiv_{BC} A \).

**Finite character.** We shall show that the following, slightly stronger statement:

**Claim 3.3.4.** If \( A \not\models^\lim_{C} B \), there are \( a \in A, b \in B \) such that for large enough \( \mathcal{N} \), \( a \not\models^{\mathcal{N}_{C \cap N}}_{C \cap N} b \).

(This is slightly stronger than finite character because we specify that in each \( \mathcal{N} \), the forking actually comes from \( b \), rather than \( \overline{b} \cap \mathcal{N} \).)

**Proof of Claim.** Suppose \( A \not\models^\lim_{C} B \), and let \( a \in A, b' \in \overline{B}, c' \in \overline{C} \), and \( \psi \) be as in condition 5 on \( \mathcal{H} \). Let \( \varphi(x, y) \in \mathcal{L}, n < \omega, \) and \( b \in B \) be such that:

\[ \mathcal{M} \models (\exists^=n \varphi)(x, b) \land \varphi(b', b). \]

Further, let \( \mathcal{N}_{\text{alg}} \) be the element of \( \mathcal{H} \) guaranteed by condition 3 such that

for \( \mathcal{N} \supseteq \mathcal{N}_{\text{alg}} \), \( \mathcal{N} \models (\exists^=n \varphi)(x, b) \land \varphi(b', b). \)
That is, for $\mathcal{N} \supseteq \mathcal{N}_{\text{alg}}$, $b' \in \text{acl}^\mathcal{N}(b)$.

Next, take $\mathcal{N}_{tv}$ such that for $\mathcal{N} \supseteq \mathcal{N}_{tv}$,
$$\mathcal{N} \models \psi(a, b', c'),$$
and $\mathcal{N}_f$ such that for $\mathcal{N} \supseteq \mathcal{N}_f$, $\psi(x, b', c')$ forks over $\overline{C} \cap N$ in $\mathcal{N}$. Then for $\mathcal{N} \supseteq \mathcal{N}_{tv} \cup \mathcal{N}_f$, $\text{tp}^\mathcal{N}(a/b'c')$ forks over $\overline{C} \cap N$ in $\mathcal{N}$, i.e.
$$a \nmid_{\overline{C} \cap N} b'.$$

If $\mathcal{N} \supseteq \mathcal{N}_{tv} \cup \mathcal{N}_f \cup \mathcal{N}_{\text{alg}}$, monotonicity implies that
$$a \nmid_{\overline{C} \cap N} \text{acl}^\mathcal{N}(b).$$

By Corollary 1.4.9,
$$a \nmid_{\overline{C} \cap N} b.$$

**Strong finite character**. Suppose $A \nmid_{\overline{C} \cap N} \text{lim} B$. Then by condition 5 on $\mathcal{H}$, there are $a \in A$, $b \in \overline{B}$, $c \in \overline{C}$, $\psi(x, y, z)$ without parameters, and $\mathcal{N}_f$ such that $\mathcal{M} \models \psi(a, b, c)$ and $\psi(x, b, c)$ forks over $\overline{C} \cap N$ in $\mathcal{N}$ for all $\mathcal{N} \supseteq \mathcal{N}_f$. The first part of strong finite character is clearly satisfied by this choice of $\psi$, $a$, $b$, and $c$. Now suppose that $\mathcal{M} \models \psi(a', b, c)$. By condition 3 on $\mathcal{H}$, there is $\mathcal{N}_{tv}$ such that for $\mathcal{N} \supseteq \mathcal{N}_{tv}$, $\mathcal{N} \models \psi(a', b, c)$. Let $\mathcal{N}_0$ be some element of $\mathcal{H}$ containing $\mathcal{N}_f \cup \mathcal{N}_{tv}$. For $\mathcal{N} \supseteq \mathcal{N}_0$, $\text{tp}^\mathcal{N}(a'/bc)$ forks over $\overline{C} \cap N$ in $\mathcal{N}$, i.e.
$$a' \nmid_{\overline{C} \cap N} b.$$

By definition, $a' \nmid_{\overline{C} \cap N} \text{lim} b$, as desired.

**Symmetry.** Suppose $B \nmid_{\overline{C} \cap N} \text{lim} A$. By Claim 3.3.4, there are $b \in B$, $a \in A$, and $\mathcal{N}_b$ such that for all $\mathcal{N}_b \subseteq \mathcal{N} \in \mathcal{H}$,
$$b \nmid_{\overline{C} \cap N} a.$$

By symmetry of $\nmid^\mathcal{N}$, $a \nmid_{\overline{C} \cap N} b$, and hence by monotonicity of $\nmid^\mathcal{N}$, $a \nmid_{\overline{C} \cap N} \overline{B} \cap N$. By definition, $A \nmid_{\overline{C} \cap N} \text{lim} B$.

**Remark 3.3.5.**

1. Our ‘strong finite character’* is somewhat weaker than standard strong finite character, as we must look in $\overline{B}$ and $\overline{C}$ for our parameters, rather than $B$ and $C$.

2. It is useful to identify where each of the hypotheses on $\mathcal{H}$ is used. The **covering condition** is used throughout. As for the others:

   - The proofs of transitivity and symmetry use the full strength of **simplicity** of each $\text{Th}(\mathcal{N})$.

   - The **automorphism invariance** condition is used in the proof of invariance.
The proofs of base monotonicity, extension, and strong finite character all depend on Lemma 3.2.5, and hence on the convergence of truth value, stabilization of non-forking, and strong finite character hypotheses.

The proofs of normality and finite character use the convergence of truth value and strong finite character hypotheses. The proof of normality also uses the normality of each $\lim_N$, and the proof of finite character uses monotonicity of $\lim_N$ and Corollary 1.4.9. These are all properties of non-forking independence in any theory.

The proof of monotonicity relies only on the monotonicity of non-forking, which is true in any theory.

The extension for types claim (Claim 3.3.3) uses only the forking stabilization hypothesis.

Proposition 3.3.6. If $\mathcal{M} \models T$ and $\mathcal{H}$ satisfies conditions 1 and 6, then $\lim$ satisfies anti-reflexivity:

$$a \lim_B a \text{ implies } a \in B.$$

Proof. Suppose $a \lim_B a$. Let $\mathcal{N}_a$ be as in the definition of $\lim$. For $\mathcal{N} \supseteq \mathcal{N}_a$,

$$a \lim_{\mathcal{N} \supseteq \mathcal{N}_a}(\overline{\{a\}} \cup \overline{B}) \cap \mathcal{N}.$$

By monotonicity of $\lim$, $a \lim_{\mathcal{N} \supseteq \mathcal{N}_a}a$. By anti-reflexivity of $\lim$, $a \in acl^\mathcal{N}(\overline{B} \cap \mathcal{N})$. By Remark 3.2.4, Condition 6 on $\mathcal{H}$ implies that $a \in acl^\mathcal{M}(\overline{B} \cap \mathcal{N})$. By Remark 3.2.4, Condition 6 on $\mathcal{H}$ implies that $a \in acl^\mathcal{M}(\overline{B} \cap \mathcal{N})$, as desired.

Remark 3.3.7. In Proposition 3.3.6, we do not need to require that the $Th(\mathcal{N})$’s are simple, since anti-reflexivity of forking independence holds in all theories.

Proposition 3.3.8. Suppose $\mathcal{M} \models T$ is $\kappa$-saturated and strongly $\kappa$-homogeneous, $\mathcal{H}$ satisfies conditions 1 through 7 (that is, $\mathcal{H}$ approximates $\mathcal{M}$ and satisfies the additional conditions of “stabilization of algebraic closure” and “homogeneity”), and the elements of $\mathcal{H}$ have theories that are simple and for which $\limf$ satisfies the Independence Theorem over algebraically closed sets. Then $\lim$ satisfies the Independence Theorem over algebraically closed sets. That is, if $A = \overline{A}$, $B \lim_A C$, and $d$ and $c$ are such that $tp^\mathcal{M}(d/A) = tp^\mathcal{M}(e/A)$, $d \lim_A B$, and $e \lim_A C$, then there is $a \models tp^\mathcal{M}(d/AB) \cup tp^\mathcal{M}(e/AC)$ with $a \lim \mathcal{A} BC$.

Proof. Suppose not. Then either $tp^\mathcal{M}(d/AB) \cup tp^\mathcal{M}(e/AC)$ is inconsistent (in which case it is eventually inconsistent, and hence trivially eventually forks, in $\mathcal{H}$), or for each realization $a$, $a \lim \mathcal{A} BC$. By Lemma 3.2.5 and the fact that $A = \overline{A}$,

$$a \lim_A BC \leftrightarrow tp^\mathcal{M}(a/\overline{A} \cup \overline{BC}) \text{ eventually forks over } \overline{A}$$

$$\leftrightarrow tp^\mathcal{M}(a/A \cup \overline{BC}) \text{ eventually forks over } A.$$
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It follows (since this is true for all realizations a), that \( \text{tp}^M(d/AB) \cup \text{tp}^M(e/AC) \) does not have an eventually non-forking (over \( A \)) extension to \( S^M(A \cup BC) \). (Note that

\[
\text{tp}^M(d/AB) \cup \text{tp}^M(e/AC)
\]

is a partial type over \( A \cup BC \), since \((A \cup B) \cup (A \cup C) = A \cup B \cup C \subseteq A \cup BC \).) By extension for eventually non-forking types (Claim 3.3.3, above), \( \text{tp}^M(d/AB) \cup \text{tp}^M(e/AC) \) itself eventually forks over \( A \). By the definition of eventual non-forking, there is a finite subtype \( \pi \subseteq \text{tp}^M(d/AB) \cup \text{tp}^M(e/AC) \) that eventually forks over \( A \). The subtype \( \pi \) is of the form

\[
\pi_1(x,a_1,b) \cup \pi_2(x,a_2,c),
\]

where \( \pi_1 \subseteq \text{tp}^M(d/AB) \), \( \pi_2 \subseteq \text{tp}^M(e/AC) \), \( b \in B \), and \( c \in C \).

Let \( \psi_1(x,a_1,b) = \bigwedge \pi_1 \) and \( \psi_2(x,a_2,c) = \bigwedge \pi_2 \). Then \( \psi_1(x,a_1,b) \land \psi_2(x,a_2,c) \) eventually forks over \( A \).

By the definitions and hypotheses above, we are guaranteed the following elements of \( \mathcal{H} \):

- \( N_f \), such that for \( N \supseteq N_f \), \( \psi_1 \land \psi_2 \) forks over \( A \cap N \) in \( \mathcal{N} \);
- \( N_{\psi_1,d} \), such that for \( N \supseteq N_{\psi_1,d} \), \( N \models \psi_1(d,a_1,b) \);
- \( N_{\psi_2,e} \), such that for \( N \supseteq N_{\psi_2,e} \), \( N \models \psi_2(e,a_2,c) \);
- \( N_{b,c,A} \), such that for \( N \supseteq N_{b,c,A} \), \( b \perp_{A \cap N} c \) (since \( B \perp_{A}^{\text{lim}} C \), \( b \perp_{A}^{\text{lim}} c \) by monotonicity);
- \( N_{d,b,A} \), such that for \( N \supseteq N_{d,b,A} \), \( d \perp_{A \cap N} b \);
- \( N_{e,c,A} \), such that for \( N \supseteq N_{e,c,A} \), \( e \perp_{A \cap N} c \);
- \( N_{\text{alg}} \), such that for \( N \supseteq N_{\text{alg}} \), \( \text{acl}^N(A \cap N) = A \cap N \).

Let \( N_0 \) be some element of \( \mathcal{H} \) containing

\[
N_f \cup N_{\psi_1,d} \cup N_{\psi_2,e} \cup N_{b,c,A} \cup N_{d,b,A} \cup N_{e,c,A} \cup N_{\text{alg}}.
\]

Note also that by Condition 7 on \( \mathcal{H} \) and the fact that \( \text{tp}^M(d/A) = \text{tp}^M(e/A) \), we have that for each \( N \in \mathcal{H} \) containing \( \{d,e\} \), \( \text{tp}^N(d/A \cap N) = \text{tp}^N(e/A \cap N) \).

Suppose \( N \supseteq N_0 \). Then we have:

- \( A \cap N = \text{acl}^N(A \cap N) \),
- \( b \perp_{A \cap N} c \),
- \( d \perp_{A \cap N} b \),
- \( e \perp_{A \cap N} c \), and
- \( \text{tp}^N(d/A \cap N) = \text{tp}^N(e/A \cap N) \).
By our hypothesis that $\downarrow^{N}$ satisfies the Independence Theorem over algebraically closed sets, $tp^{N}(d/(A\cap N)b)\cup tp^{N}(e/(A\cap N)c)$ does not fork over $A\cap N$. But since $N \supseteq N_{\psi_{1,d}}\cup N_{\psi_{2,d}}$, $tp^{N}(d/(A\cap N)b)\cup tp^{N}(e/(A\cap N)c) \vdash \psi_{1}(x,a_{1},b) \land \psi_{2}(x,a_{2},b)$, which, since $N \supseteq N_{f}$, forks over $A\cap N$ in $N$ - a contradiction.

3.4 Example

Granger’s Proposition 3.1.11 shows that $\Gamma$-non-forking in $M \models T_{\infty}$ is equivalent to eventual non-forking in the finite dimensional substructures of $M$. Above, we showed that under certain assumptions (in particular, the strong finite character assumption), failure of limit independence corresponds to eventual forking of a type. Unfortunately, the strong finite character condition does not always hold for $\downarrow^{lim}$ in models of $T_{\infty}$ (with $H$ taken to be the collection of finite dimensional substructures). When we restrict to single elements on the left and finite dimensional sets on the right and in the base, however, $\downarrow^{lim}$ does satisfy our assumptions, and is related to $\downarrow^{\Gamma}$.

In the following, let $M$ be a $\kappa$-saturated, strongly $\kappa$-homogeneous model of $T_{\infty}$ (for some large $\kappa$), and $H$ the collection of finite dimensional $K^{M}$-subspaces of $M$ (that is, the substructures $N$ of $M$ such that $K^{N} = K^{M}$ and $\dim_{K}(V^{N}) < \infty$). We shall break our convention of referring to tuples with no overline, and so shall revert to writing out “acl$^{M}(A)$” rather than “$A$”. We say that a set $A$ is “finite dimensional” if $\langle A \rangle$ is finite dimensional.

We begin by reviewing some of Granger’s results on the relationship between a model of $T_{\infty}$ and its finite dimensional subspaces. Recall that $L_{\theta}$ is the language that results from adding the “linear independence” predicates $\theta_{n}$ to the two-sorted vector space language, and that if $\tilde{K}$ is an algebraically closed field, $m \in \mathbb{N} \cup \{\infty\}$, and $F \in \{S,A\}$, the theory $F T_{m}^{\tilde{K}}$ has elimination of quantifiers in $L_{\theta}$. It turns out that there is some local uniformity to this quantifier elimination.

Lemma 3.4.1 ([9], Proposition 9.3.3). 1. For any $L$-sentence, $\sigma$, there is $R \in \omega$ such that

$T_{\infty} \models \sigma$ if and only if $T_{m} \models \sigma$ for all $m \geq R$.

2. Given any $L$-formula $\psi$, there is a quantifier-free $L_{\theta}$-formula $\varphi$ and some $R < \omega$ such that $T_{m} \models \varphi \iff \psi$ for all $m \geq R$. (In particular, $T_{\infty} \models \varphi \iff \psi$.)

3. Let $F \in \{S,A\}$ and $K$ be a field with square roots. Every $L$-formula is equivalent, modulo $F T_{m}^{K}$, to a boolean combination of quantifier-free $L_{\theta}$-formulae and sentences of the form

$\Theta_{n} := \exists X_{1}, \ldots, X_{n}\theta_{n}(X_{1}, \ldots, X_{n})$. 
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Remark 3.4.2. Given $\mathcal{M} \models T_\infty$, the finite dimensional subspaces of $\mathcal{M}$ are in fact $\mathcal{L}_\theta$-substructures of $\mathcal{M}$: for $\mathcal{N} \in \mathcal{H}$ and $a_1, \ldots, a_n \in V^\mathcal{N}$,

$$\mathcal{N} \models \theta_n(a_1, \ldots, a_n) \text{ if and only if } \mathcal{M} \models \theta_n(a_1, \ldots, a_n).$$

Remark 3.4.3. For $\mathcal{N}^\prime \subseteq \mathcal{N} \in \mathcal{H}$ and $A \subset \mathcal{N}^\prime$, $\text{dcl}^{\mathcal{N}^\prime}(A) = \text{dcl}^{\mathcal{N}}(A) = \text{dcl}^{\mathcal{M}}(A)$, and the same quantifier-free $\mathcal{L}_\theta$-formulae can be used to define the elements of $\text{dcl}(A)$ in each of these structures.

Proof. This follows from the proof of Granger’s Lemma 9.2.6 [9].

Lemma 3.4.4. If $A \subset N$ and $\dim((A)) < \dim(V^N) - 1$, $\text{acl}^{\mathcal{M}}(A) \subseteq N$. In fact, $\text{acl}^{\mathcal{M}}(A) = \text{acl}^{\mathcal{N}}(A)$.

Proof. By Remark 3.1.5, $\text{acl}^{\mathcal{M}}(A) = (\text{span}_{K_A^{\text{alg}}}(A_V), K_A^{\text{alg}})$, that is, the field-theoretic algebraic closure of $K_A$ and the $K_A^{\text{alg}}$-span of the vector elements of $A$. Since $K^N = K^M$, $K_A^{\text{alg}} \subseteq K^N$. Since $A_V \subseteq V^N$, $\text{span}_{K_A^{\text{alg}}}(A_V) \subseteq \langle A \rangle \subseteq V^N$. Further, as noted above, $K_A^{\text{alg}}$ is the same whether computed in $\mathcal{M}$ or in $\mathcal{N}$, and so the $K_A^{\text{alg}}$-span of $A_V$ is the same in $\mathcal{M}$ and in $\mathcal{N}$.

Remark 3.4.5. If $A$ is finite dimensional, then for large enough $\mathcal{N}$, $A \subseteq N$, and by Lemma 3.4.4, $\text{acl}^{\mathcal{N}}(A) = \text{acl}^{\mathcal{M}}(A)$.

Lemma 3.4.6. Suppose $\mathcal{N} \in \mathcal{H}$, $C \subseteq B \subset N$, $\overline{a} \in N$, and $\langle B \rangle \neq V^N$. Then $\text{tp}^N(\overline{a}/B)$ does not $\Gamma$-fork over $C$ (in $\mathcal{N}$) if and only if $\text{tp}^M(\overline{a}/B)$ does not $\Gamma$-fork over $C$ (in $\mathcal{M}$).

Proof. We begin by considering the case where $\text{lg}(\overline{a}) = 1$. If $\text{tp}^N(a/B)$ does not $\Gamma$-fork over $C$ then, since $\langle B \rangle \neq V^N$, it follows that $K_{Ca} \downarrow_{K_C} K_B$ and one of the following conditions holds:

1. $a \in K^N = K^M$.
2. $a \in \langle C \rangle$.
3. $a \notin \langle B \rangle$ and whenever $\{b_1, \ldots, b_n\} \subset B_V \setminus (\langle C \rangle \cap B)$ is linearly independent modulo $\langle C \rangle$, $\{[a, b_1], \ldots, [a, b_n]\}$ is algebraically independent over $K_B(K_{Ca})$.

First, note that $K_{Ca} \downarrow_{K_C} K_B$ if and only if $K_{Ca} \downarrow_{K_C} K_B$, since $K^N = K^M$. Secondly, note that each of the three conditions above is true in $\mathcal{N}$ if and only if it is true in $\mathcal{M}$. It follows that $\text{tp}^M(a/B)$ does not $\Gamma$-fork over $C$. The argument in the other direction is exactly the same.

Now suppose that the result holds for tuples of length $n$, and $\text{lg}(\overline{a}a) = n + 1$. By the inductive definition of $\Gamma$-forking, $\text{tp}^N(\overline{a}a/B)$ does not $\Gamma$-fork over $C$ if and only if $\text{tp}^N(\overline{a}/B)$ does not $\Gamma$-fork over $C\overline{a}$. By the inductive hypothesis
and the argument above, this is the case if and only if \( \text{tp}^M(\bar{a}/B) \) does not \( \Gamma \)-fork over \( C \) and \( \text{tp}^M(a/B\bar{a}) \) does not \( \Gamma \)-fork over \( C\bar{a} \), that is, if and only if \( \text{tp}^M(\bar{a}a/B) \) does not \( \Gamma \)-fork over \( C \).

\( \square \)

**Remark 3.4.7.** Working in \( \mathcal{M} \) or any \( \mathcal{N} \in \mathcal{H} \), suppose \( C \subset B \) are finite dimensional sets.

1. \( a \in \langle B \rangle \setminus \langle C \rangle \) just in case the following formula is in \( \text{tp}(a/B) \) (where \( \{b_1', \ldots, b_t'\} \) is a basis for \( \langle C \rangle \)):

\[
\varphi_2(X, \bar{b}', \bar{c}') := \lnot \theta_{t+1}(b_1', \ldots, b_t', X) \land \theta_{k+1}(c_1', \ldots, c_k', X) \land \]

2. Suppose there are \( b_1, \ldots, b_n \) from \( B \setminus C \), linearly independent modulo \( \langle C \rangle \), such that \( \{[a, b_1], \ldots, [a, b_n]\} \) is algebraically dependent over \( K_B(Kc_a) \). Then there is a polynomial \( q'(x_1, \ldots, x_n) \) with coefficients \( \bar{d}' \) from \( K_B(Kc_a) \) such that

\[
q'([a, b_1], \ldots, [a, b_n]) = 0.
\]

Let \( q \) be such that \( q(x_1, \ldots, x_n, \bar{d}') = q'(x_1, \ldots, x_n) \). Since \( \bar{d}' \in K_B(Kc_a) \), each \( d_i' \) is the result of a rational function (say, \( \frac{f_i(z, \bar{w})}{g_i(z, \bar{w})} \)) applied to elements of \( K_B \) and \( Kc_a \), say, \( \bar{d}' \) and \( \bar{c}' \). Each \( b_i' \) is defined over \( B \), say by \( \chi_i(z_i, \bar{d}, \bar{c}) \) (where \( \bar{d} \in B \setminus C \) and \( \bar{c} \in C \)), and each \( c_i' \) is defined over \( Ca \), say by \( \xi_i(w_i, \bar{c}, a) \). We may assume \( \chi_i \) and \( \xi_i \) are quantifier-free. As above, let \( \{c_1', \ldots, c_k'\} \) be a basis for \( \langle C \rangle \). It should be clear, then, that the following formula is in \( \text{tp}(a/B) \) (where we have abbreviated \( (b_1, \ldots, b_n) \) by \( \bar{b} \)).

\[
\varphi_3(X; \bar{c}, \bar{d}, \bar{b}, \bar{c}'): = \bigwedge_{i=1}^r \exists z_i \chi_i(z_i, \bar{d}, \bar{c}) \land \bigwedge_{i=1}^s \exists w_i \xi_i(w_i, \bar{c}, X) \land \bigwedge_{i=1}^t y_i = f_i(\bar{z}, \bar{w}) \land q([X, b_1], \ldots, [X, b_n], y_1, \ldots, y_t) = 0 \land \theta_{m+n}(c_1', \ldots, c_k', b_1 \ldots, b_n) \land \]

Now suppose \( a' \models \varphi_3(X; \bar{d}, \bar{c}, \bar{b}, \bar{c}') \). Witnesses to the \( z_i \) are field elements defined over \( B \) - that is, elements of \( K_B \). Witnesses to the \( w_i \) are field elements defined over \( Ca' \) - that is, elements of \( Kc_a' \). Witnesses to the \( y_i \) are field elements obtained by applying rational functions to the \( z_i \) and \( w_i \) - that is, elements of \( K_B(Kc_a') \). It follows that \( \{[a', b_1], \ldots, [a', b_n]\} \) satisfies a polynomial with the \( y_i \) as coefficients - that is, \( \{[a', b_1], \ldots, [a', b_n]\} \) is algebraic over \( K_B(Kc_a') \). Further, by the last part of the formula, \( \{b_1, \ldots, b_n\} \) is linearly independent modulo \( \langle C \rangle \), and so \( a' \not\models \varphi_3(X; \bar{d}, \bar{c}, \bar{b}, \bar{c}') \).
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The next lemma is a version of Granger’s Proposition 3.1.11. We restrict its scope by considering only 1-types over finite dimensional sets, and alter it to fit into the “directed system” framework of our approximating substructures.

Lemma 3.4.8. Suppose $B$ is finite dimensional, $C \subset B$, and $p(x) \in S^M(B)$. Then the following are equivalent:

1. $p(x)$ does not $\Gamma$-fork over $C$ (in $M$).
2. For all $\varphi(x, \overline{b}) \in p$ there is $N_\varphi \in \mathcal{H}$ such that for $N \supseteq N_\varphi$, $\varphi(x, \overline{b})$ does not fork over $C \cap N$ in $N$.
3. Let $\pi(x) = \{\chi(x) : \chi(x) \in p(x) \text{ and } \chi(x) \text{ is an } \mathcal{L}_\theta \text{ quantifier-free formula} \}$. There is $N_\pi \in \mathcal{H}$ such that for $N \supseteq N_\pi$, $\pi \upharpoonright (B \cap N)$ does not fork over $C \cap N$ in $N$.

Proof. 2 $\rightarrow$ 1: Suppose that $p(x)$ does $\Gamma$-fork over $C$, and let $M \models p(a)$. One of the following holds:

1. $K_{C_B} \subsetneq K_C K_B$,
2. $a \notin \langle B \rangle \setminus \langle C \rangle$, or
3. $a \notin \langle B \rangle$ and there is $\{b_1, \ldots, b_n\} \subseteq B \setminus \langle C \rangle$ linearly independent modulo $\langle C \rangle$ and $\{[a,b_1], \ldots, [a,b_n]\}$ is algebraically independent over $K_B(K_{C_B})$.

In the first case, $tp^{K^M}(K_{C_B}/K_B)$ forks over $K_C$ in the structure $K^M$. Since $K^M = K^N$ for all $N \in \mathcal{H}$, this dependence holds in all $N$ containing $B \cup \{a\}$. Let $N'$ contain $B \cup \{a\}$. Suppose $\psi(\overline{x}, \overline{d}) \in tp^{K^{N'}}(K_{C_B}/K_B)$ divides over $K_C$. By the stable embeddedness of $K^{N'}$, $\psi(\overline{x}, \overline{d})$ divides over $K_C$ in $N'$, as well. It is realized by some $\overline{c}$ from $K_{C_B}$. Since $\overline{c} \in \text{dcl}(N')(Ca)$, there is a quantifier-free $\mathcal{L}_\theta$-formula $\chi(\overline{x}, \overline{c}, a)$ defining $\overline{c}$. By Remark 3.4.3, $\chi(\overline{x}, \overline{c}, a)$ does not depend on $N'$. Similarly, there is $\xi(\overline{y}, \overline{b}) \mathcal{L}_\theta$ quantifier-free defining $\overline{d}$. The following formula divides over $K_C$ in $N'$:

$$\varphi_1(X; \overline{b}, \overline{c}) := \Gamma(\exists^{=1_{\overline{d}}} \chi(\overline{x}, \overline{c}, X) \land (\exists^{=1_{\overline{y}}} \xi(\overline{y}, \overline{b}) \land \exists \overline{x} \overline{y} (\chi(\overline{x}, \overline{c}, X) \land \xi(\overline{y}, \overline{b}) \land \psi(\overline{x}, \overline{y})), ^-$$

If $(\overline{d}_i : i \in I)$ is a $K_C$-indiscernible sequence (with $\overline{d}_0 = \overline{d}$) witnessing the division of $\psi(\overline{x}, \overline{d})$, and for each $i$, $\sigma_i$ is a $K_C$-automorphism sending $\overline{d}$ to $\overline{d}_i$, then the sequence $(\overline{b}_i : i \in I) = (\sigma_i(\overline{b}) : i \in I)$ is a $K_C$-indiscernible sequence witnessing the division of $\varphi_1(X; \overline{b}, \overline{c})$. Furthermore, since the $\overline{d}_i$ lie outside of $K_C$, we can extract a $C$-indiscernible sequence from $(\overline{d}_i : i \in I)$, and hence from $(\overline{b}_i : i \in I)$. We see that $\varphi_1(X; \overline{b}, \overline{c})$ divides over $C$ in $N'$. Since none of $\chi(\overline{x}, \overline{c}, X)$, $\xi(\overline{y}, \overline{b})$, or $\psi(\overline{x}, \overline{y})$ depended on $N'$, $\varphi_1(X; \overline{b}, \overline{c})$ divides over $C$ in all $N \supseteq N'$. It is clear that $\varphi_1(X; \overline{b}, \overline{c}) \in p(x)$.

In the second case, we recall that $B$ (and thus, $C$) are finite dimensional. Let $\{c'_1, \ldots, c'_k\}$ be a basis for $\langle C \rangle$, and $\{b'_1, \ldots, b'_k\}$ a basis for $\langle B \rangle$. By Remark 3.4.7, $\varphi_2(X; \overline{b}, \overline{c}') \in p(x)$. Let $N' \in \mathcal{H}$ contain $B$, and suppose $N' \models \varphi_2(a'; \overline{b}, \overline{c}')$. Then $a' \notin \langle B \rangle \setminus \langle C \rangle$, and by
definition $\text{tp}^N(a'/B) \Gamma$-forks over $C$ in $N'$. But since $\Gamma$-forking is the same as dividing in $N'$, $\text{tp}^N(a'/B)$ divides over $C$ in $N'$. We chose $a'$ to be an arbitrary realization of $\varphi_2(X; \vec{b}', \vec{c}')$, so $\varphi_2(X; \vec{b}', \vec{c}')$ divides over $C$ in $N'$ (and in any $N \supseteq N'$).

In the third case, we refer again to Remark 3.4.7, and see that for some quantifier-free $L_0$-formulae $\chi_i(z_i, \vec{b}, \vec{c})$ and $\xi_i(w_i, \vec{z}, X)$, and some (parameter-free) polynomials $f_i(\vec{z}, \vec{w})$, and $q(\vec{x}, y)$, the formula $\varphi_3(X; \vec{z}, \vec{d}, \vec{b}, \vec{c}') \in p(x)$. If $N' \supseteq B$ and $N' \models \varphi_3(a'; \vec{d}, \vec{b}, \vec{c}')$, then (as noted in 3.4.7), $\text{tp}^N(a'/B)$ $\Gamma$-forks (and so divides) over $C$ in $N'$. It follows that $\varphi_3(X; \vec{z}, \vec{b}, \vec{c}')$ divides over $C$ in $N'$, and for all $N \supseteq N'$. This finishes our proof that (2) implies (1), since each possible way $p(x)$ could $\Gamma$-fork gives rise to a formula from $p(x)$ that eventually forks in $H$.

3 $\rightarrow$ 2: Using the fact that there are $N_{qf} \in H$ and $\psi(x, \vec{b})$ a quantifier-free $L_0$-formula such that for $N' \supseteq N_{qf}$, $\varphi(x, \vec{b})$ forks over $C$ in $N'$, or there is $N_{qf}$ such that $C \cup \{\vec{b}\} \subset N_{qf}$ and for $N' \supseteq N_{qf}$, $\varphi(x, \vec{b})$ does not fork over $C$ in $N'$. (That is, every formula either forks over $C$ or eventually does not fork over $C$.)

Lemma 3.4.9. If $C$ is finite dimensional, and $\varphi(x, \vec{b})$ is a formula, then either there is $N_{qf}$ such that $C \cup \{\vec{b}\} \subset N_{qf}$ and for $N' \supseteq N_{qf}$, $\varphi(x, \vec{b})$ forks over $C$ in $N'$, or there is $N_{qf}$ such that $C \cup \{\vec{b}\} \subset N_{qf}$ and for $N' \supseteq N_{qf}$, $\varphi(x, \vec{b})$ does not fork over $C$ in $N'$. (That is, every formula either forks over $C$ or eventually does not fork over $C$.)

Proof. Let $\varphi(x, \vec{b})$ and $C$ be as in the statement of the lemma. If $\varphi(x, \vec{b})$ is not realized in $M$, then it is eventually not realized in $H$ (for large enough $N$, $N' \models \forall x \varphi(x, \vec{b})$), and hence trivially eventually forks over $C$. Otherwise, we may consider whether or not $\varphi(x, \vec{b})$ $\Gamma$-forks over $C$ in $M$: if $\varphi(x, \vec{b})$ has a realization whose type over $C\vec{b}$ does not $\Gamma$-fork over $C$, then we say $\varphi(x, \vec{b})$ does not $\Gamma$-fork over $C$.

Consider $N' \in H$ such that $C \cup \{\vec{b}\} \subset N'$, $\langle C \cup \{\vec{b}\} \rangle \neq V_{N'}$, and if $\psi(x, \vec{b})$ is a quantifier-free $L_0$-formula such that $M \models \varphi(x, \vec{b}) \iff \psi(x, \vec{b})$, then $N' \models \varphi(x, \vec{b}) \iff \psi(x, \vec{b})$. (Such an $N'$ exists by finite dimensionality of $C$ and Lemma 3.4.1.) Suppose $N' \models \varphi(a, \vec{b})$. Then, since $N'$ is an $L_0$-substructure of $M$ (see Remark 3.4.2) and $N' \models \psi(a, \vec{b})$, $M \models \psi(a, \vec{b})$, and so $M \models \varphi(a, \vec{b})$. By assumption, $\text{tp}^M(a/C\vec{b}) \Gamma$-forks over $C$. By Lemma 3.4.6, $\text{tp}^{N'}(a/C\vec{b}) \Gamma$-forks over $C$ in $N'$. It follows that every realization of $\varphi(x, \vec{b})$ in $N$ $\Gamma$-forks - and hence, forks over $C \cap N = C$ in $N$, so $\varphi(x, \vec{b})$ forks over $C \cap N$ in $N$. This holds in all $N \supseteq N'$, so $\varphi(x, \vec{b})$ eventually forks over $C$, as desired.

Remark 3.4.10. Let $M$ be a $\kappa$-saturated, strongly $\kappa$-homogeneous model of $T_\infty$ for some large $\kappa$. We conjecture that one can find an element $a$ and sets $B, C$ with dim($\langle BC \setminus C \rangle$) infinite such that $a \Upsilon^\text{lim}_{C} B$, but $a \Upsilon^\Gamma_{\text{acl}(C)} \text{acl}(B) \text{acl}(C)$ (so in particular, by Lemma 3.4.8, there
Claim 3.4.12. If $\beta$, then we have:

Proof of Claim. Suppose $\alpha$, and $B = \{b_i : i < \omega\}$ such that for arbitrarily large $N \in \mathcal{H}$, there are $n$ and $b_{i_0}, \ldots, b_{i_{n-1}}$ with $\{b_{ij} : j < n\}$ linearly independent modulo $\langle acl^\mathcal{M}(C) \cap N \rangle$ and $\{[a, b_{i_0}], \ldots, [a, b_{i_{n-1}}]\}$ algebraically dependent over $K_{(acl^\mathcal{M}(C))\cap acl^\mathcal{M}(B)\cap N}(K_\mathcal{U}(a))$, yet there are no $n$ and $b_{i_0}, \ldots, b_{i_{n-1}} \in B$ with $\{b_{ij} : j < n\}$ linearly independent modulo $\langle acl^\mathcal{M}(C) \rangle$ and $\{[a, b_{i_0}], \ldots, [a, b_{i_{n-1}}]\}$ algebraically dependent over $K_{acl^\mathcal{M}(C)\cap acl^\mathcal{M}(B)}(K_{acl^\mathcal{M}(C)a})$. In other words, the third condition of $\Gamma$-non-forking would hold in arbitrarily large $N$, but be witnessed by different tuples from $B$ each time.

Lemma 3.4.11. For any set $C$ and any $N \in \mathcal{H}$, $K_{acl^\mathcal{M}(C)\cap N} = K_C^{alg} = K_{acl^\mathcal{M}(C)}$.

Proof. It is clear from the definitions that $K_C^{alg} \subseteq K_{acl^\mathcal{M}(C)\cap N} \subseteq K_{acl^\mathcal{M}(C)}$. It remains to show that $K_{acl^\mathcal{M}(C)} \subseteq K_C^{alg}$, for which it suffices to show that the generators of $K_{acl^\mathcal{M}(C)}$ are elements of $K_C^{alg}$. The generators of $K_{acl^\mathcal{M}(C)}$ are:

- $(acl^\mathcal{M}(C))_K = K_C^{alg}$
- $\{[a, b] : a, b \in (acl^\mathcal{M}(C))_V = span_{K_C^{alg}}(C_V)\}$
- for each $n$, the sets $\{\alpha_1, \ldots, \alpha_n\}$ such that there are $b_1, \ldots, b_n, b \in (acl^\mathcal{M}(C))_V$ with $b_1, \ldots, b_n$ linearly independent and $b = \Sigma_{i=1}^n \alpha_i b_i$.

As noted above, the set of field elements from $acl^\mathcal{M}(C)$ is exactly $K_C^{alg}$. We must show:

1. if $a, b \in span_{K_C^{alg}}(C_V)$, $[a, b] \in K_C^{alg}$;
2. if $b_1, \ldots, b_n, b \in span_{K_C^{alg}}(C_V)$ with $b_1, \ldots, b_n$ linearly independent and $b = \Sigma_{i=1}^n \alpha_i b_i$, then $\alpha_1, \ldots, \alpha_n \in K_C^{alg}$.

Claim 3.4.12. If $a, b \in span_{K_C^{alg}}(C_V)$, $[a, b] \in K_C^{alg}$.

Proof of Claim. Suppose $a, b \in span_{K_C^{alg}}(C_V)$. Then, for some $m$, there are $c_1, \ldots, c_m \in C_V$ and $\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_m \in K_C^{alg}$ such that

$$a = \Sigma_{i=1}^m \beta_i c_i$$
$$b = \Sigma_{i=1}^m \gamma_i c_i$$

Then we have:

$$[a, b] = [\beta_1 c_1 + \ldots + \beta_m c_m, \gamma_1 c_1 + \ldots + \gamma_m c_m]$$
$$= \Sigma_{i=1}^m \Sigma_{j=1}^m [\beta_i c_i, \gamma_j c_j]$$
$$= \Sigma_{i=1}^m \Sigma_{j=1}^m \beta_i \gamma_j [c_i, c_j].$$
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Since \( \{c_1, \ldots, c_m\} \subseteq C_V \), for any \( 1 \leq i, j \leq m \), \([c_i, c_j] \in K_C \). Since

\[
\{\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_m\} \subseteq K_C^{\text{alg}}
\]

and \( K_C^{\text{alg}} \) is a field, \( \sum_{i=1}^m \sum_{j=1}^m \beta_{i,j} [c_i, c_j] \in K_C^{\text{alg}} \), as desired. \(\square\)

Claim 3.4.13. If \( b_1, \ldots, b_n, b \in \text{span}_{K_C^{\text{alg}}}(C_V) \), \( \{b_1, \ldots, b_n\} \) is linearly independent, and \( b = \sum_{i=1}^n \alpha_i b_i \), then \( \alpha_1, \ldots, \alpha_n \in K_C^{\text{alg}} \).

Proof of Claim. Let \( b_1, \ldots, b_n, b \) and \( \alpha_1, \ldots, \alpha_n \) be as above. Since \( b_1, \ldots, b_n, b \in \text{span}_{K_C^{\text{alg}}}(C_V) \), there are \( c_1, \ldots, c_m \in C_V \) and \( \{\beta_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{\gamma_i : 1 \leq i \leq m\} \subseteq K_C^{\text{alg}} \) such that for \( 1 \leq i \leq n \),

\[
b_i = \beta_{i,1} c_1 + \ldots + \beta_{i,m} c_m
\]

and

\[
b = \gamma_1 c_1 + \ldots + \gamma_m c_m.
\]

Remark 3.4.14. We may assume \( \{c_1, \ldots, c_m\} \) is linearly independent.

Proof of Remark. Reordering if necessary, let \( \{c_1, \ldots, c_k\} \) be a maximal linearly independent subset of \( \{c_1, \ldots, c_m\} \). Suppose \( k < m \). Then for \( 1 \leq i \leq m - k \) and \( 1 \leq j < k \), there are \( \lambda_{i,j} \in K_C^{\text{alg}} \) such that \( c_{k+i} = \lambda_{i,1} c_1 + \ldots + \lambda_{i,k} c_k \). By the definition of \( K_C^{\text{alg}} \) (and the fact that \( c_1, \ldots, c_k \) are linearly independent vectors from \( C \)), each \( \lambda_{i,j} \) is an element of \( K_C^{\text{alg}} \), and hence of \( K_C^{\text{alg}} \). It follows that each \( b_i \) (and \( b \)) can be written as a \( K_C^{\text{alg}} \)-linear combination of \( c_1, \ldots, c_k \). \(\square\)

We may now write the original linear equation in terms of the \( c_i \)'s:

\[
b = \alpha_1 b_1 + \ldots + \alpha_n b_n
\]

\[
\sum_{i=1}^m \gamma_i c_i = \alpha_1 (\sum_{i=1}^m \beta_{1,i} c_i) + \ldots + \alpha_n (\sum_{i=1}^m \beta_{n,i} c_i)
\]

\[
\sum_{i=1}^m \gamma_i c_i = (\sum_{j=1}^n \alpha_j \beta_{j,1}) c_1 + \ldots + (\sum_{j=1}^n \alpha_j \beta_{j,m}) c_m
\]

By the linear independence of \( \{c_1, \ldots, c_m\} \), we obtain the following system of equations:

\[
\gamma_1 = \alpha_1 \beta_{1,1} + \ldots + \alpha_n \beta_{n,1}
\]

\[
\vdots
\]

\[
\gamma_m = \alpha_1 \beta_{1,m} + \ldots + \alpha_n \beta_{n,m}
\]

which we may represent with the following matrix equation:
Proof. If and is an element of $\langle C \rangle$ or span_{K_C^{alg}}(C_V)$, but it is a basis of a vector space containing \{b_1,\ldots,b_n\}. Since \{b_1,\ldots,b_n\} is linearly independent, the columns of $B$ are linearly independent, and $B$ has rank $n$. Since $B$ is an $m \times n$ matrix of rank $n$, $B$ has a left-inverse - call it $A$. From the previous matrix equation, we obtain

$$\begin{bmatrix} \beta_{1,1} & \beta_{2,1} & \cdots & \beta_{n,1} \\ \beta_{1,2} & \beta_{2,2} & \cdots & \beta_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1,m} & \beta_{2,m} & \cdots & \beta_{n,m} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix}. $$

Since each $\gamma_i$ and all the entries of $A$ are from $K_C^{alg}$, each $\alpha_i$ is from $K_C^{alg}$, as desired.

Claims 3.4.12 and 3.4.13 complete the proof of the lemma.

Lemma 3.4.15. For $\mathcal{M} \models T_{\infty}$ and $\mathcal{H}$ as above, $\downarrow_{lim}$ satisfies the strong finite character condition with respect to tuples of length one on the left, finite dimensional sets $B$ on the right, and small sets $C$ in the base. (We are not assuming that $C \subseteq B$. In particular, $C$ might be infinite dimensional.) That is, if $a$ is an element of $M$, $B$ a finite dimensional subset of $M$, and $a \downarrow_{lim} B$, then there are $\psi(x,\bar{y},\bar{z}) \in \mathcal{L}$, $\bar{b} \in \text{acl}^M(B)$ and $\bar{c} \in \text{acl}^M(C)$ such that $\mathcal{M} \models \psi(a,\bar{b},\bar{c})$ and $\psi(x,\bar{b},\bar{c})$ eventually forks over $\text{acl}^M(C)$ in $\mathcal{H}$.

Proof. If $a \downarrow_{lim} B$, then for arbitrarily large $N \in \mathcal{H}$, $a \downarrow_{\text{acl}^M(C) \cap N} \text{acl}^M(B) \cap N$, that is, $\text{tp}^N(a/\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N)$ forks over $\text{acl}^M(C) \cap N$ in $\mathcal{N}$. Recall that $\Gamma$-non-forking is the same as non-forking in the finite dimensional substructures, so in arbitrarily large $N \in \mathcal{H}$, $\text{tp}^N(a/\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N)$ $\Gamma$-forks over $\text{acl}^M(C) \cap N$.

In what follows, we shall restrict our attention to $\mathcal{N}$ large enough such that the following hold.

- $\{a\} \cup B \subset N$. (This is possible because $\langle B \rangle$ is finite dimensional.)
- If $a \in \langle C \rangle$, then $a \in \langle C \cap N \rangle$. (If $a \in \langle C \rangle$, then $a$ is a linear combination of finitely many vectors from $C$. For large enough $N$, those vectors are in $C_V \cap N$, and $a \in \langle C \cap N \rangle$.)
- $\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N \neq V^N$. (Since $C$ is small and $B$ is finite dimensional, $\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle = \langle CB \rangle \neq V^M$.

Let $v \in V^M \setminus \langle CB \rangle$. For $\mathcal{N}$ containing $v$, $\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N \neq V^N.$
\(\langle C \cap N \rangle \cap \langle B \rangle = \langle C \rangle \cap \langle B \rangle\). As \(\langle B \rangle\) is finite dimensional, \(\langle C \rangle \cap \langle B \rangle\) is, as well. It is generated by finitely many elements from \(C\) - say, \(c'_1, \ldots, c'_n\). Once \(\{c'_1, \ldots, c'_n\} \subset N\), then, \(\langle C \cap N \rangle \cap \langle B \rangle = \langle C \rangle \cap \langle B \rangle\).

Consider one such \(N'\). By the definition of \(\Gamma\)-forking, either

\[
K_{(\text{acl}^M(C) \cup \{a\}) \cap N'} \not\subseteq \bigcup_{\langle a \rangle \cap \langle M \rangle \cap N'} K_{(\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'}
\]

or

- \(a \notin K\), and
- \(a \notin \text{acl}^M(C) \cap N'\), and
- either \(a \in (\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'\) or there is a linearly independent (modulo \(\langle \text{acl}^M(C) \cap N' \rangle\)) set \(\{b_1, \ldots, b_m\}\) from \((\text{acl}^M(C) \cup \text{acl}^M(B))_V \cap N'\) such that \(\{[a, b_1], \ldots, [a, b_m]\}\) is algebraically dependent over

\[
K_{(\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'}(K_{(\text{acl}^M(C) \cup \{a\}) \cap N'}),
\]

and

- either \(a \notin (\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'\) \(\not\subset \langle \text{acl}^M(C) \cap N' \rangle\) or

\[
(\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N') \not\subseteq V^{N'}
\]

or there is a linearly independent (modulo \(\langle \text{acl}^M(C) \cap N' \rangle\)) set \(\{b_1, \ldots, b_m\}\) from \((\text{acl}^M(C) \cup \text{acl}^M(B))_V \cap N'\) such that \(\{[a, b_1], \ldots, [a, b_m]\}\) is algebraically dependent over \(K_{(\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'}(K_{(\text{acl}^M(C) \cup \{a\}) \cap N'}).

By the third assumption on \(N'\) (that \(\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N' \not\subseteq V^{N'}\)), the fourth condition holds trivially. Taking that into account,

\[
a \not\subseteq K_{\text{acl}^M(C) \cap N'} \text{acl}^M(C) \text{acl}^M(B) \cap N'
\]

if and only if one of the following holds:

1. \(K_{(\text{acl}^M(C) \cup \{a\}) \cap N'} \not\subseteq \bigcup_{\langle a \rangle \cap \langle M \rangle \cap N'} K_{(\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'}\);
2. \(a \in (\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N') \setminus \langle \text{acl}^M(C) \cap N' \rangle\);
3. \(a \notin (\langle \text{acl}^M(C) \cup \text{acl}^M(B) \rangle \cap N')\) and there is a linearly independent (modulo \(\langle \text{acl}^M(C) \cap N' \rangle\)) set \(\{b_1, \ldots, b_m\} \subset ((\text{acl}^M(C) \cap \text{acl}^M(B)) \cap N')_V\) such that \(\{[a, b_1], \ldots, [a, b_m]\}\) is algebraically dependent over

\[
K_{(\text{acl}^M(C) \cup \text{acl}^M(B)) \cap N'}(K_{(\text{acl}^M(C) \cup \{a\}) \cap N'}).
\]
CHAPTER 3. LIMIT INDEPENDENCE

Case 1: $K_{acl^m(C)\cap\{a\}\cap N'} \cup_{acl^m(C)\cap\{a\}\cap N'} K_{acl^m(C)\cup acl^m(B)} \cap N'$. (We have removed the superscript $'K^{N'}$ because the field, and hence dependence at the level of the field, is the same in all the structures in question.) By Lemma 3.4.11, $K_{acl^m(C)\cap N'} = K_{alg}$, so we may rewrite the dependence relation as

$$K_{acl^m(C)\cap\{a\}\cap N'} \cup_{acl^m(C)\cap\{a\}\cap N'} K_{acl^m(C)\cup acl^m(B)} \cap N' = K_{alg} K_{acl^m(C)\cup acl^m(B)} \cap N'.$$

We mimic the argument in the first case of the proof of (2) $\rightarrow$ (1) in Lemma 3.4.8. We know that $tp^{N'}(K_{acl^m(C)\cup\{a\}\cap N'} / K_{acl^m(C)\cup acl^m(B)} \cap N')$ divides over $K_{alg}$ in $N'$. Suppose that $\psi(\overline{\pi}, \overline{d})$ witnesses this (where $\overline{d} \in K_{acl^m(C)\cup acl^m(B)} \cap N'$), and suppose $N' \models \psi(\overline{\pi}, \overline{d})$ (where $\overline{\pi} \in K_{acl^m(C)\cup\{a\}\cap N'}$). Then there are $\overline{b} \in (acl^m(C) \cup acl^m(B)) \cap N'$, $\overline{c'} \in acl^m(C) \cap N'$, and quantifier-free $L_\phi$-formulae $\chi, \xi$ such that

$$N' = (\exists! \overline{x}) \chi(\overline{x}, \overline{c'}, a) \land (\exists! \overline{y}) \xi(\overline{y}, \overline{b}') \land \chi(\overline{\pi}, \overline{c'}, a) \land \xi(\overline{d}, \overline{b}').$$

It follows that the formula

$$\varphi_1(X; \overline{b}', \overline{c'}) := \neg(\exists! \overline{x}) \chi(\overline{x}, \overline{c'}, X) \land (\exists! \overline{y}) \xi(\overline{y}, \overline{b}') \land \exists \overline{y} \chi(\overline{x}, \overline{c'}, X) \land \xi(\overline{y}, \overline{b}')$$

is in $tp^{N'}(a / acl^m(B) \cup acl^m(C)) \cap N'$ and divides over $K_{alg}$. As above, given a $K_{alg}$-indiscernible sequence $(d_i : i \in I)$ (with $d_0 = \overline{d}$) (potentially from a sufficiently saturated elementary extension of $N'$) witnessing the division of $\psi(\overline{d}, \overline{c'})$, the corresponding $K_{alg}$-indiscernible sequence $(\overline{b}_i : i \in I)$ (with $\overline{b}_0 = \overline{b}$) witnesses division of $\varphi_1(X; \overline{b}', \overline{c'})$. Also as above, we can extract a $C \cap N'$-indiscernible sequence from $(\overline{b}_i : i \in I)$, witnessing that $\varphi_1(X; \overline{b}', \overline{c'})$ divides over $C \cap N'$ in $N'$. Since none of $\chi(\overline{x}, \overline{c'}, X)$, $\xi(\overline{y}, \overline{b}')$, $\psi(\overline{\pi}, \overline{y})$ depended on $N'$, $\varphi_1(X; \overline{b}', \overline{c'})$ divides over $C \cap N$ in all $N \supseteq N'$ and $\varphi_1(X; \overline{b}', \overline{c'}) \in tp^{m}(a / acl^m(B) acl^m(C)).$

Case 2: $a \in \langle acl^m(C) \cup acl^m(B) \rangle \cap N' \setminus \langle acl^m(C) \cap N' \rangle$. Let $\{b'_1, \ldots, b'_k\}$ be a basis for $\langle acl^m(C) \cup acl^m(B) \rangle \cap N'$. Note that $\langle b'_1, \ldots, b'_k \rangle \cap \langle C \rangle$ is a finite dimensional vector space. Let $C' \subseteq C$ be a basis of this intersection. By our second assumption on $N'$, $a \notin \langle C \rangle$, and the following formula is in $tp^{m}(a / acl^m(B) acl^m(C)).$

$$\varphi_2(X; \overline{b}', \overline{c'}) := \neg \theta_{k+1}(b'_1, \ldots, b'_k, X) \land \theta_{\ell+1}(c'_1, \ldots, c'_\ell, X)$$

Any realization of this formula is an element of

$$\langle acl^m(C) \cup acl^m(B) \rangle \cap N' \setminus \langle acl^m(C) \cap N' \rangle,$$

and so $\varphi_2(X; \overline{b}', \overline{c'})$ divides over $acl^m(C) \cap N'$ in $N'$. The same holds in all $N \supseteq N'$ (since $\{c'_1, \ldots, c'_\ell\}$ spans $\langle b'_1, \ldots, b'_k \rangle \cap \langle C \rangle$, realizations of $\varphi_2(X; \overline{b}', \overline{c'})$ in any $N$ containing $\overline{b}' \overline{c'}$ do not belong to $\langle C \rangle$), and so $\varphi_2(X; \overline{b}', \overline{c'})$ eventually forks over $acl^m(C)$.

Case 3: There is a linearly independent (modulo $\langle acl^m(C) \cap N' \rangle = \langle C \cap N' \rangle$) set $\{b_1, \ldots, b_m\} \subset (acl^m(C) \cup acl^m(B)) \cap N'$ such that $\{[a, b_1], \ldots, [a, b_m]\}$ is algebraically
It follows that for all \(W\) and \(\xi\).

Claim 3.4.16. A linearly independent set of vectors \(X\) is linearly independent modulo the vector space \(W\) if and only if \(\langle X \rangle \cap W = \{0\}\).

Proof. Suppose \(X = \{x_i : i \in I\}, W = \langle w_j : j \in J \rangle\) (that is, \(\{w_j : j \in J\}\) is a basis for \(W\)), and \(v \in \langle X \rangle \cap W\). Then, reordering if necessary, there are \(x_1, \ldots, x_n, w_1, \ldots, w_m\), and scalars \(\lambda_1, \ldots, \lambda_{n+m}\) such that

\[
v = \lambda_1 x_1 + \ldots + \lambda_n x_n = \lambda_{n+1} w_1 + \ldots + \lambda_{n+m} w_m
\]

and hence,

\[
\lambda_1 x_1 + \ldots + \lambda_n x_n - (\lambda_{n+1} w_1 + \ldots + \lambda_{n+m} w_m) = 0
\]

Since they come from a basis for \(W\), \(w_1, \ldots, w_m\) are linearly independent, and so \(X\)'s linear independence modulo \(W\) implies that \(\{x_1, \ldots, x_n\} \cup \{w_1, \ldots, w_m\}\) is linearly independent. It follows that for all \(i\), \(\lambda_i = 0\), and thus that \(v = 0\).
Suppose \( \langle X \rangle \cap W = \{0\} \). We must show that if \( X_0 \subseteq X \) is finite, there is no (finite) linearly independent set \( W_0 \) of vectors from \( W \) such that \( X_0 \cup W_0 \) is linearly dependent. Suppose \( X_0 = \{x_1, \ldots, x_n\} \), and let \( W_0 = \{v_1, \ldots, v_m\} \) for some \( v_1, \ldots, v_m \in W \) linearly independent. Suppose, further, that there are \( \lambda_1, \ldots, \lambda_{n+m} \) such that

\[
\lambda_1 x_1 + \ldots + \lambda_n x_n + \lambda_{n+1} v_1 + \ldots + \lambda_{n+m} v_m = 0
\]

It follows that

\[
\sum_{i=1}^n \lambda_i x_i = -\sum_{i=1}^m \lambda_{n+i} v_i.
\]

Call this vector \( v \). Since \( x_1, \ldots, x_n \in X \) and \( v_1, \ldots, v_m \in W \), \( v \in \langle X \rangle \cap W \). It follows that \( v = 0 \). Since \( \{x_1, \ldots, x_n\} \) and \( \{v_1, \ldots, v_m\} \) are linearly independent, we must have \( \lambda_i = 0 \) for each \( i \). It follows that \( X \) is linearly independent modulo \( W \).

The last conjunct of \( \varphi_3 \) states exactly that \( \langle b_1, \ldots, b_m \rangle \cap \langle C \rangle = \{0\} \), and hence that \( \langle b_1, \ldots, b_m \rangle \cap \langle C \cap N \rangle = \{0\} \) for all \( N \) under consideration.

Remark 3.4.17. We would like to generalize the previous result to tuples \( \vec{a} \) of length greater than one, but unfortunately we conjecture that there are finite tuples \( \vec{a}, \vec{b} \), and an infinite dimensional set \( C \) such that \( \vec{a} \upharpoonright \lim_{C} \vec{b} \), and this dependence is not witnessed by any single formula. In particular, we suspect a problem might arise in the following way: there are \( a_1, a_2, b, C \) such that \( a_1 a_2 \upharpoonright_{\acl^M(C)} \acl^M(C) \acl^M(b) \) (i.e. \( a_1 \upharpoonright_{\acl^M(C)} \acl^M(C) \acl^M(b) \) and \( a_2 \upharpoonright_{\acl^M(C)a_1} \acl^M(C) \acl^M(b) a_1 \), but for arbitrary large \( N \),

\[
a_2 \upharpoonright_{\acl^M(C) \cup \{a_1\} \cap N} \acl^M(C) \cup \acl^M(b) \cup \{a_1\} \cap N,
\]

and hence \( a_1 a_2 \upharpoonright_{\lim_{C}} \vec{b} \). (The fact that in the base, \( a_1 \) does not lie in the scope of the algebraic closure suggests that one might find

\[
K_{\acl^M(C) \cup \{a_1, a_2\} \cap N} \upharpoonright_{K_{\acl^M(C) \cup \{a_1\} \cap N}} K_{\acl^M(C) \cup \acl^M(b) \cup \{a_1\} \cap N},
\]

even though \( K_{\acl^M(C)a_1 a_2} \downarrow_{K_{\acl^M(C)a_1}} K_{\acl^M(C) \acl^M(b) a_1} \).

Proposition 3.4.18. Let \( \mathcal{M} \models T_\infty \), and let \( \mathcal{H} \) be the collection of finite dimensional subspaces of \( \mathcal{M} \) (viewed as \( L \)-substructures of \( \mathcal{M} \)). When \( \downarrow_{\lim} \) is restricted to tuples of length one on the left and finite dimensional sets on the right and in the base, \( \mathcal{H} \) approximates \( \mathcal{M} \) and each instance of limit independence corresponds to an instance \( \Gamma \)-independence. More precisely, for an element \( a \) and finite dimensional sets \( B, C \),

\[
a \downarrow_{\lim} B \text{ if and only if } a \downharpoonright_{\acl^M(C)} \acl^M(B) \acl^M(C).
\]

Proof. 1. \( \mathcal{H} \text{ covers } \mathcal{M} \): Every vector element generates a one dimensional subspace of \( V^\mathcal{M} \).
2. **Automorphism invariance**: the automorphic image of a finite dimensional subspace of \( V^M \) is a finite dimensional subspace of \( V^M \).

3. **Convergence of truth value**: Suppose \( a \in M, \psi(x) \in L \), and \( M \models \psi(a) \). We must show that \( \psi(a) \) is eventually true in \( H \). By Lemma 3.4.1, there is an \( L_0 \)-formula \( \varphi \) and an \( R < \omega \) such that \( T_m \models \varphi \iff \psi \) for all \( m \geq R \) (and, also by Lemma 3.4.1, \( T_\infty \models \varphi \iff \psi \)). Since \( T_\infty \models \varphi \iff \psi \), \( M \models \varphi(a) \). Take any \( N' \in H \) containing \( a \) and of dimension at least \( R \). By Remark 3.4.2, \( N' \models \varphi(a) \) and, since \( N' \models T_m \) for some \( m \geq R, N' \models \forall x(\psi(x) \iff \varphi(x)) \), so \( N' \models \psi(a) \). The same holds of all \( N \supseteq N' \), so \( \psi(a) \) is eventually true in \( H \), as desired.

4. **Stabilization of forking**: This is Lemma 3.4.9.

5. **Strong finite character**: This is Lemma 3.4.15.

6. **Stabilization of algebraic closure (for small sets)**: Suppose that
\[
A = acl^M(A) = (\text{span}_{K^\text{alg}}(A_V), K^\text{alg}_A).
\]
Since \( A \) is small, there are \( v_1, v_2 \in V^M \) such that \( \{v_1, v_2\} \) is linearly independent modulo \( (A) \). In any \( N \) containing \( \{v_1, v_2\} \), \( \dim(A \cap N) < \dim(V^N) - 1 \). Consider one such \( N' \). By Remark 3.1.5, \( acl^N(A \cap N) = (\text{span}_{K^\text{alg}}(A_V \cap N), K^\text{alg}_{A \cap N}) \). We show that \( acl^N(A \cap N)_V = A_V \cap N \) and that \( acl^N(A \cap N)_K = A_K \cap N \).

Suppose \( a \in acl^N(A \cap N)_V = \text{span}_{K^\text{alg}_{A \cap N}}(A_V \cap N) \). That is, there are \( b_1, \ldots, b_n \in A_V \cap N \) and \( \alpha_1, \ldots, \alpha_n \in K^\text{alg}_{A \cap N} \) such that \( a = \sum_{i=1}^n \alpha_i b_i \). Since \( A_V \cap N \subseteq A_V \) and \( K^\text{alg}_{A \cap N} \subseteq K^\text{alg}_A \), \( a \in \text{span}_{K^\text{alg}_A}(A_V) = acl^M(A)_V = A_V \). Since \( V^N \) is a vector space, \( a \in N \). Hence, \( a \in A \cap N \). Now suppose \( \alpha \in acl^N(A \cap N)_K = K^\text{alg}_{A \cap N} \). Since \( K^\text{alg}_{A \cap N} \subseteq K^\text{alg}_A \),
\[
\alpha \in K^\text{alg}_A = acl^M(A)_K = A_K.
\]
As \( \alpha \in N, \alpha \in A \cap N \), as desired.

7. **Homogeneity**: Suppose \( tp^M(a/A) = tp^M(b/A) \). In particular, the quantifier-free \( L_\theta \)-type of \( a \) over \( A \) in \( M \) - call it \( p(x) \) - is the same as the quantifier free \( L_\theta \)-type of \( b \) over \( A \) in \( M \). Let \( N \in H \) with \( a, b \in N \). by Remark 3.4.2,
\[
N \models p \upharpoonright_N (a)
\]
and
\[
N \models p \upharpoonright_N (b).
\]
Since \( Th(N) \) has quantifier elimination in \( L_\theta \), \( tp^N(a/A \cap N) = tp^N(b/A \cap N) \).
By Lemma 3.2.5, \( a \lim\downarrow_{C} B \) if and only if \( \text{tp}^M(a/\text{acl}^M(B)\text{acl}^M(C)) \) eventually does not fork over \( \text{acl}^M(C) \). By Lemma 3.4.8, this happens if and only if \( \text{tp}^M(a/\text{acl}^M(B)\text{acl}^M(C)) \) does not \( \Gamma \)-fork over \( \text{acl}^M(C) \). It follows that

\[
a \lim\downarrow_{C} B \text{ if and only if } a \lim\downarrow_{\text{acl}^M(C)} \text{acl}^M(C) \text{acl}^M(B).
\]
Bibliography


