Inverse problems with rough data

by

Boaz Eliezer Haberman

A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate Division of the University of California, Berkeley

Committee in charge:

Professor Daniel I. Tataru, Chair
Professor Michael M. Christ
Professor Chung-Pei Ma

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Abstract

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In this dissertation, we will discuss some inverse problems associated to elliptic equations with rough coefficients.

The first results in this thesis are based on the papers [HT13, Hab14], and relate to Calderón’s problem for the conductivity equation

$$\text{div}(\gamma \nabla u) = 0.$$ 

Given a domain $\Omega$ and a positive real-valued function $\gamma$ defined on $\Omega$, we may define the Dirichlet-to-Neumann map $\Lambda_\gamma$, which maps the Dirichlet data of solutions to the conductivity equation to the corresponding Neumann data. Calderón’s problem is to determine $\gamma$ from $\Lambda_\gamma$. Uniqueness in Calderón’s problem for smooth conductivities in dimensions three and higher was established by Sylvester and Uhlmann. Later, this result was extended by various authors to less regular conductivities which have at least one and a half derivatives. In Chapter 4, we will show that $\Lambda_\gamma$ uniquely determines a $C^1$ conductivity in dimensions three and higher. In Chapter 6, we will improve this result in dimensions $n = 3, 4$ to $\gamma \in W^{1,n}(\Omega)$. We also have a new result for conductivities in $W^{s,p}(\Omega)$ in higher dimensions for certain $(s, p)$ such that $s > 1$ but $W^{s,p}(\Omega) \not\subset W^{1,\infty}(\Omega)$.

In Chapter 7, we will prove a new result on recovering an unbounded magnetic potential $A$ from the Dirichlet-to-Neumann map $\Lambda_{A,q}$ for the Schrödinger equation

$$(D + A)^2 u + qu = 0.$$ 

Previous results on this problem assumed that the potential $A$ is bounded. We will show that, in three dimensions, the integrability assumption on $A$ can be considerably relaxed in exchange for slightly more regularity. Namely, we will show that the Cauchy data uniquely determine curl $A$ if $q \in W^{-1,3}(\Omega)$ and $A$ is small in $W^{s,3}(\Omega)$, where $s > 0$. We will also show that $q$ is determined by the Dirichlet-to-Neumann map if $A$ is small in $W^{s,3}(\Omega)$ and $q \in L^\infty(\Omega)$. 
To my mother, Elinor Haberman; may her memory be a blessing.
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Chapter 1

Calderón's problem

1.1 The Dirichlet-to-Neumann map

Let $\Omega \subset \mathbb{R}^n$ be some bounded domain with Lipschitz boundary, and let $\gamma \in L^\infty(\Omega)$ be a real-valued function defined on $\Omega$ satisfying

$$\gamma(x) > c > 0,$$

which we interpret as the conductivity of an object at a given point. An electrical potential $u$ in this situation satisfies the conductivity equation

$$L_\gamma u = 0,$$

where

$$L_\gamma u := \text{div}(\gamma \nabla u).$$

Our assumptions on $\gamma$ imply that $L_\gamma$ is uniformly elliptic. By the theory of elliptic equations, there exists, for any smooth $f$, a unique solution $u_f$ to the Dirichlet problem

$$L_\gamma u_f = 0 \quad \text{in } \Omega,$$

$$u_f|_{\partial\Omega} = f,$$

and hence we may formally define the Dirichlet-to-Neumann map $\Lambda_\gamma$ by

$$\Lambda_\gamma f := \gamma \frac{\partial u_f}{\partial \nu}|_{\partial\Omega},$$

where $\partial / \partial \nu$ is the outward unit normal derivative at the boundary.

Suppose $v$ is a smooth function such that $v|_{\partial\Omega} = g$. By the divergence theorem and the fact that $L_\gamma u_f = 0$, we have

$$\int_\Omega \gamma \nabla u_f \cdot \nabla \overline{v} \, dx = \int_\Omega \text{div}(\gamma \nabla u_f \cdot \overline{v}) \, dx$$

$$= \int_{\partial\Omega} \gamma \partial_\nu u_f \cdot \overline{g} \, dS$$

$$= \langle \Lambda_\gamma f, g \rangle_{L^2(\partial\Omega)}, \quad (1.1)$$
By elliptic regularity theory, we have the \textit{a priori} estimate
\[ \|u_f\|_{H^1(\Omega)} \lesssim \|f\|_{H^{1/2}(\partial\Omega)}. \]
The trace map \( \text{Tr} u = u|_{\partial\Omega} \) extends to a continuous surjection from \( H^1(\Omega) \) to \( H^{1/2}(\partial\Omega) \), and thus we can identify \( H^{1/2}(\partial\Omega) \) with the quotient
\[ H^{1/2}(\partial\Omega) = H^1(\Omega)/H^1_0(\Omega), \]
where \( H^1_0(\Omega) = \ker \text{Tr} \). Its dual is the space \( H^{-1/2}(\partial\Omega) \). From (1.1) we have
\[ |\langle \Lambda_\gamma f, g \rangle_{L^2(\partial\Omega)}| \lesssim \|u_f\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \lesssim \|f\|_{H^{1/2}(\partial\Omega)} \|v\|_{H^1(\Omega)}, \]
for any \( v \) such that \( \text{Tr} v = f \), and thus
\[ |\langle \Lambda_\gamma f, g \rangle| \lesssim \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}, \]
so that \( \Lambda_\gamma \) extends to a continuous map
\[ \Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega). \]
The main topic of this thesis will be Calderón’s problem: [Cal80] Can we recover \( \gamma \) from \( \Lambda_\gamma \)?

1.2 Recovering a potential and inverse scattering

Consider now a Schrödinger Hamiltonian of the form
\[ L_{A,q} = (D + A)^2 + q, \]
defined on a domain \( \Omega \). Here \( A \) is a magnetic vector potential, \( q \) is a scalar electric potential, and \( D = -i\nabla \). Since \( L_{A,q} \) can have nontrivial solutions vanishing on \( \partial\Omega \), the Dirichlet-to-Neumann map is a multivalued relation. We define it by
\[ \Lambda_{A,q} = \{(u|_{\partial\Omega}, \nu \cdot (\nabla + iA)u|_{\partial\Omega}) : L_{A,q}u = 0 \text{ in } \Omega\}. \]
This problem has a gauge invariance: If \( \phi \) is smooth function vanishing at \( \partial\Omega \), then
\[ \Lambda_{A + \nabla\phi, q} = \Lambda_{A,q}. \]
This means that we can only hope to recover \( A \) up to a gradient term. Thus a natural problem is to show that
\[ \Lambda_{A_1, q_1} = \Lambda_{A_2, q_2} \Rightarrow \text{curl } A_1 = \text{curl } A_2 \text{ and } q_1 = q_2. \]
One reason why this problem is interesting is because it is closely related to inverse scattering at fixed energy. In quantum mechanics, a wavefunction $u$ satisfies the time-dependent Schrödinger equation

$$i\partial_t u = L_{A,q} u.$$

(1.2)

For a solution with initial data $u_0$, we write

$$u = e^{-itL_{A,q}}u_0.$$

Under suitable conditions on $A$ and $q$ (namely, that their influence is localized to a compact region in space), the solution will eventually escape the influence of the potential and scatter to a solution to the free Schrödinger equation with no potential, i.e.

$$\lim_{t \to \infty} \|e^{-itL_{A,q}}u_0 - e^{-it(-\Delta)}u_0^+\| = 0.$$

Since the Schrödinger equations is time reversible, we can similarly define $u_0^-$ by

$$\lim_{t \to -\infty} \|e^{-itL_{A,q}}u_0 - e^{-it(-\Delta)}u_0^-\| = 0.$$

Here the interpretation is that every particle which interacts with the potential was once very far away from its region of influence, and at that time it was governed by the free evolution. Now, the map $u_0 \mapsto u_0^-$ is unitary and thus invertible, so we may define the \textit{scattering matrix} corresponding to the system by

$$S(u_0^-) = u_0^+.$$

The scattering matrix encodes the observations of the system that can be made from a distance. An observation is made by sending a free wave into the system from far away. The free wave interacts with the system for some time, but eventually leaves the influence of the system and scatters to a different free wave. The problem of inverse scattering is to determine curl $A$ and $q$ from the scattering matrix $S$.

Although we have expressed the scattering matrix in terms of the time-independent Schrödinger equation, there is a stationary formulation of this problem as well. Consider the stationary Schrödinger equation

$$L_{A,q} u = Eu.$$

(1.3)

Here the energy $E$ is a nonnegative real number. If $A$ and $q$ are zero, then solutions to this equation are plane waves $e^{ik \cdot x}$, where $k \in \mathbb{R}^n$ satisfies $|k|^2 = E$. According to stationary scattering theory, solutions to Schrödinger equation (1.3) for the perturbed Laplacian have the asymptotic form

$$u(x) = e^{ik \cdot x} + |x|^{-(n-1)/2}e^{i|k||x|}\left(a\left(\frac{x}{|x|}, k\right) + O(|x|^{-1})\right).$$

The function $a : S^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ is known as the scattering amplitude, and completely determines the scattering matrix $S$. 
Faddeev [Fad56] and Berezanskii [Ber58] showed that a potential $q$ can be recovered from the corresponding scattering amplitude at high energy. This problem is formally overdetermined, however, as the domain of $a$ is $(2n - 1)$-dimensional, while the domain of $q$ is $n$-dimensional.

The problem of inverse scattering at fixed energy $E$ is to determine $\text{curl } A, q$ from the scattering amplitudes $a_E(\theta, \omega) = a(\theta, \sqrt{E}\omega)$ where $\omega \in S^{n-1}$. Faddeev [Fad66] noticed that the Schrödinger equation $L_{A,q}u = Eu$ has many nonphysical exponentially increasing solutions which are asymptotically equal to $e^{x \cdot \zeta}$, for $\zeta \in \mathbb{C}^n$. In particular, we can take $|\zeta|$ to be as large as we please, even though $E$ is fixed. These solutions constructed by Faddeev are the same as the complex geometrical optics solutions introduced later by Sylvester and Uhlmann in connection with Calderón’s problem. The problems are closely related; in fact, for compactly supported potentials the Dirichlet-to-Neumann map $\Lambda_{A,q}$ can be recovered from the scattering amplitude $a_E$ at fixed energy and vice-versa.

In [Fad76], Faddeev introduced a nonphysical scattering amplitude connected with the exponentially increasing solutions, and showed that this amplitude can be recovered from the physical scattering amplitude $a_E(\theta, \omega)$, where $E$ is fixed. A similar method was discovered independently by Newton [New74,New82]. Faddeev’s ideas were developed further and made rigorous in the work of [NA84,BC85,NK87]. We summarize some uniqueness results for the inverse scattering problem for exponentially decreasing potentials. Since the problem for compactly supported potentials is equivalent to Calderón’s problem, we also include results for the Dirichlet-to-Neumann map. We indicate in this case that the potential is supported on a compact set $\Omega$ by the notation $X(\Omega)$:
From the scaling of the problem, the natural assumptions are to take $A \in L^n$ and $q \in W^{-1,n}$. So far there are no results in any dimension for $A \in L^p$ with $p < \infty$. We will prove a result in this direction, which requires a little bit more regularity and only works when $A$ is assumed to be small.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, and suppose that $A_i \in W^{s,3}(\Omega)$ (with $s > 0$) and $q_i \in W^{-1,3}(\Omega)$ for $i = 1, 2$. There exists some $\epsilon > 0$, depending on $\Omega$ and $s$, such that if $\|A_i\|_{W^{s,3}(\Omega)} \leq \epsilon$ for $i = 1, 2$ then

$$L_{A_1,q_1} = L_{A_2,q_2}$$

implies that $\text{curl} A_1 = \text{curl} A_2$. If $q_1, q_2 \in L^\infty$ then we also have $q_1 = q_2$.

### 1.3 Boundary determination and the anisotropic problem

From a physical point of view, one would expect that the value of the conductivity at the boundary would be most accessible to boundary measurements. In [KV84], Kohn and Vogelius showed that for smooth conductivities, the map $\gamma \mapsto \Lambda_\gamma$ determines the values of $\gamma$ and all of its normal derivatives on $\partial \Omega$. This was improved by Alessandrini to
Theorem 1.3.1 ([Ale90]). Suppose Ω is a Lipschitz domain and γ₁, γ₂ are Lipschitz in Ω. If γ₁ − γ₂ is Cᵏ is a neighborhood of ∂Ω, then
\[
Λ_{γ₁} = Λ_{γ₂} ⇒ ∂ⁿγ₁|_{∂Ω} = ∂ⁿγ₂|_{∂Ω}
\]
for any multiindex η with |η| ≤ k.

Brown [Bro13] proved that the boundary values can be recovered whenever this makes sense. In particular, his result applies when γ is continuous or γ ∈ W¹¹(Ω).

Theorem 1.3.2 ([Bro13]). Let Ω be a Lipschitz domain, and suppose that γ is continuous at almost every boundary point, in the sense that it is equal almost everywhere to its nontangential limit at the boundary. Then Λγ determines γ|∂Ω almost everywhere.

It is possible to determine γ at the boundary even if γ is replaced by a positive-definite matrix {γᵢⱼ}. If we define
\[
L_γ = ∑_{i,j} ∂_i (γᵢⱼ ∂_j u),
\]
then the Dirichlet-to-Neumann map is defined by
\[
Λ_γ(u|_{∂Ω}) = ∑_{i,j} γᵢⱼ ν_i ∂_j u
\]
where u satisfies L_γu = 0. The Dirichlet-to-Neumann map is invariant under diffeomorphisms: If φ : Ω → Ω is a diffeomorphism such that φ|∂Ω = id, then
\[
Λ_{φ∗γ} = Λ_γ,
\]
where
\[
φ∗γ(x) = \left( \frac{1}{\det Dφ} Dφ · γ · Dφ^T \right) (φ⁻¹(x))
\]
The Calderón problem for an anistropic conductivity is to recover γ from Λγ up to this diffeomorphism invariance.

In this context, it is natural to consider the following geometric problem, which is equivalent to Calderón’s problem in dimensions three and higher.

Let (M, g) be a compact Riemannian manifold with boundary, and define Λₙ by
\[
Λₙ(u|_{∂M}) = ρ∂ₙu|_{∂M},
\]
where ∂ₙ is again the outward unit normal, ρ is the volume density associated to g, and u solves the Laplace-Beltrami equation
\[
Δₙu = 0.
\]
We can reformulate this problem as follows:
**Conjecture.** Let \((M_1, g_1)\) and \((M_2, g_2)\) be complete connected Riemannian manifolds with boundary of dimension \(n \geq 3\). Suppose that \(\partial M_1\) is diffeomorphic to \(\partial M_2\), and that the Dirichlet-to-Neumann maps agree with respect to this identification. Then there exists a diffeomorphism \(\phi : M_1 \to M_2\) such that \(\phi^* g_2 = g_1\).

This conjecture is open in dimensions three and higher, except for some results under very restrictive assumptions on the geometry (see for example [DSFKSU09]). However, Lee and Uhlmann [LU89] showed that \(\Lambda_g\) does determine the metric and all of its normal derivatives at the boundary. Their argument is based on the following idea: Choose a neighborhood \(U\) of \(X = \partial M\) which admits boundary normal coordinates. That is, we assume there is a diffeomorphism
\[
\phi : [0, \epsilon] \times X \to U
\]
with \(\phi(0, \cdot) = \text{id}\). We can foliate \(U\) into hypersurfaces
\[
\Sigma_t = \{\phi(t, x) : x \in M\}
\]
and we assume that the curves \(t \mapsto \phi(t, x)\) are normal to the \(\Sigma_t\), so that the metric takes the form
\[
g = dt^2 + g_t
\]
where \(t\) denotes projection onto the first coordinate and \(g_t\) is a metric on \(X\). For \(t \geq 0\), the open submanifold
\[
M_t = M \setminus \{\phi(s, x) : s < t\}
\]
is bounded by \(\Sigma_t\) and determines a Dirichlet-to-Neumann map \(\Lambda_g(t)\). An easy formal calculation using the definition of the Dirichlet-to-Neumann map gives the operator Riccati equation
\[
\partial_t \Lambda_g(t) + \Lambda_g(t) \rho^{-1} \Lambda_g(t) + \rho \Delta_{g_t} = 0.
\]
Lee and Uhlmann showed that \(t \mapsto \Lambda_g(t)\) is a smooth function from the interval \([0, \epsilon]\) to the space of elliptic pseudodifferential operators of order 1 on \(X\). Using the symbol calculus for pseudodifferential operators, we can deduce that
\[
\sigma(\Lambda_g(t))^2(x, \xi) = \rho^2 g_t(x)(\xi, \xi)
\]
where \(\sigma(\Lambda_g(t))\) is the principal symbol of \(\Lambda_g(t)\). From this it follows that \(g_t\) can be recovered from \(\Lambda_g(t)\) by looking at the principal symbol. Similarly, all of the derivatives \(\partial_t^k g_t\) can be formally recovered from the asymptotic expansion of the symbol of \(\Lambda_g(t)\) by equating terms of equal homogeneity in the Riccati equation. Conversely, the terms in the asymptotic expansion of the symbol of \(\Lambda_g(t)\) are completely determined by the metric and its normal derivatives, so that any information about the interior values of the metric must lie in the smoothing part of symbol.
1.4 Calderón’s problem in the plane

As a linear operator from $H^{1/2}(\partial \Omega)$ to $H^{-1/2}(\partial \Omega)$, the Dirichlet-to-Neumann map can be expressed in terms of a Schwartz kernel, i.e.

$$\Lambda_\gamma f(x) = \int_{\partial \Omega} K(x,y) f(y) dS(y).$$

Since $\partial \Omega$ is an $(n-1)$-dimensional space, the domain of $K : \partial \Omega \times \partial \Omega \rightarrow \mathbb{R}$ has dimension $2n - 2$. On the other hand, the conductivity $\gamma$ which we are trying to recover is defined on an $n$-dimensional domain. Thus the problem of recovering a conductivity from the Dirichlet-to-Neumann map is formally determined in two dimensions, and formally overdetermined in three dimensions and higher.

This heuristic suggests that in order to solve Calderón’s problem in two dimensions, we must use all of the information contained in the Dirichlet-to-Neumann map. In higher dimensions, on the other hand, there is more wiggle room, and we can get away with ignoring much of this information. This freedom will be crucial to proving uniqueness in higher dimensions, especially as the conductivity becomes less regular.

Sylvester and Uhlmann proved uniqueness in the isotropic problem for conductivities close to the identity in [SU86]. Sylvester [Syl90] established uniqueness in the anisotropic problem for conductivities close to the identity, by using a version of isothermal coordinates to reduce it to the isotropic problem.

Nachman [Nac96] proved the first uniqueness result in two dimensions with no smallness assumption. We summarize some subsequent results in the following table:

<table>
<thead>
<tr>
<th>Isotropic</th>
<th>Anistropic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nachman [Nac96]</td>
<td>Nachman [Nac96]</td>
</tr>
<tr>
<td>Brown-Uhlmann [BU97]</td>
<td>Sun-Uhlmann [SU03]</td>
</tr>
<tr>
<td>Astala-Päivārinta [AP06]</td>
<td>Astala-Lassa-Päivärinta [ALP05]</td>
</tr>
<tr>
<td>$\gamma \in W^{2,1}$</td>
<td>$\gamma_{ij} \in C^3$</td>
</tr>
<tr>
<td>$\gamma \in W^{1,2+}$</td>
<td>$\gamma_{ij} \in W^{1,2+}$</td>
</tr>
<tr>
<td>$\gamma \in L^\infty$</td>
<td>$\gamma_{ij} \in L^\infty$</td>
</tr>
</tbody>
</table>

In [ALP11], the assumption that $\gamma_{ij}$ is bounded was relaxed slightly to a BMO-type bound, and some counterexamples were given to show that their assumptions are essentially sharp. Thus uniqueness and nonuniqueness for Calderón’s problem in the plane is fairly well-understood.

1.5 An equivalent Schrödinger equation

Using the Leibnitz rule, we may write the equation $L_\gamma u = 0$ as a perturbation of the Laplace equation, i.e.

$$- \Delta u - \nabla \log \gamma \cdot \nabla u = 0.$$ 

We wish to eliminate the first-order term $\nabla \log \gamma \cdot \nabla u$ and replace it with a zero-order potential term $qu$. We can do this, using a Liouville transformation, because the coefficient
\(\nabla \log \gamma\) is a gradient. Indeed, for any \(\phi\) we have
\[
e^{-\phi} \circ (-\Delta) \circ e^\phi = -\Delta - 2\nabla \phi \cdot \nabla - |\nabla \phi|^2 - \Delta \phi.
\]
Setting \(\phi = \frac{1}{2} \log \phi\), we obtain
\[
-\text{div}(\gamma \nabla u) = -\Delta u - \nabla \log \gamma \cdot \nabla u = -\gamma^{-1/2} \Delta (\gamma^{1/2} u) + \left(\frac{1}{4} |\nabla \log \gamma|^2 + \frac{1}{2} \Delta \log \gamma\right) u
\]
\[
-\gamma^{1/2} \text{div}(\gamma \nabla u) = -\Delta (\gamma^{1/2} u) + q(\gamma^{1/2} u),
\]
where
\[
q = \frac{1}{4} |\nabla \log \gamma|^2 + \frac{1}{2} \Delta \log \gamma,
\]
and we find that the function \(v = \gamma^{1/2} u\) satisfies the time-independent Schrödinger equation
\[
(-\Delta + q)v = 0,
\]
(1.5)
By setting \(u = 1\), we may observe that
\[
q = \gamma^{-1/2} \Delta \gamma^{1/2}.
\]
Since the conductivity equation is equivalent to a Schrödinger equation and \(\Lambda_{\gamma}\) determines \(\gamma\) and its derivatives at the boundary, it is easy to see that knowledge of the Dirichlet-to-Neumann map \(\Lambda_{\gamma}\) is equivalent to knowledge of the corresponding Dirichlet-to-Neumann map \(\Lambda_{q}\) for the Schrödinger equation.
Suppose that \(q\) can be recovered from \(\Lambda_{q}\). Then for two conductivities \(\gamma_1, \gamma_2\) we have
\[
\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2,
\]
where \(q_i = \gamma_i^{-1/2} \Delta \gamma_i^{1/2}\). A priori, this is not enough to show that \(\gamma_1 = \gamma_2\). In particular, multiplying each \(\gamma_i\) by a constant will not change the \(q_i\). Thus we need to combine the determination of \(q_i\) in the interior with the determination of \(\gamma_i\) at the boundary discussed above. Indeed, the function \(\log \gamma_1 - \log \gamma_2\) satisfies the divergence-form elliptic equation
\[
\text{div}((\gamma_1 \gamma_2)^{1/2} \nabla (\log \gamma_1 - \log \gamma_2)) = 0.
\]
Since the assumption \(\Lambda_{\gamma_1} = \Lambda_{\gamma_2}\) implies that \(\log \gamma_1 - \log \gamma_2 = 0\) on \(\partial \Omega\), this implies that \(\gamma_1 = \gamma_2\) in \(\Omega\). By this argument, Sylvester and Uhlmann [SU87] reduced the problem to that of recovering a potential and showed that, for smooth conductivities, \(\Lambda_{\gamma_1} = \Lambda_{\gamma_1}\) implies that \(\gamma_1 = \gamma_2\).
1.6 Rough coefficients

In the above discussion, we have implicitly assumed that $\gamma$ is sufficiently smooth, so that $q$ is well-defined and the boundary determination results apply. One reason to consider the inverse problem for the Schrödinger equation with potentials in Sobolev spaces of negative regularity index is that when $\gamma$ has less than two derivatives, the potential $q$ will lie in such a space. This line of inquiry was initiated by Brown [Bro96], who showed uniqueness in Calderón’s problem for conductivities in $C^{3/2+}$. We summarize some uniqueness results for Calderón’s problem in the following table:

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Interior</th>
<th>$n$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kohn-Vogelius [KV84]</td>
<td>Sylvester-Uhlmann [SU87]</td>
<td>$\geq 3$</td>
<td>$C^\infty$</td>
</tr>
<tr>
<td>Alessandri [Ale90]</td>
<td>Nachman-Sylvester-Uhlmann [NSU88]</td>
<td>$\geq 3$</td>
<td>$C^{1,1}$</td>
</tr>
<tr>
<td>Nachman [Nac96]</td>
<td>Jerison-Kenig [Cha90]</td>
<td>$\geq 3$</td>
<td>$W^{2,n/2+}$</td>
</tr>
<tr>
<td></td>
<td>Brown [Bro96]</td>
<td>$\geq 3$</td>
<td>$C^{3/2+}$</td>
</tr>
<tr>
<td></td>
<td>Päivärinta-Panchenko-Uhlmann [PPU03]</td>
<td>$\geq 3$</td>
<td>$W^{3/2,\infty}$</td>
</tr>
<tr>
<td></td>
<td>Brown-Torres [Bro03]</td>
<td>$\geq 3$</td>
<td>$W^{3/2,2n+}$</td>
</tr>
<tr>
<td></td>
<td>Haberman-Tataru [HT13]</td>
<td>$\geq 3$</td>
<td>$C^1,W^{1,\infty}$ small</td>
</tr>
<tr>
<td></td>
<td>Caro-Rogers [CR14]</td>
<td>$\geq 3$</td>
<td>$W^{1,\infty}$</td>
</tr>
<tr>
<td></td>
<td>Nguyen-Spirn [NS14]</td>
<td>$3$</td>
<td>$W^{3/2+}$</td>
</tr>
<tr>
<td></td>
<td>Haberman [Hab14]</td>
<td>$3,4$</td>
<td>$W^{1,n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5,6$</td>
<td>$W^{1+\theta(\frac{1}{2}-\frac{2}{n}),\frac{n}{\theta}}$</td>
</tr>
</tbody>
</table>

In this thesis, we will first prove the following:

**Theorem 2.** [HT13] Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with Lipschitz boundary. For $i = 1, 2$, let $\gamma_i \in W^{1,\infty}(\Omega)$ be real valued functions, and assume there is some $c$ such that $0 < c < \gamma_i < c^{-1}$. Then there exists a constant $\epsilon_{d,\Omega}$ such that if each $\gamma_i$ satisfies either $\|\nabla \log \gamma_i\|_{L^\infty(\Omega)} \leq \epsilon_{d,\Omega}$ or $\gamma_i \in C^1(\overline{\Omega})$ then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.

The smallness condition for Lipschitz conductivities can be removed [CR14], but we will not address this issue. We will also prove a stronger theorem where the regularity assumption in three and four dimensions corresponds to the $L^\infty$ scaling of the problem:

**Theorem 3.** [Hab14] Suppose $3 \leq n \leq 6$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\gamma \in W^{s,p}(\Omega)$ be real-valued functions, where

$$ (s,p) = \begin{cases} (1,n) & n = 3,4 \\ (1+(1-\theta)(\frac{1}{2}-\frac{2}{n}),\frac{n}{1-\theta}) & n = 5,6 \end{cases} $$

and $\theta \in [0,1)$. Assume in addition that there is some $c$ such that $0 < c < \gamma_i < c^{-1}$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies that $\gamma_1 = \gamma_2$. 
Chapter 2

Complex geometrical optics solutions

2.1 An integral identity

A weak definition of the DN map  Calderón’s approach for recovering the values of $\gamma$ in the interior was to use a weak formulation of the problem. By (1.1), we may express the map $\Lambda_{\gamma}$ by

$$\langle \Lambda_{\gamma}(u|_{\partial \Omega}), v|_{\partial \Omega} \rangle = \int_{\Omega} \gamma \nabla u \cdot \nabla v \, dx,$$

where $u$ is a solution to $\text{div}(\gamma \nabla u) = 0$ in $\Omega$, and $v$ is arbitrary. If $\gamma_1$ and $\gamma_2$ are two conductivities such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then this implies that

$$\int_{\Omega} (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx = 0 \quad (2.1)$$

when $\text{div}(\gamma_1 \nabla u_1) = \text{div}(\gamma_2 \nabla u_2) = 0$. Thus if the set

$$V = \text{span}\{ \nabla u_1 \cdot \nabla u_2 : \text{div}(\gamma_1 \nabla u_1) = \text{div}(\gamma_2 \nabla u_2) = 0 \}$$

is dense in some sense, then we can conclude that $\gamma_1 = \gamma_2$.

Calderón observed that $V$ is dense in $L^2(\Omega)$ (for example) when $\gamma_1 = \gamma_2 = 1$. If $\zeta \in \mathbb{C}^n$ satisfies $\zeta \cdot \zeta = 0$, then

$$\text{div}(\nabla e^{\zeta \cdot x}) = \Delta e^{\zeta \cdot x} = \zeta \cdot \zeta e^{\zeta \cdot x} = 0.$$ 

Choose $k \in \mathbb{R}^n$ and $\zeta_1, \zeta_2 \in \mathbb{C}^n$ such that

$$\zeta_i \cdot \zeta_i = 0$$
$$\zeta_1 + \zeta_2 = ik$$
$$\zeta_1 \cdot \zeta_2 \neq 0.$$ 

For example, when $k \neq 0$ we can take

$$\zeta_1 = ik + Z$$
$$\zeta_2 = ik - Z.$$
where \( Z \in \mathbb{R}^n \) satisfies \( |Z| = |k| \) and \( Z \cdot k = 0 \). Then we have
\[
\nabla e^{\zeta_1 \cdot x} \cdot \nabla e^{\zeta_2 \cdot x} = (\zeta_1 \cdot \zeta_2) e^{ik \cdot x},
\]
and thus \( V \) contains the complex exponentials \( e^{ik \cdot x} \) for any \( k \in \mathbb{R}^n \). This together with (2.1) is enough to conclude that \( \gamma_1 = \gamma_2 \), which is also a direct consequence of the assumption \( \gamma_1 = \gamma_2 = 1 \). By using a slightly more sophisticated argument, Calderón showed that the linearization of the map \( \gamma \mapsto \Lambda_\gamma \) about \( \gamma = 1 \) is injective, and his paper gave an important clue as to how boundary measurements could yield interior identifiability.

**Reduction to the Schrödinger equation** It is useful to reformulate the identity (2.1) in terms of solutions to the Schrödinger equations
\[
(\Delta + q_i) u_i = 0 \quad (2.2)
\]
where
\[
q_i = \g_i^{-1/2} \Delta \g_i^{1/2} = \frac{1}{4} |\nabla \log \g_i|^2 + \frac{1}{2} \Delta \log \g_i. \quad (2.3)
\]
If \( \Lambda_{\g_1} = \Lambda_{\g_2} \), then we will show that
\[
\int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx = 0 \quad (2.4)
\]
for any \( u_1, u_2 \) satisfying (2.2). Thus we can reduce the problem to showing that the set
\[
W = \text{span} \{ u_1 u_2 : (-\Delta + q_i) u_i = 0 \text{ for } i = 1, 2 \}
\]
is dense in a suitable sense.

**Some technical lemmas** The cost of replacing the conductivity equation with a Schrödinger equation is that we must take two derivatives of the conductivity. This means that if the conductivity is not twice-differentiable, the potential will lie in a space of negative regularity index. In order to make sense of this, we will need some technical results, which we take from [Bro03].

We define, for \( f \in W^{1,n}(\mathbb{R}^n) \) compactly supported and \( u, v \in H^1_{\text{loc}} \),
\[
\langle \Delta f \cdot u, v \rangle_{L^2(\mathbb{R}^n)} = \int \nabla f \cdot (\nabla u \nabla v + u \nabla v) \, dx. \quad (2.5)
\]
First we show that (2.4) makes sense for \( \g_i \in W^{1,n}(\mathbb{R}^n) \).

**Lemma 2.1.1.** Suppose \( n \geq 3 \). Suppose that \( \g \) is a positive function on \( \mathbb{R}^n \) such that \( \log \g \in L^\infty(\mathbb{R}^n) \), \( \nabla \log \g \in L^n(\mathbb{R}^n) \) and \( \g = 1 \) outside of a large ball \( B \). Then
\[
|\langle qu, v \rangle| \lesssim_{\g} \|u\|_{H^1(B)} \|v\|_{H^1(B)},
\]
where \( q = \g^{-1/2} \Delta \g^{1/2} \) and \( \langle qu, v \rangle \) is defined using (2.5).
Proof. Using the above definition and Hölder’s inequality, we have

\[
\langle q u, v \rangle \lesssim \|\nabla \log \gamma\|^2_{L^\infty} \|uw\|_{L^{n/(n-2)}} + \|\nabla \log \gamma\|_{L^n} \|u\nabla v + \nabla uv\|_{L^{n/(n-1)}} \\
\lesssim \gamma \|u\|_{L^{2n/(n-2)}} \|v\|_{L^{2n/(n-2)}} + \|u\|_{L^{2n/(n-2)}} \|\nabla v\|_{L^2} + \|\nabla u\|_{L^2} \|v\|_{L^{2n/(n-2)}}
\]

If \( n \geq 3 \), we have the Sobolev embedding \( H^1 \subset L^{2n/(n-2)} \), and we are done, since \( q \) is supported in \( B \).

Next, we establish (2.4) for conductivities which agree outside \( \Omega \):

**Lemma 2.1.2.** Suppose \( n \geq 3 \). Let \( \gamma_1, \gamma_2 \in W^{1,n}(\mathbb{R}^n) \) be functions such that \( 0 < c \leq \gamma_i \leq c^{-1} \) for some \( c \) and \( \gamma_i = 1 \) outside a large ball. If \( \gamma_1 = \gamma_2 \) outside \( \Omega \) and \( \Lambda_{\gamma_1} = \Lambda_{\gamma_2} \), then for \( q_i = \gamma_i^{-1/2} \Delta \gamma_i^{1/2} \), we have

\[
\langle q_1 - q_2, v_1 v_2 \rangle = 0
\]

when each \( v_i \) is a weak solution in \( H^1_{\text{loc}}(\mathbb{R}^n) \) to \( \Delta v_i - q_i v_i = 0 \).

**Proof.** By (1.4), the functions \( u_i = \gamma_i^{-1/2} v_i \) satisfy the conductivity equation

\[
\text{div}(\gamma_i \nabla u_i) = 0.
\]

We claim that

\[
\langle \Lambda_{\gamma_i} u_1|_{\partial \Omega}, u_2|_{\partial \Omega} \rangle = \int_{\Omega} \left[ \nabla v_1 \nabla v_2 - \nabla \gamma_i^{1/2} \cdot \nabla (\gamma_i^{-1/2} v_1 v_2) \right] dx. \tag{2.6}
\]

We assume \( i = 1 \), and will use the fact that \( \gamma_1 = \gamma_2 \) outside \( \Omega \). This will imply that

\[
(\gamma_1^{-1/2} - \gamma_2^{-1/2}) v_2|_{\partial \Omega} = 0. \tag{2.7}
\]

Assuming that this statement is justified, we can apply the formula (1.1) to \( \gamma_1^{-1/2} v_1 \) and \( \gamma_1^{-1/2} v_2 \), to obtain

\[
\langle \Lambda_{\gamma_1} u_1|_{\partial \Omega}, u_2|_{\partial \Omega} \rangle = \int_{\Omega} \gamma_1 \nabla (\gamma_1^{-1/2} v_1) \cdot \nabla (\gamma_1^{-1/2} v_2) dx \\
= \int_{\Omega} \left[ \nabla v_1 \cdot \nabla v_2 + \gamma_1 \nabla \gamma_1^{-1/2} \cdot (\nabla \gamma_1^{-1/2} v_1 v_2 + \gamma_1^{-1/2} v_2 \nabla v_1 + \gamma_1^{-1/2} v_1 \nabla v_2) \right] dx \\
= \int_{\Omega} \left[ \nabla v_1 \cdot \nabla v_2 - \nabla \gamma_1^{1/2} \cdot \nabla (\gamma_1^{-1/2} v_1 v_2) \right] dx.
\]

Since \( \Lambda_{\gamma_1} = \Lambda_{\gamma_2} \), we have

\[
0 = \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) u_1|_{\partial \Omega}, u_2|_{\partial \Omega} \rangle \\
= -\int_{\Omega} \left[ \nabla \gamma_1^{1/2} \cdot \nabla (\gamma_1^{-1/2} v_1 v_2) - \nabla \gamma_2^{1/2} \cdot \nabla (\gamma_2^{-1/2} v_1 v_2) \right] dx
\]
On the other hand, since $\gamma_1 = \gamma_2$ outside $\Omega$, we have
\[
0 = \int_{\mathbb{R}^n \setminus \Omega} \left[ \nabla \gamma_1^{1/2} \cdot \nabla (\gamma_1^{-1/2} v_1 v_2) - \nabla \gamma_2^{1/2} \cdot \nabla (\gamma_2^{-1/2} v_1 v_2) \right] dx
\]
Adding these two equalities we obtain the desired identity
\[
0 = \langle q_1 - q_2, v_1 v_2 \rangle.
\]

In order to complete the proof of Lemma 2.1.2, we must justify the assertion (2.7). We do so with the following

**Lemma 2.1.3.** Suppose $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$, where $n \geq 2$, and $\beta \in L^\infty \cap W^{1,n}$. If $\beta |_{\partial \Omega} = 0$, then $u \mapsto \beta u$ maps $H^1(\Omega)$ to $H^1_0(\Omega)$.

**Proof.** We replace $\beta$ with a function supported away from $\partial \Omega$. For each $\epsilon$, let $\eta_\epsilon$ be a cutoff function such that $\eta_\epsilon = 0$ when $d(x, \partial \Omega) < \epsilon$ and $\eta_\epsilon = 1$ when $d(x, \partial \Omega) > 2\epsilon$. Since $\partial \Omega$ is Lipschitz, we can choose $\eta_\epsilon$ such that $|\eta_\epsilon| + \epsilon |\nabla \eta_\epsilon| \lesssim 1$ uniformly in $\epsilon$. In particular, we have $|\nabla \eta_\epsilon| \lesssim d(x, \partial \Omega)^{-1}$.

Now set $\beta_\epsilon = \eta_\epsilon \beta$. First, we claim that $\beta_\epsilon \to \beta$ in $W^{1,n}(\Omega)$ as $\epsilon \to 0$. This follows from Hardy’s inequality and the dominated convergence theorem, since
\[
\|\nabla \eta_\epsilon \beta\|_n \lesssim \|d(x, \partial \Omega)^{-1} \beta\|_n \lesssim \|\nabla \beta\|_n.
\]

Now we claim that $\beta_\epsilon u \to \beta u$ in $H^1(\Omega)$ for any $u \in H^1(\Omega)$. First, we have $\beta_\epsilon u \to \beta u$ in $L^2$ and $\beta_\epsilon \nabla u \to \beta \nabla u$ in $L^2(\Omega)$ by the dominated convergence theorem, since $\|\beta_\epsilon\|_\infty \lesssim \|\beta\|_\infty$. It remains to show that $\nabla \beta_\epsilon u \to \nabla \beta u$ in $L^2(\Omega)$. This follows from Hölder’s inequality and Sobolev embedding:
\[
\|\nabla (\beta_\epsilon - \beta) u\|_2 \lesssim \|\beta - \beta_\epsilon\|_{W^{1,n}} \|u\|_{2n/(n-2)} \lesssim \|\beta - \beta_\epsilon\|_{W^{1,n}} \|u\|_{H^1} \to 0.
\]
Since $\beta_\epsilon u$ is obviously in $H^1_0(\Omega)$ for each $\epsilon$, this shows that $\beta u \in H^1_0(\Omega)$. 

**Boundary identification** According to the boundary identification Theorems 1.3.1 and 1.3.2, the assumption $\Lambda \gamma_1 = \Lambda \gamma_2$ is enough to ensure that $\gamma_1$ and $\gamma_2$ admit extensions which agree outside of $\Omega$. In what follows we will justify this assertion. The details depend on our assumptions on the $\gamma_i$.

**$C^1$ conductivities:** We use Whitney’s extension theorem

**Theorem 2.1.4** ([Whi34]). Let $A$ be a closed subset of $\mathbb{R}^n$, and suppose that $f$ is uniformly $C^k$ on $A$. Then $f$ admits a $C^k$ extension to $\mathbb{R}^n$. 

For a general closed set $A$, we must specify in what sense the function is $C^k$; however, for our applications this is obvious.

Suppose now that $\gamma_i \in C^1(\bar{\Omega})$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then Theorem 1.3.1 implies that $\gamma_1 = \gamma_2$ and $\partial \gamma_1 = \partial \gamma_2$ on $\partial \Omega$. By Whitney’s extension theorem, we may extend the functions $\gamma_i$ to functions $\tilde{\gamma}_i \in C^1_0(\mathbb{R}^n)$. Furthermore, we can arrange that the $\tilde{\gamma}_i$ are bounded above and below and are equal to 1 outside a compact set.

Now, since $\tilde{\gamma}_1 - \tilde{\gamma}_2 = \partial \tilde{\gamma}_1 - \partial \tilde{\gamma}_2 = 0$ on $\partial \Omega$, the function $\chi_{\mathbb{R}^n \setminus \Omega}(\tilde{\gamma}_1 - \tilde{\gamma}_2)$ is also $C^1$. Thus we may replace the extension $\tilde{\gamma}_2$ by $\tilde{\gamma}_2 + \chi_{\mathbb{R}^n \setminus \Omega}(\tilde{\gamma}_1 - \tilde{\gamma}_2)$, which agrees with $\tilde{g}_1$ outside of $\Omega$.

**Lipschitz conductivities:** In this case it is easy to construct Lipschitz extensions $\tilde{\gamma}_i$ (for example, by reflection across the boundary). Theorem 1.3.1 implies that $\gamma_1 = \gamma_2$ on $\partial \Omega$.

Again we can arrange that they agree outside of $\Omega$ by using the fact that a Lipschitz function on $\mathbb{R}^n \setminus \Omega$ vanishing on $\partial \Omega$ extends by zero to a Lipschitz function on $\mathbb{R}^n$.

**Conductivities in $W^{s,p}$:** If $1 < p < \infty$, a theorem of Calderón [Cal61] guarantees extensions of the $\gamma_i$ which are bounded in $W^{s,p}(\mathbb{R}^n)$. By Theorem 1.3.2, we have $\gamma_1 = \gamma_2$ on $\partial \Omega$ as long as the $\gamma_i$ are in $W^{1,1}(\Omega)$. To construct extensions which agree outside of $\Omega$, we use a theorem on extensions by zero, which follows from [Mar87, Theorem 1].

**Theorem 2.1.5.** Let $\Omega$ be a Lipschitz domain. Suppose that $1 < p < \infty$, $s > 1/p$ and $s - 1/p \leq 1$. If $f \in W^{s,p}(\Omega)$ and $f|_{\partial \Omega} = 0$, then the extension of $f$ by zero is bounded in $W^{s,p}(\mathbb{R}^n)$.

By the above discussion and Lemma 2.1.2, we have

**Lemma 2.1.6.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded Lipschitz domain. Suppose that $0 < c < \gamma_i < c^{-1}$ and one of the following holds for $i = 1, 2$

- $\gamma_i \in C^1(\bar{\Omega})$
- $\gamma_i \in \text{Lip}(\bar{\Omega})$
- $\gamma_i \in W^{s,p}$, where $1 < p < \infty$, $s \geq 1$ and $s - 1/p \leq 1$.

Then if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then there exist extensions of $\gamma_1$ and $\gamma_2$ which agree outside $\Omega$, which satisfy the same assumptions as the original $\gamma_i$, such that for $q_i = \gamma_i^{-1/2} \Delta \gamma_i^{1/2}$, we have

$$\langle q_1 - q_2, v_1 v_2 \rangle = 0$$

when each $v_i$ is a weak solution in $H^1_{\text{loc}}(\mathbb{R}^n)$ to $\Delta v_i - q_i v_i = 0$. 
2.2 The Sylvester-Uhlmann argument

We now construct complex geometrical optics (CGO) solutions to the Schrödinger equations \((-\Delta + q_i)v_i = 0\) in order to exploit the integral identity \(\langle q_1 - q_2, v_1 v_2 \rangle = 0\). Since the function \(e^{x \xi}\) is harmonic when \(\xi \cdot \xi = 0\), we are led to look for solutions of the form

\[
v_i = e^{x \zeta_i}(1 + \psi_i),
\]

where \(\zeta_i \in \mathbb{C}^n\) are such that \(\zeta_i \cdot \zeta_i = 0\). Writing

\[
0 = (\text{Re} \zeta + i \text{Im} \zeta)^2 = |\text{Re} \zeta|^2 - |\text{Im} \zeta|^2 + 2i \text{Re} \zeta \cdot \text{Im} \zeta,
\]

we see that this condition implies that \(|\text{Re} \zeta| = |\text{Im} \zeta|\) and \(\text{Re} \zeta \cdot \text{Im} \zeta = 0\).

We will show that \(q_1 = q_2\) by showing that \((\hat{q}_1 - \hat{q}_2)(k) = \langle q_1 - q_2, e^{ik \cdot x} \rangle = 0\) for each \(k \in \mathbb{R}^n\). To this end, we would like the \(\zeta_i\) to satisfy \(\zeta_1 + \zeta_2 = ik\), so that

\[
\zeta_1 = \eta_1 + i\left(\frac{1}{2}k + \eta_2\right),
\]

\[
\zeta_2 = -\eta_1 + i\left(\frac{1}{2}k - \eta_2\right),
\]

where \(\eta_1, \eta_2 \in \mathbb{R}^n\). In order that \((\frac{1}{2}k + \eta_2) \cdot \eta_1 = (\frac{1}{2}k - \eta_2) \cdot \eta_1 = 0\), we must have \(k \cdot \eta_1 = \eta_2 \cdot \eta_1 = 0\). The condition \(|\frac{1}{2}k + \eta_2| = |\frac{1}{2}k - \eta_2|\) implies that \(k \cdot \eta_2 = 0\). Finally, we have

\[
|\eta_1|^2 = |\frac{1}{2}k + \eta_2| = \frac{1}{4}|k|^2 + |\eta_2|^2.
\]

It is here that we may exploit the overdeterminacy of the problem. The vectors \(k, \eta_1\) and \(\eta_2\) are orthogonal, and \(|\eta_1| \geq |k|\). In \(\mathbb{R}^2\), we cannot have three orthogonal nonzero vectors, so if \(k \neq 0\) this forces \(\eta_2 = 0\). Note also that \(\eta_1\) is then completely determined by \(k\) up to reflection through the origin. The situation in \(\mathbb{R}^n\) for \(n \geq 3\) is completely different. Once \(k\) is chosen, we have \(n - 1\) degrees of freedom in choosing \(\eta_2\), and an additional \(n - 3\) degrees of freedom in choosing \(\eta_1\). In particular, for any \(k \in \mathbb{R}^n\), we can choose \(\zeta_i\) such that \(|\zeta_i|\) is as large as we please.

We shall construct CGO solutions (2.8) such that

\[
\|\psi_i\| \to 0 \text{ as } |\zeta_i| \to \infty
\]

in an appropriate norm. Once we have done this, we have

\[
0 = \langle q_1 - q_2, v_1 v_2 \rangle
= \langle q_1 - q_2, e^{i(\zeta_1 + \zeta_2) \cdot x}(1 + \psi_1)(1 + \psi_2) \rangle
= \langle q_1 - q_2, e^{ik \cdot x}(1 + \psi_1)(1 + \psi_2) \rangle
\]

Taking the limit as \(|\zeta_i| \to \infty\) we have

\[
0 = \langle q_1 - q_2, e^{ik \cdot x} \rangle = (\hat{q}_1 - \hat{q}_2)(k),
\]

and we have shown that \(q_1 = q_2\). To conclude that \(\gamma_1 = \gamma_2\), we prove the following:
**Lemma 2.2.1.** Let $\gamma_i, q_i$ be as in Lemma 2.1.2, and suppose that $q_1 = q_2$ in the sense of distributions. Then $\gamma_1 = \gamma_2$.

**Proof.** First, have $q_i \in H^{-1}(\mathbb{R}^n)$ for each $i$. This is a consequence of Lemma 2.1.1. In fact, choosing $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi q_i = q_i$, we have

$$|\langle q_i, u \rangle| = |\langle q_i, \phi u \rangle| \lesssim \|\phi\|_{H^1} \|u\|_{H^1} \lesssim \|u\|_{H^1}$$

uniformly in $u \in H^1$.

It follows that we may test $q_1 - q_2$ against the function $g_1 g_2 (\log g_1 - \log g_2) \in H^1(\mathbb{R}^n)$, where $g_i = \gamma_i^{1/2}$. This gives

$$0 = \int [\nabla g_1 \cdot \nabla (g_2 (\log g_1 - \log g_2)) - \nabla g_2 \cdot \nabla (g_1 (\log g_1 - \log g_2))] \, dx$$

$$= \int (g_2 \nabla g_1 - g_1 \nabla g_2) \cdot \nabla (\log g_1 - \log g_2) \, dx$$

$$= \int g_1 g_2 |\nabla (\log g_1 - \log g_2)|^2 \, dx$$

which implies that $g_1 = g_2$. \qed

### 2.3 The operator $\Delta_\z$ and the spaces $\dot{X}^b_\z$

It remains to construct CGO solutions $e^{x \cdot \z} (1 + \psi)$ with $\|\psi\| \to 0$ as $|\z| \to \infty$. The function $v = e^{x \cdot \z} (1 + \psi)$ solves the Schrödinger equation $(-\Delta + q)v = 0$ if

$$0 = e^{-x \cdot \z} (-\Delta + q) e^{x \cdot \z} (1 + \psi)$$

$$= -\Delta_\z (1 + \psi) + q\psi + q$$

where

$$\Delta_\z = e^{-x \cdot \z} \circ \Delta \circ e^{x \cdot \z} = \Delta + 2\z \cdot \nabla + \z \cdot \z.$$ 

If $\z \cdot \z = 0$, then $\Delta_\z 1 = 0$ and our equation for $\psi$ is

$$(\Delta_\z - q)\psi = q. \tag{2.11}$$

The operator $\Delta_\z$ is a Fourier multiplier with symbol

$$p_\z(\xi) = -|\xi|^2 + 2i\xi \cdot \z.$$ 

We define an inverse $\Delta_\z^{-1}$ by

$$\widetilde{\Delta_\z^{-1}} f(\xi) = p_\z^{-1} \hat{f}(\xi),$$

and we write (2.11) as

$$(I - \Delta_\z^{-1} \circ q)\psi = \Delta_\z^{-1} q,$$
where we interpret the $q$ on the left hand side as the map $u \mapsto qu$. We will construct $\psi$ in a Banach space $X$, by showing that
\[ \| \Delta_{\zeta}^{-1} \circ q \|_{X \to X} \leq 1/2, \]  
(2.12)

In order to obtain a useful bound for $\| \psi \|_X$, we will need to show that
\[ \| \Delta_{\zeta}^{-1} q \|_X \to 0 \text{ as } |\zeta| \to \infty. \]
(2.13)

Finally, in order to justify taking limits in (2.10), we will need to show that
\[ |\langle q_i, e^{ik \cdot x} \psi_j \rangle| \lesssim \| \psi_j \|_X \]
(2.14)
\[ |\langle q_i, e^{ik \cdot x} \psi_j \rangle| \lesssim \| \psi_j \|_X. \]
(2.15)

In Sylvester and Uhlmann’s original proof, the space $X$ was taken to be a weighted Sobolev space. In fact, one can show that something like (2.12) holds for
\[ X = H^1_{1/2}(\Omega), \]
where
\[ \| u \|_{H^1_{1/2}(\Omega)} = \tau \| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\partial \Omega)}. \]

Using this type of argument, is was shown in [PPU03,KU14] that complex geometrical optics solutions can be constructed when the conductivity $\gamma$ is merely Lipschitz. However, solutions which are constructed in $H^1_{1/2}$ will not satisfy (2.13), because this would require
\[ \| \nabla \Delta_{\zeta}^{-1} q \|_{L^2} \to 0. \]

Since $q$ contains the term $\Delta \log \gamma$, to show (2.13) for Lipschitz $\gamma$ would require
\[ \| \nabla^2 \Delta_{\zeta}^{-1}(\log \gamma) \|_{L^2} \to 0. \]

Because $\Delta_{\zeta}$ is not elliptic, this will not hold in general. Using some averaging arguments which we will discuss in Chapter 4, it was shown by [NS14] that (2.13) holds for a large set of $\zeta$ when $\gamma \in C^{1,\epsilon}$ for $\epsilon > 0$. However, their argument fails for $\gamma \in C^1$.

Following the philosophy of Bourgain [Bou93], the author and Tataru [HT13] constructed solutions in Banach spaces which are adapted to the operator $\Delta_{\zeta}$ (as in [Tat96]). The space $\dot{X}^b_{\zeta}$ is defined by the norm
\[ \| u \|_{\dot{X}^b_{\zeta}} = \| \Delta_{\zeta}^b u \|_{L^2}, \]
where $\Delta_{\zeta}^b$ is the Fourier multiplier operator with symbol $p_{\zeta}(\xi)^b$. Then (2.12) is equivalent to
\[ \| q \|_{\dot{X}^b_{\zeta} \to \dot{X}^{b-1}_{\zeta}} \leq 1/2 \]
(2.16)

On the other hand, since the spaces $\dot{X}^b_{\zeta}$ and $\dot{X}^{-b}_{\zeta}$ are dual to each other, an estimate (2.14) is equivalent to
\[ \| q_i \|_{\dot{X}^b_{\zeta} \to \dot{X}^{-b}_{\zeta}} \lesssim 1. \]
(2.17)

These two conditions will essentially coincide if we choose $b = 1/2$. Similarly, we will need to show that $\| \Delta_{\zeta}^{-1} q_i \|_{\dot{X}^{1/2}_{\zeta}} = \| q_i \|_{\dot{X}^{-1/2}_{\zeta}} \to 0$, which corresponds to the condition (2.15). This will require a new argument of [HT13] based on averaging over choices of $\zeta$. 
Chapter 3

$L^2$ estimates

3.1 The operator $\Delta_\zeta$

In what follows, it will be convenient to take

$$\zeta = \tau (e_1 - i \eta),$$

where $|e_1| = 1$ and $|\eta| \leq 1$. The conjugated Laplacian $\Delta_\zeta = e^{-x \zeta} \Delta e^{x \zeta}$ is a Fourier multiplier with symbol

$$p_\zeta (\xi) = -(\xi - i \zeta)^2 = -(\xi - \tau \eta)^2 + 2i \tau e_1 \cdot (\xi - \tau \eta) + \tau^2.$$

We want to prove estimates that are uniform in $\tau$, which we will take to be large. When $|\xi| \gg \tau$, we have

$$|p_\zeta| \sim |\xi|^2,$$

so that $\Delta_\zeta$ is elliptic in this region. Thus the interesting behavior of $\Delta_\zeta$ occurs at frequencies that are less than or comparable to $\tau$. In this case multiplication by $\tau$ is a worse operator than differentiation, so we will consider the term $2i \xi \cdot \zeta$ to be of second order, and thus part of the principal symbol of $p_\zeta$.

The characteristic set $\Sigma_\zeta = \{ \xi : p_\zeta (\xi) = 0 \}$ is the intersection of the plane perpendicular to $e_1$ and a sphere centered at $\tau \eta$:

$$\Sigma_\zeta = \{ \xi : \xi \cdot e_1 = 0, |\xi - \tau \eta| = \tau \}.$$

We will refer to the distance $d(\xi, \Sigma_\zeta)$ from this set as the modulation. We have

$$|p_\zeta|^2 = [(\tau + |\xi - \tau \eta|)(\tau - |\xi - \tau \eta|)]^2 + 4\tau^2 ((\xi - \tau \eta) \cdot e_1)^2,$$

so symbol $p_\zeta$ is elliptic at high modulation and vanishes simply on $\Sigma_\zeta$:

$$|p_\zeta| \sim \begin{cases} \tau d(\xi, \Sigma_\zeta) & \text{when } d(\xi, \Sigma_\zeta) \leq \tau/8 \\ \tau^2 + |\xi|^2 & \text{when } d(\xi, \Sigma_\zeta) \geq \tau/8 \end{cases} \quad (3.2)$$
3.2 Dyadic projections

In order to study the operator $\Delta_\zeta$, it will be useful to decompose Fourier space into dyadic regions where the modulation $d(\xi, \Sigma_\zeta)$ does not vary too much. We will use the Greek letters $\lambda, \mu, \nu$ to represent dyadic integers of the form $2^k$, where $k \in \mathbb{Z}$.

Fix a dyadic partition of unity, that is, a smooth nonnegative function $m : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ supported in $[1/4, 1]$ such that
\[
\sum_{\lambda \in 2\mathbb{Z}} m_\lambda(\rho) = 1 \text{ for } \rho > 0,
\]
where
\[
m_\lambda(\rho) = m(\rho/\lambda).
\]
For example, we may take $m = \psi(x) - \psi(2x)$, where $\psi$ is smooth on $[0, 1]$ and $\psi = 1$ on $[0, 1/2]$.

For any dyadic integer $\lambda$, we define
\[
m_{\leq \lambda} = \sum_{\mu \leq \lambda} m_\mu.
\]
By construction, $m_{\leq \lambda}$ is smooth, bounded, and supported in $[0, \lambda]$. Similarly, we define $m_{\geq \lambda}, m_{< \lambda}, m_{> \lambda}$ and so on.

Now let $\Sigma \subset \mathbb{R}^n$ be a closed set. We define $E^\Sigma_\lambda$ to be the set of $\xi$ with modulation comparable to $\lambda$:
\[
E^\Sigma_\lambda := \{ \xi : d(\xi, \Sigma) \in (\lambda/4, \lambda] \}.
\]
Then the function $m^\Sigma_\lambda$ defined by
\[
m^\Sigma_\lambda(\xi) = m_\lambda(d(\xi, \Sigma))
\]
is supported in $E^\Sigma_\lambda$.

Let $Q^\Sigma_\lambda, Q^\Sigma_{\leq \lambda}$ be the Fourier multipliers with symbols $m^\Sigma_\lambda, m^\Sigma_{\leq \lambda}$. We will wish to distinguish the cases $\lambda \leq \tau/8$ and $\lambda \gtrsim \tau/8$, so we define projections onto low and high modulation by
\[
Q^\Sigma_l = \sum_{\lambda \leq \tau/8} Q^\Sigma_\lambda, \quad Q^\Sigma_h = \sum_{\lambda > \tau/8} Q^\Sigma_\lambda,
\]
where the $\lambda$ vary over dyadic integers.

Now take $\Sigma = \Sigma_\zeta$. By (3.2), we have
\[
\|Q_h u\|_{H^1_\tau} \sim \|u\|_{\dot{X}^{1/2}_\zeta}, \tag{3.3}
\]
where
\[
\|v\|_{H^1_\tau} = \tau \|u\|_{L^2} + \|\nabla u\|_{L^2}.
\]
so the $\dot{X}^{1/2}_c$ norm is only interesting at low modulation.

Since the functions $m_\lambda$ and $m_\mu$ have disjoint support unless $\lambda \sim \mu$, we have the approximate reproducing formula

$$1 = \left( \sum_\lambda m_\lambda \right) \left( \sum_\mu m_\mu \right) = \sum_{\lambda \sim \mu} m_\lambda m_\mu.$$ 

In particular, by Cauchy-Schwartz we have

$$1 \lesssim \sum_\lambda (m_\lambda)^2.$$ 

Thus we can write the $\dot{X}^b_c$ norms in the dyadic form

$$\|u\|_{\dot{X}^b_c}^2 \sim \sum_{\lambda \in \mathbb{Z}} \lambda^{2b} \|Q_\lambda u\|_{L^2}^2.$$  

3.3 Littlewood-Paley theory

If we take $\Sigma$ to be the origin, then we have

$$m_{\lambda}^{(0)}(\xi) = m(|\xi/\lambda|).$$

By abuse of notation we will use $m_\lambda$ to denote the function $m_\lambda(|\xi|)$. We define the Littlewood-Paley projection $P_\lambda$ to be the Fourier multiplier operator with symbol $m_\lambda$. These projections have a particularly nice structure, because they are obtained by dilating a single function $m$. In particular, by the dilation invariance of the Fourier transform, we have

$$P_\lambda f = \phi_\lambda * f,$$

where $\phi_\lambda = \lambda^b \phi(\lambda x)$ and $\hat{\phi} = m$. The Littlewood-Paley projections satisfy

- **Almost orthogonality:** The operators $P_\lambda$ are self-adjoint and satisfy $P_\lambda P_\mu = 0$ unless $\lambda \sim \mu$. In particular,

$$\|u\|_{L^2}^2 \sim \sum_\lambda \|P_\lambda u\|_{L^2}^2.$$ 

- **$L^p$ boundedness:** For any interval $J$, let $P_J = \sum_{\lambda \in J} P_\lambda$. Then for $p \in [1, \infty]$ we have

$$\|P_J f\|_{L^p} \lesssim \|f\|_{L^p}.$$ 

- **Finite band property:** For $p \in [1, \infty]$ we have

$$\|P_\lambda f\|_{L^p} \sim \lambda^{-1} \|\nabla P_\lambda f\|_{L^p}.$$ 

In two dimensions we also have

$$\|P_\lambda f\|_{L^p} \sim \lambda^{-1} \|ar{\partial} P_\lambda f\|_{L^p},$$

where $\bar{\partial} = \partial_1 + i \partial_2$. 
• **Bernstein’s inequality:** For \( q \geq p \geq 1 \) we have

\[
\| \phi^\lambda \ast f \|_{L^q} \lesssim \lambda^{n(1/p-1/q)} \| f \|_{L^p},
\]

which holds not only for the Littlewood-Paley projections but more generally for any \( \phi \) such that \( \| \phi \|_{L^1} \sim 1 \).

• **Square function estimate:** For \( p \in (1, \infty) \),

\[
\| f \|_{L^p} \sim \left( \sum_{\lambda} |P^\lambda f(x)|^2 \right)^{1/2},
\]

**3.4 Localization estimates**

In this section we will show that functions in \( \dot{X}^{1/2}_\zeta \) are locally in \( L^2 \). More precisely, we will establish that

\[
\| u \|_{L^2(B)} \lesssim_B \tau^{-1/2} \| \Delta^{1/2}_\zeta u \|_{L^2}, \tag{3.5}
\]

where \( B \) is a ball. This will allow us to show that

\[
q : \dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta,
\]

where \( q \) acts on \( X^{1/2}_\zeta \) by multiplication. For example, if \( q \) is bounded and supported in some ball \( B \), then Hölder’s inequality and (3.5) give

\[
|\langle qu, v \rangle| \lesssim_B \| q \|_{L^\infty} \| u \|_{L^2_B} \| v \|_{L^2_B} \lesssim \tau^{-1} \| q \|_{L^\infty} \| u \|_{\dot{X}^{1/2}_\zeta} \| v \|_{\dot{X}^{1/2}_\zeta}.
\]

By duality, this shows that

\[
\| q \|_{\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta} \lesssim \tau^{-1} \| q \|_{L^\infty}.
\]

The presence of the factor \( \tau^{-1} \) suggests that there is some room for improvement in this estimate. Since \( \tau \) behaves like a derivative, we will be able to show that, essentially,

\[
\| q \|_{\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta} \lesssim \| q \|_{W^{-1,\infty}}.
\]

The idea behind (3.5) is the uncertainty principle. Essentially, the main obstacle to proving an elliptic estimate is the presence of the characteristic set, but once we localize in space the singularity at the characteristic set is mitigated.

To see why this should be true, note that by (3.2) we have \( |p^\lambda_\zeta|^{1/2} \gtrsim \tau^{1/2} d(\zeta, \Sigma^\lambda_\zeta)^{1/2} \).
Localization in space should blur the Fourier variable so that distances less than 1 are not distinguished. This means that after localization, we have \( |p^\lambda_\zeta|^{1/2} \gtrsim \tau^{1/2} \), which is exactly what we need to establish (3.5).
More precisely, localization to the unit ball allows us to replace $u$ with $\phi u$, where $\phi = \phi(x)$ for some fixed Schwartz function $\phi$ that is equal to one on the unit ball. The Fourier transform of $\phi u$ is $\hat{\phi} \ast \hat{u}$. We can interpret convolution with $\hat{\phi}$ as averaging on the unit scale.

Let $v$ and $w$ be two weights on $\mathbb{R}^d$. Defining $Tf = \phi \ast f$, where $\phi$ is a rapidly decreasing function, we would like to find sufficient conditions for $\|T\|_{L^2_v \rightarrow L^2_w}$ to be bounded. This is equivalent to bounding $\|Sf\|_{L^2_v \rightarrow L^2_w}$, where $Sf = w^{1/2}(\xi) \phi(\xi)^{-1/2} \ast (v(\xi)^{-1/2} f)$.

We prove the following lemma:

**Lemma 3.4.1.** Let $v$ and $w$ be nonnegative weights defined on $\mathbb{R}^d$. If $\phi$ is a fixed rapidly decreasing function, then

$$\|\phi \ast f\|_{L^2_w} \leq C_\phi \min \left\{ \sup_{\xi} \sqrt{\int J(\xi, \eta) d\eta}, \sup_{\eta} \sqrt{\int J(\xi, \eta) d\xi} \right\} \|f\|_{L^2_v},$$

where

$$J(\xi, \eta) = |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)}$$

and

$$C_\phi = C_d \langle \xi \rangle^N \|\phi\|_{L^\infty}$$

for some large $N$ depending on $d$.

**Proof.** We can write

$$\|Sf\|_{L^2_v}^2 = \int \left( \int \phi(\xi - \eta)v(\eta)^{-1/2}f(\eta)d\eta \right) \left( \int \phi(\xi - \zeta)v(\zeta)^{-1/2}f(\zeta)d\zeta \right) w(\xi)d\xi.$$  

Applying the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ we have

$$\|Sf\|_{L^2_v}^2 \leq \iiint |\phi(\xi - \eta)\phi(\xi - \zeta)| v(\eta)^{-1} |f(\eta)|^2 w(\xi) \, d\eta \, d\zeta \, d\xi.$$  

Integrating first in $\zeta$, we find that

$$\|Sf\|_{L^2_v}^2 \lesssim \iint |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)} |f(\eta)|^2 \, d\eta \, d\xi.$$  

Equivalently, we may bound the adjoint $S^*$. To describe the adjoint, we compute

$$\langle Sf, g \rangle = \int \left( \int \phi(\xi - \eta)v(\eta)^{-1/2}f(\eta)d\eta \right) w(\xi)^{1/2}g(\xi)d\xi$$

$$= \int f(\eta) \left( \int \frac{w(\xi)^{1/2}}{\phi(\xi - \eta)} g(\xi) \frac{w(\xi)}{v(\eta)^{1/2}} d\xi \right) d\eta$$

$$= \langle f, S^*g \rangle,$$
so that
\[ S^* g(\eta) = \int \overline{\phi(\xi - \eta)} g(\xi) \frac{w(\xi)^{1/2}}{v(\eta)^{1/2}} \, d\xi. \]

This means that bounding $T$ from $L_v$ to $L_w$ is the same as bounding $T^* g = \overline{\phi(-\xi)} * g$ from $L_{w-1}^2$ to $L_{v-1}^2$. To do this it suffices to show that
\[ \int |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)} \, d\eta = \int J(\xi, \eta) \, d\eta \]
is uniformly bounded. \hfill \square

We use the preceding lemma to show that the homogeneous spaces $\dot{X}_\zeta^{\pm 1/2}$ are equivalent, after localization, to the inhomogeneous spaces $X_\zeta^{\pm 1/2}$ defined by
\[ \|u\|_{X_\zeta} = \|(\tau + |p_\zeta(\xi)|^b \hat{u}(\xi))\|_{L^2}. \]

More precisely, we have

**Lemma 3.4.2.** Suppose that $\phi$ is a smooth function such that \[ C_\phi = C_d \sum_{k+l \leq N(n)} \|\langle x \rangle^k \partial^l \phi\|_{L^\infty} < \infty, \]
where $N(d)$ is a universal constant depending on the dimension. Suppose that $u$ is a tempered distribution and that $\hat{u}$ is a function. Then
\[ \|\phi u\|_{X_\zeta} \lesssim \|\phi u\|_{\dot{X}_\zeta}, \]
\[ \|\phi u\|_{\dot{X}_\zeta} \lesssim \|\phi u\|_{X_\zeta}, \]
where
\[ C_\phi = C_d \sum_{k+l \leq N(d)} \|\langle x \rangle^k \partial^l \phi\|_{L^\infty} \]

*Proof.* It suffices to establish (3.6), as the estimate (3.7) follows by duality.

The basic idea is that localization by $\phi$ smooths out the integrable singularity that arises on the characteristic set $\Sigma_\zeta$. To make this precise, we first show that
\[ \int \langle \xi - \eta \rangle^{-M} \frac{1}{d(\xi, \Sigma_\zeta)} \, d\xi \lesssim 1, \]
for large $M$. This is true for any nice codimension 2 hypersurface $\Sigma_\zeta$, as can easily be seen by using a partition of unity and flattening out the surface. We want to show that this holds independently of $\zeta$, which will be true, roughly speaking, because the surface $\Sigma_\zeta$ only gets
flatter as $\tau \to \infty$. For our purposes it will suffice to treat the case at hand. By translation and rotation, we can replace $\Sigma_\zeta$ by

$$\Sigma = \{ \xi : |\xi| = \tau, \xi_1 = 0 \}.$$  

We split the integral as $\int_{\mathbb{R}^d - \Sigma_1} + \int_{\Sigma_1}$, where

$$\Sigma_1 = \{ \xi : d(\xi, \Sigma) \leq 1 \}.$$  

For the first term, we have

$$\int_{\mathbb{R}^d - \Sigma_1} d(\xi, \Sigma)^{-1} \langle \xi - \eta \rangle^{-M} d\xi \lesssim \int \langle \xi - \eta \rangle^{-M} d\xi \lesssim 1.$$  

Next we treat the integral over $\Sigma_1$. Write $\xi = (\xi_1, \xi')$, and pass to polar coordinates in $\xi'$. Then for $\xi = \xi_1 e_1 + r \omega$ and $\eta = \eta_1 e_1 + t \nu$, we have

$$\langle \xi - \eta \rangle^{-M} \lesssim \langle r \omega - t \nu \rangle^{-M} \lesssim \langle (\tau - 1) \omega - (\tau - 1) \nu \rangle^{-M}$$

(by the elementary inequality $|r \omega - t \nu| \geq 1/2 \sqrt{r^2 + t^2} |\omega - \nu|$.) Also, note that

$$d(\xi, \Sigma_\zeta) \gtrsim |r - \tau| + |\xi_1|,$$

so that

$$\int_{\Sigma_\zeta, 1} \lesssim \int_{\mathcal{S}^{n-2}} \int_0^{\tau+1} |r \omega - r \nu|^{-M} (|r - \tau| + |\xi_1|)^{-1} r^{n-2} dr d\xi_1 d\omega.$$  

Since $(|r - \tau| + |\eta_1|)^{-1}$ is integrable with respect to $dr d\eta_1$,

$$\int_{\Sigma_\zeta, 1} \lesssim (\tau + 1)^{n-2} \int_{\mathcal{S}^{n-2}} \langle (\tau - 1) \omega - (\tau - 1) \nu \rangle^{-M} d\omega.$$  

The quantity $\tau^{n-2} \int_{\mathcal{S}^{n-2}} \langle \tau \omega - \tau \nu \rangle^{-M} d\omega$ is uniformly bounded in $\tau$, so $\int_{\Sigma_\zeta, 1} \lesssim 1$.

In order to prove (3.6) and the adjoint estimate (3.7), we need to show that

$$\int |\phi(\xi - \eta)| \frac{|p(\eta)| + \tau}{|p(\xi)|} d\xi \lesssim 1,$$  

(3.9)

where $p = p_\zeta$. We estimate $\phi(\xi - \eta)$ by $\langle \xi - \eta \rangle^{-M}$ with $M$ large and split this into two integrals $\int_{|\xi| > 100 \tau}$ and $\int_{|\xi| \leq 100 \tau}$.

When $|\xi| \geq 100 \tau$, we have $|p(\xi)| \sim |\xi|^2$, so

$$\int_{|\xi| > 100 \tau} \lesssim \int_{|\xi| > 100 \tau} \langle \xi - \eta \rangle^{-M} \frac{|p(\eta)| + \tau}{|\xi|^2} d\xi.$$  

When $|\eta| > 8 \tau$, we have $|p(\eta)| \lesssim |\eta|^2 \lesssim |\xi|^2 + |\xi - \eta|^2$, so it is easy to see that the integral is bounded uniformly in $\tau$. 


On the other hand, when $|\eta| < 4\tau$, we use that $|p(\eta)| \lesssim \tau^2$, and $|\xi - \eta| \gtrsim \tau$ in the domain of integration, and so the integral is bounded above by

$$\tau^{-N} \int_{|\xi| > 100\tau} \frac{\tau^2}{|\xi|^2} (\xi - \eta)^{-M} d\xi \lesssim 1$$

We now turn to the second integral, where we have

$$\int_{|\xi| \leq 100\tau} \lesssim \int_{|\xi| \leq 100\tau} |\phi(\xi - \eta)| \frac{|p(\eta)| + \tau}{\tau d(\xi, \Sigma_\zeta)} d\xi.$$  

When $|\eta| > 200\tau$, we have $|\xi - \eta| \geq \tau$, so the integral is bounded by

$$\tau^{-N} \int \frac{|\xi|^2 + |\xi - \eta|^2 + \tau}{\tau d(\xi, \Sigma_\zeta)} (\xi - \eta)^{-M} d\xi \lesssim 1,$$

by (3.8). On the other hand, when $|\eta| \leq 200\tau$, we have $|p(\eta)| \sim \tau d(\eta, \Sigma_\zeta)$, and by the triangle inequality,

$$\frac{|p(\eta)| + \tau}{|p(\xi)|} \lesssim \frac{|\xi - \eta| + d(\xi, \Sigma_\zeta) + 1}{d(\xi, \Sigma_\zeta)} \lesssim 1 + d(\xi, \Sigma_\zeta)^{-1} (\xi - \eta),$$

and our integral is bounded by

$$\int ((\xi - \eta)^{-M} d\xi + \int d(\xi, \Sigma_\zeta)^{-1} (\xi - \eta)^{-M} d\xi.$$  

The first integral is obviously finite, and the second integral is finite by (3.8).

By Plancherel's theorem and Lemma 3.4.1, the estimate (3.9) implies (3.6).

This lemma implies that we can replace the spaces $\dot{X}_\zeta^{1/2}$ with $X_\zeta^{1/2}$ in (2.16) and (2.17):

**Corollary 3.4.3.** Suppose $q$ is supported in a compact set $B$, and suppose $\zeta_i \cdot \zeta_i = 0$. Then

$$\|q\|_{\dot{X}_\zeta^{1/2} \to \dot{X}_\zeta^{-1/2}} \lesssim_B \|q\|_{X_\zeta^{1/2} \to X_\zeta^{-1/2}}.$$  

**Proof.** Let $\phi$ be a Schwartz function that it equal to one on $B$. Then

$$|\langle qu, v \rangle| = |\langle q\phi u, \phi v \rangle| \lesssim \|q\|_{X_\zeta^{1/2} \to X_\zeta^{-1/2}} \|\phi u\|_{X_\zeta^{1/2}} \|\phi v\|_{X_\zeta^{-1/2}} \lesssim \|q\|_{X_\zeta^{1/2} \to X_\zeta^{-1/2}} \|u\|_{\dot{X}_\zeta^{1/2}} \|v\|_{\dot{X}_\zeta^{-1/2}}.$$  

**Definition 3.4.1.** We say that $u \in \dot{X}_\zeta^{1/2}$ if $u$ is a tempered function on $\mathbb{R}^n$ whose Fourier transform is a function satisfying $p_\zeta^{1/2} \hat{u} \in L^2(\mathbb{R}^n)$. 
Corollary 3.4.4. The space $\dot{X}_\zeta^{1/2}$ with the norm described above is a Banach space contained in $H^1_{\text{loc}}$ and
\[ \|\phi u\|_{H^1} \lesssim \tau^{1/2} \|u\|_{\dot{X}_\zeta^{1/2}}. \tag{3.10} \]
for $\phi$ as in Lemma 3.4.2.

Proof. If $u \in \dot{X}_\zeta^{1/2}$, then by (3.7), the definition of the $X_\zeta^{1/2}$ norm and (3.2)
\[ \|\phi u\|_{L^2} \lesssim \tau^{-1/2} \|\phi u\|_{X_\zeta^{1/2}} \lesssim \tau^{-1/2} \|u\|_{\dot{X}_\zeta^{1/2}}. \tag{3.11} \]
To estimate $\|\nabla(\phi u)\|_{H^1}$, note that $E_{\leq \tau/8} \subset B(0,3\tau)$. Thus (3.5) and (3.3) imply that
\[
\|\nabla(\phi Q_i u)\|_{L^2} \lesssim \tau^{-1/2} \|Q_i u\|_{X_\zeta^{1/2}} + \tau^{-1/2} \|\nabla Q_i u\|_{X_\zeta^{1/2}} \\
\lesssim \tau^{1/2} \|Q_i u\|_{X_\zeta^{1/2}} \\
\|\nabla(\phi Q_h u)\|_{L^2} \lesssim \|Q_h u\|_{H^1} \lesssim \|u\|_{\dot{X}_\zeta^{1/2}}
\]
Thus the estimate (3.10) holds. In particular, if $\|u\|_{\dot{X}_\zeta^{1/2}} = 0$, then $u = 0$.

Now suppose that $u_n$ is a Cauchy sequence in $\dot{X}_\zeta^{1/2}$. We need to show that it has a limit in $\dot{X}_\zeta^{1/2}$. By (3.11), the sequence $u_n$ is Cauchy in $L^2(\langle x \rangle^{-N} \, dx)$ for $N$ sufficiently large. Thus there is some $u \in L^2(\langle x \rangle^{-N} \, dx)$ such that $u_n \to u$ in $L^2(\langle x \rangle^{-N} \, dx)$. In particular this convergence holds in the sense of tempered distributions. On the other hand, the sequence $\hat{u}_n$ is a Cauchy sequence in $L^2(|p_\zeta| \, d\xi)$, and converges to some $g \in L^2(|p_\zeta| \, d\xi)$.

We are done if we can show that $g = \hat{u}$. Indeed, for any Schwartz function $\phi$, we have $\phi u_n \to \phi u$ in $L^2$, which implies that $\hat{\phi} * \hat{u}_n \to \hat{\phi} * \hat{u}$ in $L^2$. At the same time, the proof of Lemma 3.4.2 implies that $\hat{\phi} * \hat{u}_n \to \hat{\phi} * \hat{g}$ in the $L^2((|p_\zeta| + \tau) \, dx)$ norm and thus in $L^2$. Thus $\hat{\phi} * g = \hat{\phi} * \hat{u}$ for any test function $\phi$, which implies that $g = \hat{u}$. \qed

In view of (2.16) and (2.17), we now show that

Lemma 3.4.5. Let $f \in L^\infty(\mathbb{R}^n)$. For $\zeta_1, \zeta_2 \in \mathbb{C}^n$ satisfying $\text{Re} \, \zeta_i = \tau$ and $|\text{Im} \, \zeta_i| \leq \tau$, we have
\[ \|f\|_{\dot{X}_{\zeta_1}^{1/2} \to \dot{X}_{\zeta_2}^{-1/2}} \lesssim \tau^{-1} \|f\|_\infty \tag{3.12} \]
and
\[ \sup_j \|\partial_j f\|_{\dot{X}_{\zeta_1}^{1/2} \to \dot{X}_{\zeta_2}^{-1/2}} \lesssim \|f\|_\infty. \tag{3.13} \]
Proof. By duality, it will suffice to show that
\[ \|fu\|_{L^1} \lesssim \tau^{-1} \|f\|_{L^\infty} \|u\|_{X^{1/2}_{\zeta_1}} \|v\|_{X^{1/2}_{\zeta_2}} \]  
(3.14)

\[ \|f\partial_i(uv)\|_{L^1} \lesssim \|f\|_{L^\infty} \|u\|_{X^{1/2}_{\zeta_1}} \|v\|_{X^{1/2}_{\zeta_2}} \]  
(3.15)

The first estimate (3.14) follows from Hölder’s inequality and the trivial estimate
\[ \|w\|_{L^2} \lesssim \tau^{-1/2} \|w\|_{X^{1/2}_{\zeta}}. \]  
(3.16)

For the second estimate (3.15), we write \( u = Q_h u + Q_l u = u_h + u_l \), and similarly for \( v \). By the product rule, we have
\[ \partial_i(uv) = \partial_i u_h \bar{v} + u \partial_i \bar{v} + \partial_i u_l \bar{v} + u \partial_i \bar{v}. \]

By (3.2), we have
\[ \|\partial_i w_h\|_{L^2} \lesssim \|w_h\|_{X^{1/2}_{\zeta}}. \]

Together with (3.16), this suffices to control \( \|\partial_i u_h \bar{v} + u \partial_i \bar{v}\|_{L^1} \).

For the last two terms, we use the fact that \( \hat{u}_l \) and \( \hat{v}_l \) are supported at frequencies less than or comparable to \( \tau \), so the derivative operator is bounded by \( \tau \), i.e.
\[ \|\partial_i w_l\|_{L^2} \lesssim \tau \|w_l\|_{L^2}. \]

Combined with (3.16), this suffices to control \( \|\partial_i u_l \bar{v} + u \partial_i \bar{v}\|_{L^1} \).

If \( f \) is continuous, then we can show a bit more:

Lemma 3.4.6. Suppose that \( f \) in Lemma 3.4.5 is taken to be uniformly continuous. Then
\[ \|\partial_i f\|_{X^{1/2}_{\zeta_1} \to X^{-1/2}_{\zeta_2}} = o_{\tau \to \infty}(1). \]

Proof. Let \( P_\lambda \) denote the Littlewood-Paley projections. Write \( f = f_{\leq \tau^{1/2}} + f_{> \tau^{1/2}} \), where \( f_{\leq \tau^{1/2}} = P_{\leq \tau^{1/2}} f \). By Lemma 3.4.5 and the finite band property,
\[ \|\partial_i f\|_{X^{1/2}_{\zeta_1} \to X^{-1/2}_{\zeta_2}} \lesssim \|\partial_i f_{\leq \tau^{1/2}}\|_{X^{1/2}_{\zeta_1} \to X^{-1/2}_{\zeta_2}} + \|\partial_i f_{> \tau^{1/2}}\|_{X^{1/2}_{\zeta_1} \to X^{-1/2}_{\zeta_2}} \]
\[ \lesssim \tau^{-1} \|f_{\leq \tau^{1/2}}\|_{L^\infty} + \|f_{> \tau^{1/2}}\|_{L^\infty} \]
\[ \lesssim \tau^{-1/2} \|f\|_{L^\infty} + \|f_{> \tau^{1/2}}\|_{L^\infty} \]

Since \( f \) is uniformly continuous, \( \|f_{> \tau^{1/2}}\|_{L^\infty} \to 0 \) as \( \tau \to \infty \). Taking limits, we conclude that
\[ \|m_{\partial_i f}\| = o_{\tau \to \infty}(1). \)
Chapter 4

Calderón's problem for Lipschitz conductivities

In this chapter we will prove

**Theorem 2.** [HT13] Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with Lipschitz boundary. For $i = 1, 2$, let $\gamma_i \in W^{1,\infty}(\Omega)$ be real valued functions, and assume there is some $c$ such that $0 < c < \gamma_i < c^{-1}$. Then there exists a constant $\epsilon_{d,\Omega}$ such that if each $\gamma_i$ satisfies either $\|\nabla \log \gamma_i\|_{L^\infty(\Omega)} \leq \epsilon_{d,\Omega}$ or $\gamma_i \in C^1(\Omega)$ then $\Lambda \gamma_1 = \Lambda \gamma_2$ implies $\gamma_1 = \gamma_2$.

### 4.1 Existence of CGO solutions

Using the results of the previous section, we can construct solutions to the CGO equation (2.11).

**Lemma 4.1.1.** Let $n \geq 3$. Suppose that $\gamma \in W^{1,\infty}$ satisfies $0 < c < \gamma < c^{-1}$ and $\gamma = 1$ outside a fixed compact set $B$. Let $q = \gamma^{-1/2} \Delta \gamma^{1/2}$. If $\|\nabla \log \gamma_i\|_{L^\infty}$ is sufficiently small or $\gamma_i \in C^1$, then for $|\text{Re} \zeta|$ sufficiently large, the equation

$$(\Delta_\zeta - q)\psi = h \quad (4.1)$$

has a solution $\psi$ in $\dot{X}^{1/2}_{\zeta}$ for any $h \in \dot{X}^{-1/2}_{\zeta}$. Furthermore, $\psi$ satisfies the estimate

$$\|\psi\|_{\dot{X}^{1/2}_{\zeta}} \lesssim \|h\|_{\dot{X}^{-1/2}_{\zeta}}.$$ 

**Proof.** We write (4.1) as

$$(I - \Delta^{-1}_\zeta \circ q)\psi = \Delta^{-1}_\zeta h.$$ 

We show that $I - \Delta^{-1}_\zeta q$ has a bounded inverse on $\dot{X}^{1/2}_{\zeta}$. For this it suffices to show that

$$\|\Delta^{-1}_\zeta \circ q\|_{\dot{X}^{1/2}_{\zeta} \rightarrow \dot{X}^{1/2}_{\zeta}} \leq 1/2.$$
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Since \( \Delta^{-1} \) is an isometry from \( \dot{X}^{1/2} \) to \( \dot{X}^{-1/2} \), this is the same as showing that
\[
\|q\|_{\dot{X}^{1/2}} \leq 1/2.
\]

By Corollary 3.4.3, it suffices to show that \( \|q\|_{X^{1/2}} \) is small. We have
\[
q = \frac{1}{4} |\nabla \log \gamma|^2 + \frac{1}{2} \text{div}(\nabla \log \gamma).
\]

By Lemma (3.4.5), we have
\[
\|q\|_{X^{1/2}} \lesssim \tau^{-1} \|\nabla \log \gamma\|_{L^\infty} + \|\nabla \log \gamma\|_{L^\infty}.
\]

We can make this small by taking \( \tau \) large and \( \|\nabla \log \gamma\|_{L^\infty} \) small. If \( \gamma \in C^1 \), then
\[
\|q\|_{X^{1/2}} \lesssim \tau^{-1} \|\nabla \log \gamma\|_{L^\infty} + o_{\tau \to \infty}(1)
\]
by Lemma 3.4.6.

Recently, Caro and Rogers [CR14] constructed local solutions (which satisfy \((-\Delta + q)\psi = q\) in the set \( B \)) for \( \gamma \in W^{1,\infty} \) with no smallness condition, by conjugating the operator \( -\Delta + q \) by a convex weight of the form \( e^{-M(\text{e}^{\text{e}})^2} \). Since this adds some technicalities, we will not pursue this direction.

4.2 An averaged estimate

To obtain control of our solutions to the equation \((-\Delta - m_q)\phi = q\) in \( \dot{X}^{-1/2} \), it remains to estimate \( \|q\|_{X^{-1/2}} \). The worst part of \( q \) looks like \( \Delta \log g = \nabla \cdot (\nabla \log g) \), and so we are led to bound expressions of the form
\[
\|\nabla f\|_{X^{-1/2}}^2 := \sum_i \|\partial_i f\|_{X^{-1/2}}^2,
\]
where \( f \) is some continuous function with compact support. At high modulation, \( p(\xi)^{-1/2} \sim (|\xi|+\tau)^{-1} \), so \( \|Q_h \nabla f\|_{X^{-1/2}} \leq \|f\|_2 \). At low modulation, however, \( p(\xi) \) could be small. If we use the fact that \( q \) is localized to replace the \( \dot{X}^{-1/2} \) norm with the \( X^{-1/2} \) norm, then we would have the straightforward estimate
\[
\|Q_t \nabla f\|_{X^{-1/2}} \lesssim \tau^{-1/2} \|Q_t \nabla f\|_{L^2}.
\]

In general, this will not be bounded unless \( f \in H^{1/2} \), which would require the conductivity \( \gamma \) to be in \( H^{3/2} \). This is why the previous work on Calderón’s problem for low regularity (i.e. [Bro96, PPU03, Bro03]) was restricted to conductivities with 3/2 derivatives.
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To overcome this problem, we will exploit the fact that we have at least two degrees of freedom in choosing $\zeta$. If we average over these parameters, we can obtain a better estimate that does not involve a factor of $\tau^{1/2}$.

Intuitively, this is possible because the bad case is when $\hat{f}$ concentrates near the characteristic set $\Sigma_{\zeta}$, which is a codimension 2 sphere. If we fix a point $\xi \neq 0$ and vary the parameter $\zeta$, then $\xi$ will be far away from most of the spheres $\Sigma_{\zeta}$.

To make this precise, let us define the sense in which we are varying $\zeta$. Given $U$ in the orthogonal group $O(n)$ and $\tau > 0$, we define

$$\zeta(\tau, U) = \tau U (e_1 - i e_2),$$

where $e_1$ and $e_2$ are the standard basis vectors in $\mathbb{R}^n$. Let $\mu$ denote Haar measure on $O(n)$, normalized so that if $\sigma$ is the usual spherical measure on $S^{n-1}$ and $f : S^{n-1} \to \mathbb{R}$ is integrable, then for any $\theta \in S^{n-1}$ we have

$$\int_{O(n)} f(U \cdot \theta) \, d\mu(U) = \int_{S^{n-1}} f(\omega) \, d\sigma(\omega). \quad (4.2)$$

The following key lemma was first proved in a slightly different form in [HT13]. The proof was substantially simplified in [NS14], and we present the following streamlined version from [Hab14].

**Lemma 4.2.1.** If $f \in \dot{H}^{-1}$, then

$$M^{-1} \int_M^{2M} \int_{O(n)} \|f\|_{H^{-1/2} \chi_{\zeta(\tau, U)}}^2 \, d\mu(U) \, d\tau \lesssim \|P_{\geq 100M} f\|_{H^{-1}}^2 + M^{-1} \|P_{<100M} f\|_{H^{-1/2}}^2.$$  

**Proof.** This is true if $f$ is supported at frequencies $|\xi| \geq 100M$, because there we have $|p_\zeta(\xi)| \geq |\xi|^2$. Thus we may assume that $f$ is supported at frequencies $|\xi| \lesssim M$, where we have $|p_\zeta(\xi)| \gtrsim 2\tau |\xi \cdot (U e_1)| + |\xi|^2 + 2\tau |\xi \cdot (U e_2)|$. Here we use Plancherel and the identity $U^* = U^{-1}$ and estimate by

$$\begin{aligned}
\left\| \langle \nabla \rangle^{-1/2} f \right\|_2^2 \cdot \sup_{|\xi| \leq 100M} \left| \frac{\xi}{M} \int_M^{2M} \int_{O(n)} (2\tau |(U^{-1} \xi) \cdot e_1| + |\xi|^2 + 2\tau |U^{-1} \xi \cdot e_2|))^{-1} \, d\mu(U) \, d\tau \right.
\end{aligned}$$

By (6.8), the quantity inside the supremum is given by

$$\begin{aligned}
\frac{1}{M} \int_M^{2M} \int_{S^{n-1}} (2\tau |\omega \cdot e_1| + |\xi| + 2\tau \omega \cdot e_2|)^{-1} \, d\sigma(\omega) \, d\tau
\end{aligned}$$

We view $(\tau, \omega)$ as polar coordinates and change variables to $u = \tau \omega$. Then in the region $\tau \in [M, 2M]$ the volume element $du$ is bounded below by $M^{n-1} \, d\sigma(\omega) \, d\tau$, so this integral is bounded by

$$\begin{aligned}
\frac{1}{M^n} \int_{|u| \in [M, 2M]} (2|u_1| + |\xi| + 2u_2)^{-1} \, du.
\end{aligned}$$
Writing $v = (u_1, u_2)$, and integrating over the remaining variables, we bound by
\[
\frac{1}{M^n} M^{n-2} \int_{B(0,2M)} (2|v_1| + |\xi| + 2v_2)^{-1} dv \leq \frac{1}{M^2} \int_{B(0,2M)} |v|^{-1} dv \sim \frac{1}{M}
\]

4.3 Proof of uniqueness

We now prove Theorem 2, which we state below for convenience. To simplify things, we first record the following observation:

**Lemma 4.3.1.** Suppose $\zeta, \tilde{\zeta} \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = \tilde{\zeta} \cdot \tilde{\zeta} = 0$. Then
\[
\|u\|_{X^b_\zeta} \lesssim (1 + |\zeta - \tilde{\zeta}|)||u||_{X^b_{\tilde{\zeta}}}
\]

**Proof.** We have
\[
|p_\zeta| \leq |p_{\tilde{\zeta}}| + 2|\zeta - \tilde{\zeta}| \cdot \xi|
\]
\[
\leq |p_{\tilde{\zeta}}| + 2|\zeta - \tilde{\zeta}||\xi|
\]
\[
\lesssim (1 + |\zeta - \tilde{\zeta}|)(|p_{\tilde{\zeta}}| + \tau)
\]
by (3.2).

**Theorem 2.** [HT13] Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with Lipschitz boundary. For $i = 1, 2$, let $\gamma_i \in W^{1,\infty}(\Omega)$ be real valued functions, and assume there is some $c$ such that $0 < c < \gamma_i < c^{-1}$. Then there exists a constant $\epsilon_{d,\Omega}$ such that if each $\gamma_i$ satisfies either $\|\nabla \log \gamma_i\|_{L^\infty(\Omega)} \leq \epsilon_{d,\Omega}$ or $\gamma_i \in C^1(\Omega)$ then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.

**Proof.** Fix $r > 0$ and three orthonormal vectors $\{e_1, e_2, e_3\}$, and define
\[
\zeta_1(\tau, U) = \tau U(e_1 - ie_2)
\]
\[
\zeta_2(\tau, U) = -\zeta_1(\tau, U)
\]
\[
\tilde{\zeta}_1(\tau, U) := \tau U e_1 + i(r U e_3 - \sqrt{\tau^2 - r^2} U e_2)
\]
\[
\tilde{\zeta}_2(\tau, U) := -\tau U e_1 + i(r U e_3 + \sqrt{\tau^2 - r^2} U e_2)
\]

In what follows, all of inequalities will implicitly depend on $r$. For example, we have $|\zeta_i - \tilde{\zeta}_i| \lesssim 1$. In particular, by Lemma 4.3.1, the spaces $X^b_{\zeta_i}$ and $X^b_{\tilde{\zeta}_i}$ have equivalent norms.

Using Lemma 4.1.1, there exist solutions $\psi_i$ to the equations the equations $(\Delta_{\zeta_i(\tau, U)} - m_{q_i})\psi_i = q_i$, satisfying
\[
\|\psi_i\|_{X^{1/2}_{\zeta_i(\tau, U)}} \lesssim \|q_i\|_{X^{-1/2}_{\zeta_i(\tau, U)}}.
\]
Note that by (3.10), such a solution lies in $H^1_\text{loc}(\mathbb{R}^n)$. This implies that the corresponding solution $v_i = e^{i \xi_i(r,U)}(1 + \psi_i)$ to the Schrödinger equation $(\Delta - q_i) v_i$ lies in $H^1_\text{loc}(\mathbb{R}^n)$ as well.

Let $k = 2rUe_3$. By Lemma 2.1.6,

$$0 = \langle q_1 - q_2, e^{ik \cdot x}(1 + \psi_1)(1 + \psi_2) \rangle$$

$$= \langle q_1 - q_2, e^{ik \cdot x} \rangle + \langle q_1 - q_2, e^{ik \cdot x} \psi_1 \psi_2 \rangle + \langle q_1 - q_2, e^{ik \cdot x}(\psi_1 + \psi_2) \rangle$$

We need to show that the second and third terms are small. Let $\phi$ be a Schwartz function that is equal to one on the support of $q$. Then

$$|\langle q_1, e^{ik \cdot x} \psi_1 \psi_2 \rangle| = |\langle q_1 e^{-ik \cdot x} \bar{\psi}_2, \psi_1 \rangle|$$

$$\lesssim \|e^{ik \cdot x} \phi \bar{\psi}_2\|_{X^{1/2}_{\xi_1(r,U)}} \|\phi \psi_1\|_{X^{1/2}_{\xi_1(r,U)}}$$

$$= \|e^{-ik \cdot x} \phi \bar{\psi}_2\|_{X^{1/2}_{\xi_1(r,U)}} \|\phi \psi_1\|_{X^{1/2}_{\xi_1(r,U)}}$$

$$\lesssim \|\psi_2\|_{\dot{X}^{1/2}_{\xi_1(r,U)}} \|\psi_1\|_{\dot{X}^{1/2}_{\xi_1(r,U)}}$$

$$\lesssim \|q_2\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}} \|q_1\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}}$$

since the seminorms of $e^{-ik \cdot x} \phi$ are bounded with a bound depending only on $r$. We can bound the $q_2$ term in the same way. On the other hand, we have

$$|\langle q_1, e^{ik \cdot x} \psi_1 \rangle| \lesssim \|q_1\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}} \|\psi_1\|_{\dot{X}^{1/2}_{\xi_1(r,U)}}$$

$$\lesssim \|q_1\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}} \|q_1\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}}$$

by duality of $\dot{X}^{1/2}_{\xi_1(r,U)}$ and $\dot{X}^{-1/2}_{\xi_1(r,U)}$. The terms with $\bar{\psi}_2$ are the same. In summary, we obtain

$$|\langle \hat{q}_1 - \hat{q}_2 \rangle(2rUe_3)| \lesssim \sum_{1 \leq i,j,k,l \leq 2} \|q_1\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}} \|q_k\|_{\dot{X}^{-1/2}_{\xi_1(r,U)}}.$$  \hspace{1cm} (4.3)

Recalling that $q_i = \frac{1}{2} \Delta \log \gamma_i + \frac{1}{4} |\nabla \log \gamma_i|^2$, we can apply (3.6) and Lemma 4.2.1 to obtain

$$M^{-1} \int_M^{2M} \int_{O(n)} \|q_i\|_{\dot{X}^{-1/2}_{\xi_j(r,U)}}^2 \, d\mu(U) \, d\tau \lesssim \|P_{\geq 100M} \Delta \log \gamma_i\|_{\dot{H}^{-1/2}}^2 + M^{-1} \|P_{<100M} \Delta \log \gamma_i\|_{\dot{H}^{-1/2}}^2$$

$$+ M^{-1} \|\nabla \log \gamma_i\|_{L^2}^2$$

$$\lesssim M^{-1/2} \|P_{<M^{1/2}} \nabla \log \gamma_i\|_{L^2}^2$$

$$+ M^{-1/2} \|P_{>M^{1/2}} \nabla \log \gamma_i\|_{L^2}^2$$

$$\rightarrow 0$$

as $M \rightarrow \infty$, because $\nabla \log \gamma_i \in L^2 \cap L^4$. Now we integrate (6.15) and apply Hölder’s inequality to obtain

$$2M^{-1} \int_M^{2M} \int_{S^{n-1}} |\langle \hat{q}_1 - \hat{q}_2 \rangle| \, (r,\omega) \, d\omega \, d\tau = M^{-1} \int_M^{2M} \int_{O(n)} \|\hat{q}_1 - \hat{q}_2\|_{(2rUe_3)} \, d\mu(U) \, d\tau$$

$$\rightarrow 0$$
as $M \to \infty$. The right hand side does not depend on $M$, so this implies that

$$\int_{S^{n-1}} |(\hat{q}_1 - \hat{q}_2)(2r\omega)| \, d\omega = 0.$$ 

Since $r$ is arbitrary, this implies that $\hat{q}_1 = \hat{q}_2$, so that $q_1 = q_2$. By Lemma 2.2.1, we conclude that $\gamma_1 = \gamma_2$. \qed
Chapter 5

Unique continuation and Carleman estimates

5.1 Unique continuation

Suppose that $f$ is a real-analytic function on a connected open set $\Omega \subset \mathbb{R}^n$. If $f$ vanishes to infinite order at some point $x_0$, then it is easy to see that it vanishes identically in $\Omega$. Since solutions to the Laplace equation $\Delta u = 0$ are real-analytic, the Laplace operator satisfies the strong unique continuation property (SUCP):

If $\Delta u = 0$ in $\Omega$ and $u$ vanishes to infinite order at a point $x_0 \in \Omega$, then $u$ vanishes identically in $\Omega$.

If we perturb the Laplacian by adding smooth lower order terms, then solutions are no longer analytic. However, the property SUCP still holds. The strongest theorem of this type is due to Koch and Tataru. A simplified version of their results is as follows

**Theorem 5.1.1** ([KT01]). Let $P$ be an elliptic differential operator of the form

$$Pu = \sum_{ij} \partial_i (g^{ij}(x) \partial_j u) + W_1(x) \cdot \nabla u + \nabla (W_2(x) u) + V(x) u,$$

where the coefficients $g, W_i, V$ are defined in a ball $B = B(x_0, R)$. Assume $g^{ij}$ satisfies the uniform ellipticity condition

$$c|\xi|^2 \leq g^{ij} \xi_i \xi_j \leq C|\xi|^2 \text{ for } \xi \in \mathbb{R}^n \setminus 0$$

and that $g^{ij} \in W^{1,\infty}(B)$, $W_i \in L^n(B)$ and $V \in L^{n/2}(B)$. If $u \in H^1(B)$ is such that $Pu = 0$ in $B$ and

$$\int_{B(0, r)} |u|^2 \, dx \leq c_N r^N$$

for $r < R$ and $N > 0$, then $u$ vanishes identically in $B$. 
CHAPTER 5. UNIQUE CONTINUATION AND CARLEMAN ESTIMATES

These assumptions are essentially sharp, in that the result is false when \( n \geq 3 \) if the assumption on \( g^{ij} \) is replaced with a Hölder condition of order less than 1 [Pli63, Mil73, Man98] or if the exponent in the \( L^p \) assumptions on \( W_i \) and \( V \) is lowered (as can be seen by computing the Laplacian of the function \( \exp[-|\ln x|^{1+\epsilon}] \)).

5.2 The Carleman method

The first results on unique continuation which did not rely on analyticity are due to Carleman [Car39]. Carleman’s method is based on weighted estimates of the form

\[
\|e^{\tau \phi} u\| \lesssim \|e^{\tau \phi} \Delta u\|,
\]

where \( \phi \) is a real-valued function and \( \tau \) is a large parameter. In this section we will illustrate how to use such estimates to prove unique continuation theorems.

For simplicity we will focus on the weak unique continuation property (WUCP) for lower-order perturbations of the Laplacian. We say an elliptic operator \( \mathcal{P} \) satisfies WUCP if

\[ \mathcal{P}u = 0 \text{ in } \Omega, \text{ and } u \text{ vanishes on a nonempty open set } \Rightarrow u \text{ vanishes identically in } \Omega. \]

Let us first prove the following basic theorem:

**Theorem 5.2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a connected open set, with \( n \geq 3 \). Let \( A \) be a bounded vector field on \( \Omega \), and let \( V \) be a bounded function. If

\[ (\Delta + A \cdot \nabla + V)u = 0 \]

on \( \Omega \), and \( u \in H^1(\Omega) \) vanishes in an open subset of \( \Omega \), then \( u \) vanishes in \( \Omega \).

The proof of this theorem will be based on the following Carleman estimate

**Lemma 5.2.2.** Under the assumptions of Theorem 5.2.1, there exists some \( \delta = \delta(A, V) > 0 \) such that

\[
\|e^{\tau x_1} u\|_{L^2} \lesssim \|e^{\tau x_1} (\Delta + A \cdot \nabla + V)u\|_{L^2}
\]

whenever \( u \) is supported in a ball of radius \( \delta \).

Before proving this, let us show how this implies a unique continuation theorem. First we will show that unique continuation holds across spheres. The point is that a linear weight is well-suited to proving unique continuation across planes, and the region in between a sphere and a plane just inside the sphere is a small compact set. Given an open set \( U \), define

\[ U_+ = \{ x \in U : |x| > 1 \} \]
\[ U_0 = \{ x \in U : |x| = 1 \} \]
\[ U_- = \{ x \in U : |x| < 1 \}. \]
Lemma 5.2.3. Suppose that \( u \in H^1(U) \) and \((\Delta + A \cdot \nabla + V)u = 0 \) in \( U \). If \( u = 0 \) in \( U_+ \) then \( u = 0 \) in a neighborhood of \( U_0 \).

Proof. By compactness of the sphere and rotation invariance, it suffices to prove that if \( U \) contains the point \((1,0,\ldots,0)\), then \( u \) vanishes in a neighborhood of \((1,0,\ldots,0)\).

In order to apply Lemma 5.2.2 to \( u \), we need to produce an approximate solution \( u \in H^1(\mathbb{R}^n) \) to \((\Delta + A \cdot \nabla + V)u = 0 \) which is localized to a small ball. Fortunately, the Carleman estimate (5.1) is compatible with such a localization.

Let \( \chi(x_1) \) be a smooth cutoff function, such that \( \chi = 1 \) when \( x_1 \geq 1 - \epsilon \) and \( \chi = 0 \) when \( x_1 \leq 1 - 2\epsilon \). By choosing \( \epsilon \) sufficiently small, we can ensure that the support of \( \chi u \) is contained in \( U_- \). In particular, the function \( \chi u \) extends by zero to a function in \( H^1(\mathbb{R}^n) \).

By taking \( \epsilon \) sufficiently small we can arrange that the support of \( \chi u \) is contained in a very small ball, such that Lemma 5.2.2 applies.

Since \( \chi = 1 \) when \( x_1 \geq 1 - \epsilon \), we have

\[
\|e^{\tau x_1} u\|_{L^2(x_1 \geq 1 - \epsilon)} \lesssim \|e^{\tau x_1} \chi u\|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim \|e^{\tau x_1} (\Delta + A \cdot \nabla + V)(\chi u)\|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim \|e^{\tau x_1} (\Delta \chi u + 2\nabla \chi \cdot \nabla u + \nabla \chi \cdot Au)\|_{L^2(\mathbb{R}^n)},
\]

since \((\Delta + A \cdot \nabla + V)u = 0\) in the support of \( \chi \). Now we use the fact that \( \nabla \chi = 0 \) away from \( \{x_1 \in [1 - \epsilon, 1 - 2\epsilon]\} \):

\[
\|e^{\tau x_1} u\|_{L^2(x_1 \geq 1 - \epsilon)} \lesssim e^{\tau(1-\epsilon)}\|u\|_{H^1(U)}
\]

In particular,

\[
\|u\|_{L^p(x_1 \geq 1 - \epsilon/2)} \lesssim e^{-\epsilon \tau/2},
\]

with a constant independent of \( \tau \). Letting \( \tau \to \infty \), we find that \( u \) vanishes when \( x_1 \geq 1 - \epsilon/2 \). In particular, \( u \) vanishes in a neighborhood of \((1,0,\ldots,0)\).

Using the conformal inversion \( u \mapsto |x|^{-(n-2)}u(|x|^{-2}x) \), we can prove that unique continuation holds across spheres in the opposite direction as well.

Lemma 5.2.4. Suppose \( 0 \notin U \). Under the assumptions of Lemma 5.2.3, if \( u = 0 \) in \( U_- \) then \( u = 0 \) in a neighborhood of \( U_0 \).

Proof. Let

\[
\tilde{u}(x) = |x|^{-(n-2)}u(|x|^{-2}x)
\]

A computation using polar coordinates shows that

\[
\Delta \tilde{u}(x) = |x|^{-n}\Delta u(|x|^{-2}x).
\]

(5.2)
Let $\tilde{U}$ be the image of $U$ under the inversion $x \mapsto |x|^{-2}x$. Then $\tilde{U}_0 = U_0$ and $\tilde{U}_\pm = U_\pm$. Thus $\tilde{u}$ vanishes in $\tilde{U}_\pm$. By (5.2), we have

$$(\Delta + \tilde{A} \cdot \nabla + \tilde{V})\tilde{u} = 0,$$

where $\tilde{A}$ and $\tilde{V}$ are bounded. Applying Lemma 5.2.3, we find that $\tilde{u}$ vanishes in a neighborhood of $\tilde{U}_0$. This implies, in turn, that $u$ vanishes in a neighborhood of $U_0$.

Finally, we conclude that WUCP holds for $\Delta + A \cdot \nabla + V$:

**Proof of Theorem 5.2.1.** Let $U$ be the maximal open set on which $u$ vanishes. We will show that $\partial U \subset \partial \Omega$. This implies that $\Omega = U \cup (\Omega \setminus U)$. Since $\Omega$ is connected and $U$ is nonempty, we can conclude that $\Omega \setminus U = \emptyset$.

For each $x \in U$, we have $x \in B(x, r(x)) \subset U$ with $r(x) = d(x, \partial U)$. In particular, we may write

$$U = \bigcup_{x \in U} B(x, r(x)),$$

so that

$$\partial U \subset \bigcup_{x \in U} \partial B(x, r(x)).$$

Now let $y_0 \in \partial U$ be arbitrary, and choose some $x_0 \in U$ such that $y_0 \in \partial B(x_0, r(x_0))$. By construction, $u$ vanishes in $B(x_0, r(x_0))$. If $y_0$ is an interior point of $\Omega$, then Lemma 5.2.4 implies that $u$ vanishes in a neighborhood of $y_0$. But this means that $y_0 \in U$, a contradiction. Thus $y_0 \in \partial \Omega$, and we have shown that $\partial U \subset \partial \Omega$.

### 5.3 $L^2$ Carleman estimates

In this section we will explain how to prove estimates of the form

$$\|e^{\tau x_1}u\|_{L^2} \lesssim \|e^{\tau x_1}(\Delta + A \cdot \nabla + V)u\|_{L^2}.$$

To prove such an estimate, we show that the lower order terms are perturbative. This will follow from estimates of the form

$$\tau \|e^{\tau x_1}u\|_{L^2} + \|e^{\tau x_1}\nabla u\|_{L^2} \ll \|e^{\tau x_1}\Delta u\|_{L^2}.$$ 

By making the change of variables $v = e^{\tau x_1}u$, we can reduce to showing that

$$\|v\|_{H^1_\tau} \lesssim \|\Delta_{\tau x_1}v\|_{L^2},$$

where

$$\|v\|_{H^1_\tau} = \tau \|v\|_{L^2} + \|\nabla v\|_{L^2},$$

and

$$\Delta_{\tau x_1} = e^{\tau x_1} \Delta e^{-\tau x_1}.$$
is exactly the type of operator $\Delta_\zeta$ which we studied in Chapter 3. Suppose that $u$ supported in the unit ball $B(0, 1)$. Then by (3.5), (3.10) and (3.7), we have
\[ \|u\|_{H^1_\tau} \lesssim \tau^{1/2} \|u\|_{X^{1/2}_\tau} \]
\[ = \tau^{1/2} \|\Delta_{\tau e_1} u\|_{X^{-1/2}_\tau} \]
\[ \lesssim \tau^{1/2} \|\Delta_{\tau e_1} u\|_{X^{-1/2}_\tau} \]
\[ \lesssim \|\Delta_{\tau e_1} u\|_{L^2}. \]
This shows that
\[ \tau \|e^{\tau x_1} u\|_{L^2} + \|e^{\tau x_1} \nabla u\|_{L^2} \lesssim \|e^{\tau x_1} \Delta u\|_{L^2}. \tag{5.3} \]
uniformly in $\tau$.

We will get a better estimate for the gradient term if $u$ is supported in a smaller ball. Indeed, if $u$ is supported in $B(x, r)$, then $u_r(x) = u(rx)$ is supported in the unit ball. Thus
\[ r \|e^{\tau x_1} \nabla u(r x)\|_{L^2} \lesssim r^2 \|e^{\tau x_1} \Delta u(r x)\|_{L^2}, \]
and by rescaling we have
\[ \|e^{\tau x_1/r} \nabla u(x)\|_{L^2} \lesssim r^{\tau x_1/r} \|e^{\tau x_1} \Delta u\|_{L^2}. \tag{5.4} \]

Now suppose that $A, V \in L^\infty$. If $r$ is sufficiently small and $\tau$ is sufficiently large, then (5.3) and (5.4) imply the perturbed estimate
\[ \|e^{\tau x_1} u\|_{L^2} \lesssim \|e^{\tau x_1}(\Delta + A \cdot \nabla u + V) u\|_{L^2} \]
whenever $u$ is supported in a ball of radius $r$. Thus we have proven Lemma 5.2.2 and the proof of Theorem 5.2.1 is concluded.

### 5.4 $L^p$ Carleman estimates

The $L^2 \to L^2$ Carleman estimates of the previous section require the coefficients to be bounded, since otherwise multiplication by the coefficient is not a bounded operator on $L^2$. In order to get unique continuation for equations with unbounded coefficients, we will need to prove Carleman estimates in $L^p$ spaces.

For operators of the form $\Delta + V$, strong unique continuation under the sharp assumption $V \in L^{n/2}_{\text{loc}}$ was first proven by Jerison and Kenig [JK85]. Later, Kenig, Ruiz and Sogge gave a simpler proof of the weak unique continuation property based on the following $L^p$ Carleman estimate for linear weights:

**Theorem 5.4.1** ([KRS87]). Let $n \geq 3$. The inequality
\[ \|e^{\tau x_1} u\|_{L^p} \lesssim \|e^{\tau x_1} \Delta u\|_{L^{p'}} \tag{5.5} \]
holds uniformly in $\tau$ for $u \in W^{2,p'}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$ and $1/p' - 1/p = 2/n$ (i.e. $p' = 2n/(n + 2)$ and $p = 2n/(n - 2)$.)
The numerology is such that by localizing this estimate, we can replace the Laplacian by $\Delta + V$:

**Corollary 5.4.2.** Let $V \in L^{n/2}(\mathbb{R}^n)$ be fixed. Under the assumptions of Theorem 5.4.1, there exists $\delta(V) > 0$ such that
\[
\|e^{\tau x_1} u\|_{L^p} \lesssim \|e^{\tau x_1} (\Delta + V) u\|_{L^p'}
\]
whenever $u$ is supported in a ball of radius $\delta$.

**Proof.** Let $\epsilon > 0$, to be chosen later. By the uniform continuity of Lebesgue integrals, there exists $\delta$ such that $\|V\|_{L^{n/2}(B)} < \epsilon$ for any ball $B$ of radius $\delta$. For $u$ supported in such a ball, we have
\[
\|e^{\tau x_1} u\|_{L^p(B)} \lesssim \|e^{\tau x_1} \Delta u\|_{L^p'(B)} \lesssim \|e^{\tau x_1} (\Delta + V) u\|_{L^p'(B)} + \|e^{\tau x_1} V u\|_{L^p'(B)} \lesssim \|e^{\tau x_1} (\Delta + V) u\|_{L^p'(B)} + \epsilon \|e^{\tau x_1} u\|_{L^p(B)}.
\]
By choosing $\epsilon$ sufficiently small, we can absorb the second term into the left hand side of the equation to obtain the desired estimate. 

By replacing Lemma 5.2.2 with Corollary 5.4.2, we can prove unique continuation for the operator $-\Delta + V$ with $V \in L^{n/2}_{\text{loc}}$ by the exact method which we used to prove Theorem 5.2.1.

As before, the estimate (5.5) is equivalent to
\[
\|u\|_{L^p} \lesssim \|\Delta_{\tau e_1} u\|_{L^p'},
\]
where $\Delta_{\tau e_1} = e^{\tau x_1} \Delta e^{-\tau x_1}$.

The proof of Theorem 5.4.1 is based on an idea of Hörmander [Hör83], who observed that since the symbol $p_\tau = p_{\tau e_1}$ vanishes on a codimension-2 sphere, the Tomas-Stein restriction theorem gives improved $L^p$ bounds for $u$ when $\hat{u}$ is localized near $\Sigma_\tau = \{\xi : p_\tau(\xi) = 0\}$.

The usual statement of the Tomas-Stein theorem relates to the Fourier transform of measures supported on the sphere.

**Theorem 5.4.3** ([Ste93, Tom75]). Suppose $p \geq (2d + 2)/(d - 1)$. Let $\sigma$ denote the surface measure on $S^{d-1}$. Then
\[
\|\hat{f} d\sigma\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(S^{d-1})}.
\]

Let $\tau S^{d-1}$ denote the sphere of radius $\tau$. Given a set $E$ we define its $\lambda$-neighborhood
\[
N_\lambda(E) := \{\xi : d(\xi, E) \leq \lambda\}.
\]
We use the following rescaled and localized variant of the restriction theorem:
Corollary 5.4.4. Let $p$ be as above. Suppose that $\hat{g}$ is supported in $N_\lambda(\tau S^{d-1})$, where $\lambda \leq \tau/8$. Then

$$\|g\|_{L^p} \lesssim \lambda^{1/2} \tau^{(d-1)/2 - d/p} \|\hat{g}\|_{L^2(N_\lambda(\tau S^{d-1}))}.$$ 

Proof. By Fourier inversion, we have

$$g(x) = \int \hat{g}(\xi) e^{ix \cdot \xi} d\xi$$

$$= \int_{\tau - \lambda}^{\tau + \lambda} \int_{S^{d-1}} \hat{g}(\rho \omega) e^{i\rho(x,\omega)} \rho^{d-1} d\rho d\sigma$$

$$= \int_{\tau - \lambda}^{\tau + \lambda} \rho^{d-1} \hat{g}(\rho \omega) d\sigma (\rho x) d\rho$$

By Minkowski’s inequality, the restriction theorem, Cauchy-Schwarz, and Plancherel this implies that

$$\|g\|_{L^p} \lesssim \int_{\tau - \lambda}^{\tau + \lambda} \|\hat{g}(\rho \omega) d\sigma (\rho x)\|_{L^p(\mathbb{R}^d)} \rho^{d-1} d\rho$$

$$\lesssim \int_{\tau - \lambda}^{\tau + \lambda} \rho^{-d/p} \|\hat{g}(\rho \omega) d\sigma (\rho x)\|_{L^p(\mathbb{R}^d)} \rho^{d-1} d\rho$$

$$\lesssim \tau^{-d/p} \int_{\tau - \lambda}^{\tau + \lambda} \|\hat{g}(\rho \omega)\|_{L^2(S^{d-1})} \rho^{d-1} d\rho$$

$$\lesssim \tau^{-d/p} (\lambda \tau^{d-1})^{1/2} \left( \int_{\tau - \lambda}^{\tau + \lambda} \|\hat{g}(\rho \omega)\|_{L^2(S^{d-1})}^2 \rho^{d-1} d\rho \right)^{1/2}$$

$$= \lambda^{1/2} \tau^{(d-1)/2 - d/p} \|\hat{g}\|_{L^2(N_\lambda(\tau S^{d-1}))}.$$ 

We now apply the restriction theorem to the dyadic regions

$$E_\lambda = \{ \xi : d(\xi, \Sigma) \in (\lambda/4, \lambda] \}$$

defined in Section 3.2. Recall that $Q_\lambda$ is a Fourier multiplier which localizes to $E_\lambda$.

Lemma 5.4.5. Let $p = 2n/(n-2)$, $\lambda \leq \tau/8$. Let $\zeta = \tau (e_1 - i\eta)$, where $|e_1| = 1$ and $|\eta| \leq 1$. Then

$$\|Q_\lambda f\|_p \lesssim (\lambda/\tau)^{1/n} \|\Delta_\zeta^{1/2} f\|_{L^2}. \quad (5.6)$$

$$\|f\|_p \lesssim \|\Delta_\zeta^{1/2} f\|_{L^2}. \quad (5.7)$$
Proof. Note that $p_\zeta(\xi) = p_{te_1}(\xi - \tau \eta)$. Since $L^p$ is invariant under multiplication by complex exponentials, and $L^2$ is invariant under translation, we may assume that $\eta = 0$.

By rotation invariance, we may assume $e_1 = (1, 0, \ldots, 0)$. We use the notation $\xi = (\xi_1, \xi')$.

For (5.6), write $g = Q_\lambda f$. Note that $E_\lambda \subset \{ \xi : |\xi_1| \leq c\lambda, ||\xi'|| = \tau \leq c\lambda \}$.

We can write $g = \phi_\lambda *_{x_1} g$, where $\phi_\lambda(x_1) = \lambda \phi(\lambda x_1)$ for some Schwartz function $\phi$, and the convolution is taken in the $x_1$ variable only. By Minkowski’s inequality and Young’s inequality, we have

$$
\|g\|_p \approx \left\| \int \phi_\lambda(x_1 - y_1) g(y_1, x') \, dy_1 \right\|_p \\
\lesssim \left\| \int |\phi_\lambda(x_1 - y_1)| \|g(y_1)\|_{L^p_{x'}} \, dy_1 \right\|_{L^p_{x'}} \\
\lesssim \lambda^{1/2-1/p} \|g\|_{L^2_{x_1}, L^p_{x'}}.
$$

If we regard $g$ as a function in the $x'$ variable, we see that its Fourier transform lies in $N_{c\lambda}(\tau S^{n-2})$. By Corollary 5.4.4, we have $\|g(x_1)\|_{L^p_{x'}} \leq \lambda^{1/2-1/(n-2)} \|\hat{g}(x_1)\|_{L^2_{x'}}$ for each $x_1$.

It follows that

$$
\|g\|_p \lesssim \lambda^{1/2} \lambda^{1/2-1/p} \tau^{1/2-1/n} \|\hat{g}\|_{L^2} \lesssim \lambda^{1/n} \tau^{-1/n} \|\Delta^{1/2}_\tau f\|_{L^2},
$$

since $|p_\tau| \sim \tau \lambda$ on $E_\lambda$.

For (5.7), we apply (5.6) near $\Sigma_\zeta$ and Sobolev embedding away from $\Sigma_\zeta$. On $E_{\leq \tau/8}$ we have

$$
\|Q_\lambda f\|_p \lesssim \sum_{\lambda \leq \tau/8} \|Q_\lambda f\|_p \\
\lesssim \sum (\lambda/\tau)^{1/n} \|Q_\lambda \Delta^{1/2}_\tau f\|_{L^2} \\
\lesssim \left( \sum \|Q_\lambda \Delta^{1/2}_\tau f\|_{L^2} \right)^{1/2} \\
\leq \|\Delta^{1/2}_\tau f\|_{L^2}
$$

On $E_{> \tau/8}$ we have the Sobolev embedding

$$
\|Q_\lambda f\|_p \lesssim \|Q_\lambda f\|_{H^1} \lesssim \|\Delta^{1/2}_\tau f\|_{L^2},
$$

since $|p_\tau| \sim (\tau + |\xi|)^2$ when $\xi \in E_{> \tau/8}$. Combining these estimates gives the claimed inequality. \qed

Now we have essentially proven Theorem 5.4.1:
Proof of Theorem 5.4.1. It suffices to show that the operator
\[ \Delta^{-1} \tau : L^p' (\mathbb{R}^n) \rightarrow L^p (\mathbb{R}^n) \]
is bounded. We have already shown that \( \Delta^{-1/2} \tau \) is bounded as a map from \( L^2 \rightarrow L^p \), so it suffices to show that it is bounded as a map from \( L^p' \rightarrow L^2 \) as well. By duality, this is equivalent to showing that \( \Delta^{-1/2} \tau \) is bounded as a map from \( L^p' \rightarrow L^2 \) as well. By duality, this is equivalent to the corresponding bound on \( \Delta^{-1/2} \tau \) by reflection invariance.

\[ \square \]

5.5 Unbounded gradient terms

We now discuss some issues that arise in the unique continuation problem for operators of the form \( \Delta + A \cdot \nabla u + V \), where \( A \) is unbounded. Suppose for example that we assume that \( A \) is merely bounded in \( L^q_{\text{loc}} \) for some \( q < \infty \). In order to apply the Carleman method as before, we would need to prove a gradient Carleman estimate of the form
\[ \| e^{\tau \phi} \nabla u \|_{L^r} \lesssim \| e^{\tau \phi} \Delta u \|_{L^{p'}} \tag{5.8} \]
where \( 1/p' - 1/r = 1/q \).

Barcelo, Kenig, Ruiz and Sogge [BKRS88] observed that this strategy runs into some problems when \( q \) is close to the optimal exponent \( n \). They showed that no gradient Carleman estimate can hold for linear weights \( \phi = x_1 \) unless \( r = p' = 2 \). In contrast, they proved unique continuation for \( A \in L^{(3n-2)/2} \) and \( V \in L^{n/2+} \) by establishing a gradient Carleman estimate with the convex weight \( \phi = x_1 + x_1^2 \). Furthermore, they showed that for any weight \( \phi \), the estimate (5.8) cannot hold uniformly in \( \tau \) and \( u \) unless \( 1/p' - 1/r \leq 2/(3n - 2) \).

The essential problem is that there are two competing phenomena associated with Carleman estimates. Suppose that \( u \) is localized to a ball of radius \( r \). By using \( L^2 \) estimates, which are based on localization, we have shown that for \( \tau > r^{-1} \), the following estimate holds:
\[ \| e^{\tau x_1} \nabla u \|_{L^2} \lesssim r \| e^{\tau x_1} \Delta u \|_{L^2}. \]
This estimate is good because it doesn’t blow up as \( \tau \rightarrow \infty \) and gets better when \( u \) is highly localized. However, it doesn’t allow us to multiply by functions in \( L^q \) for \( q < \infty \).

On the other hand, suppose we are trying to use the Fourier restriction theorem to prove a Carleman estimate. Recall that in order to prove an estimate for the gradient we will need to control \( \tau \| u \|_{L^2} \). If \( \hat{u} \in E_\lambda \), then by (5.6) we have
\[ \tau \| \Delta^{-1} \tau \|_{L^2} = \tau (\lambda \tau)^{-1} \| u \|_{L^2} \]
\[ \lesssim \tau (\lambda/\tau)^{1/n} (\lambda \tau)^{-1/2} \| u \|_{L^{p'}} \tag{5.9} \]
This estimate would allow us to multiply by functions in \( L^{n/2} \), but for fixed \( \lambda \) it blows up like \( \tau^{1/2 - 1/n} \) as \( \tau \rightarrow \infty \).
In [Wol92], Wolff proved WUCP for the operator $\Delta + A \cdot \nabla + V$ assuming only that $A \in L^n$ and $V \in L^{n/2}$. Wolff’s result relies on an argument that combines the localization and Fourier restriction aspects of the Carleman estimate. This idea was later adapted by Koch and Tataru to prove SUCP for a more general class of operators (Theorem 5.1.1).

We will briefly describe some of the ideas involved in Wolff’s argument. We would like to prove an estimate of the form

$$
\|e^{\tau x_1} \nabla u\|_{L^2(E)} \leq C(\tau, E)\|e^{\tau x_1} \Delta u\|_{L^{p'}(\mathbb{R}^n)},
$$

where $E$ is a rectangle centered at the origin. We write $u = Q_hu + Q_lu$. For $Q_hu$ this is just Sobolev embedding (as long as $C(1, \tau E) \geq 1$). On the other hand, for $Q_lu$ we can just use Hölder’s inequality and the finite band property and reduce this to the Kenig-Ruiz-Sogge estimate:

$$
\|Q_lu\|_{H^1(E)} \lesssim |E|^{1/n} \|Q_lu\|_{W^{1,p}(E)}
\lesssim |E|^{1/n} \|\Delta x_1 Q_lu\|_{W^{1,p'}(\mathbb{R}^n)}
\lesssim \tau |E|^{1/n} \|\Delta x_1 u\|_{L^{p'}(\mathbb{R}^n)}
$$

Thus we have

$$
\|e^{\tau x_1} \nabla u\|_{L^2(E)} \lesssim (1 + \tau |E|^{1/n})\|e^{\tau x_1} \Delta u\|_{L^{p'}}.
$$

(5.10)

This estimate has the correct $L^{p'}$ exponents, but blows up as $\tau \to \infty$. However, we may observe that the estimate is useful when $u$ is highly localized, i.e. when $\tau \sim |E|^{-1/n}$.

One might hope to patch together these localized Carleman estimates and deduce a unique continuation theorem. Suppose that $u$ satisfies the differential inequality

$$
|\Delta u| \leq A|\nabla u| + q|u|,
$$

(5.11)

with $A \in L^n$ nonnegative, and that $u$ vanishes in an open set. By inversion, we may reduce to the case where $u : \mathbb{R}^n \setminus B(0, 1) \to \mathbb{R}$ satisfies a similar equation and has compact support in $\mathbb{R}^n \setminus B(0, 1)$. We need to show that this is not possible unless $u = 0$.

Let $K$ be the convex hull of $\text{supp } u$. By multiplying by an exponential weight, we can localize the problem to $\partial K$, as follows: Let $\chi$ be a cutoff function such that $\chi = 1$ near $\partial K$. By (5.11),

$$
|\Delta (\chi u)| \leq A|\nabla (\chi u)| + q\chi u + \varepsilon,
$$

(5.12)

where $\varepsilon \lesssim |u| + |\nabla u|$ and $\text{supp } \varepsilon$ is contained in the interior of $K$.

Clearly we can replace $A$ by a function which is bounded below by 1 without changing (5.11). Thus we can use the function

$$
F = A|\nabla (\chi u)| + q\chi u
$$

as a substitute for $\chi u$. In particular, we have

$$
\text{supp } F = \text{supp}(\chi u).
$$
Instead of applying the Carleman method to $\chi u$ measured in some Sobolev norm, we replace $\chi u$ with $F$. This allows us to apply the Carleman method in a scale-invariant way.

For any $\theta \in S^{n-1}$, the function $x \cdot \theta$ restricted to the convex set $K$ will have its largest value on $\partial K$. It follows that

$$\|e^{\lambda x \cdot \theta} \varepsilon\|_{L^p} \leq \|e^{\lambda x \cdot \theta} F\|_{L^p}$$

for large $\lambda$, since $\text{supp} \varepsilon$ is contained in the interior of $K$. By a compactness argument one can show that in fact

$$\|e^{k \cdot x} \varepsilon\|_{L^p} \leq \|e^{k \cdot x} F\|_{L^p}, \quad (5.13)$$

whenever $|k|$ is sufficiently large.

Now we localize further by restricting to some small set $E$. By (5.5), (5.10), (5.12) and (5.13), we have, for $k \in \mathbb{R}^n$ and $|k| > |E|^{-1/d}$,

$$\|e^{k \cdot x} F\|_{L^p(E)} \lesssim \|A\|_{L^n(E \cap \text{supp} \chi)} \|e^{k \cdot x} \nabla (\chi u)\|_{L^2(E)} + \|q\|_{L^{n/2}(\text{supp} \chi)} \|e^{k \cdot x} \chi u\|_{L^2(E)}$$

$$\lesssim (|k||E|^{1/n}) \|A\|_{L^n(E \cap \text{supp} \chi)} + \|q\|_{L^{n/2}(\text{supp} \chi)} \|e^{k \cdot x} \Delta (\chi u)\|_{L^p'(\mathbb{R}^n)}$$

$$\lesssim (|k||E|^{1/n}) \|A\|_{L^n(E \cap \text{supp} \chi)} + \|q\|_{L^{n/2}(\text{supp} \chi)} \|e^{k \cdot x} F\|_{L^p'(\mathbb{R}^n)},$$

Thus we obtain a lower bound of the form

$$\|A\|_{L^n(E \cap \text{supp} \chi)} \gtrsim \left(\frac{\|e^{k \cdot x} F\|_{L^p'(E)}}{\|e^{k \cdot x} F\|_{L^{p'}(\mathbb{R}^n)}} - \|q\|_{L^{n/2}(\text{supp} \chi)}\right) \cdot \frac{1}{|k||E|^{1/n}}, \quad (5.14)$$

We want to use this to show that $\|A\|_{L^n(\text{supp} \chi)}$ is bounded below uniformly in $\chi$, which is a contradiction since the support of $\chi$ can have arbitrarily small measure. This is difficult because the right hand side of the Carleman estimate (5.10) is no longer localized to the set $E$, and the factor $\|e^{k \cdot x} F\|_{L^{p'}(E)}$ could be much smaller than $\|e^{k \cdot x} F\|_{L^{p'}}$. Furthermore, the factor $|k|^{-1}|E|^{-1/n}$ may be very small.

Wolff’s key idea is that we can say something about where the mass of the functions $e^{k \cdot x} F$ is concentrated, merely by virtue of the fact that we are multiplying a fixed compactly supported function by an exponential weight. Namely, we know that multiplying $F$ by $e^{k \cdot x}$ will tend to push the mass of $F$ in the direction $k$.

Suppose that most of the mass of $e^{k \cdot x} F$ is concentrated on a rectangle $E_k$. We expect that for $k_1$ and $k_2$ sufficiently separated, the sets $E_{k_1}$ and $E_{k_2}$ will be disjoint.

To gain some intuition for the scales involved, consider the Gaussian measures $\mu = e^{-\frac{1}{2}|x|^2} dx$. The family

$$\mu_k = e^{k \cdot x} e^{-\frac{1}{2}|x|^2} dx = e^{\frac{1}{2}k^2} e^{-\frac{1}{2}|x-k|^2} dx$$

consist of translated Gaussians centered at $k$, and most of the mass of $\mu_k$ is concentrated near $k$. In particular, if $E_k = B(k, 1)$, then

$$\mu_k(E_k) \gtrsim \mu_k(\mathbb{R}^n). \quad (5.15)$$
If $|k_1 - k_2| > 2$, then $E_{k_1}$ and $E_{k_2}$ are disjoint. Thus if we restrict ourselves to vectors $k$ such that $|k| \in [M, 2M]$, we can obtain $M^n$ disjoint sets $E_{k_j}$ satisfying (5.15). In particular

$$
\sum |E_{k_j}|^{-1} \gtrsim M^n.
$$

More generally, let $\mu$ be a measure on $\mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus B(0, R))$ decays faster than any exponential as $R \to \infty$. Wolff showed that for fixed $M$, there exist $k_j \in B(M, 2M)$ and disjoint convex sets $E_{k_j}$ such that for $\mu_k = e^{k \cdot x} \mu$, we have

$$
\mu_k(E_k) \gtrsim \mu_k(\mathbb{R}^n)
$$

$$
\sum |E_{k_j}|^{-1} \gtrsim M^n.
$$

Now take $\mu = F^\nu \, dx$, and apply (5.14) to the resulting sets $E_{k_j}$. Since $\|q\|_{L^{n/2}(\text{supp } \chi)}$ can be made arbitrarily small, we obtain a lower bound

$$
\|A\|_{L^n(\text{supp } \chi)}^n \geq \sum_j \|A\|_{L^n(\text{supp } \chi \cap E_{k_j})}^n \gtrsim M^{-n} \sum |E_{k_j}|^{-1} \gtrsim 1,
$$

which is a contradiction, since $\text{supp } \chi$ has arbitrarily small measure.
Chapter 6

Conductivities with unbounded gradient

In this chapter we will prove

Theorem 3. [Hab14] Suppose $3 \leq n \leq 6$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\gamma \in W^{s,p}(\Omega)$ be real-valued functions, where

$$(s,p) = \begin{cases} (1,n) & n = 3,4 \\ (1 + (1 - \theta)(\frac{1}{2} - \frac{2}{n}), \frac{n}{1-\theta}) & n = 5,6 \end{cases}$$

and $\theta \in [0,1)$. Assume in addition that there is some $c$ such that $0 < c < \gamma_i < c^{-1}$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies that $\gamma_1 = \gamma_2$.

6.1 Scaling considerations

Recall that the conductivity equation takes the form

$$\text{div}(\gamma \nabla u) = 0.$$ 

We can rescale this equation by replacing $\gamma$ by $c \gamma$ or by replacing $u(x)$ and $\gamma(x)$ by $u(\lambda x)$ and $\gamma(\lambda x)$. Since these two operations preserve $\|\log \gamma\|_{BMO}$, it is natural to consider the equation with coefficients satisfying $\log \gamma \in BMO$. From this point of view, the requirement in Chapter 4 that $\nabla \log \gamma \in L^\infty$ seems too stringent.

Uniqueness holds in two dimensions under a condition of the type $\log \gamma \in BMO$. This parallels the situation in unique continuation, which holds for elliptic equations in divergence form with bounded coefficients [Ale92].

On the other hand, it is doubtful whether we can expect to relax the regularity index in higher dimensions. The failure of unique continuation and Carleman estimates for elliptic equations with coefficients of Hölder regularity less than one suggests that one cannot construct CGO solutions when $\gamma$ has less than one derivative.
On the other hand, the conductivity equation has the form
\[ \Delta u + A \cdot \nabla u = 0, \]
where \( A = \nabla \log \gamma \). Since the strong unique continuation property holds for such an equation with \( A \in L^N \), it is reasonable to conjecture that uniqueness in Calderón’s problem holds under the assumption that \( \nabla \log \gamma \in L^N \). Note that the norm \( \| \nabla \log \gamma \|_{L^N} \) is invariant under the scaling of the equation.

6.2 Estimates at fixed modulation

The conductivity equation is somewhat better than the generic equation \( (\Delta + A \cdot \nabla)u = 0 \), because the gradient term \( A \) is of the form \( \nabla f \). As we have discussed earlier, this allows us to transform the first-order term into a zero-order term and obtain an equation
\[ (-\Delta + q)v = 0, \]
where \( v = \gamma^{1/2}u \) and \( q = \frac{1}{4} |\nabla \log \gamma|^2 + \frac{1}{2} \Delta \log \gamma \).

Although this equation does not contain a gradient term, we will still need to deal with the failure of \( L^p \) Carleman estimates for the gradient, because \( q \) has negative regularity. It is unclear whether one can apply Wolff’s ideas in this context. Wolff’s method relies on localization and the fact that in unique continuation we are trying to rule out the existence of some fixed function \( u \). In contrast, in order to solve the CGO equation
\[ (-\Delta_\zeta + q)\psi = q, \]
it appears necessary to prove estimates which hold uniformly for any \( \psi \) lying in some Banach space. Also, one must solve this equation globally for a fixed choice of \( \zeta \), so it is not clear how the problem can be localized.

Instead of relying on improvements in \( L^p \) Carleman estimates which come from localization, we will take a dual point of view, and take advantage of improvements which come at high modulation. The uncertainty principle gives a connection between these two phenomena, since spatial localization at the scale \( \mu^{-1} \) effectively makes modulation that is less than \( \mu \) indistinguishable from modulation that is equal to \( \mu \). If \( \hat{u}_\mu \) is supported in the region \( \{ \xi : d(\xi, \Sigma_\zeta) \sim \mu \} \), then by Lemma 5.4.5 we have
\[ \|u_\mu\|_{2n/(n-2)} \lesssim (\mu/\tau)^{1/n} \|u_\mu\|_{X^{1/2}_\zeta}. \]  
(6.1)
and by definition we have
\[ \|u_\mu\|_2 \lesssim (\mu \tau)^{-1/2} \|u_\mu\|_{X^{1/2}_\zeta}. \]  
(6.2)
Assume we are given \( v_\nu \) satisfying a similar condition, with \( \mu \leq \nu \). Let \( f = \nabla \log \gamma \). Then
\[ | \int \nabla f \cdot u_\mu \nabla v_\nu | \lesssim \|\nabla f\|_n (\mu/\nu)^{1/n} (\nu \tau)^{-1/2} \|u_\mu\|_{X^{1/2}_\zeta} \|v_\nu\|_{X^{1/2}_\zeta}. \]
CHAPTER 6. CONDUCTIVITIES WITH UNBOUNDED GRADIENT

Now we exploit the fact that the Fourier transform of \( u_\mu v_\nu \) is supported in \( \{ \xi : |\xi| \lesssim \tau, |\xi \cdot e_1| \lesssim \nu \} \). By orthogonality, we may restrict \( f \) to this region, so that the above becomes

\[
\int \nabla f \cdot u_\mu v_\nu \lesssim \|D^{1/2-1/n} D_1^{1/n-1/2} f\|_n \|u_\mu\|_{X_1^{1/2}} \|v_\nu\|_{X_1^{1/2}}.
\]

where \( D \) and \( D_1 \) are operators with symbols \(|\xi|\) and \(|\xi \cdot e_1|\), respectively. An argument along these lines gives an estimate of the form

\[
\|\nabla f\|_{X_1^{1/2}} \lesssim \|D^{1/2-1/n} D_1^{1/n-1/2} f\|_n.
\]

Although we have lost \( 1/2 - 1/n \) derivatives in this estimate, this is counterbalanced by a gain of \( 1/2 - 1/n \) derivatives in the \( e_1 \) direction. This gain is useless if the Fourier support of \( f \) concentrates near the plane perpendicular to \( e_1 \). However, we expect that this behavior does not occur on average, and we can take advantage of this by exploiting our freedom in choosing \( \zeta \).

It is easiest to average over all choices of \( e_1 \in S^{n-1} \). In \( L^2 \) we have

\[
\int_{e_1 \in S^{n-1}} \|D^\alpha D_1^{-\alpha} f\|_2^2 d\sigma(e_1) \lesssim \|f\|_2.
\]

for \( \alpha < 1/2 \). Heuristically, we can interpolate this with the trivial observation that

\[
\sup_{e_1 \in S^{n-1}} \|f\|_\infty \lesssim \|f\|_\infty,
\]

to obtain

\[
\left( \int_{e_1 \in S^{n-1}} \|D^\beta D_1^{-\beta} f\|_p^p d\sigma(e_1) \right)^{1/p} \lesssim \|f\|_p,
\]

when \( \beta < 1/p \). In three dimensions, we have \( 1/2 - 1/3 < 1/3 \), and we find that \( \|\nabla f\|_{X_1^{1/2} \to X_1^{-1/2}} \) is bounded on average. In four dimensions, we have \( 1/2 - 1/4 = 1/4 \). This causes a logarithmic divergence, which turns out to be harmless. For \( n \geq 5 \), however, we do not have a way to avoid losing derivatives.

### 6.3 Bilinear estimates

Now we would like to control \( \|f\|_{X_1^{1/2} \to X_1^{-1/2}} \), where \( f \) is some distribution acting on \( X_1^{1/2} \) by multiplication. By duality, this is equivalent to establishing a bilinear estimate of the form

\[
|\langle f, u \rangle| \lesssim \|u\|_{X_1^{-1/2}} \|v\|_{X_1^{1/2}}.
\]

Suppose that \( f \in L^{n/2} \). By (5.7), we have

\[
|\langle f, u \rangle| \lesssim \|f\|_{n/2} \|u\|_{2n/(n-2)} \|v\|_{2n/(n-2)} \lesssim \|f\|_{n/2} \|u\|_{X_1^{1/2}} \|v\|_{X_1^{1/2}}
\]

(6.3)
We also have

\[ |\langle fu, v \rangle| \lesssim \|f\|_{\infty} \|u\|_2 \|v\|_2 \lesssim \tau^{-1} \|f\|_{\infty} \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}. \]  

(6.4)

A more difficult task is to control \( \|\nabla f\|_{X^{1/2}_\xi \rightarrow X^{-1/2}_\xi}. \) To simplify the notation, we will make the convention here that

\[ Q_1 = Q_{\leq 1}. \]

This reflects the fact that our problem is localized to the unit scale, so we do not distinguish frequencies which are separated on the unit scale.

**Lemma 6.3.1.** Let \( s, p, \theta \) be as in Theorem 3. Let \( 1/q = 1/2 - 1/p. \) There is some \( \alpha > 0 \) such for fixed \( \lambda \leq 100 \tau, \) we have

\[ \sum_{1 \leq \mu \leq \nu \leq \tau/8} \frac{(\nu/\lambda)^{(1-\theta)/n}}{\lambda^{1-\theta/n}} \|Q_\mu u\|_q \|Q_\nu v\|_2 \lesssim (\lambda/\tau)^{\alpha} \lambda^{n-2} \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}, \]  

(6.5)

and

\[ \sum_{\lambda \leq \nu < \tau/8} \|Q_\mu u\|_q \|Q_\nu v\|_2 \lesssim \lambda^{-1} (\lambda/\tau)^{\alpha} \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}, \]  

(6.6)

where we define \( Q_1 = Q_{\leq 1}. \)

**Proof.** We may interpolate (5.6) with the trivial estimate \( \|Q_\mu u\|_2 \lesssim (\mu\tau)^{-1/2} \|Q_\mu u\|_{X^{1/2}_\xi} \) to obtain

\[ \|Q_\mu u\|_q \lesssim (\mu/\tau)^{(1-\theta)/n} (\mu\tau)^{-\theta/2} \|Q_\mu u\|_{X^{1/2}_\xi}, \]

where \( 1/q = 1/2 - 1/p \) and \( \theta \) is such that \( p = n/(1-\theta). \) Combining this with the trivial \( L^2 \) estimate for \( v, \) we obtain

\[ \|Q_\mu u\|_q \|Q_\nu v\|_2 \lesssim B_{\mu, \nu} \|Q_\mu u\|_{X^{1/2}_\xi} \|Q_\nu v\|_{X^{1/2}_\xi}, \]

where

\[ B_{\mu, \nu} := \tau^{-1/2} (\mu^{1-\theta}/n)^{1/2} (\mu^{1-\theta}/n)\nu^{-1/2}. \]

Set

\[ \beta := \frac{1-\theta}{n} - \frac{\theta}{2}. \]

Suppose first that \( \beta > 0. \) When \( \nu \geq \lambda \) we have

\[ B_{\mu, \nu} \lesssim \tau^{-1/2} (\mu^{1-\theta}/n)^{1/2} (\mu/\nu)^{\beta} \]

\[ = \lambda^{-\theta-1} (\lambda/\tau)^{(1-\theta)/n+\theta/2+1/2} (\mu/\nu)^{\beta}, \]

We take \( \alpha \leq (1-\theta)/n + (1+\theta)/2 \) and use the discrete Young’s inequality to establish (6.6).
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Suppose now that $\nu < \lambda$. When $n = 3$, we set $\theta = 0$,

$$
(\nu/\lambda)^{1/3} B_{\mu,\nu} = (\nu/\lambda)^{1/3} \tau^{-5/6} \mu^{1/3} \nu^{-1/2} \\
= (\mu/\nu)^{1/3} (\nu/\lambda)^{1/6} (\lambda/\tau)^{2/3} \lambda^{-1}.
$$

By Young’s inequality we have (6.5) for $\alpha \leq 2/3$. When $n = 4$ we take $\theta$ to be zero and obtain

$$
(\nu/\lambda)^{1/4} B_{\mu,\nu} = (\nu/\lambda)^{1/4} \tau^{-3/4} \mu^{1/4} \nu^{-1/2} \\
= (\mu/\nu)^{1/4} (\lambda/\tau)^{3/4} \lambda^{-1}.
$$

Applying Young’s inequality we have (6.5) for $\alpha \leq 3/4$.

When $n > 4$, we have

$$
(\nu/\lambda)^{1-\theta/n} B_{\mu,\nu} = (\mu/\nu)^{\beta} \nu^{-1/2+2(1-\theta)/n-\theta/2} \lambda^{-2} (\lambda/\tau)^{1-\theta/n+\theta/2+1/2}.
$$

In this case we have (6.5) for $\alpha \leq (1 - \theta)/n + \theta/2 + 1/2$

In higher dimensions, we also want to consider the case $(1 - \theta)/n - \theta/2 \leq 0$. For $\nu \geq \lambda$ we have

$$
B_{\mu,\nu} \leq \mu^{\beta} \lambda^{-1/2} \tau^{-(1-\theta)/n-\theta/2-1/2} \\
\lesssim \lambda^{-1} \tau^{-(1-\theta)/n-\theta/2}.
$$

Then we have (6.6) for $\alpha < (1 - \theta)/n + \theta/2$, since there are only $\sim \log \tau$ possible values of $\mu, \nu$.

For $\lambda \geq \nu$ we have

$$
(\nu/\lambda)^{1-\theta/n} B_{\mu,\nu} \lesssim \nu^{(1-\theta)/n-1/2} \lambda^{-2} (\lambda/\tau)^{1-\theta/n+\theta/2+1/2}.
$$

Thus we have (6.5) for $\alpha \leq (1 - \theta)/n + \theta/2 + 1/2$. \qed

Let $P_\lambda$ denote the Littlewood-Paley projections, and let $P_\mu^1$ denote the Littlewood-Paley projections in the $e_1$ direction. Then

**Lemma 6.3.2.** Let $s, p$ be as in Theorem 3. Then for any $f \in W^{s-1,p}(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$,

$$
\|\nabla f\|_{X_\xi^{1/2} \to X_\xi^{-1/2}} \lesssim \|f\|_p + \sup_{1 \leq \nu \leq \lambda \leq 100r} (\lambda/\tau)^{\beta} (\lambda/\nu)^{1/p} \lambda^{s-1} \|P_\lambda P_\mu^1 f\|_p
$$

where $\beta > 0$.

**Proof.** Write

$$
\langle (\nabla f)u, v \rangle = \langle (\nabla f)Q_h u, Q_h v \rangle + \langle (\nabla f)Q_l u, Q_l v \rangle + \langle (\nabla f)Q_{1l} u, Q_h v \rangle + \langle (\nabla f)Q_{1l} u, Q_l v \rangle.
$$
We can treat all but the last term using (3.3), (5.7). Integrating by parts,
$$
|\langle (\nabla f)Q_h u, Q_h v \rangle| \lesssim \|f\|_n \|Q_h \nabla u\|_2 \|Q_h v\|_{2(n/2)} + \|f\|_n \|Q_h u\|_{2(n/2)} \|Q_h \nabla v\|_2 
$$
$$
\lesssim \|f\|_n \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}.
$$

Since $Q_l v$ is supported in $|\xi| \lesssim \tau$,
$$
|\langle (\nabla f)Q_h u, Q_l v \rangle| \lesssim \|f\|_n \|Q_h \nabla u\|_2 \|Q_l v\|_{2(n/2)} + \|f\|_n \|Q_l u\|_{2\nu} \|Q_l \nabla v\|_{2(n/2)} 
$$
$$
\lesssim \|f\|_n \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}.
$$

It remains to estimate $\langle (\nabla f)Q_l u, Q_l v \rangle$. We have

$$
\langle (\nabla f)Q_l u, Q_l v \rangle = \sum_{\mu, \nu, \lambda} \int \nabla P_\lambda f \cdot Q_\mu u \cdot \overline{Q_\nu v} \, dx, \quad (6.7)
$$

where we define

$$
Q_1 = Q_{\leq 1}.
$$

Suppose $\mu \leq \nu$ (the case $\mu > \nu$ is identical). Because $Q_\mu u \cdot \overline{Q_\nu v}$ has Fourier support in $\{\xi : |\xi| \leq 2\nu\}$, Plancherel’s theorem and Hölder’s inequality give

$$
\left| \int \nabla P_\lambda f \cdot Q_\mu u \cdot \overline{Q_\nu v} \, dx \right| \leq \left| \int \nabla P_\lambda f \cdot Q_\mu u \cdot \overline{Q_\nu v} \, dx \right| 
$$
$$
\lesssim \|P_{\leq 2\nu} \nabla P_\lambda f\|_p \|Q_\mu u\|_q \|Q_\nu v\|_2.
$$

Furthermore, since $Q_\mu u \cdot \overline{Q_\nu v}$ has Fourier support in $\{\xi : |\xi| \leq 100\tau\}$, we can assume $\lambda \leq 100\tau$ in this sum. Applying Lemma 6.3.1, we get

$$
|\langle (\nabla f)Q_l u, Q_l v \rangle| \lesssim \sum_{\nu \geq \lambda} \|\nabla P_\lambda f\|_p \|Q_\mu u\|_q \|Q_\nu v\|_2 
$$
$$
+ \sum_{\nu<\lambda<100\tau} \sum_{\mu \leq \nu} \langle (\lambda/\nu)^{1/p} (\nu/\lambda)^{1/p} \|\nabla P_\lambda P_{\leq 8\nu} f\|_p \|Q_\mu u\|_q \|Q_\nu v\|_2 
$$
$$
\lesssim \sum_{\lambda \leq 100\tau} \{ (\lambda/\tau)^{\alpha} \lambda^{-1} \|\nabla P_\lambda f\|_p 
$$
$$
+ \sup_{\nu \leq \lambda \leq 100\tau} \langle (\lambda/\nu)^{\alpha}(\nu/\lambda)^{1/p} \lambda^{s-2} \|\nabla P_\lambda P_{\leq 8\nu} f\|_p \rangle 
$$
$$
\times \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi} 
$$
$$
\lesssim \left( \|f\|_p + \sup_{\nu \leq \lambda \leq 100\tau} \langle (\lambda/\nu)^{\alpha/2}(\nu/\lambda)^{1/p} \lambda^{s-1} \|P_\lambda P_{\leq 8\nu} f\|_p \rangle \right) 
$$
$$
\times \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}.
$$
6.4 Averaging

Given a tuple of orthonormal vectors $F = (e_1, \ldots, e_k)$ with $1 \leq k < n$, we define $P^F_\nu$ to be Littlewood-Paley projection in directions $e_1, \ldots, e_k$. Let $\mu$ denote Haar measure on $O(n)$, normalized so that if $\sigma$ is the usual spherical measure on $S^{n-1}$ and $f : S^{n-1} \to \mathbb{R}$ is integrable, then for any $\theta \in S^{n-1}$ we have

$$\int_{O(n)} f(U \cdot \theta) \, d\mu(U) = \int_{S^{n-1}} f(\omega) \, d\sigma(\omega). \tag{6.8}$$

Lemma 6.4.1. Suppose $p \in [2, \infty]$. Let $f \in L^p(\mathbb{R}^n)$. For $U \in O(n)$ and $1 \leq \nu \leq \lambda$, define

$$A_{\lambda, \nu}(U) = (\lambda/\nu)^{k/p} \|P^U \|_{\leq \nu} f\|_p.$$ 

Then

$$\|A_{\lambda, \nu}\|_{L^p(O(n))} \lesssim \|f\|_p.$$

Proof. We define an operator $T$ mapping functions on $\mathbb{R}^n$ to functions on $O(n) \times \mathbb{R}^n$ by

$$Tf(U, x) = P^U \|\leq \nu \| x.$$ 

The lemma asserts that this operator is bounded from $L^p(\mathbb{R}^n)$ to $L^p(O(n) \times \mathbb{R}^n)$. By interpolation, it suffices to establish this at the endpoints $p = 2$ and $p = \infty$.

When $p = \infty$ this is just the fact that the Littlewood-Paley projections are bounded on $L^\infty$.

When $p = 2$ we use Plancherel’s theorem and Fubini.

$$\|Tf\|_{L^2}^2 = \int_{O(n)} \int_{\mathbb{R}^n} |\phi(\xi/\lambda) \chi(\xi \cdot (Ue_1)/\nu, \ldots, \xi \cdot (Ue_k)/\nu) \hat{f}(\xi)|^2 \, d\xi \, d\mu(U)$$

$$\leq \left( \sup_{\xi} \int_{O(n)} |\phi(\xi/\lambda) \chi(\xi \cdot (Ue_1)/\nu, \ldots, \xi \cdot (Ue_k)/\nu)|^2 \, d\mu(U) \right) \|f\|_2^2$$

Here $\phi$ is supported on an annulus, and $\chi$ is supported on a ball. We estimate the last integral using (6.8):

$$\int_{O(n)} |\phi(\xi/\lambda) \chi(\xi \cdot (Ue_1)/\nu, \ldots, \xi \cdot (Ue_k))|^2 \, d\mu(U) \lesssim \sup_{|\xi| \sim \lambda} \int_{S^{n-1}} |\chi(|\xi|\omega \cdot e_1/\nu, \ldots, |\xi|\omega \cdot e_k)/\nu)|^2 \, d\sigma(\omega)$$

The last integral is the area of the intersection of the unit sphere with a rectangle centered at the origin of dimensions $(\nu/\lambda)^k \cdot 10^{n-k}$. Since $1 \leq k < n$ this shows that

$$\|Tf\|_2 \lesssim (\nu/\lambda)^{k/2} \|f\|_2,$$

which completes the proof.
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Now we summarize our estimates in the following

**Theorem 6.4.2.** Let $s, p$ be as in Theorem 3, and let $\gamma$ be a positive real-valued function on $\mathbb{R}^n$ such that $\nabla \log \gamma \in W^{s-1,p}$ and $\gamma = 1$ outside of a large ball $B$. For $q = \gamma^{-1/2} \Delta \gamma^{1/2}$, we have

\[
M^{-1} \int_{M/2}^{2M} \int_{O(n)} \|q\|_{X^{-1/2}_2}^2 d\mu(U) d\tau \to 0
\]  

(6.9)

Furthermore,

\[
\sup_{\tau \in [M/2, 2M]} \|q\|_{X^{-1/2}_2} \leq A_M(U),
\]  

(6.10)

where

\[
\liminf \|A_M\|_{L^p(O(n))} = 0
\]  

(6.11)

as $M \to \infty$.

**Proof.** First, we write

\[
\gamma^{-1/2} \Delta \gamma^{1/2} = \frac{1}{2} \Delta \log \gamma + \frac{1}{4} |\nabla \log \gamma|^2 = \sum_i \nabla_i f_i + h,
\]

where $f_i \in W^{s-1,p}$ and $h \in L^{p/2}$.

We decompose each term into a good part and a bad part. Let $\phi_\epsilon = \epsilon^{-n} \phi(x/\epsilon)$, where $\phi$ is a $C_0^\infty$ function supported on the unit ball and $\int \phi = 1$. Define $f_\epsilon = f * \phi_\epsilon$.

By (6.3), we have

\[
\|\nabla f_\epsilon\|_{X^{1/2}_2 \to X^{-1/2}_2} + \|h_\epsilon\|_{X^{1/2}_2 \to X^{-1/2}_2} \lesssim \tau^{-1} (\|\nabla f_\epsilon\|_\infty + \|h_\epsilon\|_\infty)
\]

\[
\lesssim \tau^{-1} \epsilon^{-2} (||f||_n + ||h||_{n/2}).
\]  

We also have

\[
\|\nabla f_\epsilon\|_{X^{-1/2}_2} \lesssim \tau^{-1/2} \|\nabla f_\epsilon\|_2
\]

\[
\lesssim \tau^{-1/2} \epsilon^{-1} ||f||_2
\]

\[
\lesssim \tau^{-1/2} \epsilon^{-1} ||f||_n
\]  

since $n > 2$ and $f$ is compactly supported. For $n \geq 4$ we have

\[
\|h_\epsilon\|_{X^{-1/2}_2} \lesssim \tau^{-1/2} ||h_\epsilon||_2
\]

\[
\lesssim \tau^{-1/2} ||h_\epsilon||_{n/2}
\]  

(6.12)

and for $n = 3$ we have

\[
\|h_\epsilon\|_{X^{-1/2}_2} \lesssim \tau^{-1/2} ||h_\epsilon||_2
\]

\[
\lesssim \tau^{-1/2} \epsilon^{-1/2} ||h||_{3/2}
\]
Taking $\epsilon = M^{-1/4}$, we find that if we replace $q$ with $q_\epsilon$ then the left hand sides of (6.9) and (6.10) vanish as $\tau \to \infty$.

It remains to treat the bad part $q - q_\epsilon$. Let $g = f - f_\epsilon$, and define

$$A(\tau, U) = \|\nabla g\|_{X^{1/2}_{\xi(\tau, U)} \to X^{-1/2}_{\xi(\tau, U)}}$$

Using Lemma 6.3.2, we have

$$\sup_{\tau \in [M/2, 2M]} A(\tau, U) \lesssim \|g\|_{L^p} + \left( \sum_{1 \leq \nu \leq \lambda \leq M/4} [(\lambda/\mu)^{\beta} \lambda^{s-1} A_{\lambda, \nu}(U)]^p \right)^{1/p}$$

where $A_{\lambda, \nu}(U) = (\lambda/\nu)^{1/\beta} \|P_{\sim \lambda} f_{\leq 8\nu} g\|_{L^p}$. We take $A_M(U)$ to be the right hand side of this inequality, which is clearly a measurable function on $O(n)$. Now, $P_{\sim \lambda} g = P_{\sim \lambda} f_{\sim \lambda} g$, where $P_{\sim \lambda} g = \sum_{16 \leq \mu \leq 16\lambda} P_{\mu} g$. Applying Lemma 6.4.1, we have

$$\|A_M(U)\|_{L^p(O(n))} \lesssim \|g\|_{L^p} + \sum_{1 \leq \nu \leq \lambda \leq M/4} [(\lambda/\mu)^{\beta} \lambda^{s-1} \|P_{\sim \lambda} f\|_{L^p}]^p$$

As $M \to \infty$, we have $\epsilon = M^{-1/4} \to 0$, so $\|g\|_{L^p} \to 0$. To control the second term, we use a trick from [NS14]. Let

$$b_k = \sum_{1 \leq \lambda \leq 2^k} (\lambda/2^k)^{\beta} [\lambda^{s-1} \|P_{\sim \lambda} f\|_{L^p}]^p.$$ 

The Littlewood-Paley square function estimate implies that

$$\sum_{k=0}^{\infty} b_k \lesssim \sum_{\lambda \geq 1} [\lambda^{s-1} \|P_{\sim \lambda} f\|_{L^p}]^p \lesssim \|f\|_{W^{s-1, p}}^p$$

In particular, we must have $\lim \inf_{k \to \infty} kb_k = 0$. Taking $M = 2^k$, we obtain (6.11).

By Lemma 4.2.1, we have

$$M^{-1} \int_{M/2}^{2M} \int_{O(n)} \|\nabla g\|^2 d\mu(U) d\tau \lesssim \|g\|^2_{L^2} \lesssim \|g\|^2_{L^p} \to 0.$$

Next we treat $h - h_\epsilon$. When $n \geq 4$ we have $\|h - h_\epsilon\|_{X^{1/2}_{\xi}} \to 0$ by (6.12). When $n = 3$, we use (3.2) and Sobolev embedding

$$\|h - h_\epsilon\|_{X^{-1/2}_{\xi}} \lesssim \|h - h_\epsilon\|_{H^{-1/2}} \lesssim \|h - h_\epsilon\|_{3/2} \to 0.$$
Finally, by (6.3) we have

\[ \| h - h_\epsilon \|_{X_\zeta^{1/2} \to X_\zeta^{-1/2}} \lesssim \| h - h_\epsilon \|_{n/2} \to 0. \]

Now we can prove Theorem 3:

**Theorem 3.** [Hab14] Suppose \( 3 \leq n \leq 6 \), and let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with Lipschitz boundary. Let \( \gamma \in W^{s,p}(\Omega) \) be real-valued functions, where

\[ (s,p) = \begin{cases} (1,n) & n = 3, 4 \\ (1 + (1 - \theta)(\frac{1}{2} - \frac{2}{n}), \frac{n}{1-\theta}) & n = 5, 6 \end{cases} \]

and \( \theta \in [0,1) \). Assume in addition that there is some \( c \) such that \( 0 < c < \gamma_i < c^{-1} \). Then \( \Lambda_{\gamma_i} = \Lambda_{\gamma_2} \) implies that \( \gamma_1 = \gamma_2 \).

**Proof.** Let \( q_1, q_2 \) be as in Lemma 2.1.6, so that

\[ \langle q_1, v_1 v_2 \rangle = \langle q_2, v_1 v_2 \rangle \quad (6.13) \]

when each \( v_j \) is a solution in \( H_{\text{loc}}^1(\mathbb{R}^n) \) to \( \Delta v_j - q_j v_j = 0 \). By Lemma 2.2.1 we will be done once we show that \( q_1 = q_2 \).

Fix \( r > 0 \) and three orthonormal vectors \( \{e_1, e_2, e_3\} \), and define

\[ \zeta_1(\tau, U) = \tau U(e_1 - ie_2) \]
\[ \zeta_2(\tau, U) = -\zeta_1(\tau, U) \]
\[ \tilde{\zeta}_1(\tau, U) := \tau U e_1 + i(r U e_3 - \sqrt{\tau^2 - r^2} U e_2) \]
\[ \tilde{\zeta}_2(\tau, U) := -\tau U e_1 + i(r U e_3 + \sqrt{\tau^2 - r^2} U e_2) \]

In what follows, all of inequalities will implicitly depend on \( r \). For example, we have \( |\zeta_i - \tilde{\zeta_i}| \lesssim 1 \). In particular, by Lemma 4.3.1, the spaces \( X_{\zeta_i}^b \) and \( X_{\tilde{\zeta}_i}^b \) have equivalent norms.

Now we use a compactness argument of [NS14] to select \( U \) and \( \tau \). First we fix \( \epsilon > 0 \). By restricting the integrals in Theorem 6.4.2 to the ball \( B_\epsilon = \{ U \in O(n) : \| U - I \| < \epsilon \} \), we may choose, for a sequence of \( M = M_l \) such that \( M_l \to \infty \), some \( \tau = \tau_{\epsilon,l} \in [M_l, 2M_l] \), \( U = U_{\epsilon,l} \in B_\epsilon \) and \( \delta = \delta_{\epsilon,l} > 0 \) such that

\[ \sum_{i,j} (\| q_i \|_{X_{\zeta_j(\tau,U)}^{1/2} \to X_{\zeta_j(\tau,U)}^{-1/2}} + \| q_i \|_{X_{\tilde{\zeta}_j(\tau,U)}^{-1/2} \to X_{\tilde{\zeta}_j(\tau,U)}^{1/2}}) \leq \delta \quad (6.14) \]

where \( \delta_{\epsilon,l} \to 0 \) as \( l \to \infty \).

By (3.4.3), we have

\[ \| q_i \|_{X_{\zeta_j(\tau,U)}^{1/2} \to X_{\tilde{\zeta}_j(\tau,U)}^{-1/2}} \lesssim \| q_i \|_{X_{\zeta_j(\tau,U)}^{1/2} \to X_{\tilde{\zeta}_j(\tau,U)}^{-1/2}} \]

\[ \| q_i \|_{X_{\zeta_j(\tau,U)}^{-1/2} \to X_{\tilde{\zeta}_j(\tau,U)}^{1/2}} \lesssim \| q_i \|_{X_{\zeta_j(\tau,U)}^{-1/2} \to X_{\tilde{\zeta}_j(\tau,U)}^{1/2}} \]
It follows that,
\[ \|q_i\|_{\dot{X}^{1/2}_{\zeta_i(r,U)} \rightarrow \dot{X}^{-1/2}_{\zeta_i(r,U)}} \lesssim \delta_{e,l} \]
Since \( \delta_{e,l} \rightarrow 0 \) as \( l \rightarrow \infty \), we can choose \( l \) large enough that the left hand side is less than 1/2. Since \( \|\Delta^{-1}_{\zeta} \|_{\dot{X}^{-1/2}_{\zeta} \rightarrow \dot{X}^{1/2}_{\zeta}} = 1 \) for any \( \zeta \), we can use the contraction mapping principle to construct solutions \( \psi \in \dot{X}^{1/2}_{\zeta_i(r,U)} \) to the equations \( (\Delta_{\zeta_i(r,U)} - q_i)\psi_i = q_i \), satisfying
\[ \|\psi_i\|_{\dot{X}^{1/2}_{\zeta_i(r,U)}} \lesssim \|q\|_{\dot{X}^{-1/2}_{\zeta_i(r,U)}}. \]
Note that by (3.10), such a solution lies in \( H^1_{\text{loc}}(\mathbb{R}^n) \). This implies that the corresponding solution \( v_i = e^{x \cdot \zeta_i(r,U)}(1 + \psi_i) \) to the Schrödinger equation \( (\Delta - q) v_i \) lies in \( H^1_{\text{loc}}(\mathbb{R}^n) \) as well.

Let \( k = 2rUe_3 \). By (6.13),
\[ 0 = \langle q_1 - q_2, e^{ik \cdot x}(1 + \psi_1)(1 + \psi_2) \rangle \]
\[ = \langle q_1 - q_2, e^{ik \cdot x} \rangle + \langle q_1 - q_2, e^{ik \cdot x} \psi_1 \psi_2 \rangle + \langle q_1 - q_2, e^{ik \cdot x} (\psi_1 + \psi_2) \rangle. \]
We need to show that the second and third terms are small. Let \( \phi \) be a Schwartz function that is equal to one on the support of \( q \). Then
\[ \|q_1, e^{ik \cdot x} \psi_1 \psi_2 \| \leq \|q_1, e^{-ik \cdot x} \psi_2, \psi_1 \| \]
\[ \lesssim \|e^{-ik \cdot x} \phi \psi_2 \|_{\dot{X}^{1/2}_{\zeta_1(r,U)}} \|\phi \psi_1 \|_{\dot{X}^{1/2}_{\zeta_1(r,U)}} \]
\[ = \|e^{ik \cdot x} \phi \psi_2 \|_{\dot{X}^{1/2}_{\zeta_1(r,U)}} \|\phi \psi_1 \|_{\dot{X}^{1/2}_{\zeta_1(r,U)}} \]
\[ \lesssim \|\psi_2 \|_{\dot{X}^{1/2}_{\zeta_2(r,U)}} \|\psi_1 \|_{\dot{X}^{1/2}_{\zeta_1(r,U)}} \]
\[ \lesssim \|q_2 \|_{\dot{X}^{-1/2}_{\zeta_2(r,U)}} \|q_1 \|_{\dot{X}^{-1/2}_{\zeta_1(r,U)}}. \]
since the seminorms of \( e^{-ik \cdot x} \phi \) are bounded with a bound depending only on \( r \). We can bound the \( q_2 \) term in the same way. On the other hand, we have
\[ \|q_1, e^{ik \cdot x} \psi_1 \| \lesssim \|q_1 \|_{\dot{X}^{-1/2}_{\zeta_1(r,U)}} \|\psi_1 \|_{\dot{X}^{1/2}_{\zeta_1(r,U)}} \]
\[ \lesssim \|q_1 \|_{\dot{X}^{-1/2}_{\zeta_1(r,U)}} \|q_1 \|_{\dot{X}^{-1/2}_{\zeta_1(r,U)}}. \]
by duality of \( \dot{X}^{1/2}_{\zeta_1(r,U)} \) and \( \dot{X}^{-1/2}_{\zeta_1(r,U)} \). The terms with \( \psi_2 \) are the same. In summary, we obtain
\[ \|q_1 - q_2\|_{(2rUe_3)} \lesssim \sum_{1 \leq i,j,k,l \leq 2} \|q_i\|_{\dot{X}^{-1/2}_{\zeta_i(r,U)}} \|q_k\|_{\dot{X}^{-1/2}_{\zeta_i(r,U)}} \lesssim \delta^2 \]
by (6.14).
Since the $U_{\epsilon,l}$ are contained in a compact set in $O(n)$, we may pass to a subsequence such that $U_{\epsilon,l} \to U_{\epsilon}$ for some $U_{\epsilon} \in O(n)$. Since the $\hat{q}_i$ are continuous, we may pass to the limit in (6.15) to obtain

$$|\hat{q}_1 - \hat{q}_2| (2rU_{\epsilon}e_3) \lesssim \lim_{l \to \infty} \delta_{\epsilon,l}^2 = 0.$$ 

As $\epsilon \to 0$, we have $U_{\epsilon} \to I$, so that this implies that $(\hat{q}_1 - \hat{q}_2)(2re_3) = 0$. Since $e_3 \in S^{n-1}$ and $r$ were arbitrary, this means that $\hat{q}_1 - \hat{q}_2 = 0$. \qed
Chapter 7

Recovering a magnetic potential

7.1 Cauchy data for Schrödinger operators

Let $\Omega$ be some bounded open set. Let $A \in L^n(\Omega)$ be a vector field on $\Omega$, and let $q \in W^{-1,n}(\Omega)$ be a scalar-valued distribution. We define the Schrödinger Hamiltonian $L_{A,q}$ by

$$L_{A,q} = (D + A)^2 + q,$$

where $D = -i\nabla$. Thus

$$L_{A,q} u = -\Delta u - i \text{div}(Au) - iA \cdot \nabla u + A^2 u + qu,$$

If we define

$$Q_{A,q}(u,v) = \int_{\Omega} (\nabla u \cdot \nabla v + iA(u\nabla v - v\nabla u) + (A^2 + q)uv) \, dx,$$

then a function $u \in H^1(\Omega)$ is a weak solution to $L_{A,q} u = 0$ if

$$Q_{A,q}(u,v) = 0 \text{ when } v \in H^1_0(\Omega).$$

By Sobolev embedding, it is easy to see that

$$|Q_{A,q}(u,v)| \lesssim_{A,q} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},$$

so $Q_{A,q}^u(v) = Q_{A,q}(u,v)$ defines a linear functional on $H^1(\Omega)/H^1_0(\Omega)$. If $\Omega$ is Lipschitz, then the trace map $u \mapsto u|_{\partial\Omega}$ is a surjection from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ whose kernel is $H^1_0(\Omega)$, so by abuse of notation we will denote by $v|_{\partial\Omega}$ the image of $v$ under the quotient map $H^1(\Omega) \to H^1(\Omega)/H^1_0(\Omega)$. If everything is smooth, then for $v$ arbitrary we have

$$0 = \int_{\Omega} (- \text{div} \nabla u - i \text{div}(Au) - iA \cdot \nabla u + A^2 + q)v \, dx$$

$$= \int_{\Omega} (\nabla u \cdot \nabla v + iA(u\nabla v - v\nabla u) + (A^2 + q)uv) \, dx$$

$$- \int_{\partial\Omega} v \cdot (\nabla + iA)u \cdot v \, dx, \quad (7.1)$$
so that
\[ Q_{A,q}^u(v|_{\partial \Omega}) = \langle \nu \cdot (\nabla + iA)u|_{\partial \Omega}, \overline{v}|_{\partial \Omega} \rangle_{L^2(\Omega)}. \]
In other words, we may identify \( Q_{A,q}^u \) with the function \( \nu \cdot (\nabla + iA)u|_{\partial \Omega} \). Unlike the conductivity equation, the Schrödinger equation \( L_{A,q}u = 0 \) may have nontrivial solutions in \( H_0^1(\Omega) \), so \( Q_{A,q}^u \) is not necessarily determined by \( u|_{\partial \Omega} \). Thus the Dirichlet-to-Neumann map is a multivalued relation
\[ \Lambda_{A,q} = \{(u|_{\partial \Omega}, Q_{A,q}^u) : u \in H^1(\Omega) \text{ and } L_{A,q}u = 0\}. \]
This problem has a gauge invariance property:

**Lemma 7.1.1.** Let \( \Omega \) be a Lipschitz domain, and suppose \( \psi \in W^{1,n}(\Omega) \cap L^\infty(\Omega) \). If \( \psi|_{\partial \Omega} = 0 \), then \( \Lambda_{A+\nabla \psi,q} = \Lambda_{A,q} \).

**Proof.** We have
\[ e^{-i\psi} L_{A,q} e^{i\psi} = (D + A + \nabla \psi)^2 + q. \]
Thus the map \( u \mapsto e^{-i\psi}u \) is a bijection between solutions to \( L_{A,q}u = 0 \) and solutions to \( L_{A+\nabla \psi,q}u = 0 \), and
\[ \langle Q_{A,q}^u, \overline{v}|_{\partial \Omega} \rangle = \langle Q_{A+\nabla \psi,q}^{e^{-i\psi}u}, e^{-i\psi}v|_{\partial \Omega} \rangle. \]
By Lemma 2.1.3, multiplication by \( e^{i\psi} \) leaves \( u|_{\partial \Omega} \) unchanged (if \( \psi \) is compactly supported then this is obvious), so we can conclude that \( \Lambda_{A,q} = \Lambda_{A+\nabla \psi,q} \).

Because of this gauge invariance, we can only expect to recover the magnetic potential \( A \) up to a gradient term. We will show that

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^3 \) be a Lipschitz domain, and suppose \( A_i \in W^{s,3}(\Omega) \) (with \( s > 0 \)) and \( q_i \in W^{-1,3}(\Omega) \) for \( i = 1, 2 \). There exists some \( \epsilon > 0 \), depending on \( \Omega \) and \( s \), such that if \( \|A_i\|_{W^{s,3}(\Omega)} \leq \epsilon \) for \( i = 1, 2 \) then
\[ L_{A_1,q_1} = L_{A_2,q_2} \]
implies that \( \text{curl } A_1 = \text{curl } A_2 \). If \( q_1, q_2 \in L^\infty \) then we also have \( q_1 = q_2 \).

Our proof is based on the methods of [Sun93a, ER95, KU14].

### 7.2 An integral identity

As before, the condition that \( \Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \) can be written as an integral identity:

**Lemma 7.2.1.** If \( \Lambda_{A_1,q_2} = \Lambda_{A_2,q_2} \), then
\[ \int_\Omega [i(A_1 - A_2) \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) + (A_1^2 - A_2^2 + q_1 - q_2) u_1 u_2] \, dx = 0 \]
for any \( u_i \in H^1(\Omega) \) solving \( L_{A_1,q_1} u_1 = 0 \) and \( L_{A_2,q_2} u_2 = 0 \) in \( \Omega \).
**Proof.** If $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$, then there is some $v_2$ in $H^1(\Omega)$ such that $L_{A_2,q_2}v_2 = 0$ and

$$(u_1|_{\partial\Omega}, Q_{A_1,q_1}^{u_1}) = (v_2|_{\partial\Omega}, Q_{A_2,q_2}^{v_2}).$$

Thus

$$Q_{A_1,q_1}(u_1,u_2) = Q_{A_2,q_2}(v_2,u_2) = Q_{-A_2,q_2}(u_2,v_2) = Q_{-A_2,q_2}(u_2,u_1) = Q_{A_2,q_2}(u_1,u_2).$$

This is exactly (7.2).

Suppose that we can construct CGO solutions $u_i \sim e^{x \cdot \zeta_i}$ to $L_{A_1,q_1}u_1 = L_{-A_2,q_2}u_2 = 0$, where

$$\zeta_1 \sim \tau (e_1 - ie_2),$$
$$\zeta_2 \sim -\tau (e_1 - ie_2),$$
$$\zeta_1 + \zeta_2 = ik,$$
$$e_1 \perp e_2 \perp k,$$
$$|e_1| = |e_2| = 1.$$  

Plugging these solutions into the identity yields

$$0 \sim 2i\tau (e_1 - ie_2) \cdot \int_{\Omega} (A_1 - A_2)e^{ik \cdot x} \, dx + \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2)e^{ik \cdot x} \, dx.$$  

When $\tau$ is large, the first integral dominates, and we have

$$0 = (e_1 - ie_2) \cdot (A_1 - A_2(k))$$

for $e_1, e_2$ orthonormal vectors perpendicular to $k$. This implies that curl $A_1 = $ curl $A_2$. In particular, if $A_1$ and $A_2$ agree near the boundary $\partial\Omega$, then there is a gauge transform $\psi$ such that $A_1 - A_2 = \nabla \psi$. This means that $\Lambda_{A_2,q_2} = \Lambda_{A_1,q_2}$. If we now construct CGO solutions to $L_{A_1,q_1}u_1 = L_{-A_1,q_2}u_2 = 0$, then this time we obtain

$$0 \sim \int_{\Omega} (q_1 - q_2)e^{ik \cdot x} \, dx,$$

and we can conclude that $q_1 = q_2$.  

7.3 A transport equation

The construction of CGO solutions for the Schrödinger equation is much less straightforward in the presence of a magnetic potential $A$. If $u = e^{x \cdot \zeta}(1 + \psi)$ solves $L_{A,q}u = 0$, then $\psi$ solves $L_{A,q,\zeta}(1 + \psi) = 0$,

where

$$L_{A,q,\zeta} = e^{-x \cdot \zeta}L_{A,q}e^{x \cdot \zeta}$$

$$= (D + A - i\zeta)^2 + q$$

$$= -\Delta + D \cdot A + 2A \cdot D - 2i\zeta \cdot A + A^2 + q.$$ 

The first problem is to invert the operator $L_{A,q,\zeta}$. As we have seen in the preceding chapters, all of the terms in $L_{A,q,\zeta}$ except for $2\zeta \cdot A$ are perturbative, at least when $A$ and $q$ are sufficiently smooth. However, the term $\zeta \cdot A$ causes new difficulties, and so far we have only established that it is perturbative if $\|A\|_{L^\infty}$ is sufficiently small. We will refine our methods to treat $A \in W^{s,3}(\mathbb{R}^3)$ for $s > 0$, but we will not remove the smallness assumption in this work.

A second problem is that the inhomogeneous term $L_{A,q,\zeta}(1)$ has the form

$$L_{A,q,\zeta}(1) = D \cdot A - 2i\zeta \cdot A + A^2 + q.$$ 

Again, the term $\zeta \cdot A$ causes problems. We need our solutions to satisfy

$$\tau \langle A_1 - A_2, \psi_1 \psi_2 \rangle = o(\tau).$$

For bounded $A$ we have shown that

$$\tau |\langle A, \psi_1 \psi_2 \rangle| \lesssim \|\psi_1\|_{X_{\zeta_1}^{1/2}} \|\psi_2\|_{X_{\zeta_2}^{1/2}},$$

and in order to control this we would need to have

$$\|\zeta_i \cdot A\|_{X_{\zeta_i}^{-1/2}} = o(\tau^{1/2}).$$

In general, we only have

$$\|\zeta_i \cdot A\|_{X_{\zeta_i}^{-1/2}} = O(\tau^{1/2}),$$

and the averaging argument in Lemma 4.2.1 does not help to control the low frequencies. Thus we need some way to to get rid of the term $\zeta_i \cdot A$.

We will do this by constructing CGO solutions of the more general form

$$u = e^{x \cdot \zeta}(a + \psi),$$ \hspace{1cm} (7.3)
where \( a = e^{-i\phi} \) for a suitable function \( \phi \) depending on \( \zeta \). Then \( \psi \) will solve the equation
\[
L_{A,q,\zeta}\psi = -\Delta a - 2\zeta \cdot \nabla a + (D \cdot A)a + 2A \cdot Da - 2i\zeta \cdot Aa + A^2a + qa. \tag{7.4}
\]
Roughly speaking, we want to choose \( \phi \) such that \( a \) solves the transport equation
\[
\zeta \cdot \nabla a = -i\zeta \cdot Aa,
\]
which is equivalent to
\[
\zeta \cdot \nabla \phi = \zeta \cdot A.
\]
Since \( \zeta = \tau(e_1 - ie_2) \), where \( e_1 \) and \( e_2 \) are orthonormal vectors, this just a \( \overline{\partial} \) equation for \( \phi \) in the plane determined by \( e_1 \) and \( e_2 \).

### 7.4 The \( \overline{\partial} \) equation

Let \( \overline{\partial} \) be the operator
\[
e = e_1 + ie_2,\]
where \( e_1 \) and \( e_2 \) are orthogonal unit vectors, define
\[
\overline{\partial}_e = e_1 \cdot \nabla + ie_2 \cdot \nabla.
\]
We may assume without loss of generality that \( e_1 \) and \( e_2 \) are the standard basis vectors, and we write
\[
\overline{\partial} = \partial_1 + i\partial_2.
\]
Let \( f \) be a function defined on the complex plane, which we identify with \( \mathbb{R}^2 \) by writing \( z = z_1 + iz_2 \). The equation
\[
\overline{\partial} u = f
\]
is of Cauchy-Riemann type. If \( f \) is smooth and compactly supported, then it has a solution given by
\[
\overline{\partial}_e^{-1} f(w) = \frac{1}{2\pi} \int \frac{f(w - z)}{z} \, dz,
\]
and

**Lemma 7.4.1.** If \( f : \mathbb{C} \to \mathbb{R} \) is supported in a ball of radius \( R \), then
\[
\|\langle w \rangle \overline{\partial}_e^{-1} f(w)\|_{L^\infty} \lesssim_R \|f\|_{L^\infty}.
\]

**Proof.** We write
\[
|\overline{\partial}_e^{-1} f(w)| \lesssim \|f\|_{L^\infty} \int \chi_{B(0,R)}(z - w)|z|^{-1} \, dz.
\]
When $|w| \leq 2R$, we estimate this integral by

$$\int_{B(0,3R)} |z|^{-1} dz \sim R.$$ 

When $|w| > 2R$, we have $|z| \geq |w|$ in the region of integration, so we estimate instead by

$$|w|^{-1} \int_{\chi(B(0,R))} dz \sim R^2 |w|^{-1}.$$ 

\[ \square \]

### 7.5 A lemma of Eskin and Ralston

The addition of $e^{i\phi}$ in the CGO solutions (7.3) will change the main term in (7.2) to

$$i(\zeta_1 - \zeta_2) \cdot \int_{\Omega} (A_1 - A_2)e^{i(\phi_1 - \phi_2)}e^{ix \cdot k} dx.$$ 

The next lemma shows that we can remove the factor $e^{i(\phi_1 - \phi_2)}$ from this integral and recover the Fourier transform

**Lemma 7.5.1** ([ER95]). Let $e = e_1 + ie_2 \in \mathbb{C}^n$ and $k \in \mathbb{R}^n$, where $|e_1| = |e_2| = 1$ and $e_1 \cdot e_2 = e_1 \cdot k = e_2 \cdot k = 0$. Suppose that $A \in C_0^\infty(\mathbb{R}^n)$ and $\phi = \overline{e}^{-1}(e \cdot A)$. Then

$$(e_1 + ie_2) \cdot \int A e^{i\phi} e^{ix \cdot k} dx = (e_1 + ie_2) \cdot \int A e^{ix \cdot k} dx.$$ 

**Proof.** We assume that $e_1$ and $e_2$ are the first two standard basis vectors. Since

$$e \cdot A e^{-i\phi} = i\overline{e}(e^{-i\phi}),$$ 

and $k = (0, 0, k')$, we can write

$$(e_1 + ie_2) \cdot \int A e^{-i\phi(x)} e^{ix \cdot k} dx = i \int \overline{e}(e^{-i\phi(x_1,x_2,x')} e^{ix' \cdot k'} dx_1 dx_2 dx'.$$

By the divergence theorem, we have

$$\int (\partial_1 + i\partial_2)(e^{-i\phi(x_1,x_2,x')}) dx_1 dx_2 dx = \lim_{R \to \infty} \int_{\partial B(0,R)} (\nu_1 + i\nu_2)e^{-i\phi(x_1,x_2,x')} dS,$$

where $\nu$ is the outward unit normal on the circle $\partial B(0,R)$. By Lemma 7.4.1, we have $|\phi| = O(1/(x_1 + ix_2))$ uniformly in $x'$. Thus we have the Taylor expansion

$$e^{-i\phi} = 1 - i\phi + O((x_1 + ix_2)^{-2}).$$
Then
\[ \int_{\partial B(0,R)} (\nu_1 + i\nu_2) e^{i\phi(x_1,x_2,x')} dS = \int_{\partial B(0,R)} (\nu_1 + i\nu_2) dS - i \int_{\partial B(0,R)} (\nu_1 + i\nu_2) \phi dS + \int_{\partial B(0,R)} O(R^{-2}) dS \]
\[ = \int_{B(0,R)} \partial \phi(1) dx_1 dx_2 \]
\[ = \int_{B(0,R)} \partial \phi dx_1 dx_2 + O(R^{-1}) \]

Taking the limit as \( R \to \infty \) we get
\[ \int (\partial_1 + i\partial_2) (e^{i\phi(x_1,x_2,x')}) dx_1 dx_2 dx = -i \int \partial \phi dx_1 dx_2 \]
\[ = -i \int e \cdot A dx_1 dx_2 \]

Plugging this into (7.5) gives the identity. \( \square \)

### 7.6 Phase estimates

In this section we will show that the phases \( \phi = \phi_c \) that we construct will be bounded on average. Let \( f \) be an integrable function on \( \mathbb{R}^2 \). If \( P_\lambda \) denotes the Littlewood-Paley projections, then by the finite band property we have
\[ \| P_\lambda \partial^{-1} f \|_{L^p(\mathbb{R}^2)} \lesssim \lambda^{-1} \| P_\lambda \partial \partial^{-1} f \|_{L^p(\mathbb{R}^2)} = \lambda^{-1} \| P_\lambda f \|_{L^p(\mathbb{R}^2)} \]  
(7.6)

for \( p \in [1, \infty] \).

Let \( \eta : \mathbb{R}^2 \to \mathbb{R} \) be a smooth compactly supported bump function, such that \( \int \eta = 1 \) and
\[ \int x^\alpha \eta dx = 0 \]
when \( 1 \leq |\alpha| \leq M \). Write \( \tilde{P}f = \eta * f \). It is not hard to see that for any \( N \),
\[ \| \eta * \psi_\lambda \|_{L^1} \lesssim \lambda^{-N} \]
for \( \lambda \geq 1 \), where \( \psi_\lambda \) is the kernel of \( P_\lambda \). Thus by Young’s inequality, we have, for any \( p \in [1, \infty] \),
\[ \| P_\lambda \tilde{P}u \|_{L^p(\mathbb{R}^2)} \lesssim \lambda^{-N} \| u \|_{L^p(\mathbb{R}^2)} \]  
(7.7)

when \( \lambda \geq 1 \). On the other hand, the moment condition on \( \eta \) implies that
\[ \|(1 - \tilde{P})u\|_{L^p(\mathbb{R}^2)} \lesssim \| \nabla^N u \|_{L^p(\mathbb{R}^2)}. \]
Thus by the finite band property, we have
\[ \| P(1 - \bar{P})u \|_{L^p(\mathbb{R}^2)} \lesssim \lambda^N \| u \|_{L^p(\mathbb{R}^2)} \]  
when \( \lambda \leq 1 \).

The following lemma is essentially the fact that we can apply the Sobolev embedding \( W^{1/p+\rho} \subset L^\infty \) in each direction separately to obtain, heuristically,
\[ \| \tilde{\partial}_e^{-1} f \|_{L^\infty} \lesssim \| \langle D \rangle^{2/p} \langle D \rangle^{1/p+} \tilde{\partial}_e^{-1} f \|_{L^p} \lesssim \| \langle D \rangle^{2/p-1} \langle D \rangle^{1/p} f \|_{L^p}. \]

**Lemma 7.6.1.** For \( p > 3 \) and \( f \) supported in \( B \subset \mathbb{R}^3 \), we have
\[ \| \tilde{\partial}_e^{-1} f \|_{L^\infty(\mathbb{R}^3)} \lesssim_{B,p} \| f \|_{L^p} + \sum_{1 \leq \nu \leq \lambda} \nu^{2/p-1} \lambda^{1/p} \| P_{\leq \nu} f \|_{L^p}, \]
where the \( P_\lambda \) are Littlewood-Paley projections, and the \( P_{\leq \nu}^e \) are Littlewood-Paley projections in the \( e \) plane.

**Proof.** Since \( \bar{P}^e f \) is supported in a fixed compact set, we have, by Lemma 7.4.1, some applications of Bernstein’s inequality and (7.7),
\[ \| \tilde{\partial}_e^{-1} \bar{P}^e f \|_{L^\infty} \lesssim \| \bar{P}^e f \|_{L^\infty} \lesssim \| f \|_{L^p} + \| \bar{P}^e P_{\geq 1} f \|_{L^\infty} \lesssim \| f \|_{L^p} + \sum_{\lambda \geq 1} \left( \| P_{\leq \lambda} f \|_{L^\infty} + \sum_{\nu \geq 1} \| \bar{P}^e P_{\nu}^e P_{\lambda} f \|_{L^\infty} \right) \lesssim \| f \|_{L^p} + \sum_{\lambda \geq 1} \left( \lambda^{1/p} \| P_{\leq \lambda} f \|_{L^p} + \sum_{\nu \geq 1} \nu^{-N+2/3} \lambda^{1/p} \| P_{\nu}^e P_{\lambda} f \|_{L^p} \right), \]

For \( \tilde{\partial}_e^{-1} (1 - \bar{P}^e) f \), we split the estimate into three parts:
\[ \| \tilde{\partial}_e^{-1} (1 - \bar{P}^e) f \|_{L^\infty} \lesssim \sum_{\nu,\lambda} \| \tilde{\partial}_e^{-1} (1 - \bar{P}^e) P_{\lambda}^e P_{\lambda} f \|_{L^\infty} \lesssim \sum_{\nu \leq 1} \sum_{1 \leq \nu \leq \lambda} + \sum_{1 \leq \nu} \cdot \]
When \( \nu \leq 1 \) we use (7.6) and (7.8) to obtain
\[ \sum_{\nu \leq 1} \lesssim \sum_{\nu \leq 1} \nu^{-N-1} \lambda^{1/p} \| P_{\leq \lambda} f \|_{L^p} \lesssim \sum_{\lambda} \lambda^{1/p} \| P_{\leq \lambda} f \|_{L^p}. \]
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When \( \nu \geq 1 \), we can discard the factor \((1 - \tilde{P}^e)\), and we estimate by
\[
\begin{align*}
\sum_{1 \leq \nu \leq \lambda} \nu^{-1} \| P_{\leq \nu}^e \lambda f \|_{L^\infty} & \lesssim \sum_{1 \leq \nu \leq \lambda} \nu^{2/p - 1} \lambda^{1/p} \| P_{\leq \nu}^e \lambda f \|_{L^p} \\
\sum_{\nu > \lambda} \nu^{-1} \| P_{\leq \nu}^e \lambda f \|_{L^\infty} & \lesssim \sum_{\nu \geq 1} \nu^{2/p - 1} \nu^{1/p} \| f \|_{L^p} \\
& \lesssim \| f \|_{L^p}
\end{align*}
\]
since \( p > 3 \).

By averaging this estimate we find that \( \overline{\partial}_{e}^{-1} f \) will be bounded for most choices of \( e \):

**Corollary 7.6.2.** Under the assumptions of the lemma,
\[
\int_{U \in \mathcal{O}(3)} \| \overline{\partial}_{U}^{-1} f \|_{L^\infty(\mathbb{R}^3)}^p d\mu(U) \lesssim_{B,p} \| f \|_{L^p}^p.
\]

**Proof.** By (6.4.1) we have
\[
\begin{align*}
\sum_{1 \leq \nu \leq \lambda} \left( \int_{U \in \mathcal{O}(3)} \nu^{2/p - 1} \lambda^{1/p} \| P_{\leq \nu}^e \lambda f \|_{L^p}^p d\mu(U) \right)^{1/p} & \lesssim \sum_{1 \leq \nu \leq \lambda} \nu^{2/p - 1} \lambda^{1/p} (\nu/\lambda)^{2/p} \| f \|_{L^p} \\
& \lesssim \sum_{1 \leq \nu \leq \lambda} \nu^{3/p - 1} (\nu/\lambda)^{1/p} \| f \|_{L^p} \\
& \lesssim \| f \|_{L^p}
\end{align*}
\]
since \( p > 3 \).

\[\square\]

### 7.7 Solvability for \( L_{A,q,\zeta} \)

Now we show that on average, the terms in \( L_{A,q,\zeta} + \Delta_\zeta \) are all perturbative.

**Lemma 7.7.1.** For \( q \in L^{n/2}(\mathbb{R}^n) \), we have
\[
\| m_q \|_{X^{1/2} \to X^{-1/2}^{1/2}} \lesssim \| q \|_{L^{n/2}}.
\]

Let \( \theta > 0 \) be small. For \( A \in L^p(\mathbb{R}^3) \) supported in a ball \( B \) with \( p = (1 - \theta)/3 \) and \( s > 0 \), we have
\[
\tau \| A \|_{X^{1/2} \to X^{-1/2}^{1/2}} + \| A \cdot \nabla \|_{X^{1/2} \to X^{-1/2}^{1/2}} \lesssim \| A \|_{L^p} + \sup_{\mu \leq \nu \leq \lambda \leq 8r} a_{\nu,\lambda,\tau}(\lambda/\nu)^p \lambda^s \| P_1^{\nu} P_{\leq \nu} A \|_{L^p},
\]
where \( \sup_{\tau} \sum_{\nu \leq \lambda \leq 8r} a_{\nu,\lambda,\tau} < C \).
Proof. The first estimate follows from
\[ |\langle qu, v \rangle| \lesssim \|q\|_{L^{n/2}} \|u\|_{L^{2n/(n-2)}} \|v\|_{L^{2n/(n-2)}} \lesssim \|q\|_{L^{n/2}} \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}. \]

Next, we estimate the bilinear form \( \langle A \cdot \nabla u, v \rangle \). Write
\[ \langle A \cdot \nabla u, v \rangle = \langle A \cdot \nabla Q_h u, v \rangle + \langle A \cdot \nabla Q_t u, Q_h v \rangle + \langle A \cdot \nabla Q_t u, Q_t v \rangle \]
The terms with \( Q_h \) are easy to estimate. In fact, for \( A \in L^n(\mathbb{R}^n) \) we have
\[ |\langle A \cdot \nabla Q_h u, v \rangle| \lesssim \|A\|_n \|A\|_{L^n} \|\nabla Q_h u\|_{L^2} \|v\|_{X^{1/2}_\xi} \]
\[ \lesssim \|A\|_n \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi} \]
\[ |\langle A \cdot \nabla Q_t u, Q_h v \rangle| \lesssim \|A\|_n \|\nabla Q_t u\|_{L^2} \|Q_h v\|_{L^2} \]
\[ \lesssim \|A\|_n \|Q_t u\|_{L^2} \|Q_h v\|_{H^1} \]
\[ \lesssim \|A\|_n \|u\|_{X^{1/2}_\xi} \|v\|_{X^{1/2}_\xi}. \]

It remains to estimate the low modulation terms. Write
\[ \langle A \cdot \nabla Q_t u, Q_t v \rangle = \sum_{\mu, \nu, \lambda \geq 1} \int P_{\lambda} A \cdot \nabla Q_\mu u \cdot \overline{Q_\nu v} \, dx, \tag{7.9} \]
where we make the notational convention that
\[ Q_1 = Q_{\leq 1} \]
\[ P_1 = P_{\leq 1}. \]

We will assume that \( \mu \leq \nu \), as the case \( \mu > \nu \) is treated in exactly the same way. Since \( \nabla Q_\mu u \cdot \overline{Q_\nu v} \) has Fourier support in \( \{ |\xi_1| \leq 2\nu \} \), we have
\[ \int P_{\lambda} A \cdot \nabla Q_\mu u \cdot \overline{Q_\nu v} \, dx = \int P_{\leq 8\nu}^1 P_{\lambda} A \cdot \nabla Q_\mu u \cdot \overline{Q_\nu v} \, dx. \]

Suppose first that \( \lambda > (\mu \tau)^{1/2} \). We have
\[ \left| \int P_{\leq 8\nu}^1 P_{\lambda} A \cdot \nabla Q_\mu u \cdot \overline{Q_\nu v} \right| \lesssim B_{\mu, \nu} \|P_{\leq 8\nu}^1 P_{\lambda} A\|_{L^p} \|Q_\mu u\|_{X^{1/2}_\xi} \|Q_\nu v\|_{X^{1/2}_\xi} \]
where
\[ B_{\mu, \nu} = \tau^{1/2-\theta/3} \mu^{1-\theta/3} \nu^{-1/2} \]
When \( \nu > \lambda \) we have
\[ B_{\mu, \nu} \lesssim \tau^{1/2-\theta/3} \mu^{1-\theta/3} \nu^{-1/2} \mu^{1-\theta/3} \nu^{-1/2} (\nu \tau)^{-1/4} \]
\[ \lesssim \tau^{1/4-\theta/3} \mu^{1-\theta/3} \nu^{-2/4} \mu^{1-\theta/3} \nu^{-1/4} \]
\[ \lesssim \tau^{-\theta/2} \]
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Since there are only log τ choices of μ, ν, λ, we can just sum over all possible values.

When ν ≤ λ, we have

\[(ν/λ)^{(1-θ)/3}B_{μ,ν} = ν^{(1-θ)/3-1/2}μ^{(1-θ)/3-θ/2}\lambda^{-(1-θ)/3}τ^{1/2}-(1-θ)/3\theta/2
\]

\[= ν^{(1-θ)/3-1/2}μ^{(1-θ)/6-θ/2}τ^{1/2}-(1-θ)/2-θ/2(λ/(μτ))^{1/2}-(1-θ)/3\]

\[≤ ν^{-θ/3}(μ/ν)^{1/6}(λ/τ^{1/2})-(1-θ)/3\]

In this case we take \(a_{ν,λ,τ} = \chi_{λ≥τ^{1/2}}(λ/τ^{1/2})-(1-θ)/6ν^{-θ/3}\) and apply Young’s inequality.

Finally, we turn to the case when λ ≤ (μτ)^1/2. We divide the torus \(E = \{d(ξ, Σ_ξ) ≤ τ/8\}\) into \([τ^{1/2}/μ^{1/2}]\) thickened arcs \(A_k\) of equal length. Let \(R_k\) be the Fourier projection onto the kth arc. The distance between two points in \(E\) is bounded below by \(τ\theta\), where \(θ\) is the angular separation. Since the \(R_k\) each subtend an angle of approximately \(μ^{1/2}/τ^{1/2}\), we have \(d(A_j, A_k) > λ\) unless \(|j - k| ≤ C\), where \(C\) is a fixed constant. Since \(P_λA \cdot R_kQ_μu\) has Fourier support in the set \(\{A_k + B(0, 2λ)\}\), we find that \(\langle P_λA \cdot R_k∇Q_μu, R_jQ_νv⟩ = 0\) unless \(|j - k| ≤ C\), so that

\[||\langle P_{≤σμ}^1P_λA \cdot ∇Q_μu, Q_νv⟩|| \lesssim \sum_{|j - k| ≤ C}||\langle P_{≤σμ}^1P_λA \cdot R_k∇Q_μu, R_jQ_νv⟩||\].

Let \(A_{k,μ}\) denote the set \(A_k ∩ E_μ\). By rotation and translation in Fourier space, we may arrange that \(A_{k,μ} ⊂ C[0, μ] × [0, μ] × [0, μ^{1/2}τ^{1/2}]\). Now, we can apply Bernstein’s inequality in each direction, to obtain

\[||P_{≤σμ}^1P_λA \cdot R_kQ_μu||_L^2 \lesssim ||P_{≤σμ}^1P_λA||_L^{2,1}||R_kQ_μu||_L^{2,1} \lesssim λ^{1/3}μ^{2/3}||P_{≤σμ}^1P_λA||_L^3||R_kQ_μu||_L^2.\]

Set \(f = ∇u\), sum over \(j, k\) and apply Cauchy-Schwarz, to obtain

\[||\langle P_{≤σμ}^1P_{≤μ^{1/2}τ^{1/2}}A \cdot ∇Q_μu, Q_νv⟩|| \lesssim μ^{2/3}λ^{1/3}||P_{≤σμ}^1P_λA||_L^3||∇Q_μu||_L^2||Q_νv||_L^2 \lesssim ν^{-1/6}μ^{1/6}(λ/ν)^{1/3}||P_{≤σμ}^1P_λA||_L^3||Q_μu||_{X^{1/2}_ς}||Q_νv||_{X^{1/2}_ς} \]

Then we can take \(a_{ν,λ,τ} = λ^{-8}\) and apply Young’s inequality.

Thus we have established the bound for \(A \cdot ∇\). The bound for \(A\) is established in exactly the same way.

\[\square\]

Lemma 7.7.2. Suppose \(A \in W^{s,p}(ℝ^n)\) and \(q \in W^{-1,3}(ℝ^n)\) are compactly supported. Let \(||T||\) denote \(||T||_{X^{1/2}_ς → X^{-1/2}_ς}\), and let

\[F(M, U) = \sup_{τ \in [M, 2M]}(||D \cdot A|| + ||A \cdot ∇|| + τ||A|| + ||A^2|| + ||q||).\]

Then

\[||F(M, U)||_L^1(μ(3)) = O_{M → ∞}(||A||_{W^{s,p}(ℝ^n)}) + o_{M → ∞}(1).\]
Proof. We have
\[
\sup_{\tau} \int_{O(3)} \sup_{\nu \leq \lambda \leq \tau} a_{\nu, \lambda, \tau}(\lambda/\nu)^p \lambda^s \|P_{\lambda} P_{\nu}^f A\|_{L^p} \, d\mu(U) \lesssim \sup_{\tau} \sum \nu \leq \lambda \leq \tau a_{\nu, \lambda, \tau} \int_{O(3)} \lambda^s (\lambda/\nu)^p \|P_{\lambda} P_{\nu} A\|_{L^p} \, d\mu(U)
\]
\[
\lesssim \sum a_{\nu, \lambda, \tau} \|A\|_{W^{s,p}} \lesssim \|A\|_{W^{s,p}}.
\]
Thus we can control the norm of $A \cdot D$ and $\tau A$.

The terms $\nabla A$, $A^2$ and $q$ are controlled as in Theorem 6.4.2. \qed

7.8 Recovering the magnetic potential

Theorem 7.8.1. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, and suppose that $A_i \in W^{s,3}(\Omega)$ (with $1/3 > s > 0$) and $q_i \in W^{-1,3}(\Omega)$ for $i = 1, 2$. There exists some $\epsilon > 0$, depending on $\Omega$ and $s$, such that if $\|A_i\|_{W^{s,3}(\Omega)} \leq \epsilon$ for $i = 1, 2$ then

$\Lambda_{A_1, q_1} = \Lambda_{A_2, q_2}$

implies that $\text{curl } A_1 = \text{curl } A_2$.

Proof. Since $s < 1/3$ and $\Omega$ is Lipschitz, we may extend the $A_i$ and $q_i$ by zero to $\mathbb{R}^3$, so that $\|A_i\|_{W^{s,3}(\mathbb{R}^3)}$ is still small and $\|q_i\|_{W^{-1,3}(\mathbb{R}^3)}$ is unchanged. By Sobolev embedding, we can also assume that $A \in W^{s,p}(\mathbb{R}^3)$ for $p > 3$ by taking $s$ to be slightly smaller. We now show how to construct solutions in $H^1(\Omega)$ to

$u = e^{\mp \tilde{\zeta}}(a + \psi)$.

to $L_{A, q} u = 0$, where $A \in W^{s,p}(\mathbb{R}^3)$ and $q \in W^{-1,3}(\mathbb{R}^3)$ are supported in $\Omega$. To do this we solve

$L_{A, q} \tilde{\zeta} \psi = F + G,$

(7.10)

where

$F = -\Delta a + (D \cdot A) a + 2A \cdot Da + A^2 a + qa,$

(7.11)

and

$G = -2\tilde{\zeta} \cdot \nabla a - 2i\tilde{\zeta} \cdot A.$

(7.12)

In order to control $G$ we let $a = e^{-i\phi}$, where

$\phi = \nabla^{-1}_\zeta (\zeta \cdot A \leq M^\sigma),$

with $\sigma$ small but positive and $|\zeta - \tilde{\zeta}| = O(1)$. Here $A \leq \lambda = \eta \ast A$, where $\eta_\lambda(x) = \lambda^3 \eta(\lambda x)$, and $\eta$ is a compactly supported mollifier. Thus

$-2\tilde{\zeta} \cdot \nabla a - 2i\tilde{\zeta} \cdot A a = -2i\zeta \cdot (A - A \leq M^\sigma) a + (\tilde{\zeta} - \zeta) \cdot (2\nabla a + 2iAa).$
Fix \( \chi \in C^\infty_0 \) such that \( \chi = 1 \) on \( \Omega \). We need to check that \( \chi^2(F + G) \) is in \( X^{-1/2}_\zeta \). We claim that in fact
\[
\| \chi^2(F + G) \|_{X^{-1/2}_\zeta} \lesssim \| a \|_{C_N} (\tau^{-1/2} + \| \nabla A \|_{X^{-1/2}_\zeta} + \| A \|_{L^2} + \| A \|_{L^2}^2 + \| q \|_{X^{-1/2}_\zeta})
+ \tau^{1/2} \| a \|_{L^\infty} \| A - A_{\leq M} \|_{L^2},
\]
where \( N \) is a large integer. For \( G \) we have the trivial estimate
\[
\| \chi^2G \|_{X^{-1/2}_\zeta} \lesssim \tau^{-1/2} - 2i\zeta \cdot (A - A_{\leq M}) \chi^2a \|_{L^2} + \tau^{-1/2} \| \nabla a \|_{L^\infty} + \tau^{-1/2} \| a \|_{L^\infty} \| A \|_{L^2}
\lesssim \tau^{1/2} \| a \|_{L^\infty} \| A - A_{\leq M} \|_{L^2} + \| a \|_{C^1} (\tau^{-1/2} + \tau^{-1/2} \| A \|_{L^2}).
\]
For \( F \) we apply Lemma 3.4.2. The Schwartz seminorms of \( \chi^2 \nabla^k a \) are bounded by \( \| a \|_{C_N} \) for some large \( N \), and hence
\[
\| \chi^2 \nabla^k a \|_{X^{-1/2}_\zeta} \lesssim \| a \|_{C_N} \| u \|_{X^{-1/2}_\zeta}
\]
for any \( u \). Thus
\[
\| \chi^2 F \| \lesssim \| a \|_{C_N} (\| \chi \|_{X^{-1/2}_\zeta} + \| \nabla A \|_{X^{-1/2}_\zeta} + \| A \|_{X^{-1/2}_\zeta} + \| A \|_{X^{-1/2}_\zeta} + \| q \|_{X^{-1/2}_\zeta})
\]
By the definition of \( X^{-1/2}_\zeta \) we have
\[
\| \chi \|_{X^{-1/2}_\zeta} + \| A \|_{X^{-1/2}_\zeta} \lesssim \tau^{-1/2}(\| \chi \|_{L^2} + \| A \|_{L^2}),
\]
and by (3.2) and Sobolev embedding we have
\[
\| A^2 \|_{X^{-1/2}_\zeta} \lesssim \| A^2 \|_{H^{-1/2}_\zeta} \lesssim \| A^2 \|_{L^{3/2}} = \| A \|_{L^3}^2.
\]
Thus we have proven (7.13).

Now we proceed as in the proof of Theorem 3. Fix \( r > 0 \) and three orthonormal vectors \( \{ e_1, e_2, e_3 \} \), and define
\[
\zeta_1(\tau, U) = \tau U(e_1 - ie_2)
\]
\[
\zeta_2(\tau, U) = -\zeta_1(\tau, U)
\]
\[
\tilde{\zeta}_1(\tau, U) := \tau U e_1 + i(rU e_3 - \sqrt{\tau^2 - r^2}U e_2)
\]
\[
\tilde{\zeta}_2(\tau, U) := -\tau U e_1 + i(rU e_3 + \sqrt{\tau^2 - r^2}U e_2)
\]
In what follows, all of inequalities will implicitly depend on \( r \). For example, we have \( |\zeta_i - \tilde{\zeta}_i| \lesssim 1 \). In particular, by Lemma 4.3.1, the spaces \( X^b_\zeta \) and \( X^b_{\tilde{\zeta}} \) have equivalent norms.

Fix \( \epsilon > 0 \). Let \( \tilde{A}_1 = A_1 \) and \( \tilde{A}_2 = -A_2 \), and \( \phi_i = \partial_{\zeta_i}^{-1}(\zeta \cdot A_{\leq M}) \). Let
\[
B(U, M) = \sum_{i=1,2} \left( \| \phi_i \|_{X^{-1/2}_\zeta} + \| \nabla \tilde{A}_i \|_{X^{-1/2}_\zeta} + \| \phi_i \|_{L^\infty} + M^{-N} \| \phi_i \|_{C^N} \right)
\]

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By Lemmas 7.7.2 and 4.2.1 and Corollary 7.6.2, we may choose, as in the proof of Theorem ??, a sequence of $M = M_l$ such that $M_l \to \infty$, some $\tau = \tau_{\epsilon,l} \in [M_l, 2M_l]$, $U = U_{\epsilon,l} \in B_\epsilon$ such that

$$B(U, M) \lesssim 1$$

and $F_i(U, M)$ is small, where $F_i(U, M)$ is as in Lemma 7.7.2. Thus

$$\|L_{\tilde{A}_i, \tilde{q}, \tilde{\zeta}_i} + \Delta_{\tilde{\zeta}_i}\|_{X^{1/2}_{\tilde{\zeta}_i} \to X^{-1/2}_{\tilde{\zeta}_i}} \leq c,$$

where $c$ is small.

By (3.4.3), we have

$$\|L_{\tilde{A}_i, \tilde{q}, \tilde{\zeta}_i} + \Delta_{\tilde{\zeta}_i}\|_{\dot{X}^{1/2}_{\tilde{\zeta}_i} \to \dot{X}^{-1/2}_{\tilde{\zeta}_i}} \leq 1/2,$$

when $c$ is sufficiently small. For $F_i$ and $G_i$ given by (7.11) and (7.12), the estimate (7.13) implies that

$$\|x^2(F_i + G_i)\|_{\dot{X}^{-1/2}_{\tilde{\zeta}_i}} \lesssim_{A,q} M^{N\sigma} + M^{1/2}\|A_i - (A_i)\|_{L^2} = o_{M \to \infty}(M^{1/2}).$$

Thus by the contraction mapping principle there are $u_i = e^{x\tilde{\zeta}_i}(e^{-i\phi_i} + \psi)$ solving $L_{\tilde{A}_i, \tilde{q}, \tilde{\zeta}_i} u_i = 0$ in $\Omega$ such that

$$\|\psi_i\|_{\dot{X}^{1/2}_{\tilde{\zeta}_i}} = o(M^{1/2}).$$

Now we apply the integral identity (7.2), and claim that this gives

$$0 = i(\tilde{\zeta}_2 - \tilde{\zeta}_1) \int e^{-i(\phi_1 + \phi_2)} e^{ix\cdot k} (A_1 - A_2) \, dx + o(M),$$

where $k = rUe_3$.

We will show how to estimate the various error terms that arise.

When there are no phase factors, we estimate using the boundedness of $F_i(U, M)$ and the fact that $\|\psi_i\|_{X^{1/2}_{\tilde{\zeta}_i}} = o(M^{1/2})$.

The terms with no $\psi_i$ factors are all bounded by $\|A\|_{L^1} \|e^{-i\phi}\|_{C^1} = o(M^\sigma)$.

For terms that look like $\zeta \cdot \int A a \psi_i \, dx$, we estimate by

$$M \left| \int A a \psi_i \, dx \right| \lesssim \|Aa\|_{X^{-1/2}_{\tilde{\zeta}_i}} \|\psi_i\|_{X^{1/2}_{\tilde{\zeta}_i}} \lesssim M^{1/2} \|a\|_{L^\infty} \|A\|_{L^2} \|\psi_i\|_{X_{\tilde{\zeta}_i}} = o(M)$$

For terms that look like $\int A \nabla a \psi_i \, dx$, we estimate by $\|A\|_{L^2} \|a\|_{C^1} \|\psi_i\|_{X^{1/2}_{\tilde{\zeta}_i}} = o(M^{1/2+\sigma})$.

For terms that look like $\int \nabla A a \psi_i \, dx$ we have

$$\left| \int \nabla A a \psi_i \, dx \right| \lesssim \|\nabla A\|_{X^{-1/2}_{\tilde{\zeta}_i}} \|\psi_i\|_{X^{1/2}_{\tilde{\zeta}_i}}.$$
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Since $\nabla A$ is bounded in $X_{\xi_i}^{-1/2}$, this will be bounded by $M^{N^2\sigma+1/2} = o(M)$ by Lemma 3.4.2.

For terms that look like $\int Aa\nabla \psi_i \, dx$, we can integrate by parts to transfer derivatives onto the phase factor or onto $A$.

Now we turn to the main term

$$i(\tilde{\xi}_2 - \tilde{\xi}_1) \cdot \int e^{-i(\phi_1 + \phi_2)} e^{ix-k}(A_1 - A_2) \, dx = i(\xi_2 - \xi_1) \cdot \int e^{-i(\phi_1 + \phi_2)} e^{ix-k}((A_1)_{\leq M^\sigma} - (A_2)_{\leq M^\sigma}) \, dx$$

$$+ \sum_i O(\|A_i\|_{L^1} + M\|A_i\|_{M^\sigma} - A_i\|_{L^1}).$$

Let $\xi = \xi_1 = -\xi_2$. We have $\phi_1 = \overline{\partial}_\xi^{-1}(\xi \cdot A_1)_{\leq M^\sigma}$ and $\phi_2 = \overline{\partial}_\xi^{-1}(-\xi \cdot (-A_2))_{\leq M^\sigma}$. So $\phi_1 + \phi_2 = \overline{\partial}_\xi^{-1}(\xi \cdot (A_1 - A_2)_{\leq M^\sigma})$, and we may apply Lemma 7.5.1 to conclude that

$$U(e_1 - ie_2) \cdot \int e^{ix-k}((A_1)_{\leq M^\sigma} - (A_2)_{\leq M^\sigma}) \, dx = o(1) \tag{7.14}$$

Since the $U = U_{\epsilon, l}$ are contained in a compact set in $O(3)$, we may pass to a subsequence such that $U_{\epsilon, l} \to U_{\epsilon}$ for some $U_{\epsilon} \in O(3)$. Since the $\hat{A}_i$ are continuous, we may pass to the limit in (7.14) to obtain

$$U_{\epsilon}(e_1 - ie_2) \cdot (\hat{A}_1 - \hat{A}_2)(rU_{\epsilon}e_3) = 0.$$

As $\epsilon \to 0$, we have $U_{\epsilon} \to I$, so that this implies that $(e_1 - ie_2) \cdot (\hat{A}_1 - \hat{A}_2)(re_3) = 0$. Since $e_1, e_2, e_3 \in S^2$ and $r$ were arbitrary, this means that $\text{curl} A_1 = \text{curl} A_2$.

Now we show that the conclusion of this theorem implies the existence of a gauge transformation $\psi$ such that $\nabla \psi = A_1 - A_2$.

**Lemma 7.8.2.** Suppose that $A_1, A_2 \in L^n(\mathbb{R}^n)$ are compactly supported, and that

$$\text{curl} A_1 = \text{curl} A_2.$$

Then there exists $\psi \in W^{1,n}(\mathbb{R}^n)$ supported in $\text{supp}(A_1 - A_2)$ such that

$$\nabla \psi = A_1 - A_2.$$

**Proof.** If $\psi$ exists then

$$\Delta \psi = \text{div}(\nabla \psi) = \text{div}(A_1 - A_2).$$

Consider the equation

$$\Delta \psi_\lambda = \text{div} \eta_\lambda \ast (A_1 - A_2). \tag{7.15}$$

where $\eta_\lambda = \lambda^n \eta(\lambda x)$ and $\eta$ is a compactly supported mollifier.
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Since the right hand side of (7.15) is compactly supported and integrable, standard potential theory gives a unique solution to (7.15) which vanishes at infinity. By Theorem 7.8.1, we have \( \text{curl } A_1 = \text{curl } A_2 \). The vector Laplacian satisfies the identity

\[ \Delta = \text{grad} \circ \text{div} - \text{curl} \circ \text{curl}. \]

Applying this to \( \eta_\lambda \ast (A_1 - A_2) - \nabla \psi_\lambda \), we have

\[ \Delta(\eta_\lambda \ast (A_1 - A_2) - \nabla \psi_\lambda) = \nabla(\eta_\lambda \ast (\text{div } A_1 - \text{div } A_2) - \Delta \psi_\lambda) = 0. \]

Since \( \eta_\lambda \ast (A_1 - A_2) - \nabla \psi_\lambda \) vanishes at infinity, this implies that \( \nabla \psi_\lambda = \eta_\lambda \ast (A_1 - A_2) \). In particular, \( \psi_\lambda \) is constant outside of \( \text{supp} \eta_\lambda \ast (A_1 - A_2) \). Since \( \psi_\lambda \) vanishes at infinity, this implies that \( \psi_\lambda = 0 \) outside of \( \text{supp} \eta_\lambda \ast (A_1 - A_2) \).

From the Calderón-Zygmund estimates for Riesz potentials, the equation (7.15) implies that

\[ \| \nabla \psi_\lambda \|_{L^n} \lesssim \| A_1 - A_2 \|_{L^n}. \]

The function \( \psi_\lambda \) are all supported in some fixed bounded set \( B \). By Friedrichs’ inequality, this implies that the sequence \( \{ \psi_k \}_{k=1}^\infty \) is bounded in \( W^{1,n} \). Thus there is a subsequence converging weakly to some \( \psi \in W^{1,n} \). This \( \psi \) is clearly supported in \( \text{supp}(A_1 - A_2) \) and satisfies \( \nabla \psi = A_1 - A_2 \).

7.9 Recovering the scalar potential

If the scalar potential \( q \) is bounded, then we can recover \( q \) as well.

**Theorem 7.9.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a Lipschitz domain, and suppose that \( A_i \in W^{s,3}(\Omega) \) (with \( 1/3 > s > 0 \)) and \( q_i \in L^\infty(\Omega) \) for \( i = 1, 2 \). There exists some \( \epsilon > 0 \), depending on \( \Omega \) and \( s \), such that if \( \| A_i \|_{W^{s,3}(\Omega)} \leq \epsilon \) for \( i = 1, 2 \) then

\[ \Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \]

implies that \( \text{curl } A_1 = \text{curl } A_2 \) and \( q_1 = q_2 \).

**Proof.** As in the proof of Theorem 7.8.1 we extend \( A_i \) and \( q_i \) by zero to all of \( \mathbb{R}^3 \). It is clear by the definition of \( \Lambda_{A,q} \) that if we extend \( A_1, A_2 \) and \( q \) by zero to a larger domain then \( \Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \) on the larger domain as well. Thus we may replace \( \Omega \) by a large ball \( B \) and assume that the \( A_i \) are compactly supported in \( B \).

Now we can use Theorem 7.8.1 to produce a gauge transformation \( \psi \in W^{1,3}_0(B) \) such that

\[ \nabla \psi = A_1 - A_2. \]

Since the \( A_i \) are in \( W^{s,3} \) with \( s > 0 \), we have \( \psi \in L^\infty \) by Sobolev embedding. By the gauge invariance (Lemma 7.1.1), we have \( \Lambda_{A_2,q_2} = \Lambda_{A_1,q_2} \). Thus we may assume that \( A_1 = A_2 \).
Now we construct solutions \( u_i = e^{ix \cdot \tilde{\zeta}}(e^{-i\phi_i} + \psi_i) \) to \( L_{\tilde{A}_i,q_i} u_i \) as before, where \( \tilde{A}_1 = A_1, \tilde{A}_2 = -A_1 \), and \( \phi_i = \tilde{\zeta}^{-1}(\zeta \cdot (\tilde{A}_i)_{\leq M^*}) \). Applying the integral identity as before, and noting that \( A_1 = A_2 \) and \( \phi_1 + \phi_2 = 0 \), we obtain

\[
0 = \int (q_1 - q_2)e^{ix \cdot k}(1 + e^{-i\phi_1}\psi_2 + e^{-i\phi_2}\psi_1 + \psi_1\psi_2) \, dx.
\]

Thus if the \( \phi_i \) are bounded and the \( \psi_i \) satisfy \( \|\psi_i\|_{X_{\zeta}^{1/2}} = o(\tau^{1/2}) \), then by the \( L^2 \) estimate \( \|\psi_i\|_{L^2} \lesssim \tau^{-1/2}\|\psi_i\|_{X_{\zeta}^{1/2}} \) we can conclude that

\[
(q_1 - q_2)(k) = o(1).
\]

This shows that the averaging argument of Theorem 7.8.1 yields the conclusion that \( q_1 = q_2 \).

7.10 Open questions

We conclude this chapter with some remarks on our results for magnetic potentials in three dimensions. Using the tools we have, it should be possible to show that

**Conjecture 1.** Suppose that \( \Omega \subset \mathbb{R}^3 \) is a bounded open set and that \( A_i \in L^{3+}(\Omega) \) and \( q_i \in W^{-1,3}(\Omega) \),

\[
\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \Rightarrow \text{curl } A_1 = \text{curl } A_2 \text{ and } q_1 = q_2.
\]

Our results for recovering \( A \) are close to this conjecture; for our results, we require that the \( A_i \) possess slightly more regularity and that the \( A_i \) are small in a suitable sense. However, because the error terms in the CGO solutions that we construct grow like \( \tau^{1/2} \), we cannot recover \( q \) in any space near \( W^{-1,3} \) (or even \( L^{3/2} \)) in the presence of a potential, because we are not able to control the error terms.

The reason for the \( \tau^{1/2} \) growth is that our error \( \psi \) satisfies an equation of the form

\[
\Delta_\zeta \psi = -\Delta a + D \cdot A a + 2A \cdot Da + A^2 a + qa - 2\zeta \cdot \nabla a - 2i\zeta \cdot A.
\]

We can eliminate the terms with \( \zeta \) by choosing \( a \) appropriately, but no matter how we choose \( a \) we will end up with terms of the form \( \nabla W a \), with \( W \in L^3 \). If \( a = 1 \) then the estimate

\[
\|\nabla W a\|_{X_{\zeta}^{-1/2}} \lesssim \|W\|_{L^2}
\]

holds on average. The same will be true if \( a \) lives at frequency 1, since then multiplication by \( a \) preserves \( X_{\zeta}^{-1/2} \). On the other hand, if \( a \) lives at frequency \( \tau \), then we can transfer the derivatives from \( \hat{W} \) to \( a \), and use the dual of (5.7) and Hölder’s inequality to obtain

\[
\|\nabla W a\|_{X^{-1/2}} \lesssim \|W\|_{L^3}\|\nabla a\|_{L^2}.
\]
It is not hard to see that \( \| \nabla a \|_{L^2} \lesssim 1 \) on average when \( \nabla a = Aa \) and \( A \in W^{s,3} \) with \( s > 0 \). Thus the solutions that we construct should really be bounded in \( X^{1/2}_\zeta \). A rigorous proof of this would allow the recovery of potentials \( q \) in \( W^{-1,3} \) in the presence of a magnetic potential \( A \in W^{s,3} \).

The need for extra regularity comes from a logarithmic divergence in Lemma 7.7.1, which makes it impossible to invert the operator \( \Delta_\zeta + A \cdot D \) when \( A \) is only assumed to be in \( L^{3+} \). On the other hand, we know how to treat zero-order terms of the form \( \nabla B \), where \( B \in L^3 \). To convert the first-order term \( A \cdot \nabla \) into a zero-order term of the form \( \nabla B \), we would need to conjugate the equation with a pseudodifferential operator of the form \( e^{-i\phi(x,D)} \), as in [NU94, NSU95, ER95]. This conjugation technique would also get rid of the smallness requirement on \( A \). We believe that this strategy is feasible for \( A \in L^{3+} \), but it involves some considerable technical difficulties, which we hope to address in future work.

Another question is what happens at the endpoint when \( A \in L^3 \) and what happens in higher dimensions. The results of [Wol92, KT01] suggest that we should have

**Conjecture 2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, and suppose that \( A_i \in L^n(\Omega) \) and \( q_i \in W^{-1,n}(\Omega) \). Then

\[
\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \Rightarrow \text{curl} A_1 = \text{curl} A_2 \text{ and } q_1 = q_2.
\]

To prove this seems significantly more difficult, even in three dimensions, and would require new ideas beyond the methods of this work.
Bibliography


Daniel Tataru, *The \(X_s^\theta\) spaces and unique continuation for solutions to the semilinear wave equation*, Communications in Partial Differential Equations 21 (1996), no. 5-6, 841–887.

