The embedded contact homology of nontrivial circle bundles over Riemann surfaces

by

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Abstract

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The embedded contact homology (ECH) of a 3-manifold Y is a topological invariant defined using a contact form on Y which counts certain pseudoholomorphic curves in $\mathbb{R} \times Y$. We compute the ECH of nontrivial circle bundles over Riemann surfaces.

To my grandfather,
John Richard Farris Jr.
(1919-2010)

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Chapter 1

Introduction

1.1 Embedded contact homology

Embedded contact homology is a 3-manifold invariant due to Hutchings ([12] and [27] are good overviews of ECH and its context). ECH is defined in terms of a contact structure on an oriented 3-manifold Y. Recall that a contact structure ξ on a 3-manifold Y is a maximally nonintegrable rank two subbundle of TY. We will consider only co-orientable contact structures, which may be expressed as $\xi = ker(\alpha)$ for some $\alpha \in \Omega^1(Y)$. In terms of the *contact form* α, the bundle ξ is a contact structure iff $\alpha \wedge d\alpha > 0$. Martinet [20] showed that every 3-manifold has a contact structure. A contact 3-manifold with a contact form (Y, ξ, α) gives rise to the symplectic 4-manifold $(\mathbb{R}_t \times Y, (e^t\alpha))$, called the symplectization of Y.

A contact form α on Y uniquely defines a nonvanishing vector field $R \equiv R_\alpha$ by the conditions $i_R(d\alpha) = 0$ and $\alpha(R) = 1$. R is the *Reeb vector field* of α. The closed (not necessarily embedded) trajectories of R are called *Reeb orbits*. The Reeb vector field is closely related to Hamiltonian vector fields on symplectic manifolds, and so it is of great dynamical interest.

The flow of R preserves α and hence ξ. Thus, given a Reeb orbit a of length l and $p \in a$, if we integrate R to obtain the time-l map ϕ_l, then $D_p\phi_l : \xi_p \to \xi_p$ is an isomorphism. We say that a is nondegenerate if 1 is not an eigenvalue of $D_p\phi_l$, and α is nondegenerate if all the Reeb orbits of R_α are nondegenerate. Hutchings defined a chain complex $ECC_*(Y, \xi, \alpha)$ for a nondegenerate contact form whose generators are $\{(a_i, n_i)\}$, where a_i is a Reeb orbit, n_i is a positive integer, and $n_i = 1$ if a_i is a hyperbolic Reeb orbit. (We will define hyperbolic Reeb orbits in chapter 3.) Such a multiset of Reeb orbits is an *orbit set*, and orbit sets satisfying the condition on hyperbolic Reeb orbits are called *admissible* orbit sets.

To define the differential of ECC_*, we consider pseudoholomorphic curves in the symplectization $\mathbb{R} \times Y$. First, choose a generic complex structure J_ξ on the contact bundle $\xi \to Y$, i.e. a bundle map $J_\xi : \xi \to \xi$ such that $J_\xi \circ J_\xi = -Id_\xi$. We can extend this to an $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R}_t \times Y$ by setting $J(\frac{\partial}{\partial t}) = R$. A pseudoholomorphic curve in $\mathbb{R} \times Y$ is a map $u : (C, j) \to (\mathbb{R} \times Y, J)$, where C is a punctured Riemann surface and u satisfies the Cauchy-Riemann equation $\overline{\partial}_J(u) := du + J \circ du \circ j = 0$. The simplest examples of pseudoholomorphic curves in $\mathbb{R}\times Y$ are *trivial cylinders.* A trivial cylinder over a Reeb orbit $a \subset Y$ is a map $u : \mathbb{R}_s \times S^1_t \to \mathbb{R}\times Y$ with $u(s,t) = (s, a(t))$. If $u : C \to \mathbb{R}\times Y$ is an arbitrary pseudoholomorphic curve in $\mathbb{R}\times Y$, at each of the punctures of the domain C, u is asymptotic to the trivial cylinder over some Reeb orbit, at either ∞ or $-\infty$. We call these positive or negative ends of u, respectively, and refer to the corresponding punctures of C as positive or negative punctures. A pseudoholomorphic curve $u : C \to \mathbb{R}\times Y$ thus yields two collections of *asymptotic Reeb orbits*, one at ∞, one at $-\infty$. If a_i is an embedded Reeb orbit, u has positive ends at some collection $a_i^{q_1^i}, ..., a_i^{q_{k_i}^i}$ of iterates of a_i. If we remember only the total multiplicity $\sum_{j=1}^k q_j$ of the positive ends at a for each embedded Reeb orbit a_i, we obtain an orbit set $\mathbf{a} := \{(a_i, \sum_j q_j^i)\}$. Similarly, we obtain an orbit set $\mathbf{b}$ from the negative ends of C, and we write $\partial C = \mathbf{a} - \mathbf{b}$.

If $\mathbf{a}$ is an admissible orbit set, we define the differential ∂ on ECC_* by $\partial\mathbf{a} = \sum_b \langle\partial\mathbf{a}, \mathbf{b}\rangle b$. The sum is taken over admissible orbit sets $\mathbf{b}$, and $\langle\partial\mathbf{a}, \mathbf{b}\rangle b$ is the signed count of J-holomorphic curves $u : C \to \mathbb{R}\times Y$ such that $I(C) = 1$ and $\partial C = \mathbf{a} - \mathbf{b}$. $I(C)$ is the *ECH index* of C, which is a topological quantity with the property that for generic J, if $I(C) = 1$, there are only finitely many curves C with $\partial C = \mathbf{a} - \mathbf{b}$, and C is the union of an embedded J-holomorphic curve C' and the branched cover of a union of trivial cylinders that do not intersect C'. We will define $I(C)$ in chapter 3. The embedded contact homology of (Y, ξ, α) is defined as $H_*(ECC_*, \partial)$.

ECH turns out to depend only on Y, and is isomorphic to versions of Heegaard Floer homology ([9], [4]) and monopole (Seiberg-Witten) Floer homology ([28]). Furthermore, since ECH is generated by Reeb orbits, it can yield existence results for Reeb orbits in terms of the topology of Y. The first such result is the dimension 3 case of the Weinstein conjecture: the Reeb vector field for any contact form on a closed contact manifold has a closed trajectory. [26]. Better lower bounds on the number of embedded Reeb orbits are known [16], and any computation of ECH for a specific manifold holds the promise of better bounds.[1]

1.2 ECH of circle bundles over surfaces

In this paper, we compute the embedded contact homology of nontrivial circle bundles Y over a Riemann surface Σ. Take the contact form α to be a connection form with nonvanishing curvature on $Y \to \Sigma$, viewed as a principal S^1 bundle. The contact structure $\xi = ker(\alpha)$, being an S^1-equivariant collection of horizontal 2-plane fields, is a connection on $Y \to \Sigma$. The Reeb vector field of α is the derivative $\frac{\partial}{\partial\theta}$ of the S^1 action on Y. Thus, any iterate of any fiber is a Reeb orbit.

Holomorphic curves in $\mathbb{R}\times Y$ have a natural interpretation as sections of line bundles. $\mathbb{R}\times Y$ may be viewed as an $\mathbb{R}\times S^1 \cong \mathbb{C}^*$ bundle over Σ. If we choose a complex structure j_Σ on Σ and lift it to ξ, it will be S^1-invariant. Extending the complex structure on ξ to an $\mathbb{R}\times S^1$-invariant almost complex structure on $\mathbb{R}\times Y$ by $J\frac{\partial}{\partial t} = R = \frac{\partial}{\partial\theta}$ as before,

[1] Note that unlike the analogous situation in Morse theory, where the number of critical points of a smooth function bounds the Betti numbers of a manifold because Morse homology is generated by critical points, ECH is generated by *collections* of Reeb orbits, so bounds are less immediate.

we see that the projection $\pi : \mathbb{R} \times Y \to \Sigma$ is holomorphic and its $\mathbb{C}^*$ fibers have complex structures. If $u : (C, j) \to (\mathbb{R} \times Y, J)$ is a J-holomorphic curve, then at a puncture p of C, u approaches a trivial cylinder over some Reeb orbit a of Y. But since the Reeb orbit a is (an iterate of) a fiber $P^{-1}(q)$ of $P : Y \to \Sigma$, we can extend the map $\pi \circ u : C \to \Sigma$ by mapping p to q. Filling in each puncture in this manner, we obtain a closed Riemann surface $\overline{C} \supset C$, and a holomorphic map $\overline{u} : (\overline{C}, j) \to (\Sigma, j_\Sigma)$. We can pull back the bundle $\mathbb{R} \times Y \to \Sigma$ by $\overline{u}$ to obtain a $\mathbb{C}^*$ bundle $\overline{u}^*(\mathbb{R} \times Y) \to \overline{C}$. J-holomorphic curves $u : C \to \mathbb{R} \times Y$ correspond exactly to meromorphic sections of $\overline{u}^*(\mathbb{R} \times Y) \to \overline{C}$. Positive and negative ends of u correspond to zeros and poles of the section. Finding J-holomorphic curves in $\mathbb{R} \times Y$ with given asymptotic orbit sets is thus closely related to finding holomorphic sections of line bundles on closed Riemann surfaces with zeros and poles of specified degrees.

The results of this paper will not depend upon this interpretation of J-holomorphic curves in $\mathbb{R} \times Y$. However, there is a map $U : ECH_* \to ECH_{*-2}$, which is induced by the chain map on ECC_* which counts $I(C) = 2$ curves passing through a generic marked point in $\mathbb{R} \times Y$. U can be (at least partially) computed using the methods of this paper, and these computations use the complex line bundle structure on $\mathbb{R} \times Y$.

The Reeb orbits of the contact form described above are parameterized by Σ. However, these orbits are degenerate. (Nondegenerate Reeb orbits are isolated, in particular.) To compute ECH we must perturb α to a nondegenerate contact form $\alpha' = f\alpha$, where $f : Y \to \mathbb{R}$ is a nonvanishing smooth function. If we choose f to be the pullback of a perfect Morse function $\overline{f} : \Sigma \to \mathbb{R}$ on the base, α' and the almost complex structure J defined by $J\frac{\partial}{\partial t} = R_{\alpha'}$ are still S^1-invariant. This will make our computations tractable. The Reeb orbits of α' are (iterates of) the fibers above the critical points of f, along with some "long" Reeb orbits. By choosing f sufficiently close to 1, we can force the length of the orbits which are not iterates of a fiber to be arbitrarily long. For nontrivial circle bundles, ECH turns out to be relatively $\mathbb{Z}$ graded, and generators of ECC containing sufficiently long Reeb orbits have a degree which may be bounded below in terms of the perturbation. Thus, to compute ECH up to any finite degree N, we can choose f so close to 1 that only Reeb orbits over critical points of $\overline{f}$ appear any ECC generator of degree $\leq N$. By taking a direct limit as f approaches 1, we can compute ECH using only these "fiber Reeb orbits".

Furthermore, because J is S^1-invariant, the moduli space $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ of J-holomorphic curves C with $I(C) = 1$ and $\partial C = \mathbf{a} - \mathbf{b}$ inherits an S^1 action, which is locally free so long as the curves $C \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ themselves are not S^1-invariant. (i.e. as long as C is not a branched cover of a union of cylinders, which we treat separately.) If $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ and hence $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)/\mathbb{R}$ (its quotient with respect to $\mathbb{R}$-translation) have an S^1 action, then both of them, if nonempty, must have dimension ≥ 1. But for a generic J, $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ is 1-dimensional, so $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)/\mathbb{R}$ is 0-dimensional. Thus, $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ is empty, $\partial = 0$, and $ECC_*(Y, \xi, \alpha) \cong ECH_*(Y, \xi)$. Before examining the question of genericity, we state the main result of this paper:

Theorem 1.2.1 *Let Y be the total space of a nontrivial circle bundle over a Riemann surface Σ with contact structure ξ coming from a connection on the principal bundle $Y \to \Sigma$. Then*

$$ECH_*(Y, \xi) \cong \wedge^*(H_*(\Sigma, \mathbb{Z}))$$

.

Remark 1.2.2 ECH has a $\mathbb{Z}_2$ grading, and the isomorphism stated above is an isomorphism of $\mathbb{Z}_2$-graded Abelian groups. ECH splits as a direct sum indexed by $H_1(Y)$, and each summand has a relative $\mathbb{Z}$ grading refining the absolute $\mathbb{Z}_2$ grading; we explain this in chapter 3 and give a refined version of the theorem in chapter 5. The splitting over $H_1(Y)$ distinguishes between circle bundles of different Euler classes over Riemann surfaces of the same genus. Embedded hyperbolic Reeb orbits correspond to index 1 critical points of the perfect Morse function $f : \Sigma \to \mathbb{R}$ and hence to generators of $H_1(\Sigma)$, and elliptic orbits correspond to generators of $H_0(\Sigma)$ and $H_2(\Sigma)$. So, a generator of ECC corresponds to a multiset of generators of $H_*(\Sigma)$. We get the exterior algebra on $H_*(\Sigma)$ because hyperbolic orbits can appear with multiplicity at most one and elliptic orbits can appear with any multiplicity.

Remark 1.2.3 The Seiberg-Witten Floer homology of Seifert fibered 3-manifolds, including nontrivial circle bundles over Riemann surfaces, is implicit in [22], which was written before a proper definition of Seiberg-Witten Floer homology had been given. The Heegard Floer homology with $\mathbb{Z}_2$ coefficients for these three-manifolds, including the U map, is computed in [23]. We expect that the methods used in this paper will suffice to compute the ECH of Seifert fibered manifolds.

1.3 Regularity and domain-dependent almost complex structures

The discussion preceding the theorem was predicated on the assumption that J was regular, i.e. that the linearization D of the Cauchy-Riemann operator $\overline{\partial}_J$ (which we take to be a Fredholm map between appropriate Banach spaces) is surjective, which implies that the zero set of $\overline{\partial}_J$, the moduli space of J-holomorphic curves, is a smooth manifold whose dimension equals the Fredholm index of the linearized operator. This was necessary to argue that an index 1 moduli space for an S^1-invariant J would have the "wrong" dimension.

However, we cannot in general expect to find a J which is simultaneously S^1-invariant and regular. If we compose the J-holomorphic map $u : C \to \mathbb{R} \times Y$ with the projection $\pi : \mathbb{R} \times Y \to \Sigma$, we obtain a map $\pi \circ u : C \to \Sigma$ which turns out to have a well-defined nonnegative degree d, because $\pi \circ u$ extends to a map of closed surfaces If $d = 0$ or 1, we can find a regular S^1 invariant J. When $d > 1$, the projection of $u : C \to \mathbb{R} \times Y \to Y$ has intersection number d with a given S^1-orbit, and hence at least d intersections (some of them possibly at the same point, which occurs when the map u is a nontrivial branched covering of its image). The complex structure cannot be independently perturbed at those d points by an S^1-invariant perturbation, which is the key step in the standard proofs of transversality. If we drop the requirement that J be S^1-invariant, it is known that there always exists a regular J (because we can do the perturbations independently), but we lose all hope of actually understanding the moduli spaces.

To resolve this, we employ a more general notion of complex structure in which we let $J(p, q) \in Aut(T_q(\mathbb{R} \times Y))$ be a family of almost complex structures parameterized by $p \in C$. A J-holomorphic map $u : (C, j) \to (\mathbb{R} \times Y, J)$ for such a domain-dependent J satisfies a Cauchy-Riemann equation of the form

$$du_p + J(p, u(p)) \circ du_p \circ j_p = 0 \ (\forall p \in C)$$

Here, $J = J(p, u(p)) \in Aut(T_{u(p)}(\mathbb{R} \times Y))$ satisfies $J \circ J = -Id$. In the usual Cauchy-Riemann equation, J would depend only on $u(p)$, not on p. If $p, p' \in C$ map by u to the same S^1 orbit in Y, we can perturb J independently at $u(p)$ and $u(p')$ while preserving its S^1-invariance. We use this to find regular S^1-invariant *domain-dependent almost complex structures* (DDACS). In this setting, the argument sketched before the theorem works, and we can conclude that for a generic S^1-invariant DDACS J, moduli spaces of J-holomorphic curves C with $I(C) = 1$ are empty.

However, ECH is defined for a generic (domain-independent) almost complex structure, so to conclude anything about ECH, it is necessary to relate the moduli spaces of $I(C) = 1$ J_0-holomorphic curves with $\partial C = \mathbf{a} - \mathbf{b}$, for a regular S^1-invariant DDACS J_0, to the moduli space of J_1-holomorphic curves with the same $I(C)$ and the same asymptotic orbit sets, for J_1 a regular (S^1-dependent, domain-independent) almost complex structure.

To compare these moduli spaces, we take a generic one-parameter family J_t of domain-dependent almost-complex structures interpolating between J_0 and J_1. For each J_t, consider the moduli space $\mathscr{M}_t \equiv \mathscr{M}(\mathbf{a}, \mathbf{b}, J_t)$ of J_t holomorphic curves C with $\partial C = \mathbf{a} - \mathbf{b}$. $\mathscr{M}_t$ is transversely cut out except at finite number of times $t_i \in [0, 1]$. At such a nonregular J_{t_i}, ∂ can change by either the creation or destruction of a pair of holomorphic curves of opposite sign, or by a handleslide. In the former case, the signed numbers of curves in $\mathscr{M}_{t_i - \epsilon}$ and $\mathscr{M}_{t_i + \epsilon}$ are the same. In general, at a handleslide (at which a sequence of Fredholm index 1 curves C_t breaks into a "holomorphic building" consisting of an index 0 curve, an index 1 curve C', and some "connectors", which are Fredholm index 0 branched coverings of trivial cylinders), ∂ can change. By considering a local version of the (relative) adjunction formula from [10], we can show that connectors cannot occur at the top or bottom of the holomorphic building, but only between the index 0 curve and the index 1 curve. This means that a gluing theorem from [14] and [15] applies, and it describes the change in ∂ for the buildings that arise here. Namely, ∂ changes by a multiple of the signed number of elements of $\mathscr{M}(\mathbf{a}', \mathbf{b}', J_{t_i})$, where $\partial C' = \mathbf{a}' - \mathbf{b}'$. At J_0, the moduli spaces are empty so ∂ is 0. We show by an inductive argument that $\#\mathscr{M}(\mathbf{a}', \mathbf{b}', J_{t_i})$ is zero, and hence ∂ does not change at a handleslide. The induction depends on the fact that when a sequence of curves C_i degenerates in the limit to a holomorphic building of the type described above, the degree of the index 1 curve in the building is less than the common degree of the C_i.

1.4 Prospects

As is common for theories involving pseudoholomorphic curves, explicit ECH computations are quite difficult to carry out; besides this paper, the only examples done directly (i.e. not via the isomorphism with Heegaard Floer or monopole Floer homology) are S^3, T^3 and $S^1 \times S^2$ in [13], which was generalized in [19] to many T^2-bundles over S^1. One reason to do computations in the ECH framework is that certain structures common to all

3-manifold Floer homologies are more transparent in ECH. For instance, a choice of contact structure on a 3-manifold specifies an element in the Floer homology of the manifold. The definition of this "contact element" is complicated in the other two Floer theories, but in ECH, it is just the homology class of the empty collection of Reeb orbits. Another is that other structures defined using ECH may not have obvious counterparts in the other two theories, and a direct calculation of ECH is indispensible in understanding them. For instance, Hutchings [18] has defined "ECH capacities" which give embedding obstructions for certain exact symplectic 4-manifolds $(X, d\alpha)$ bounding contact 3-manifolds (Y, α); these invariants depend nontrivially on the contact form α, so it is not clear if or how these could be recovered inside Heegaard or monopole Floer homology.

Thus, we can hope that generalizing and using this work will help illuminate the structure of ECH. A natural extension of these results would be to Seifert fibered spaces. Also, it would be useful for applications to compute the "U map" for circle bundles. The author expects that the methods will suffice for both. Furthermore, we might hope that analyzing this ECH computation may result in a better lower bound on the minimal number of embedded Reeb orbits for a contact form on a nontrivial circle bundle over a surface. ([16] proves that there are at least 3 embedded Reeb orbits for any Y that is not a lens space).

1.5 Outline

The plan of this paper is as follows: in section 2, we compute and collect the topological and geometric facts we need about our contact 3-manifolds Y, and discuss complex structures and pseudoholomorphic curves in symplectizations. We also introduce the various kinds of perturbations which we will use. In section 3, we compute $I(C)$ for holomorphic curves $C \to \mathbb{R} \times Y$ and define the degree $d(C)$. Section 4 works out ECH in the case where $\Sigma \cong S^2$, which is much simpler than $g(\Sigma) > 0$. In section 5, we begin the computation of ECH for higher-genus Σ, and discuss holomorphic cylinders (which must be treated separately from holomorphic curves with stable domains). In section 6, we define domain-dependent almost complex structures and prove regularity for generic S^1-invariant DDACS. In section 7, we compare moduli spaces for domain-dependent and -independent almost complex structures, and ultimately show that $\partial = 0$.

Chapter 2

Nontrivial circle bundles over Riemann surfaces

2.1 Circle bundles

In this section we introduce the three-manifolds Y to be considered and collect some facts about their topology.

Let Y be the total space of a principal circle bundle $S^1 \to Y \to \Sigma$ over a closed oriented 2-manifold Σ, with negative Euler class $e(Y) = -e < 0$. In particular, Y is a nontrivial bundle. (We will never consider $S^1 \times \Sigma$, so "circle bundle" will be synonymous with "nontrivial circle bundle of negative Euler class" in this paper.) We warn the reader that π will generally denote the map $Y \to \Sigma$, in the course of this paper we will consider myriad auxiliary bundles, and π will often be hijacked to denote the relevant projection in context.

$H_*(Y, \mathbb{Z})$ may be computed via the Leray-Serre spectral sequence. The E_2 page is given by $H_*(\Sigma, H_*(S^1; \mathbb{Z}))$, and the only E_2 differential between nonzero groups is $d_2^e : H_2(\Sigma, H_0(S^1, \mathbb{Z})) \cong \mathbb{Z} \to H_0(\Sigma, H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}$. d_2^e is given by multiplication by $e(Y)$, so, $\frac{ker(d_2^e)}{im(d_2^e)} \cong \mathbb{Z}/e$. Thus, $H_1(Y) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/e$ and $H_2(Y) \cong \mathbb{Z}^{2g}$.

The $\mathbb{Z}^{2g}$ factors of $H_1(Y)$ and $H_2(Y)$ come from $H_1(\Sigma)$: the bundle Y restricted to any loop in Σ is orientable and therefore trivial, so given a loop in Σ whose fundamental class is a generator of $H_1(\Sigma)$, the fundamental class of a section of the bundle over a loop yields generator of $H_1(\Sigma)$ and the total space of the bundle yields a generator of $H_2(Y)$. The torsion in $H_1(Y)$ comes from the fibers of the bundle: remove a small disc D from Σ and trivialize Y over $\Sigma - D$. Take a constant nonvanishing section with respect to this trivialization, restricted to ∂D, and compare it to a trivialization over D. As you traverse ∂D, the section over $\Sigma - D$, winds $e(Y) = -e$ times around the fiber with respect to the trivialization over D. Thus, a path wrapping e around a fiber bounds a 2-chain, and the fundamental class $[F]$ of a fiber $F \cong S^1$ generates the $\mathbb{Z}/e$ factor.

The map induced on H_1 by the projection $Y \to \Sigma$ is onto, and its kernel is the $\mathbb{Z}/e$ factor. The map induced on H_2 is zero, as the generators described above map to circles. The induced map on H^2 is similarly zero, which we will use [in section] to show that the contact structure on Y we will consider is trivial as a vector bundle.

2.2 Contact three-manifolds and holomorphic curves in symplectizations

We recall the basic definitions of three-dimensional contact geometry. Let Y be a closed orientable 3-manifold. A (co-orientable) *contact structure* on Y is a real 2-plane bundle $\xi \subset TY$ which can be defined as the kernel of a *contact 1-form* $\alpha \in \Omega^1(Y)$ satisfying $\alpha_p \wedge d\alpha_p > 0$ for every $p \in Y$. α determines the *Reeb vector field* $R \equiv R_\alpha$ by $R \in ker(d\alpha)$ and $\alpha(R) = 1$. The closed trajectories of R will be referred to as *Reeb orbits*. We will identify Reeb orbits that differ by parameterization, and sometimes use the term "Reeb orbit" to refer to the image of a Reeb orbit. (But note that a cover of a Reeb orbit is again a Reeb orbit; the "multiplicity" of a Reeb orbit will mean the multiplicity with which a Reeb orbit covers the underlying embedded Reeb orbit.) The vector field R is transverse to ξ, yielding the splitting $TY \cong \langle R\rangle \oplus \xi$. Note that $\alpha, \alpha' \in \Omega^1(Y)$ determine the same contact structure ξ iff $\alpha = f\alpha'$ for some nonvanishing $f : Y \to \mathbb{R}$, but R_α and $R_{\alpha'}$, and the their sets of Reeb orbits, may be quite different.

On the noncompact 4-manifold $\mathbb{R}_t \times Y$, the closed 2-form $\omega := d(e^t\alpha)$ satisfies $\omega \wedge \omega > 0$ and so determines a *symplectic* structure known as the *symplectization* of Y. We can pull ξ, α, and R back to $\mathbb{R} \times Y$ by the projection to Y, and we'll use the same notation for the pullbacks. Thus, $T\mathbb{R} \times Y \cong \langle \frac{\partial}{\partial t}\rangle \oplus \langle R\rangle \oplus \xi$. Given an almost complex structure j on the bundle $\xi \to Y$, we can define an $\mathbb{R}$-invariant *cylindrical almost complex structure* J on $\mathbb{R} \times Y$ by letting $J|_\xi = j$ and setting $J\frac{\partial}{\partial t} = R$. $d\alpha$ gives ξ the structure of a symplectic vector bundle, and we say that j, and hence J, are $d\alpha$-compatible if $d\alpha(v, jv) \geq 0$ for any $v \in \xi_p$. We will only consider compatible almost complex structures.

A *J-holomorphic map* (or *(pseudo)holomorphic map* or *curve*) is a map

$$u : (C, j_C) \to (\mathbb{R} \times Y, J)$$

where C is a punctured Riemann surface with complex structure j_C, and u satisfies $du \circ j_C = J \circ du$. C may be disconnected, and we will always assume that u is nonconstant on each of its components.[1] We will always write $\overline{C}$ for the closed Riemann surface from which C is obtained by deleting a finite number of punctures. Note that $J\frac{\partial}{\partial t} = R$ implies that if $\gamma \subset Y$ is a Reeb orbit, the *trivial cylinder* or *orbit cylinder* $\mathbb{R} \times \gamma$ is J-holomorphic. The boundary conditions on the holomorphic curves we will consider imply that they have finite energy ($\int_C u^*d\alpha < \infty$)); by a foundational theorem from [8], for any such curve, at each puncture of C, u is asymptotic to the trivial cylinder $\mathbb{R} \times \gamma$ for some Reeb orbit γ. If the $\pi_{\mathbb{R}} \circ u(p) \to \infty$ as $p \in C$ approaches a puncture p_0, we call p_0 a positive puncture and the end a "positive" or "outgoing" end; if $\pi_{\mathbb{R}} \circ u(p) \to \infty$, p_0 is a negative puncture and the end is "negative" or "incoming". We will speak of curves as going from their outgoing ends (each component must have at least one, by applying the maximum principle to $\pi_{\mathbb{R}} \circ u$) to their incoming ends (if there are any). We often will implicitly identify $\mathbb{R} \times Y$ with $(0, 1) \times Y$ and compactify it to $[0, 1] \times Y$, and view a holomorphic map u as a cobordism

[1] In chapter 6, we show that the curves under consideration can never have constant components.

from a collection of Reeb orbits in $\{1\}\times Y$ (corresponding to the positive ends) to a (possibly empty) collection of Reeb orbits in $\{0\}\times Y$. Each Reeb orbit is a cover of a fiber, so an end of u at a Reeb orbit a determines an embedded Reeb orbit and a (positive) covering multiplicity. For a holomorphic map $u : C \to \mathbb{R}\times Y$ and an embedded Reeb orbit a to which it is asymptotic at some positive puncture, we form the collection m^j of the covering multiplicities (with repitition) of all the ends asymptotic to a at ∞, which we view as a partition of the positive integer $\sum_j m^j$. If we consider all the (fiber) Reeb orbits a_i, each with total multiplicity $m_i = \sum m_i^j$ (where m_i^j is the multiplicity of the jth end asymptotic to a_i), then we can form the total homology class $\sum m_i[a_i] \in H_1(Y;\mathbb{Z})$. If u is similarly asymptotic at $-\infty$ to the Reeb orbit b_j with total multiplicity n_j, we can similarly form $\sum_j n_j[b_j]$. In $[0,1]\times Y$, $\partial u(C) = \sum m_i a_i - \sum n_j b_j \equiv \mathbf{a}-\mathbf{b}$. This yields the equality $[\mathbf{a}] \equiv \sum m_i[a_i] = \sum n_j[b_j] \equiv [\mathbf{b}]$ of total homology classes for any two collections of Reeb orbits which are joined by a holomorphic curve.

2.3 A contact structure on nontrivial circle bundles

$S^1 \to Y \to \Sigma$ may be taken to be a principal S^1-bundle of Euler class $-e$. Given a connection 1-form $\alpha \in \Omega^1(Y)$ on $Y \to \Sigma$, it has curvature given by $\int_Y(\alpha\wedge d\alpha = -2\pi(-e) = 2\pi e$. If we choose α to be pointwise non-zero, then α will be a contact 1-form with Reeb vector field $\frac{\partial}{\partial\theta}$, where $\frac{\partial}{\partial\theta}$ is the derivative of the S^1 action on Y. Such an α is S^1-invariant, and the fibers are Reeb orbits of length 2π, and $d\alpha$ is the pullback $\pi^*(\omega)$ of some symplectic form $\omega \in \Omega^2(\Sigma)$. The n-fold iterate of a fiber is also a Reeb orbit for any n, so the collection of all Reeb orbits is parameterized by $\cup_n\Sigma$, a smooth 2-manifold.

The contact structure $\xi = ker(\alpha)$ consists exactly of the S^1-equivariant horizontal subspaces determined by the connection. Thus, a bundle map

$$\begin{array}{ccc} \xi & \longrightarrow & T\Sigma \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & \Sigma \end{array}$$

is obtained by restricting the S^1-equivariant map $D\pi : TY \to T\Sigma$ to ξ; i.e. the subbundle $\xi \subset TY$ is isomorphic to the pullback bundle $\pi^*T\Sigma$. If we choose an ω-compatible complex structure j_Σ on Σ and pull it back by this diagram, we obtain an S^1-invariant, $d\alpha$-compatible, complex structure j on ξ, and hence an S^1-invariant almost complex structure J on $T\mathbb{R}\times Y$ with respect to which the projection $\pi : \mathbb{R}\times Y \to \Sigma$ is (J, j_Σ)-holomorphic. If $u : (C, j_C) \to (\mathbb{R}\times Y, J)$ is a J-holomorphic map, the composition $\pi\circ u$ is a holomorphic map of Riemann surfaces. Since at any puncture p of C, u approaches some Reeb orbit $\gamma \subset Y$, and any Reeb orbit is a cover of a fiber of $Y \to \Sigma$, $\pi\circ u$ may be extended over p. If $\overline{C}$ is a closed Riemann surface such that we can obtain C from $\overline{C}$ by removing punctures, we thus obtain a continuous extension of $\pi\circ u$ to $\overline{u} : \overline{C} \to \Sigma$ which is a holomorphic map of Riemann surfaces. If u has an end at a Reeb orbit which is a k-fold of the embedded Reeb orbit $\pi^{-1}(p)$, then $\overline{u}$ has a an order k branch point at p. We define d to be the degree of the map $\overline{u}$. If we consider $\mathbb{R}\times Y$ as an $\mathbb{R}\times S^1 \cong \mathbb{C}^*$ bundle over Σ, we can pull it back to the $\mathbb{C}^*$ bundle $\overline{u}^*\mathbb{R}\times Y$ over $\overline{C}$, and u corresponds to a meromorphic order d

multisection of $\overline{u}^*T(\mathbb{R}\times Y)$, whose zeros and poles correspond to negative and positive ends of u. The number of zeroes minus the number of poles (both counted with multiplicity) is given by the first Chern class of the bundle; since $c_1(\mathbb{R}\times Y \to \Sigma$ is $-e$, by naturality we have $c_1(\overline{u}^*\mathbb{R}\times Y = d(-e) < 0$. Thus, a holomorphic curve u with n negative ends (with multiplicity) has $m = de + n$ positive ends.

This interpretation of holomorphic curves as meromorphic multisections of line bundles is a useful heuristic, but since we will have to perturb this picture, we do not use it directly in this paper. However, it will be used in future work in which the ECH "U map" is computed.

2.4 Perturbations

The contact form described above has a two-dimensional (Morse-Bott) family of embedded Reeb orbits parameterized by Σ. However, ECH is defined for contact forms with isolated Reeb orbits. More precisely, if γ is a Reeb orbit of length T, $p \in \gamma$, and ϕ_T is the time-T map determined by R, then $D\phi_T : \xi_p \to \xi_p$, and we say that γ is nondegenerate if 1 is not an eigenvalue of $D\phi_T$. α is said to be nondegenerate if all the Reeb orbits (including iterates) of R_α are nondegenerate.

So, we perturb α to $\alpha' = f\alpha$, where $f = e^g$ is the pullback by $\pi : Y \to \Sigma$ of a perfect Morse-Smale function $\overline{f} = e^{\overline{g}} : \Sigma \to \mathbb{R}$. By construction, α' and hence $R_{\alpha'}$ are S^1 invariant. T

The new $R_{\alpha'}$ is given by $R'_p = \frac{1}{f}\frac{\partial}{\partial\theta} + \frac{1}{f^2}X_f$, where X_f is the horizontal lift to ξ of the Hamiltonian vector field $X_{\overline{f}}$ on (Σ,ω). $(X_f)_p = 0$ iff $q = \pi(p)$ is a critical point of $\overline{f}$. Thus, on the fiber over a critical point $\pi^{-1}(q) \subset Y$, $R_{\alpha'} = \frac{1}{f(q)}\frac{\partial}{\partial\theta}$–the fiber is a closed orbit of R'_α. If $d\overline{f}_q \neq 0$, $\pi^{-1}(q)$ is no longer a Reeb orbit (because $\overline{f}$ is Morse and $R_{\alpha'}$ is S^1-invariant), so these fiber Reeb orbits are isolated. R' may have other closed trajectories, but they are long compared to the fibers; by choosing an arbitrarily small perturbation, we can bound below the shortest Reeb orbit that is not a fiber. For the manifolds under consideration, the ECH differential is action decreasing, we can below bound the action of any Reeb orbits that can contribute to a given ECH chain group. We can thus compute ECH arbitrarily far by choosing a sufficiently small perturbation of the form described above, and the entire ECH will be the direct limit as the peturbations shrink to zero. Thus, to compute ECH, we may ignore Reeb orbits which are not fibers (or covers thereof).

From now on, $R = R'_\alpha$, and the Reeb vector field for the original Morse-Bott contact form α will be denoted $\frac{\partial}{\partial\theta}$. If we need to take a smaller perturbation of α, it will be of the form $\epsilon \cdot e^g\alpha$, $\epsilon > 0$, so that the fiber Reeb orbits under consideration will be the same in all situations.

If, as in the Morse-Bott case, we choose a compatible almost complex structure j on ξ by pulling back a complex structure j_Σ on Σ, we can use the same recipe ($J\frac{\partial}{\partial t} = R$) to extend it to an S^1-invariant cylindrical almost complex structure J on $\mathbb{R}\times Y$. For this J, the projection map $\mathbb{R}\times Y \to \Sigma$ is not (J, j_Σ)-holomorphic (for instance, $\frac{\partial}{\partial t} \mapsto 0$ but $J\frac{\partial}{\partial t}_p = R \mapsto \frac{1}{f^2}X_f$, which is nonzero unless $\pi(p) \in \mathrm{crit}(\overline{f})$) nor, given a J holomorphic map $(C, j_C) \to (\mathbb{R}\times Y, J)$, is the composition $\pi \circ u$ (j_C, j_Σ)-holomorphic. However, $\pi\circ u$ may still be completed to a smooth map $\overline{u} : \overline{C} \to \Sigma$ as before, and its degree is still well-defined (as a map of closed oriented surfaces).

We can relate the degree to the number of positive and negative ends. Define the *$d\alpha$-energy* E of such a J-holomorphic map $u : (C, j) \to (\mathbb{R} \times Y, J)$ by $E(u) = \int_C u^* d\alpha'$. The integrand is pointwise non-negative because j_ξ is assumed to be $d\alpha$-compatible, and it is zero iff $Im(Du_p) = \langle \frac{\partial}{\partial t}, R \rangle$. These spaces coincide at every point iff u is an orbit cylinder, so we have $E(u) \geq 0$, with equality iff u is an orbit cylinder. If as before we view u as a map $C \to \mathbb{R} \times Y \cong (0, 1) \times Y$ and compactify the target to $[0, 1] \times Y$, and also compactify the domain to $\widehat{C}$ by replacing the punctures with circles, then u extends to a map $\widehat{u}$ which sends the boundary circles to the asymptotic Reeb orbits of their respective ends. By Stokes's theorem, $E(u) = \int_C u^*(d\alpha') = \int_{\partial \widehat{C}} u^* \alpha = 2\pi(m - n)$, where m and n are the total multiplicities of positive and negative Reeb orbits. But since $u^*(d\alpha') = u^*(\pi^*(\omega)) = (\pi \circ u)^*(\omega)$, $E(u) = d[\omega] = d(-2\pi(-e) = 2\pi ed$, so $m - n = ed$. Note that since all the Reeb orbits are (iterates of) fibers, equality of total homology classes of Reeb orbits at ∞ and $-\infty$ a priori implied that $m - n \equiv 0 \pmod{e}$.

Chapter 3

Computation of the ECH index

3.1 ECH

Let (Y, ξ, α) be a contact 3-manifold with nondegenerate contact form α, and let $(\mathbb{R} \times Y, d\alpha, J)$ be its symplectization with a generic choice of compatible cylindrical almost complex structure J. In this section, we will define the embedded contact homology $ECH(Y, \xi, \alpha, J)$.

The embedded contact homology is the homology of a chain complex generated by *admissible* collections of Reeb orbits. A finite multiset $\mathbf{a} = \{(a_i, m_i)\}$, where a_i is a Reeb orbit and m_i is a positive integer, is called an *orbit set*. ECC_* is the abelian group generated by orbit sets satisfying the following admissibility condition: for any $(a_i, m_i) \in \mathbf{a}$, $m_i = 1$ if a_i is a hyperbolic orbit (See section [3.5] for the definition). We will denote the total multiplicity $\sum m_i$ of an orbit set by $|\mathbf{a}|$ and the total homology class $\sum m_i[a_i]$ by $[\mathbf{a}]$. Note that for circle bundles, $[\mathbf{a}] = |\mathbf{a}| \bmod e \in \mathbb{Z}/e \subset H_1(Y)$.[1][2]

The differential $\partial\mathbf{a}$ is defined by a certain count of holomorphic curves $u : C \to \mathbb{R} \times Y$ whose positive ends asymptotic to the Reeb orbit a_i have total multiplicities m_i for each i. Namely,

$$\partial\mathbf{a} := \sum_{\{C:\partial C = \mathbf{a} - \mathbf{b}, I(C)=1\}} \mathbf{b}$$

, where $\mathbf{b}$ is a multiset of Reeb orbits and $I(C)$ is the "ECH index" of C. Note that if we translate a holomorphic curve $u : C \to \mathbb{R} \times Y$ in the $\mathbb{R}$ direction, we obtain a different holomorphic curve with the same asymptotics (unless C is an orbit cylinder); we are actually summing over $\mathbb{R}$-families of curves.

It is a difficult theorem that $\partial^2 = 0$ ([14],[15]). [3] We define $ECH_*(Y, \xi, \alpha) := H_*(ECH(Y, \xi, \alpha, \partial)$. This homology turns out to be independent of the auxiliary choice

[1]Note that the empty collection of Reeb orbits $\emptyset$ is an admissible orbit set. It is a cycle because every holomorphic curve has a positive end, and its homology class is the "contact element" in ECH.

[2]Besides the condition on the multiplicities of hyperbolic orbits, ECC orbit sets differ from the generators of the chain complexes of SFT-type theories in that ECH associates a single multiplicity to a given Reeb orbit, whereas SFT associates a multiplicity along with a partition of the multiplicity into positive integers.

[3]This is much simpler in SFT, whose differential preserves the multiplicities of the different ends asymptotic to a given Reeb orbit. Broken curves contributing to $\langle \partial^2_{ECH}\mathbf{a}, \mathbf{b} \rangle$ have matching total multiplicities but not matching partitions where they break, so a more difficult gluing theorem is required.

of J as well as of α. In fact, Taubes has shown ([28] and its four sequels) that ECH is a topological invariant by showing that it is isomorphic to (Seiberg-Witten) monopole Floer homology; it is via this isomorphism that J and α invariance are shown.

Before explaining $I(C)$, we note that since if $\mathbf{b}$ appears with nonzero coefficient in $\partial\mathbf{a}$, then there is some curve C with $\partial C = \mathbf{a} - \mathbf{b}$, so $[\mathbf{a}] = [\mathbf{b}]$. If for $h \in H_1(Y;\mathbb{Z})$, we let $ECC_*(Y, \xi, \alpha, J, h) = ECC_*(h)$ be generated by all admissible orbit sets $\mathbf{a}$ with $[\mathbf{a}] = h$, then ∂ respects the splitting $ECC_* = \bigoplus_{h \in H_1(Y)} ECC_*(h)$. We thus obtain homology groups $ECH_*(Y, \xi, J, h) = ECH_*(h)$. [4]

3.2 The ECH index

We now define the ECH index $I(C)$. Let $u : C \to \mathbb{R} \times Y$ be a J-holomorphic curve with $\partial C = \sum m_i a_i - \sum n_j b_j$. If a^k denotes a k-fold cover of the Reeb orbit a, then

$$I(C) = c_1^\tau(\xi) + Q^\tau(C, C) + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \mu^\tau(a_i^j) - \sum_{i=1}^{l} \sum_{j=1}^{n_i} \mu^\tau(b_i^j)$$

In this formula, c_1^τ is the relative first Chern class, Q^τ is the self-intersection number of C. and μ^τ is the Maslov index. Each term depends on a choice of trivialization τ of ξ over each embedded Reeb orbit, but their sum $I(C)$ depends only on the relative second homology class of C. We define each of these and compute them in the circle bundle case; more details and general properties can be found in [10] and [11]. Note that for the relative first Chern class and self-intersection terms, there is no need to perturb the contact form from the Morse-Bott α with which we began, but that μ is defined for non-degenerate Reeb orbits.

3.3 Trivializations

We can choose a convenient trivialization τ for ξ over each embedded Reeb orbit a using the S^1-action on Y. Namely, choose an isomorphism $T_{\pi(a)}\Sigma \cong \mathbb{R}^2$, and pull it back to an isomorphism $\xi_q \cong \mathbb{R}^2$ for every $q \in \pi^{-1}(p)$, viewing $\xi \subset TY$ as the horizontal subbundle. We will use this trivialization throughout the rest of the paper,and often drop τ from the notation for c_1, Q, μ, and w. [10]contains further discussion and a complete discussion of how all these quantities transform with respect to a change of trivialization, and we have mainly hewed to its notation.

3.4 Relative $H_2(Y)$

The ECH index $I(C)$ depends only on the ends of C and the relative second homology class of C. If we compactify $\mathbb{R} \times Y$ to $[0,1] \times Y$ and take the circle compactification $\widehat{C}$ of C so that it defines a cobordism between orbit sets $\mathbf{a}$ and $\mathbf{b}$ in $[0,1] \times Y$, then $H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$ is defined as the homology of the complex of 2-chains $C \in C_2([0,1] \times Y, \mathbb{Z})$

[4] This splitting over $H_1(Y)$ corresponds to the splitting of monopole Floer homology over $spin_c$ classes.

such that $\partial C = \mathbf{a} - \mathbf{b}$. H_2^{rel} is an $H_2(Y, \mathbb{Z})$-torsor. A holomorphic curve $C \to \mathbb{R} \times Y$ asymptotic to $\mathbf{a}$ at ∞ and $\mathbf{b}$ at $-\infty$ determines a class $[C] \in H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$ by taking the circle compactification of C.

It turns out that for circle bundles, $c_1^\tau el$, Q and hence the ECH index I are independent of the relatively homology class, and so the ECH index of a holomorphic curve C depends only on its ends. μ depends on the asymptotic Reeb orbits, but c_1^{rel} and Q will depend only on the total multiplicity of positive and negative ends.

[this is key and i need to use it somewhere...]

3.5 Relative first Chern class

$c_1^\tau(C)$ is the relative first Chern class of the bundle $u^*\xi$ over C, defined as follows: let $\widehat{C}$ be the circle compactification of C as before. Over each boundary component of $\widehat{C}$, choose a nonvanishing section s of ξ with winding number zero with respect to to the trivialization given by τ. Take any generic extension of s to a section of $u^*\xi$, and count the signed number of zeros.

$c_1^\tau(C) \equiv c_1(C)$ clearly depends only on the class $[C] \in H_2^{rel}$. If we have $Z, Z' \in H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$, then $c_1(Z) - c_1(Z') = \langle c_1(\xi), Z - Z' \rangle$, where $Z - Z' \in H_2(Y)$. For circle bundles, $c_1(\xi) = 0$. For, the bundle map

$$\begin{array}{ccc} \xi & \longrightarrow & T\Sigma \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & \Sigma \end{array}$$

implies that $c_1(\xi) = c_1(\pi^* T\Sigma)$. But the induced map $\pi^* : H^2(\Sigma) \to H^2(Y)$ is trivial. Thus, $c_1(C)$ depends only on the ends of C.

To compute relative c_1 for $u : C \to \mathbb{R} \times Y$ for a circle bundle Y, we use the above bundle map to obtain that $u^*\xi = u^*(\pi^* T\Sigma) = (\pi \circ u)^* T\Sigma$. For a Reeb orbit $a = \pi^{-1}(q)$, Choose a nonzero vector $v \in T_q\Sigma$, and lift it to $\xi|_a$ to get a nonvanishing S^1-invariant section. We can thus extend this section to a section of $\overline{u}^* T\Sigma$, which extends $u^*\xi$ to a vector field on $\overline{C}$. which does not vanish at the punctures. $c_1(C)$ equals the signed number of zeroes an extension of this section, which is $c_1(\overline{u}^* T\Sigma)$. By naturality, this is $dc_1(T\Sigma) = d\chi(\Sigma)$, where d is the degree of the map $\overline{u} : \overline{C} \to \Sigma$. Because the difference between the total multiplicities of positive ends and negative ends is de, $c_1(C)$ is determined by the number of positive and negative ends of C.

3.6 Relative self-intersection number

$Q^\tau(C, C) \equiv Q$ is the relative self-intersection number of C . To define it, consider two classes $Z, Z' \in H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$. Take two immersed representatives S, S' with $\partial S = \partial S' = \mathbf{a} - \mathbf{b}$ and $[S] = Z, [S'] = Z' \in H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$. We require that S, S' are embedded in the interior, and transverse to each other at any interior intersections and the boundary. Furthermore, require that the ends of S, S' singly cover Reeb orbits. (So that self-intersections of C do not "escape to infinity".) Finally, let the projections of S, S' to

$[0,1]$ and to rxy to be immersions near their boundaries, so that along the boundary Reeb orbits, they intersect ξ in distinct rays, which we require to have winding number zero with respect to τ. Then we define $Q(Z,Z')$ to be the signed number of interior intersections of S and S'.

$Q(Z,Z')$ is well-defined, and in general if $Z,Z',Z'' \in H_2^{rel}(Y,\mathbf{a},\mathbf{b})$, then $Q(Z,Z'')-Q(Z',Z'') = (Z-Z') \cap h$, where $h = [\mathbf{a}] = [\mathbf{b}] \in H_1(Y)$. For circle bundles, h is always e-torsion, so $Z - Z' \cap h$=0. Hence, Q is independent of relative homology class. We will exploit this to compute by choosing convenient representatives of $[C]$ to compute Q.

We first compute Q for the simplest S^3, and then generalize. View S^3 as the total space of the Hopf fibration $S^1 \to S^3 \to S^2$, with Euler class -1. (Recall that changing the sign of the Euler class reverses the orientation of the three-manifold, so we are on S^3 with orientation given by the volume form $\alpha \wedge d\alpha$.) It is well known that any two fibers form a Hopf link, with linking number 1. Equivalently, surfaces (here, they can be taken to be discs) bounding the two fibers have intersection number 1. A surface S bounding a fiber a carries the relative homology class $[S] = Z \in H_2^{rel}(Y,a,\emptyset)$. To compute $Q(Z,Z)$, we take two surfaces $D, D' \in Z$ bounding the fiber a and satisfying the conditions in the definition of Q. Glue D' to a small tube connecting a to a nearby fiber a' and intersecting neither D nor D', obtaining a new disc D'' bounding a'. Then $D \cap D' = D \cap D''$, so counting points with sign, $|D \cap D'| = |D \cap D''| = 1$.

We generalize:

Let $g(\Sigma)$ and e be arbitrary, and let Y be the bundle with base Σ and Euler class $-e < 0$. A Reeb orbit wrapping e times around a fiber $a = \pi^{-1}(p)$ is bounded by by a surface $S \subset \mathbb{R} \times Y$ whose projection $S_Y$to Y is a section of $Y \to \Sigma$ over $\Sigma - \{p\}$. If S' is similarly a section over $\Sigma - \{q\}$ that wraps e times around $b = \pi^{-1}(q)$, then $S \cap S' = e$, because the section S_Y has an order e intersection with the e-fold cover of the fiber b. A Reeb orbit wrapping de times around a, $d > 0$, is bounded by d disjoint sections of $Y \to \Sigma$ over $\Sigma - \{p\}$; call their union S, and let S' be the union of d sections over $\Sigma - \{q\}$. $S \cap S' = ed^2$ because in Y, each of the d sections in S intersects b with order de. As in the S^3 case, if we have $Z \in H_2(Y, de \cdot a, \emptyset)$, take S and S', and glue on ends to each so that the resulting surfaces satisfy the conditions in the definition of Q (in particular, so that they each bound have de simple ends at a), so that $Q(Z,Z) = ed^2$.

A general $Z = [C]$ arising from a holomorphic curve lies in $H_2(Y,\mathbf{a},\mathbf{b})$, where $\mathbf{b}$ has total multiplicity $n \geq 0$ and $\mathbf{a}$ therefore has total multiplicity $n + de$, $d \geq 0$. We can represent Z by a surface $S \cup S'$ consisting the union S of d surfaces which each bound e Reeb orbits in $\mathbf{a}$ (and have no negative boundary) and S', the union of n cylinders from an orbit of $\mathbf{a}$ to an orbit of $\mathbf{b}$. For simplicity in computing the number of intersections, , we can assume the ends of S are distinct fibers (not necessarily Reeb orbits) and that the cylinders are all trivial cylinders over fibers distinct ∂S; then as before, we glue on ends so that S and S' bound the correct Reeb orbits. By the previous computation, the self intersection of S is ed^2. S' clearly has self-intersection zero. $S \cap S' = nd$, since the n trivial cylinders has one intersection with each of the d composing S. Thus, $Q(C,C) = Q(S \cup S', S \cup S') = Q(S,S) + Q(S',S') + 2Q(S,S') = ed^2 + 2ne = \frac{(ed+n)^2 - n^2}{e}$.

To go from these intersection numbers to Q, recall that Q depends only on the number of orbits at each end. Given $\mathbf{a}$, $\mathbf{b}$, we can compute Q by taking $S \in [S] \in H_2(Y,\mathbf{a},\mathbf{b})$

which has only simply covered ends, and glue on a tube to each orbit

3.7 Conley-Zehnder Indices

The final term is the sum of Conley-Zehnder indices of certain iterates of the Reeb orbits at the positive end minus the corresponding sum at the negative end. Recall that a trivialization of ξ over a Reeb orbit α defines a path in $Sp_2(\mathbb{R})$. If the Reeb orbit is nondegenerate, this path it has a Maslov index, and the Conley-Zehnder index of a Reeb orbit a is defined to be this Maslov index and denoted $\mu_\tau = \mu$. A choice of trivialization determines a path from the identity to $\Phi(1)$. Recall that nondegenerate Reeb orbits are those whose monodromy $\Phi(1)$ do not have 1 as an eigenvalue, i.e. $det(\Phi(1) - I) \neq 0$. The locus $det(\Phi(1) - I) = 0$ is a co-oriented codimension one subvariety known as the Maslov cycle, and its complement has four components. Two are composed of elliptic matrices, which have a pair of conjugate complex eigenvalues. The component containing small positive rotations is the set of positive elliptic matrices, and the one containing small negative rotations is the set of negative elliptic matrices. The remaining components are the hyperbolic matrices, which have real eigenvalues–both positive for positive hyperbolic matrices, and both negative for negative hyperbolic matrices. Before perturbing the contact form, the path of symplectic matrices associated to a Reeb orbit in our trivialization τ is the constant path at the identity, and a small perturbation of the contact form ends on a matrix near the identity, which can only be elliptic or positive hyperbolic. Negative hyperbolic orbits will never appear in this paper, and "hyperbolic" will from now on always mean "negative hyperbolic". The Maslov index of a path from the origin to apoint not on the Maslov cycle and having only transverse intersections with is defined to be the signed number of intersections of the path with the Maslov cycle. The sign comes from comparing the orientation of the path and the co-orientation on the Maslov cycle. The initial intersection of the path with the origin counts for 1 paths pointing into the positive elliptic component, -1 for paths into the negative elliptic component, and 0 for those pointing into the (positive) hyperbolic component. [5].

The iteration of a Reeb orbit ending at A yields a path in Sp_2 ending at A^k, $k > 0$; in our case, if the perturbation is small enough, so that A is close to the identity, A^k and the path to it will lie inside the same component as A for $k < N$ (and in particular have no intersections with the Maslov cycle besides the identity), for some N which we can make as large as we like by choosing a sufficiently small perturbation. Thus, we will assume that $\mu(a^k) = \mu(a)$ for any k under consideration.

The Reeb orbits under consideration are fibers over critical points of the Morse function $g :\to \mathbb{R}$. The Maslov index of $\alpha = \pi^{-1}(p)$ is one less than the Morse index of the critical point p. Thus, for a perfect Morse function g, there will be one embedded Reeb orbit (the fiber above the maximum of g), denoted e_+, with Maslov index 1; $2g(\Sigma)$ embedded positive hyperbolic Reeb orbits (the fibers above the saddle points of g), which have Maslov index 1, denoted $h_1, ..., h_{2g}$; and one embedded negative elliptic Reeb orbits (the fiber above the minimum of g) with Maslov index -1, denoted e_-.

[5] For a full discussion and the definition and properties of the Maslov index for more general paths, see xxx

If a holomorphic curve C has j ends at a Reeb orbit a of multiplicities $i_1, ..., i_j$, with $i_1 + ... + i_j = m$, its contribution to the Maslov term is $\sum_{k=1}^{m} \mu(a^k) = \sum_{k=1}^{m} \mu(a) = m\mu(a)$. If $\mathbf{a} = \{(a_i, m_i)\}$ is an orbit set, we denote $\sum_{i=1}^{k} \sum_{j=1}^{m_i} \mu(a_i^j)$ by $\mu(\mathbf{a})$. The term appearing in $I(C)$ for $\partial C = \mathbf{a} - \mathbf{b}$ is therefore $\mu(\mathbf{a}) - \mu(\mathbf{b})$.

Note the contrast with the Maslov index term in the Fredholm index, which is $\sum_{k=1}^{j} \mu(a^{i_k})$ for each Reeb orbit a. The Fredholm index is sensitive to the separate multiplicities of the various ends at each Reeb orbit, whereas the ECH index sees only their total multiplicity.

3.8 ECH index depends only on ends; gradings

We conclude from the above discussion that the ECC of nontrivial circle bundles is generated over $\mathbb{Z}$ by $e_+^k e_-^l h_1^{i_1} ... h_{2g}^{i_{2g}}$, where $k, l \geq 0$ and $i_j \in \{0, 1\}$. ECC splits according to h, which is given by $k + l + \sum i_j (\text{mod } e)$. It will turn out that the ECH differential will vanish, so that $ECH \cong ECC$. Because they have the same Reeb orbits, the total embedded contact homologies for different e are naturally isomorphic, but the splitting according to $h \in H_1(Y)$ distinguishes 3-manifolds with different Euler classes.

Because $I(C)$ is independent of $[C] \in H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$ for circle bundles, $I(C)$ depends only on the multiplicity and type of Reeb orbits in the orbit sets $\mathbf{a}$ and $\mathbf{b}$. Therefore, unlike in the general case, there is a well-defined relative $\mathbb{Z}$ grading $I(\mathbf{a}, \mathbf{b})$ given by $I(S)$ for any surface S with $\partial S = \mathbf{a} - \mathbf{b}$, whether or not the class $[S] \in H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$ (or any other class in $H_2^{rel}(Y, \mathbf{a}, \mathbf{b})$, for that matter) is represented by a holomorphic curve. Thus, we can also speak of $d(\mathbf{a}, \mathbf{b}) = \frac{m-n}{e}$, where m and n are the total multiplicities of $\mathbf{a}$ and $\mathbf{b}$, and of $d(S) = d(\mathbf{a}, \mathbf{b})$ when $\partial S = \mathbf{a} - \mathbf{b}$. For $h = 0$, we set the degree of the canonical orbit set $[\emptyset]$ to 0, which gives an absolute $\mathbb{Z}$ grading on ECC by $|\mathbf{a}| \equiv I(\mathbf{a}, \emptyset)$. This is compatible with any grading on $ECC(Y, \xi, \alpha', J', 0)$ that also assigns the canonical element $[\emptyset]$ the degree zero. However, for $h \neq 0$, we a priori have only a *relative* $\mathbb{Z}$ grading, i.e. there is not an obvious way to compare the degree of orbit sets for α with orbit sets for some other α' (which will generally have a totally different set of Reeb orbits). [6]

3.9 Hyperbolic Reeb orbits and the parity of ECH

Proposition 3.9.1 *Let H be the total multiplicity of all hyperbolic orbits in the union of* $\mathbf{a}$ *and* $\mathbf{b}$*. Then* $I(\mathbf{a}, \mathbf{b}) \equiv H (\text{mod } 2)$

Proof.

Let $\partial S = \mathbf{a} - \mathbf{b}$. Then $I(\mathbf{a}, \mathbf{b}) = d(\mathbf{a}, \mathbf{b})\chi(\Sigma) + Q(S, S) + \mu(\mathbf{a}) - \mu(\mathbf{b})$.

$\chi(\Sigma) \equiv_2 0$. $\mu(a) \equiv_2 1$ iff a is elliptic, so $\mu(\mathbf{a}) - \mu(\mathbf{b}) \equiv_2 E$, where E is the total multiplicity of elliptic orbits in $\mathbf{a}$ and $\mathbf{b}$.

Let $d = d(\mathbf{a}, \mathbf{b})$. Then $|\mathbf{a}| - |\mathbf{b}| = de$, and $Q = \frac{|\mathbf{a}|^2 - |\mathbf{b}|^2}{e} = d(|\mathbf{a}| + |\mathbf{b}|) = d(de + 2|\mathbf{b}|) \equiv_2 d^2 e \equiv_2 de \equiv_2 |\mathbf{a}| + |\mathbf{b}| = H + E$. So, $I(C) \equiv_2 H + E + E \equiv_2 H$.

[6] Hutchings has defined absolute gradings for ECH using homotopy classes of 2-plane fields. The 2-plane field corresponding to $\mathbf{a}$ is defined by modifying ξ along the Reeb orbits of $\mathbf{a}$; see [11]. However, the relative grading here is simple enough that we will not consider absolute gradings.

□

Chapter 4

ECH Computations: circle bundles over S^2

In this section, we compute ECH for $\Sigma \cong S^2$. It is much simpler than the higher-genus case, because the differential vanishes for formal reasons, and domain-dependent perturbations are not necessary. [1] The case of $Y = S^3$ was previously computed by Hutchings ([12]). The ECH of all nontrivial circle bundles over S^2 is computed in essentially the same way as for S^3.

Choose a perfect Morse-Smale function $\overline{g}$ on S^2, i.e. a Morse function with one maximum, one minimum, and no other critical points. Pull $e^{\overline{g}}$ back to a function $\pi^*(e^{\overline{g}}) \equiv f : Y \to \mathbb{R}$. If α is the Morse-Bott contact form from the principal S^1 bundle, the contact form $f\alpha$ has two short embedded Reeb orbits e_+ and e_-. These are fibers above the maximum and minimum points, which are respectively positive and negative elliptic. Let $N \subset Y$ be the union of two small solid tori containing e_+ and e_-, and choose a complex structure on ξ by taking an S^1-invariant contact structure on $N|_\xi$ and extending generically over Y. Then, $ECC_*(Y)$ is generated by the orbit sets $e_+^k e_-^l$, with $k, l \geq 0$. Because all Reeb orbits are elliptic, $I(C)$ is even. Because ∂ counts holomorphic curves with $I(C) = 1$, it is automatically zero. Thus, $ECH(Y, \xi) \cong ECC_*(Y, \xi, f\alpha) \cong \mathbb{Z}\{e_+^a e_-^b | a, b \geq 0\}$.

For $e = -1$, Y is S^3 (-1 rather than 1 affects the orientation on S^3), and $H_1(S^3) = 0$. The degree of a generator is given by $|e_+^a e_-^b| = I(e_+^a e_-^b, \emptyset)$. For any surface S with boundary $e_+^a e_-^b$, we have $c_1(S) = d\chi(\Sigma) = 2(a+b)$ and $Q(S,S) = (a+b)^2$. The Conley-Zehnder term is $\sum_{k=1}^a CZ(e_+^k) + \sum_{k=1}^b CZ(e_-^k) = a - b$. Thus, $|e_+^a e_-^b| = 2(a+b) + (a+b)^2 + (a-b) = (a+b)^2 + 3a + b$. As observed by Hutchings, this polynomial, restricted to pairs of nonnegative integers, is a bijection $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to 2\mathbb{Z}_{\geq 0}$. So,

$$ECC_i(S^3) = ECH_i(S^3) = \begin{cases} \mathbb{Z}, \ i \text{ is even }, \\ 0, \ i \text{ is odd.} \end{cases}$$

For $e > 1$, $h = 0 \in H_2(Y)$, we have $ECH(Y) = \mathbb{Z}\{e_+^a e_-^b \ : \ e|(a+b)\}$. For any S with $\partial S = e_+^a e_-^b$, $d = deg(S) = (a+b)/e$. So $|e_+^a e_-^b| = d\chi(S^2) + ed^2 + (a-b) =$

[1] We note that the since once-and twice-punctured holomorphic spheres are not stable, we could not define domain-dependent perturbations for all the relevant curves in any case.

$2(a+b)/e+e(a+b)^2/e^2+(a-b)$. It is straightforward to verify that this expression, restricted to $a, b \in \mathbb{Z}_{\geq 0}$ with $e|(a+b)$, takes on every nonnegative even integer exactly once. The empty set of Reeb orbits has degree zero, so we have so $ECH_i(h=0) \cong \mathbb{Z}$ if i is even, and 0 if i is odd. Similarly, for any other $h = 1, 2, 3, ..., e-1 \in \mathbb{Z}/e \cong H_1$, $ECH_*(Y, \xi, f\alpha, h)$ is generated by $\{e_+^a e_-^b | a+b \cong h(\text{mod } e)\}$. e_-^h is the generator of minimal degree, for any even integer $2i$, there is a unique generator $e_+^a e_-^b$ with $d(e_+^a e_-^b, e_-^h) = 2i$, i.e. $ECH(h)$ is again a semi-infinite relatively $\mathbb{Z}$-graded sequence of alternating $\mathbb{Z}$ and 0. As before, this can be verified by calculating that the expression $2(a+b-h)/e+(a+b-h)^2/e+2h(a+b-h)/e+a-b-h$, restricted to $a, b \in \mathbb{Z}_{\geq 0}$ such that $a+b \cong h(\text{mod } e)$, attains every nonnegative even value exactly once. The total ECH is the direct sum of e such sequences of relatively $\mathbb{Z}$-graded alternating $\mathbb{Z}$ and 0, one of which is distinguished by containing $[\emptyset]$. We note that for every e, the total ECH is generated by $e_+^a e_-^b$; Y corresponding to different e are distinguished by the splittings they induce on ECH.

Chapter 5

ECH computations: circle bundles over surfaces of positive genus

If $g(\Sigma) \geq 1$, a perfect Morse function $\overline{f} = e^{\overline{g}}$ on Σ has $2g$ index 1 critical points, and thus (using the notation from section [n]), the Reeb vector field R of $f\alpha$ has $2g$ hyperbolic Reeb orbits which are fibers of $Y \to \Sigma$. As a result, nontrivial $I = 1$ holomorphic curves are possible. We will show that although such curves occur, we still have $\partial = 0$, so ECH$\cong$ ECC. This holds because of the S^1 action on Y; if we can choose a regular S^1-invariant almost-complex structure, then if a J-holomorphic curve $u : C \to \mathbb{R} \times Y$ is not a cylinder, and $0 \neq \theta \in S^1$, then the map $\theta \cdot u \neq u$ is also J-holomorphic. We thus obtain an S^1 action on the moduli spaces $\mathscr{M}(Y, \xi, \alpha, \mathbf{a}, \mathbf{b})$. (Note that the moduli space consists of (covers of) trivial cylinders iff $\mathbf{a} = \mathbf{b}$. "If" is clear; "only if" holds by applying Stokes's theorem to $u^* d\alpha$.)

However, it is generally impossible to choose a complex structure that is both regular and S^1-invariant. This splits into two cases:

5.1 Holomorphic cylinders

For $C \cong \mathbb{R} \times S^1$, a variation of the arguments in [25] show that for sufficiently small $\epsilon > 0$, the S^1-invariant complex structure determined by an S^1-invariant complex structure on ξ and $J\frac{\partial}{\partial t} = R_{e^{\epsilon g}\alpha}$ is regular. Also, as $\epsilon \to 0$, J-holomorphic curves $C \in \mathscr{M}(ka, kb)$ (where $a = \pi^{-1}(p)$ and $b = \pi^{-1}(q)$ are embedded Reeb orbits) approach broken trajectories of the following form (see [1]): a holomorphic cylinder from a to $a' = \pi^{-1}(p')$ (with respect to the complex structure determined by α), followed by a trajectory in Σ of the negative gradient vector field $-\nabla e^{\epsilon g}$, connecting p' to some p'', followed by a holomorphic cylinder connecting $\pi^{-1}(p'')$ to some other Reeb orbit $a''' = \pi^{-1}(p''')$, etc. Because the cylinders are complex with respect to J_α, the extension $\overline{u} : \overline{C} \to \Sigma$ is a holomorphic map of Riemann surfaces, and $\overline{C} \cong S^2$. Since $\Sigma \geq 1$, $\overline{u}$ is a constant map, and thus u must be a trivial cylinder.

The sum of the indices of the gradient flows equals $I(ka, kb)$. For the case relevant to computing ECH, $I = 1$, so the gradient flow part of the broken trajectory must in each case consist of a single Morse index 1 negative gradient flow. For ϵ sufficiently small and

$u \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J_\epsilon)$, $\pi \circ u$ lies in a C^0 neighborhood of that gradient flow, and due to this correspondence, the ECH differential $\langle \partial a, b \rangle$ agrees with the Morse differential $\langle \partial p, q \rangle = 0$, where the last equality holds because we chose a perfect Morse function.

5.2 Stable curves

The above argument is special to the case $C \cong \mathbb{R} \times S^1$. In the remaining cases, it is not possible to find a regular S^1-invariant J. We resolve this by broadening the notion of a complex structure to a domain-dependent almost complex structure. We define everything in the section, after giving a brief overview of the constructions and arguments here. So: in the Cauchy-Riemann equation for a map $u : C \to \mathbb{R} \times Y$, we allow J to depend on points in C, rather than on points $u(p)$ in the codomain. To define such structures, we need functions on C that are independent of reparameterization. This means that we should define functions on isomorphism classes of punctured Riemann surfaces, i.e. elements of $\mathscr{M}_{g,n}$, where we view the punctures as marked points. Since we want to vary these complex structures, we should define functions on all of $\mathscr{M}_{g,n}$, and since when a sequence of holomorphic maps in $\mathbb{R} \times Y$ has a broken curve as a limit–a building, as described in [2]–the limit of the domains is a nodal curve, so we should actually define our functions on $\overline{\mathscr{M}_{g,n}}$.

To have well-defined nontrivial functions on C, we should have C be stable (i.e. $Aut(C)$ is finite). All the remaining cases are stable: We have already taken care of $C \cong \mathbb{R} \times S^1$. All holomorphic curves $u : C \to \mathbb{R} \times Y$ have at least one positive end, so C must have at least one puncture, excluding closed curves of genus 0 and 1. The only remaining unstable domain is $\mathbb{C}$. But for such a map $u : \mathbb{C} \to \mathbb{R} \times Y$, which must be asymptotic to a unique Reeb orbit, we have $\overline{u} : S^2 \to \Sigma$, which must be nullhomotopic because $g(\Sigma) > 0$, and if we let the perturbation be sufficiently small, $\overline{u}$ must be close to a constant, which means that u is concentrated near its limiting Reeb orbit, and thus cannot bound it. (Otherwise, it would contradict the fact that faraway fibers are linked.)

So, we will be able to construct well-defined functions on C. These functions must respect the orbifold structure of $\mathscr{M}_{g,n}$, i.e. they must by invariant with regard to the (finite) symmetry groups at the orbifold points. While the derivative of such an invariant function will have a nontrivial kernel at an orbifold point, since the set of orbifold points have positive complex codimension, there is enough flexibility in the normal direction to define appropriate functions.

Chapter 6

Transversality for domain-dependent almost complex structures

In this section, we construct domain-dependent almost-complex structures, or DDACS, which are flexible enough to achieve regularity without sacrificing S^1 invariance. We thus obtain smooth moduli spaces of the expected dimension which have S^1 actions. However, ECH is defined only for a (domain-independent) almost complex structure. In the next section, we will compare the transversely-cut-out moduli spaces we construct here to domain-independent ones.[1]

The implementation of DDACS in this paper is modeled on that of [3] and [6], who studied genus 0 curves in certain symplectic manifolds of arbitrary dimension. The latter paper also studies a situation with S^1 symmetry (the symplectization of the mapping torus of a symplectic manifold with a Hamiltonian diffeomorphism), and obtains a conclusion analogous to ours. [3] and appendix D of [21] carefully describe the analytical details in the genus 0 case, and the extension to higher genus is immediate, so we refer the reader to their exposition, except for two new phenomena which we will explain. One is that higher genus curves can have finite nontrivial symmetry groups, so that the moduli space of curves, and therefore the moduli space of holomorphic map, are orbifolds. The second is that, even using domain-dependent almost complex structures, a nodal curve with a constant component of positive genus cannot be perturbed to achieve transversality near the curve. However, in dimension 4, index considerations prevent such curves from arising.

[1] The foundational results of ECH depend heavily on intersection positivity (such as the embeddedness theorem for curves with $I = 1$), and this does not hold for domain-dependent curves: at a transverse self-intersection of a J-holomorphic curve in M, the relevant tangent spaces are complex with respect to *different* complex structures on the tangent space to M , so their intersection need not be positive. It is not obvious that ECH should be defined for such complex structures (e.g. that $\partial^2 = 0$), or even if it can be defined, that it agrees with the ECH for an (domain-dependent) almost complex structure.

6.1 Domain-dependent almost complex structures

Let $\mathscr{J}'$ be the set of almost complex structures on the bundle $\xi \to Y$ which are compatible with the symplectic structure $d\alpha|_\xi$; we identify $J \in \mathscr{J}'$ with the $\mathbb{R}$-invariant almost complex structure on $T(\mathbb{R}\times Y)$ obtained by setting $J\frac{\partial}{\partial t} = R$. Let $(\mathscr{J}')^{S^1} \subset \mathscr{J}'$ denote the subset of S^1-invariant almost complex structures. Note that $j \leftrightarrow \pi^* j$ gives a correspondence between S^1-invariant complex structures on ξ and complex structures on $T\Sigma$. Fix a generic $J_0 \in \mathscr{J}^{S^1}$, and let $\{N_a\}$ be a disjoint collection of tubular neighborhoods of the Reeb orbits $\{a\}$, and let $N = \cup_a N_a$. We define $\mathscr{J} := \{J \in \mathscr{J}' : J_p = (J_0)_p \; \forall p \in N\}$, the subset of almost complex structures which agree with J_0 on N, and let $\mathscr{J}^{S^1} \subset \mathscr{J}$ consist of the S^1-invariant elements of J. These correspondend to almost complex structures on $T\Sigma$ which agree with a fixed $\pi_*(J_0)$ on $\pi(N)$.

Let $\mathscr{M}_{g,n}$ be the moduli space of stable Riemann surfaces (aka complex curves) of genus g with n (ordered) marked points. Recall that a curve $C \in \mathscr{M}_{g,n}$ is *stable* means its automorphism group is finite; this holds iff $2g + n \geq 3$. We will only ever speak of $\mathscr{M}_{g,n}$ for which this condition holds, and these spaces are smooth orbifolds. Let $\overline{\mathscr{M}_{g,n}}$ be the Deligne-Mumford compactification of $\mathscr{M}_{g,n}$ consisting of connected stable nodal Riemann surfaces with n marked points. That is, $(C, j) \in \overline{\mathscr{M}_{g,n}}$ consists of a disjoint union of $(C_i, j_i) \in \mathscr{M}_{g_i,n_i+m_i}$, where C_i is a stable curve, whose $n_i + m_i$ distinguished points consist of a subset of n_i of the marked points of C (so $\sum n_i = n$), with the induced ordering, and m_i nodes. Every node $p \in C_i$ is paired with some other node $p' \in C_{i'}$, with the stipulation that $i' \neq i$ for at least one of the nodes of each C_i. We thus obtain a connected singular surface by gluing p to p' for every pair $\{p, p'\}$ of nodes. Any sequence of curves $C^k \in \mathscr{M}_{g,n} \subset \overline{\mathscr{M}_{g,n}}$ has a subsequence whose limit is a nodal curve $C \in \overline{\mathscr{M}_{g,n}}$. Furthermore, if $p_i \in C^i$ is a marked point, a subsequence converges to some marked point $p \in C$, so $p \in C_i \in \mathscr{M}_{g_i,n_i+m_i}$ for some $C_i \subset C$.

Lemma 6.1.1 *If $C \in \overline{\mathscr{M}_{g,n}}$, for every component $C_i \in \mathscr{M}_{g_i,n_i+m_i}$ of C, $g_i \leq g$, and $g_i = g \Rightarrow n_i + m_i < n$.*

Proof. We can obtain the (nodal) limit C topologically from any smooth C^k by a sequence of the following kinds of degenerations. The first is that k marked points in some component C' can collide and form a bubble attached to C'. The genus of C' does not change, but it loses k marked points and gains a node at the point of attachment of the bubble. Thus, the total number of marked and nodal points on C' decreases by $k - 1$. The bubble itself is a genus 0 component with k marked points and one node. If the the original smooth curve C^k had genus 0, then it must have had more than k marked points. So, every new component resulting from repeated bubbling has genus 0 or g

The second kind of degeneration comes from letting the complex structure on the curve degenerate. The result topologically is that a simple closed curve on some component, the vanishing cycle, is crushed to a point. If the vanishing cycle is a nonseparating curve, it reduces the genus of a component C' by 1 without creating any new components, and if it is separating, it breaks a component C' into two pieces whose genera sum to that of C'. The case where one has genus 0 and the other has genus $g(C')$ is topologically identical to bubbling.

□

Let us order pairs (g, n) lexicographically, i.e. $(g', n') < (g, n)$ means that $g' < g$ or $g' = g$ and $n' < n$, as in the lemma. $\partial\overline{\mathscr{M}_{g,n}}$ is a stratified space whose stratum containing a nodal curve C is the product (over the $\mathscr{M}_{g',n'}$ for each component C_i of C with $C_i \in \mathscr{M}_{g',n'}$. The lemma tells us that if $\mathscr{M}_{g',n'}$ is a factor in a stratum of $\mathscr{M}_{g,n}$, then $(g', n') < (g, n)$. One of the components of a given $C \in \overline{\mathscr{M}_{g,n}}$ is distinguished by containing the nth marked point. We can use this to inductively define functions on all the $\overline{\mathscr{M}_{g,n}}$ simultaneously, in a *coherent* fashion. Namely, assume we have defined $f_{g',n'} : \mathscr{M}_{g',n'} \to X$ for all $(g', n') < (g, n)$. Each element of $\partial\overline{\mathscr{M}_{g,n}}$ is nodal curve C with n marked points. Say p_n lies on $C' \in \mathscr{M}_{g',n'}$. By the lemma, $(g', n') < (g, n)$, so by hypothesis we have a function $f_{g',n'}$ on $\mathscr{M}_{g',n'}$. We can thus define $f_{g,n}(C) := f_{g',n'}(C')$. The collection $\{f_{g',n'}\}_{(g',n')<(g,n)}$ thus determines $f_{g,n}|_{\partial\overline{\mathscr{M}_{g,n}}}$. We can extend $f_{g,n}$ to the interior $\mathscr{M}_{g,n}$ of $\overline{\mathscr{M}_{g,n}}$. We may continue in this fashion, defining $f_{g,N}$ on $\mathscr{M}_{g,N}$ for all $N > n$, and then $f_{g+1,n}$ for all n, etc. This gives a prescription for constructing functions of the following type:

Definition 6.1.2 A *domain-dependent almost complex structure or* DDACS *is a collection of* C^l *(*$l > 0$*) maps* $F = \{F_{g,n} : \overline{\mathscr{M}_{g,n}} \to \mathscr{J}\}$*,which are coherent in the sense that if* $C_k \in \overline{\mathscr{M}_{g,n}}$*, and* $C_k \to C \in \partial\overline{\mathscr{M}_{g,n}}$*, and* $C' \in \mathscr{M}_{g',n'}$ *is the component of* C *containing the* n*th marked point, then* $\lim_{k\to\infty} F_{g,n}(C_k) = F_{g,n}(C) = F_{g',n'}(C')$*. The set of all such maps will be denoted* $\mathscr{J}_D$*.*

Note that $\mathscr{M}_{g,n}$ is an orbifold. We recall that a neighborhood of a point in an n-dimensional orbifold is modeled on the quotient of $\mathbb{R}^n$ by the linear action of some finite group G, and a C^l function on an orbifold in a neighborhood modeled on $\frac{\mathbb{R}^n}{G}$ is a C^l function on R^n which is invariant under the group action. For $g > 1$, the locus of points on $\mathscr{M}_{g,n}$ without automorphisms (i.e. where the action of G on $\mathbb{R}^n$ is nontrivial) has real codimension at least two. In particular, a generic curve of genus $g > 1$ has no nontrivial automorphisms. For $g = 0$, every stable curve has a trivial automorphism group, and for $g = 1, n = 1$, a generic elliptic curve has an involution, and isolated points in $\mathscr{M}_{g,1}$ (which is 2 (real) dimensional) has extra automorphisms, and so functions on $\mathscr{M}_{g,1}$ have no constraints at generic points and respect extra symmetries at the points with extra automorphisms. The derivative of a G-invariant function will have a nontrivial kernel, but at any tangent space $T_x\mathscr{M}_{g,n}$, there is a subspace of (real) dimension at least 2 on which the derivative has no constraints, and there exists a map from a neighborhood of any point in $\mathscr{M}_{g,n}$ to a neighborhood of any point q in a manifold X (of dimension at least two) sending a two dimensional subspace of that unconstrained subspace to any 2 dimensional subspace to any two-dimensional subspace of T_qX. If we fix j and the first n marked points on $\mathscr{M}_{g,n+1}$, we may view this as a map $T_{p_{n+1}}C \to T_qX$.

The algorithm described before the definition clearly suffices to construct any DDACS. We call a DDACS generic if for every (g, n), the extension of $f_{g,n}$ from the boundary–where the values are determined by $f_{g',n'}$–to the interior of $\mathscr{M}_{g,n}$ is a generic C^l map. We work in the C^l category rather than C^∞ so that $\mathscr{J}_D$ is a Banach manifold (we will need to apply the Sard-Smale theorem).

The special role of the nth marked point is related to the definition of J-holomorphic curves for a DDACS J. Recall that there is a map $\pi : \overline{\mathscr{M}_{g,n+1}} \to \overline{\mathscr{M}_{g,n}}$ which forgets the $(n+1)$st marked point (and collapses any resulting unstable components, which are necessarily of genus 0.) The fiber over $C \in \overline{\mathscr{M}_{g,n}}$ is itself isomorphic to C. This is clear away from the marked and nodal points of C. The fiber above the kth marked point is a single nodal curve which has a genus 0 component containing the kth and $(n+1)$st marked points and a node, is glued to $p_k \in C$. This component collapses when the $(n+1)$st marked point is removed. The fiber above a node resulting from gluing $p \in C'$ to $q \in C''$ is a single curve which has a genus 0 component containing two nodes and the $(n+1)$st marked point, attached by the first node to C' at p and to C'' at q by the second node. This genus zero component similarly collapses when the marked point is removed. Thus, a point of $\overline{\mathscr{M}_{g,n+1}}$ is equivalent to a pair (C,p), where $C \in \overline{\mathscr{M}_{g,n}}$ and $p \in C$.

If $C \in \mathscr{M}_{g,n+1}$, we can delete the first n marked points to get an n-times punctured curve with one marked point. Fix the first n marked points, and let $J = \{F_{g',n'}\}$ be a DDACS. Then restricting $F_{g,n+1}$ to $C \cong \pi^{-1}(C) \subset \mathscr{M}_{g,n+1}$, we obtain a map $J_C : C \to \mathscr{J}$, i.e. a family of almost complex structures on ξ parameterized by C. Now viewing C as an n-times punctured curve, a map $u : C \to \mathbb{R} \times Y$ is J-holomorphic if it satisfies the domain-dependent Cauchy-Riemann equation $\overline{\partial}_J(u) = du + J_C \circ du \circ j = 0$. Note that J_C depends on $p \in C$ as well as on $u(p) \in \mathbb{R} \times Y$; for each $p \in C$, we have the equation $\overline{\partial}_{J_C}(u) = du_p + J(p, u(p)) \circ du_p \circ j_p = 0$, where $\overline{\partial}_J(u) : T_pC \to T_{u(p)}\mathbb{R} \times Y$. We'll write J_C as J, understanding that in the Cauchy-Riemann equation for an n-times punctured curve $(C,j) \in \mathscr{M}_{g,n}$, the domain of J is restricted to $\pi^{-1}(C)$.

Remark 6.1.3 The target in the definition of DDACS is $\mathscr{J}$. So, if $J \in \mathscr{J}_D$ and $u : (C,j) \to (\mathbb{R} \times Y, J)$ is a (j,J)-holomorphic curve, then $u|_{u^{-1}(\mathbb{R}\times N)}$ is (j, J_0)-holomorphic. In particular, since J_0 is domain-independent, the subset $u(C) \cap \mathbb{R} \times N$ of $u(C)$ satisfies intersection positivity, which will be crucial in the final chapter.

6.2 Regularity for generic S^1-invariant domain-dependent almost complex structures

We will need to consider regularity for two subsets of $\mathscr{J}_D$. The first is the set of S^1-invariant domain-dependent almost complex structures:

$$\mathscr{J}_D^{S^1} = \{F_{g,n} \in C^\infty(\mathscr{M}_{g,n+1}, \mathscr{J}^{S^1}) : \{F_{g,n}\} \text{ is a DDACS}\}$$

We will prove that a generic $J \in \mathscr{J}_D^{S^1}$ is regular, which implies the weaker statement that a generic $J \in \mathscr{J}_D$ is regular. The other subset is $\mathscr{J}$ itself; $J \in \mathscr{J}$ can be identified with the constant maps $\{\overline{\mathscr{M}_{g,n+1}} \to \{J\}\}_{g,n}$. [10] shows that a generic compatible almost complex structure is regular for curves of ECH index 1, and its argument is easily modified to show that a generic J satisfying the constraint $J|_N = J_0$ is also regular.[2]

[2]Regularity follows from the subclaim of in Lemma 9.12 of [10] that a certain projection of the linearized Cauchy-Riemann operator is surjective on a nonempty open set of the domain of some pseudoholomorphic curve $u : C \to \mathbb{R} \times Y$, but the proof obtains surjectivity on an open dense subset of C. The intersection of this open dense set with $u^{-1}(\mathbb{R} \times (Y \setminus N))$ contains a nonempty open set, so the result holds.

Theorem 6.2.1 (Generic S^1-invariant DDACS are regular) *Let* **a**, **b** *be orbit sets with deg*$(\mathbf{a},\mathbf{b}) > 0$*. Then for a generic* $J \in \mathscr{J}_D^{S^1}$*, and a J-holomorphic map* $u \in \mathscr{M}(\mathbf{a},\mathbf{b},J)$*, the linearization* D_u *of* $\overline{\partial}_J$ *at u is surjective. Therefore,* $\mathscr{M}_u$ *is a smooth orbifold whose dimension equals the Fredholm index of* $D_{u,J}$*.*

Remark 6.2.2 The statement that $\mathscr{M}$ is a smooth orbifold follows from the "folk theorem" proved in [29] (Theorem 0): if $u \in \mathscr{M}(\mathbf{a},\mathbf{b},J)$ and D_u is surjective, then a neighborhood of u in $\mathscr{M}(\mathbf{a},\mathbf{b},J)$ may be given the structure of a smooth orbifold whose dimension equals the Fredholm index. This generalizes the familiar theorems (cf. Chapter 3 of [21]) which state that the moduli space is a smooth manifold in a neighborhood of a regular *somewhere injective* curve, i.e. one which does not factor through a nontrivial branched covering. At an orbifold point $u \in \mathscr{M}(\mathbf{a},\mathbf{b},J)$, where $u : (C,j) \to (\mathbb{R}\times Y, J)$, $(C,j,p_1,...,p_{n+1})$ has a nontrivial automorphism group with respect to which u is invariant, so u factors through the branched covering $C \to \frac{C}{Aut(C,j)}$. Other multiple covers may arise which do not come from automorphisms of the domain, and one of the virtues of a domain-dependent almost complex structure is that such multiple covers may be perturbed away by choosing different perturbations at different points in $u^{-1}(u(p))$. The multiple covers coming from automorphisms are *not* perturbed away, because the functions $F_{g,n+1} : \mathscr{M}_{g,n+1} \to \mathscr{J}$ are invariant with respect to the orbifold symmetry groups. However, the subset of orbifold points of $\overline{\mathscr{M}_{g,n}}$—i.e. of curves with nongeneric symmetries—has real codimension at least 2 in $\overline{\mathscr{M}_{g,n}}$, we may conclude that the subset of holomorphic curves in the moduli space whose domains are orbifolds also has real codimension at least 2. Therefore, a generic holomorphic curve is not an orbifold point in its moduli space, and a generic path of holomorphic curves avoids the locus of orbifold points, and we need not trouble further about them.

We will give some definitions before beginning the proof:

Fix the orbit sets **a** and **b**; in this discussion, all moduli spaces under consideration are of curves (C,j) with $\partial C = \mathbf{a} - \mathbf{b}$. Let $\mathscr{B} := W^{1,p}(C, \mathbb{R}\times Y)$ be the Banach space of maps $C \to \mathbb{R}\times Y$ of Sobolev class $(1,p)$, $p > 2$; the latter condition implies that any $u \in \mathscr{B}$ can be represented by a continuous map. Let $\mathscr{E} \to \mathscr{B}$ be the Banach space bundle whose fiber over $u \in \mathscr{B}$ is $\mathscr{E}_u = L^p(\Omega^{0,1}(u^*(\mathbb{R}\times Y), C)$. (We'll often drop the Sobolev exponents below.) Then for any DDACS J, $\overline{\partial}_J : \mathscr{B} \to \mathscr{E}$, $u \mapsto \overline{\partial}_J(u) = du + J \circ du \circ j$ gives a smooth section of this bundle. Now consider $\overline{\partial}_J(u)$ as a function of both J and u. Its zero set lies in $\mathscr{B}\times\mathscr{J}$. Pull back $\mathscr{E}$ by the projection $\mathscr{B}\times\mathscr{J} \to \mathscr{B}$ to obtain a bundle $\mathscr{E} \to \mathscr{B}\times\mathscr{J}$. The universal $\overline{\partial}$ operator $\overline{\partial}(u,J) := \overline{\partial}_J(u)$ is a section $\overline{\partial} : \mathscr{E} \to \mathscr{B}\times\mathscr{J}$. Its zero set is the *universal moduli space* $\mathscr{M}(\mathscr{J}_D^{S^1}) \equiv \mathscr{M}(\mathbf{a},\mathbf{b};\mathscr{J}_D^{S^1}) = \{(u,J) : J \in \mathscr{J}_D^{S^1}, \overline{\partial}_J(u) = 0\}$. It projects to $\mathscr{J}_D$, and the fiber over $J \in \mathscr{J}_D^{S^1}$ is precisely $\mathscr{M}(\alpha,\beta,J) \equiv \mathscr{M}_J$.

If $u : (C,j) \to \mathbb{R}\times Y$ is J-holomorphic, i.e. $\overline{\partial}_J(u) = 0$, consider the differential of $\overline{\partial}$ at (u,J). It is a Fredholm map $D \equiv D_u + D_J : T_uB \oplus (T_J\mathscr{J}_D^{S^1}) \to E_u$.

At (u,J), D_u is essentially the same as the differential of the projection

$$\mathscr{M}(\mathscr{J}_D^{S^1}) \to \mathscr{J}_D^{S^1}$$

at (u,J), and the former is surjective at (u,J) for all $u \in \mathscr{M}_J$ (i.e. J is regular) iff J is a regular value of the projection. By the Sard-Smale theorem, regular values of $\mathscr{M}(\mathscr{J}_D^{S^1})$ are generic. Thus, if we can show that D is surjective, $\mathscr{M}(\mathscr{J}_D^{S^1})$ is a (Banach) manifold and the fibers $\mathscr{M}_J$ over a generic $\in \mathscr{J}_D^{S^1}$ are manifolds of dimension $ind(D_u)$.

Recalling that $T_u(B) = W^{1,p}(\Gamma(u^*(T(\mathbb{R}\times Y), C))$, for $\zeta \in T_u(B)$ we can write, as usual, $D_u(\zeta) = d\zeta + J \circ d\zeta \circ j + \nabla_\zeta \circ du \circ j \in W^p(\Omega^{0,1}(u^T(\mathbb{R}\times Y), C)$.

The domain of D_J is the tangent space at J to the space complex structures under consideration. Let $\mathscr{J}(V)$ be the set of all ω-compatible almost complex structure son the symplectic vector space (V, ω). If $J \in \mathscr{J}(V)$, then

$$T_J\mathscr{J}(V) = End^{0,1}_{J,\omega}(V) := \{A \in End(V) : AJ + JA = 0 \text{ and } \forall v, w \in V, \omega(Av, w) + \omega(v, Aw) = 0\}$$

If $J \in \mathscr{J}^{S^1}$, $T_J\mathscr{J}^{S^1} = \Gamma_N^{S^1}(End^{0,1}_{J,d\alpha}(\xi), Y)$, where $End^{0,1}_{J,d\alpha}(\xi)$ is the bundle over Y with fibers $End^{0,1}_{J(p),d\alpha_p}(\xi_p)$ and $\Gamma_N^{S^1}$ denotes its S^1-invariant sections whose restriction to N is identically zero. (Because $J|_N \equiv J_0|_N$.) Let us now restrict consideration to domains C of genus g with n marked points. Choose coherent functions $f_{g',n'}$ for all $(g', n') < (g, n+1)$ and let $\mathscr{J}^{S^1}_{D,g,n+1}$ consist of all C^l maps $f_{g,n+1} : \overline{\mathscr{M}_{g,n+1}} \to \mathscr{J}$ compatible with $\{f_{g',n'}\}_{(g',n')<(g,n+1)}$ on $\partial\overline{\mathscr{M}_{g,n+1}}$ Then if $J \in \mathscr{J}^{S^1}_{D,g,n+1}$, $T_J\mathscr{J}^{S^1}_{D,g,n+1} = \Gamma(T^*(\mathscr{M}_{g,n+1}) \otimes J^*(T\mathscr{J}^{S^1}))$; we in fact will consider the sections of this bundle of Sobolev class $(1, k)$ so that it forms a Banach space. Reverting to our earlier notation, we may view the n punctures on a curve $\overline{C}$ as fixed, with j varying on $C = \overline{C} \setminus \{p_1, ..., p_n\}$, so that the tangent space to $\mathscr{M}_{g,n+1}$ at a point $(\overline{C}, j, p_1, ..., p_n, p_{n+1})$ is $T_j\mathscr{J}(C) \oplus T_{p_{n+1}}C$. So if $u : (C, j) \to (\mathbb{R}\times Y, J)$ is a holomorphic curve, and $V = (a, A) \in T_J(\mathscr{J}^{S^1}_{D,g,n+1})$, where $A : T_j(C) \to T_J\mathscr{J}^{S^1}$ and $a \in End^{0,1}_j TC$, then $D_J(V) = A \circ du \circ j_C + J \circ du \circ a$.

Proof.

$deg(\alpha, \beta) > 0$ implies that $u(C)$ is not a union of cylinders. I is additive and positive for non-cylindrical holomorphic curves, so there is a unique noncylindrical component C' of C. Trivial cylinders are always cut out transversely, maps $\mathbb{R}\times S^1 \to \mathbb{R}\times Y$ are dealt with in the previous chapter. So we may have reduced the theorem to the case $C' = C$. The domain C of u must be a stable curve: it has an end, so it's stable if it has positive genus. If C had genus 0, we could fill in the punctures of C to get a closed Riemann surface $\overline{C} \cong S^2$ and a map $\overline{u} : S^2 \to \Sigma$ extending $\pi \circ u$. But since $g(\Sigma) > 0$, $\overline{u}$ is nullhomotopic

Since it is not a trivial cylinder, C' must have ends at (covers of) at least two distinct embedded Reeb orbits, so $u(C)$ intersects $\mathbb{R}\times N$, and in fact $u^{-1}(\mathbb{R}\times(Y \setminus N)$ contains a nonempty open set of C'.

We want to show that the linearization $D = D_u + D_J$ of $\overline{\partial}_J$ is surjective for any $(u, J) \in \mathscr{M}(\mathscr{J})$. Since D_u is Fredholm, D has a closed range, so surjectivity is equivalent to the triviality of the annihilator of $Im(D)$. The codomain of D is $E_u = L^p(\Omega(u^*(\mathbb{R}\times Y), C)$, so we can make the identification $E_u^* = L^q(\Omega(u^*(\mathbb{R}\times Y), C)$. So let $\eta \in coker(D)$. This and the splitting $D = D_u + D_J$ imply

$$\langle D_u(\zeta), \eta\rangle = 0 \text{ and } \langle D_J(V), \eta\rangle = 0$$

for all $\zeta \in T_uB$, $V \in T_J(\mathscr{J}_D^{S^1})$.

By the first equation, $\eta \in ker(D_u^*)$. This implies that η is smooth by elliptic regularity and if η vanishes on an open set, then $\eta \equiv 0$ by unique continuation. (See, e.g., Lemma 3.4.7 of [21]. To show the latter, consider the second equation.

Since $D = D_u + D_J$, $\eta \in ker(D_u^*)$ in particular. D_u^* is elliptic because D_u is, so by elliptic regularity, η is smooth. By unique continuation and the Carleman similarity principle, if $\eta \equiv 0$ on an open subset of C, then $\eta \equiv 0$ on all of C.

First, note that u cannot be a nodal curve with a constant component of positive genus. For, if such a curve C, with $ind(C) = 1$, was the union of a nodal curve C_1 and a constant component C_2, then $1 = ind(C_1) + ind(C_2)$. Because $u|_{C_2}$ is constant, the pullback of $T\mathbb{R}\times Y$ restricted to C_2 is trivial, so $c_1(u^*(T\mathbb{R}\times Y)) = c_1(T(\mathbb{R}\times Y))|_{C_1} + c_1(T(\mathbb{R}\times Y))|_{C_2} = c_1(T(\mathbb{R}\times Y))|_{C_1} \equiv c_1$. Furthermore, because u maps C_2 to a constant, all the punctures must lie on C_1. So the sum μ_0 of the Maslov indices at the ends of C and C_1 agree. By hypothesis, C_2 has positive genus, so we must have $g(C_1) < g(C)$, and $\chi(C_1) > \chi(C)$. So, we have

$$ind(C) = -\chi(C) + c_1 + \mu_0 = 1$$

$$ind(C_1) = -\chi(C_1) + c_1 + \mu_0$$

and thus $ind(C_1) < ind(C) = 1$. Therefore $ind(C_2) = 1 - ind(C_1) > 0$. But we assumed that J is a generic DDACS, which implies that all of its restrictions to $\partial(\overline{\mathscr{M}_{g,n+1}})$—which determine the almost complex structure on C_1 and C_2—are generic. But for generic almost complex structures, positive index curves of positive genus do not exist. Thus, u is not constant on a component of positive genus. Constant components of genus 0 may be eliminated by reparameterization, so they pose no obstacle, and we conclude that u is not constant on any component of C, so zeroes of du are isolated. [3]

Let $u : C \to \mathbb{R}\times Y$ be a J-holomorphic map. The set of regular points p of C such that $\pi \circ u(p)$ is a regular value of $\pi \circ u$ form an open dense subset of C. Furthermore, if we intersect this with the set of points $p \in C$ where $Im(du_p) = \xi_{u(p)}$, it remains open and dense. (The projection to Y is already open and dense by the nonintegrability of ξ.) The intersection D of this set with $u^{-1}(\mathbb{R}\times(Y \setminus N))$ contains a nonempty open set. Assume $\eta_p \neq 0$ for some $p \in D \subset C$. This implies that $\eta_p \in Hom^{0,1}_{J(p,u(p)}(T_pC, T_{u(p)}(T\mathbb{R}\times Y))$ and $du_p \circ j_p \in Hom^{1,0}_{J(p,u(p))}(T_pC, T_{u(p)}(T\mathbb{R}\times Y)$ are injective maps. So given any $0 \neq v \in T_pC$, then $0 \neq \eta_p(v), du_p \circ j_p(v) \in T_{u(p)}T\mathbb{R}\times Y$. We first wish to find some $A_p \in End^{0,1}_{J(p,u(p)),d\alpha_p}(T(\mathbb{R}\times Y))$ such that $A_p(du_p \circ j_p(v)) = \eta_p$. On D, $\xi_{u(p)}$ and $Im(du_p)$ are distinct complex subspaces of $T_{u(p)}(\mathbb{R}\times Y)$, so they span it, so the codomain of D, $Hom^{0,1}_{j,J}(T_pC, T_{u(p)}(\mathbb{R}\times Y)$, splits as the direct sum of $Hom^{0,1}_{j,J}(T_pC, \xi_{u(p)}$ and $End^{0,1}_{j,J}(T_pC)$. Split η_p into its ξ and $Im(du)$ components: $\eta_p = \eta_\xi + \eta_{TC}$. Because $J \circ du : TC \to Im(du)$ is injective, for any given $v_p \in T_pC$, we can choose $a_p \in End^{0,1}_{j(p)}(T_pC)$ so that $J \circ du \circ a(v_p) = \eta_p$. For the ξ component, note that $End^{0,1}_{J(p,u(p))}\xi_{u(p)} = T_{J(p,q)}\mathscr{J}$ is one (complex) dimensional,

[3] For holomorphic curves in symplectic manifolds of dimension $2n$, the index is given by $ind(C) = (n-3)\chi(C) + c_1 + \mu_0$. In our argument for the case $n = 2$, it was crucial that the coefficient of χ is negative. For $n > 2$, it is not possible in general to preclude constant components of positive genus, which is why the authors of [3] were forced to restrict themselves to genus 0 curves.

and for any given $v_q, w_q \in \xi_q$, there is an element $B_p \in End^{0,1}_{J(p,u(p))}\xi_{u(p)}$ sending v_p to w_p. So choose $B_p : T_p(C) \to T_{J(p,u(p))}$ sending $du \circ j(v_p)$ to η_{xi}. Thus, $A_p \equiv (a_p, A_p) : v_p \mapsto \eta_p$.

Let us suitably extend A_p to an $A \in T_J \mathscr{J}_D^{S^1}$.

When $F_{g,n}$ is restricted to $C \subset \mathscr{M}_{g,n+1}$, $V(p,q)$ depends on $p \in C$ and $\pi(q) \in \Sigma$. We want to extend A to all of $T_J \mathscr{J}_D^{S^1}$, which means letting it vary with the complex structure j on the surface C ; we thus obtain an $V(P,q)$ whose domain is $\mathscr{M}_{g,n+1} \times \Sigma$. So: define a smooth cutoff function $\kappa : \Sigma \to \mathbb{R}$ which is nonnegative, 1 at $\pi \circ u(p)$, and 0 outside some open neighborhood of $\pi \circ u(p)$ that does not contain any of the other critical points of the Morse function $\overline{f} : \Sigma \to \mathbb{R}$ used to perturb the contact form. Let $\nu : \mathscr{M}_{g,n+1} \to \mathbb{R}$ be a smooth nonnegative function which is 1 at $(C, j, p_1, ..., p_n, p)$ and zero outside an open neighborhood of it. The neighborhoods of the p_i should not intersect each other, and the $(n+1)$st neighborhood D' (of $p \in D$) should not contain any preimages of $\pi \circ u(p)$ besides p itself. (These preimages are finite in number; otherwise, they would accumulate, and since $\pi \circ u$ satisfies a perturbed Cauchy-Riemann equation, it would have to be locally and hence globally constant.)Then, choose an arbitrary smooth extension A' of A_p and shrink both neighborhoods if necessary to ensure that if $q \in supp(\kappa)$ and $(j, p_1, ..., p_{n+1}) \in supp(\nu)$, then $\langle A(j, p_1, ...p_{n+1}, q) \circ du_{p_{n+1}} \circ j_{p_{n+1}}, \eta_{p_{n+1}} \rangle > 0$. At last, we define $A(j, p_1, ..., p_{n+1}, q) := \kappa(q)\nu(j, ..., p_{n+q}) \cdot A'(j, ..., p_{n+1}, q)$.

Therefore, $\int_C \langle D_J(A)_p, \eta_p dp \rangle = \int_{D'} \langle D_J(A)_p, \eta_p dp \rangle$. As the integrand is smooth, nonnegative, and positive at $p \in D'$, $\int_C \langle D_J(A)_p, \eta_p dp \rangle > 0$, contradicting the assumption that η is orthogonal to the image of D_J. So, if $\eta \in coker(D_J) \supset coker(D)$, $\eta_p = 0$ for for any p in the nonempty open set $D \subset C$, so $\eta \equiv 0$, $coker(D) = 0$, and D is surjective.

For M sufficiently small, M contains only regular values of $\pi \circ u$, so $\pi \circ u^{-1}(M) \subset C$ consists of a finite number of disjoint homeomorphic copies $p \in M_1, ..., M_D$ of M. Adjusting ρ if necessary, we can assume $q \in M_2 \cup ... \cup M_D$ implies that $\rho(C, j, p_1, ..., p_n, q) = 0$. So $D_J(A)_q = 0$ outside M_1. Therefore, $\langle \eta, D_J(A) \rangle$ is 0 for $q \notin M_1$, nonnegative otherwise, and positive at $q = p$. So $\int_C \langle \eta, D_J(A) \rangle = \int_{M_1} \langle \eta, D_J(A) \rangle > 0$, which contradicts $\eta \in coker(D)$. Thus $\eta = 0$, and D is surjective.

□

Corollary 6.2.3 *If $deg(\mathbf{a}, \mathbf{b}) > 0$ and $I(\mathbf{a}, \mathbf{b}) = 1$, $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ is empty for generic $J \in J_D^{S^1}$.*

Proof.

Let $u : C \to \mathbb{R} \times Y$ be J-holomorphic for some regular $J \in \mathscr{J}_D^{S^1}$. Then D_J is Fredholm. If J' is any other DDACS, $D_{J'}$ has the same Fredholm index.. If J' is a generic domain-independent almost-complex structure, and $u' : C' \to \mathbb{R} \times Y$ is a J'-holomorphic curve with $\partial C' = \mathbf{a} - \mathbf{b}$, then by the ECH index inequality from [10], $ind(D_{J'}) \leq I(C) = I(\mathbf{a}, \mathbf{b}) = 1$. So $ind(D_J) \leq 1$ as well. The theorem shows that $\mathscr{M}(\mathbf{a}, \mathbf{b}, J)$, if nonempty, is a manifold of dimension $ind(D_J)$. But if $C \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J)$, $deg(\mathbf{a}, \mathbf{b}) > 0$ implies that image of C is not a union of trivial cylinders, so S^1 acts locally freely on $\mathscr{M}$. $\mathbb{R}$, as always, acts freely on $\mathscr{M}$ by translation, and these actions commute. So $dim(M_C) \geq 2$, a contradiction. Thus $\mathscr{M}(\mathbf{a}, \mathbf{b}, J) = \emptyset$.

□

Chapter 7

A comparison theorem

7.1 1-parameter families of DDACS

Let **a** and **b** be admissible orbit sets with $deg(\mathbf{a}, \mathbf{b}) > 0$ and $I(\mathbf{a}, \mathbf{b}) = 1$. The previous section shows that a generic S^1-invariant DDACS J_0 is regular, from which we concluded that $\mathscr{M}(\mathbf{a}, \mathbf{b}, J_0) = \emptyset$. Let $J_1 \in \mathscr{J}$ be a generic almost complex structure on ξ; [10] proves that J_1 is regular. By definition, $\langle \partial \mathbf{a}, \mathbf{b} \rangle = \#(\mathscr{M}(\mathbf{a}, \mathbf{b}, J_1)/\mathbb{R})$. (We will not discuss orientations here, but orientations on these moduli spaces exist, and all counts under consideration are signed.) We will show that this number is zero by comparing the moduli spaces for J_0 and J_1. So, let $J_t, t \in [0, 1]$, be a generic path in $\mathscr{J}_D^{S^1}$ connecting J_0 to J_1.

When J_t is regular, $\mathscr{M}(\mathbf{a}, \mathbf{b}, J_t)/\mathbb{R}$ is a (possibly empty) set of signed points. We cannot guarantee that J_t is regular for every t, but focusing on curves with $\partial C = \mathbf{a} - \mathbf{b}$, $\mathscr{M}(\mathbf{a}, \mathbf{b}, J_t)$ will be cut out transversely for all but finitely many $t_i \in (0, 1)$. To understand $\mathscr{M}(\mathbf{a}, \mathbf{b}, J_1)$, we need to compare $\mathscr{M}_{t_i - \epsilon}$ and $\mathscr{M}_{t_i + \epsilon}$ for a nongeneric time t_i. Two kinds of bifurcations can occur. The first is a cancellation, in which a pair of ($\mathbb{R}$-families of) oppositely signed holomorphic curves from **a** to **b** is created or destroyed at time t_i. In this case, the number of curves changes by two, but the the signed count remains the same.

7.1.1 Handleslides

The second kind of bifurcation is a handleslide. A handleslide occurs at t_i when, as $t \to t_i$, a sequence $C_t \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J_t)$ breaks into a J_{t_i} holomorphic building (see [2]) whose levels consist of an index 1 curve C_1, an index 0 curve C_0, and index zero connectors.[1] A *connector* is a branched cover a union of trivial cylinders. [2] We will show that connectors cannot occur at the top or bottom levels of the building, i.e. the broken curve actually consists of an index 1 curve $C_1 \in \mathscr{M}(\mathbf{a}, \mathbf{b}', J_{t_i})$ at the positive end, above an index 0 connector $C' \in \mathscr{M}(\mathbf{b}', \mathbf{b}', J_{t_i})$ (whose positive and negative ends consist of partitions of the

[1] Because the complex structures are domain-dependent, we cannot control the ECH index when the curve breaks. ECH is 0 for connectors, but all we can say otherwise is $I(C_0) + I(C_1) = 1$ by additivity.

[2] The curve C_1 is regular, but C_0 is not: generic index 0 moduli spaces, being 0 dimensional *before* modding out by $\mathbb{R}$ translation, must be $\mathbb{R}$ invariant, and hence cylinders. But the universal moduli space for a 1 parameter family of index 0 curves is 1 dimensional, so after modding out by $\mathbb{R}$, we expect isolated index 0 curves at distinct isolated times.

orbit set $\mathbf{b}'$), and at the bottom, an index zero curve $C_0 \in \mathscr{M}(\mathbf{b}', \mathbf{b}, J_{t_i})$. We could also have the index 0 curve on top and the index 1 curve at the bottom; everything we say below has a counterpart for this case.

The fact that connectors cannot occur at the top and bottom is crucial, because it allows us to apply the gluing analysis of [14],[15] to relate the number of curves in $\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_0-\epsilon})$ and $\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_0+\epsilon})$:

$$\#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_0+\epsilon}) = \#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_0-\epsilon}) + \#G(C_1, C_0) \cdot \#\mathscr{M}(\mathbf{a}, \mathbf{b}', J_{t_0}) \tag{7.1.1}$$

where $\#G$ is an integer depending on the partitions at the negative ends of C_1 and the positive ends of C_0. The $\#\mathscr{M}(\mathbf{a}, \mathbf{b}', J_{t_0-\epsilon})$ term is familiar from finite-dimensional Morse theory.The situation here is much subtler, because we have to glue the negative end of C_1 to the positive end of C_0 by inserting a connector $C' \in \mathscr{M}(\mathbf{a}', \mathbf{a}', J_{t_i})$. The partition of $\mathbf{a}$ at the negative end of C' must match the partition of $\mathbf{a}'$ at the positive end of C_0, and the positive partition of $\mathbf{a}'$ must match the negative partition of C_1. [14] derives an intricate combinatorial formula for $\#G(C_1, C_0)$, the number of ways to glue C_1 to C_0 using an index 0 connector, in terms of the ends of C_1 and C_0.[3] The existence of $\#G$ will suffice for the purposes of this paper, since we will always end up multiplying it by zero.

Number the handleslides $t_1, ..., t_k$; cancellation bifurcations do not change the curve counts, so we may omit them. (But note that a cancellation must occur before the first handleslide, since we start out with an empty moduli space!) We begin with an S^1-invariant DDACS J_0, so we know that $\#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_1-\epsilon}) = 0$. If we can show that $\#\mathscr{M}(\mathbf{a}, \mathbf{b}', J_{t_1-\epsilon})$ is zero for all possible $\mathbf{b}'$, the equation (7.1.1) yields $\#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_1+\epsilon}) = \#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_1-\epsilon}) = 0$. We then continue until the next handleslide at t_2. If $\#\mathscr{M}(\mathbf{a}, \mathbf{b}'', J_{t_2}) = 0$ for all $\mathbf{b}''$, we obtain $\#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_2+\epsilon}) = \#\mathscr{M}(\mathbf{a}, \mathbf{b}, J_{t_2-\epsilon}) = 0$. And so on.

Now consider $\mathscr{M}(\mathbf{a}, \mathbf{b}', J_{t_1})$, a smooth, 1-dimensional moduli space of Fredholm index 1 curves $u : C \to \mathbb{R} \times Y$ with $\partial C = \mathbf{a} - \mathbf{b}'$. We wish to show $\#\mathscr{M}(\mathbf{a}, \mathbf{b}', J_{t_1}) = 0$. Choose a generic path in $\mathscr{J}_D^{S^1}$ from J_{t_1} to some regular S^1-invariant DDACS, which might as well be J_0. We'll call this path J_t, $t \in [0, t_1]$, but note that it need not coincide with the J_t considered above and set aside for the moment. By the results of the previous section, $\mathscr{M}(\mathbf{b}', \mathbf{b}, J_0)$ is empty. As before, following the one-parameter family J_t, the moduli spaces of index 1 $\mathscr{M}(\mathbf{a}, \mathbf{b}', J_t)$ are transversely cut out except for finitely many times $\{t_i'\}$ at which a cancellation (irrelevant to the count) or handleslide occurs. Formula (7.1.1) governs the change in $\#\mathscr{M}(\mathbf{b}', \mathbf{b}, J_t)$ as t crosses a t_i. If we knew $\#\mathscr{M}(\mathbf{b}'', \mathbf{b}, J_{t_1'}) = 0$ whenever a family $C_t' \in \mathscr{M}(\mathbf{b}', \mathbf{b}, J_t)$ breaks into a building whose index 1 piece is $C_1' \in \mathscr{M}(\mathbf{b}'', \mathbf{b}, J_{t_1'})$, then $\#\mathscr{M}(\mathbf{b}', \mathbf{b}, J_t)$ does not change as t crosses t_1'. To do this, we need to consider $\mathscr{M}(\mathbf{b}'', \mathbf{b}, J_{t_1'})$, and connect $J_{t_1'}$ to an S^1-invariant regular DDACS J_0 as before, etc.

By considering the degree, we see that this process cannot continue indefinitely. Degree is additive, so the degree of the original curve is the sum of the degrees of the index 0 and index 1 curves that appear in its degeneration (a connector has degree zero). If the

[3] In [14],[15] the authors consider a slightly different situation. Rather than studying the degenerations of a 1-parameter family of index 1 curves, they consider degenerations of a sequence of $I = 2$ curves with respect to a fixed almost complex structure into two $I = 1$ curves with an index 0 connector between them. The situations are very similar analytically, and the results carry over to the present situation.

index 0 curve has degree 0, then it is a union of branched covers of cylinders, at least one of which is not an orbit cylinder (otherwise, the curve is a connector). But a nontrivial cylinder, and hence the union of cylinders including it, has positive index (the index of a nontrivial cylinder is equal to the difference of the Maslov indices of its ends, which is in turn equal to the difference of the Morse indices of the critical points over which the Reeb orbits lie, which is positive because the cylinder projects to a neighborhood of a gradient flow line if the perturbation is small).

So, the index 0 curve must have positive degree, and hence if at a handleslide, $C_t \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J_t)$ at $t = t_1$ degenerates to a building with degree 1 level $C_1 \in \mathscr{M}(\mathbf{b}', \mathbf{b})$, then $deg(C_1) < deg(C)$. If C_1' is an index 1 curve that similarly arises from studying moduli spaces $\mathscr{M}(\mathbf{b}', \mathbf{b}, J_t)$ at a handleslide, then $deg(C_1') < deg(C_1)$, etc.

We continue until either the index 1 curve $C_1'' \in \mathscr{M}(\mathbf{a}', \mathbf{b}', J)$ in a building cannot further degenerate via handleslides, or we reach a degree zero curve of index 1.[4] In the first case, we can choose a generic path in $\mathscr{J}_D^{S^1}$ from J to an S^1-invariant DDACS J_0 that has only handleslides, and thus $\#\mathscr{M}(\mathbf{a}', \mathbf{b}', J) = \#\mathscr{M}(\mathbf{a}', \mathbf{b}', J_0)0 = 0$. If C_1'' has degree zero, it is either the union of a branched cover of a nontrivial cylinder with branched covers of trivial cylinders, or the union of a branched cover of a trivial cylinder over a hyperbolic Reeb orbit with branched covers of trivial cylinders over elliptic orbits. In the former case, the curve counts correspond to a count of gradient flows on Σ between the critical points associated to the asymptotic Reeb orbits of the nontrivial cylinder. These counts will be zero because the Morse function on Σ is perfect. We will show the latter case cannot arrive due to a local adjunction formula argument. [5] So, assuming two technical lemmas to be proven in the next section, we have shown:

Theorem 7.1.1 *Let $deg(\mathbf{a}, \mathbf{b}) > 0$. Then for a generic (honest, domain-independent) almost complex structure J, the signed count $\#\mathscr{M}(\mathbf{a}, \mathbf{b}, J) = 0$. Consequently, $\langle \partial \mathbf{a}, \mathbf{b} \rangle = 0$.*

7.2 Writhe and the local adjunction formula

In this section, we prove two results about connectors needed in the proof of $\langle \partial \mathbf{a}, \mathbf{b} \rangle$ in the last section. Recall that a connector is a branched cover of a union of trivial cylinders in $\mathbb{R} \times Y$. All or some of the components may be unbranched. Any connector $u : C \to \mathbb{R} \times Y$ has degree zero, so $c_1(u^*T\mathbb{R} \times Y)$ and $Q(C)$ are zero. Connectors have the same total multiplicities of each orbit at both ends, so $\mu(C)$, and hence $I(C)$, are zero as well.

Now consider the Fredholm index $ind(C) = -\chi(C) + 2c_1(u^*T(\mathbb{R} \times Y) + \mu_0(C) = -\chi(C) + \mu_0(C)$. $\mu_0 = \mu_0^+ - \mu_0^-$, where $\mu_0^+(C) = \sum_a \mu(a)$, where the sum is taken over all

[4] It is actually enough to stop at degree 1, since transversality for degree 1 can be achieved by S^1 invariant domain-independent almost complex structures.

[5] We remark that this part of the result is true more generally. Due to the simplicity of the Reeb orbits that arise here, we can explicitly classify and rule out all index 1 connectors. In other situations such connectors might occur, but the count of them should be zero: in [7], Fabert uses an obstruction bundle argument to show that the contribution of index 1 cylinders to SFT is zero. It should be possible to adapt his argument to show that the number of ways to glue a building consisting of an index 1 hyperbolic cylinder, a connector, and an index 0 curve is also zero.

asymptotic Reeb orbits of C at ∞, and μ_0^- is defined similarly.

Lemma 7.2.1 *(Classification of connectors) Let $u : C \to \mathbb{R} \times Y$ be a connector with connected components C_i.*

i) If $ind(C) = 0$, then each C_i is either an unbranched cover of a cylinder or a branched cover of an elliptic orbit.

ii-a) Index 0 connected branched covers of positive elliptic orbits have exactly one positive end.

ii-b) Index 0 connected branched covers of negative elliptic orbits have exactly one negative end

iii) If $ind(C) = 1$, then each exactly one component of C_i is nontrivial, and it is a branched cover of the cylinder over a hyperbolic Reeb orbit which has either one positive end and two negative ends, or two positive ends and one negative end.

Proof. If a component C_i of C has k positive and l negative punctures, $-\chi(C_i) = 2g(C_i) - 2 + k + l$.

For a component C_h which covers a trivial cylinder over a hyperbolic orbit, $\mu(C_h) = 0$, so $ind(C_h) = -\chi(C_h) + \mu_0(C_h) = -\chi(C_h) = 2g(C_h) - 2 + k + l \geq 0$, with equality iff C_h is a genus 0 curve with exactly one positive and one negative end. In that case, $u|_{C_h}$ has no branch points by the Riemann-Hurewicz formula. $ind(C_h) = 1$ iff the genus is zero and $(k, l) = (1, 2)$ or $(2, 1)$.

If a component C_{e_+} covers a trivial cylinder over the positive elliptic orbit, then $\mu_0(C_{e_+}) = k - l$ and $ind(C_{e_+}) = 2g(C_{e_+}) - 2 + k + l + (k - l) = 2g - 2 + 2k \geq 2k - 2 \geq 0$, with equality iff C_{e_+} has genus zero, exactly one negative end, and any number of positive ends.

If a component C_{e_-} covers a trivial cylinder over the negative elliptic orbit, then $\mu_0(C_{e_-}) = l - k$ and $ind(C_{e_-}) = 2g(C_{e_-}) - 2 + k + l + (l - k) = 2g - 2 + 2l \geq 2l - 2 \geq 0$, with equality iff C_{e_-} has genus zero, exactly one positive end, and any number of negative ends.

Branched covers of trivial cylinders over elliptic orbits cannot have index 1 because the number of ends at hyperbolic Reeb orbits has the same parity as the Fredholm index.

By additivity of the Fredholm index under union, parts i) and iii) follow.

□

To prove that connectors cannot occur at the top and bottom of a building, and that index 1 connectors over hyperbolic orbits cannot occur, we will use a relative adjunction formula from [10]. In preparation for this, we recall the definition of the asymptotic writhe of a holomorphic curve and compute the asymptotic writhes that arise here.

If $(u : C \to \mathbb{R} \times Y) \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J)$ is a J-holomorphic curve, for $s \gg 0$, $C \cap \{s\} \times Y$ consists of a braid around an asymptotic Reeb orbit for every positive end of u. Focusing in on a single embedded Reeb orbit a, if u has k positive ends at a with multiplicities $q_1, ..., q_k$, then the intersection of C with $\{s\} \times N_a$ contains a k component braid ζ, whose

ith component ζ_i has q_i strands. We use the trivialization τ to identify N_a with $S^1 \times D^2$ in such a way that $a \subset N_a$ corresponds to $S^1 \times \{0\}$; the statement that ζ is a braid around a means that the projection $S^1 \times D^2 \to S^1$, restricted to ζ, is a submersion. This implies that if we smoothly imbed $S^1 \times D^2$ into R^3, mapping a to the standard unit circle in the xy-plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, the projection of ζ to the xy plane is an immersion. The signed count of the crossings of this projection is the writhe of ζ. (We perturb if necessary so so that the crossings are are transverse, and use the convention of [10] that a counterclockwise twist has positive writhe.) It is elementary that $w(\zeta) = \sum_i w(\zeta_i) + 2\sum_{i<j} lk(\zeta_i, \zeta_j)$, where lk is the usual linking number of oriented links in $\mathbb{R}^3$.

The sum of the corresponding writhes for each asymptotic Reeb orbit at infinity is w_+, and the analogous sum at $-\infty$ is w_-. The writhe of C is $w(C) = w_+ - w_-$.

We will need to know the writhes of the ends of a curve C, $\partial C = \mathbf{a} - \mathbf{b}$, at each embedded Reeb orbit a appearing in $\mathbf{a}$ and $\mathbf{b}$. The following lemma computes the writhe at a in terms of the partition that C induces on its total multiplicity in $\mathbf{a}$ and $\mathbf{b}$. In what follows, let e_+, e_-, and h denote positive elliptic, negative elliptic, and hyperbolic Reeb orbits whose monodromies are small perturbations of the identity.

To compute the writhe, we use the existence and some properties of an asymptotic expansion of the function $u : C \to \mathbb{R} \times Y$ near a puncture. In an open disc D around a puncture p of C, $D \setminus p \cong \mathbb{R} \times S^1$. Let $(s,t) \in \mathbb{R} \times S^1$ be local coordinates on D. Near p, u approaches a cylinder over q times some embedded Reeb orbit a. u composed with the projection to $\mathbb{R} \times a^q$ is a q-fold covering map, and considering the normal bundle to u, we can describe u for $s << 0$ as a section of the bundle $\mathbb{R}_s \times a^q_t \to \mathbb{R} \times Y$. We then can write

$$u(s,t) = \sum_{n \geq 1} a_n e^{\lambda_n s} e_n(t), \tag{7.2.1}$$

where $a_n \in \mathbb{R}$, the $e_n(t)$ are eigenfunctions with eigenvalue λ_n of the self-adjoint "asymptotic operator". See [10], section 6, for the definition of A and the proof of this expansion with some analytic assumptions; [24] gives a general proof.

At a negative end, only terms with $\lambda_n > 0$ can appear with nonzero coefficient a_n. Assume $\lambda_i \leq \lambda_{i+1}$ for all i. For $s \ll 0$, the term with the smallest λ_n for which $a_n \neq 0$ is much larger than any subsequent terms with $\lambda_{n'} > \lambda_n$. So , so the writhe may be computed by considering truncations of the asymptotic expansion of u. Nonzero eigenfunctions are nonvanishing, so the truncations have well-defined winding numbers. In citeHT2, it is shown that for a generic almost complex structure J, the coefficient a_1 does not vanish for any index 1 curve (Proposition 3.2), and for any pair of ends of an index 1 curve C, their a_1's are distinct (Proposition 3.9). We will need a slight extension of this results that follows from their methods: for a generic J, for any index 1 curve, $a_2, a_3 \neq 0$ as well.

These genericity results, together with Kato's perturbation theory (the exact statements needed are in Lemma 6.4 of [10]), also allow us to exactly compute the winding number ρ of ζ_i around a. Viewing $\zeta_i \subset S^1 \times D^2$ that does not intersect $S^1 \times \{0\}$, this is the winding number around 0 of the projection of ζ_i to D^2. At a negative end, it is $\rho_-(\zeta_i) = \lceil \mu(a^{q_i})/2 \rceil = \lceil \mu(a)/2 \rceil$ (the last equality holds because our Reeb orbits have monodromy near the identity), and at a positive end, the winding number is $\rho_+(\zeta_i) == \lfloor \mu(a)/2 \rfloor$. (If we don't know that $a_1 \neq 0$, the Maslov index expressions give only lower bounds on the winding number for negative ends and upper bounds for positive ends.)

Lemma 7.2.2 *Let ζ_i and ζ_j be connected braids around the embedded Reeb orbit a with multiplicities q_i and q_j. Then, at a positive end at a,*
i) If $a = e_+$, $w_+(\zeta_i) = 1 - q_i$, $w_-(\zeta_i) = q_i - 1$, $lk_+(\zeta_i, \zeta_j) = 0$, and $lk_-(\zeta_i, \zeta_j) = min(q_i, q_j)$
ii) If $a = e_-$, $w_+(\zeta_i) = 1 - q_i$, $w_-(\zeta_i) = q_i - 1$, $lk_+(\zeta_i, \zeta_j) = -min(q_i, q_j)$, and $lk_-(\zeta_i, \zeta_j) = 0$

Proof.

First, consider the writhe at negative ends. The proof of (Lemma 6.7 of [10]) shows that if $a = e_+$, since $\rho_-(e_+) = 1$, $gcd(q_i, \rho_-(e_+)) = 1$, so considering the first term of the asymptotic expansion (nonvanishing by the aforementioned genericity theorem of [15], it can be seen that ζ_i is isotopic to a $(q_i, \rho_-(e_+)) = (q_i, 1)$ torus braid, so it has writhe $q_i - 1$.

For $a = e_-$, $\rho_-(e_-) = 0$, so ζ_i winds 0 times around a. Therefore, ζ_i is isotopic to the cabling of a 1-stranded braid ζ_1 with winding number 0 by a q_i-stranded braid ζ_2. The extension of the genericity statement of [15] (that the terms with e_1, e_2 and e_3 in the asymptotic expansion appear with nonzero coefficients) implies that ζ_2 is isotopic to the braid swept out by the second term in the asymptotic expansion, and by the results of general perturbation theory cited in Lemma 6.4 of [10][6], ζ_2 has winding number 0, so it is in turn a cabling of a 1 stranded braid ζ_3 with winding number 0 by a q_i-stranded braid ζ_4. ζ_4 is isotopic to braid swept out by the third term of the asymptotic expansion, and by genericity, that term is the e_3 term. Perturbation theory then implies that ζ_4 has winding number 1, so it is isotopic to a $(q, 1)$ torus knot and therefore $w(\zeta_4) = q - 1$.

By the general formula for the writhes of cables, we have $w(\zeta_i) = q_i^2 w(\zeta_1) + w(\zeta_2)$, and $w(\zeta_2) = q_i^2 w(\zeta_3) + w(\zeta_4)$. $w(\zeta_1) = w(\zeta_3) = 0$, so $w(\zeta_i) = w(\zeta_2) = w(\zeta_4) = q_i - 1$.

Lemma 6.9 of [10] shows that if the leading terms of the asymptotic expansions of ζ_i and ζ_j are distinct—which is guaranteed by [15]—then

$$lk_-(\zeta_i, \zeta_j) = min(q_i \rho_-(a^{q_j}), q_j \rho_-(a^{q_i}))$$

. If $a = e_+$, this equals $min(q_i, q_j)$. If $a = e_-$, the linking number is zero.

The result for positive ends follows from that for negative ends by the proof of Lemma 6.13 of [10]. It shows that the writhe of ζ at a at ∞ is the opposite of the writhe of the mirror image (under the bijection $(t, y) \leftrightarrow (-t, y)$ of $\mathbb{R} \times Y$) OF ζ), and the mirror image of the end of a curve with a positive end at an elliptic orbit of rotation angle θ has a negative end at an elliptic orbit with rotation angle $-\theta$.

□

Lemma 7.2.3 *(No connectors at top and bottom) Let J_t, $t \in [0, 1]$, be a generic family of DDACS, J_0 be generic, and $C(t) \in \mathscr{M}(\mathbf{a}, \mathbf{b}, J_t)$ be a family of index 1 curves that, as $t \to 1$, degenerates to a building with n levels $C_{i+1} \in \mathscr{M}(\mathbf{a}_i, \mathbf{a}_{i+1}, J_1)$, $i = 0, ..., n$, where $\mathbf{a}_0 = \mathbf{a}$ and $\mathbf{a}_n = \mathbf{b}$. Then neither C_0 nor C_n are connectors.*

[6] Namely, we use the following facts: the space of eigenfunctions e with winding number $\rho(e) = n$ is 2-dimensional for any $n \in \mathbb{Z}$. Eigenfunctions $e, e'$with eigenvalues $\lambda \leq \lambda'$ have winding numbers $\rho(e) \leq \rho(e')$. Finally, the minimal winding number for a positive eigenvalue is $\lceil \mu(e_-)/2 \rceil = 0$, and the maximal winding number for a negative eigenvalue is $\lfloor \mu(e_-)/2 \rfloor = -1$. This implies that the winding number of e_2 is also 0, and the winding number of the e_3 is 1.

Proof.

Assume $C_1 \in \mathscr{M}(\mathbf{a}, \mathbf{a}_1, J_1)$ is a connector, so $\mathbf{a} = \mathbf{a}_1$, and consider an embedded Reeb orbit a appearing in $\mathbf{a}$. For some $s_0 \gg 0$ and t near 1, $([s_0, \infty) \times N_a) \cap C(t)$ may be identified with the union of the components of C_1 that cover $\mathbb{R} \times a$. Denote both by C. Viewed as a subset of $C(t)$, C is not a trivial cylinder. C is an embedding in the complement of a finite number of singular points (if it is a multiple cover, replace C by the underlying nearly-embedded curve of which it is a branched covers), so we can apply the following relative adjunction formula with singularities from [10]) (remark 3.2 following proposition 3.1):

$$c_1(u^*(T\mathbb{R} \times Y)|_C = \chi(C) + w(C) + Q(C) - 2\delta(C) \tag{7.2.2}$$

$\delta(C) \geq 0$ is a certain count of the singularities of C. In general, we can say nothing about δ for J-holomorphic curve C if J is a DDACS, because intersection positivity fails. But here, intersection positivity holds because $C \subset \mathbb{R} \times N$, and $J(p, u(p)) = J_0(u(p))$ for any $u(p) \in \mathbb{R} \times N$, so that J is domain-*in*dependent on C. Therefore, $\delta \geq 0$, with equality if and only if C is embedded. (Specifically, $\delta = \sum_{p \in Sing(C)} \delta_p$, where δ_p is the number of (transverse) double points of a generic holomorphic perturbation C in a neighborhood of p.) We will show that C_1 were a nontrivial connector, (7.2.2) would imply $\delta(C) < 0$, a contradiction.

$c_1(u^*(T\mathbb{R} \times Y)|_C$ and $Q(C)$ are zero because C_1 is a branched cover of a trivial cylinder (in particular, it has the same total multiplicities at $\pm\infty$). Dropping the C's, the equation becomes $2\delta = w + \chi$. By Lemma 2.1, index 0 branched covers of a $\mathbb{R} \times a$ must be unbranched if a is hyperbolic. So, assume there is a connector at the top, and separately consider its branched covers of trivial cylinders over $a = e_+$ and $a = e_-$.

Case 1: a is a positive elliptic orbit.

In this case, the linking number term vanishes, and the writhe of a k-stranded braid with multiplicities $q_i, i = 1, ..., k$, is given by $\sum_1^k (1 - q_i) = k - m$. So the total writhe is $w = w_\infty - w_{-\infty} = (k-m)-(l-m) = k-l$. Each component of an index 0 connector over a positive elliptic orbit has one positive end and some number ν_i of negative ends, so the k positive ends must correspond to k components, and the total number of negative ends $\sum_1^k \nu_i$ is l. Thus, $\chi(C) = \sum_1^k (2-1-\nu_i) = k-l$, and $2\delta = \chi + w = (k-l)+(k-l) = 2(k-l)$. $k \leq l$, and if there is nontrivial branching at any component, $k < l$, so $\delta < 0$.

Case 2: a is a negative elliptic orbit.

Here, the writhe at a k-stranded braid with multiplicities q_i is $w = \sum_1^k (1 - q_i) - 2\sum_{i<j} \min(q_i, q_j) = k-m-2\sum_{i<j} \min(q_i, q_j)$. Each component of an index 0 connector over a negative elliptic orbit has one negative end and some number ν_i of positive ends, so the l negative ends correspond to l components, and there are $\sum_1^l \nu_i = k$ positive ends. So $\chi(C) = \sum_1^l (2-\nu_i-1) = l-k$, and $2\delta = \chi + w_\infty - w_{-\infty} = (l-k)+[(k-m)-2\sum_{i<j} \min(m_i, m_j)] - [(l-m)-2\sum_{i<j} \min(n_i, n_j)] = -2(\sum_{i<j} \min(m_i, m_j) - \sum_{i<j} \min(n_i, n_j))$.

Note that since each component has exactly one negative end, the set $\{m_i\}$ is a subpartition of $\{n_j\}$, and the subpartition is proper unless all components are unbranched covers of trivial cylinders. The result for positive now follows from the following combinatorial lemma. Applying the relative adjunction formula at the negative ends, the symmetry properties of the writhe (Lemma 6.13 of [10]), imply that a connector at the bottom over a positive (resp. negative) elliptic orbit cannot occur because a connector at the top over a negative (resp. positive) elliptic orbit cannot occur.

□

Lemma 7.2.4 *Given a multiset $Q = \{q_i\}$ of positive natural numbers, define*

$$f(Q) = \sum_{i<j} \min(q_i, q_j)$$

. If Q, Q' are partitions of $m \in \mathbb{N}$ and $Q' = \{q'_i\}$ is a proper subpartition of Q, then $f(Q') > f(Q)$.

Proof.

Assume $q_i \le q_{i+1}, q'_i \le q'_{i+1}$ for all i. Then $f(Q) = \sum_{i=1}^{k}(k-i)q_i$. We can obtain any subpartition Q' from Q by iterating two operations: first, subdivide by replacing any q_i by $q_i - 1$ and 1 (and re-ordering the indices), and second, replace any pair q'_i, q'_j with $q'_i < q'_j - 1$which come from the same q_k (by these two operations), by the pair $q'_i + 1, q'_j - 1$. Each of these operations increases f:

For the first operation, WLOG assume that i is the smallest index with value q_i, so that $q_{i-1} < q_i$. Then the first operation replaces $Q = q_1 \le q_2 \le ... \le q_i \le ... \le q_k$ with $Q' = 1 \le q_1 \le ... \le q_{i-1} \le q_i - 1 < q_{i+1} \le ... \le q_k$. The coefficients of each term is the same for $f(Q)$ and $f(Q')$ except that $f(Q')$ begins with k and replaces the $(k-i)q_i$ of $f(Q)$ with $(k-i)(q_i - 1)$, so $f(Q') = f(Q) + k - (k-i)q_i + (k-i)(q_i - 1) = f(Q) + i > f(Q)$.

For the second operation, we are subtracting from a term with a smaller coefficient and adding to a term with larger coefficient: all the terms of f except the ith and jth are unaffected, and $(k-i)q_i$ and $(k-j)q_j$ from $f(Q)$ become $(k-i)(q_i+1)$ and $(k-j)(q_j-1)$, and hence $f(Q') = f(Q) + (k-i) - (k-j) = f(Q) + (j-i) > f(Q)$.

□

Lemma 7.2.5 *The index 1 curve C in a building that arises in a handleslide cannot be an index 1 connector.*

Proof. By Lemma 2.1, C must consist of the union of C', a single index 1 branched cover of a cylinder over a hyperbolic orbit, with an unbranched cover of a union of trivial cylinders. Furthermore, C' has two positive punctures and one negative puncture, or vice versa. Using the same reasoning as in Lemma 2.3, (7.2.2) implies that C' would have to have a negative number of singularities. As before, the c_1 and Q terms of (7.2.2) vanish. $\mu(h^k) = \mu(h) = 0$, so the writhe bound of [10], Lemma 6.7, implies that $w_+(h^k) \le 0$ and $w_-(h^k \ge 0$ for all k, so $w_+ - w_- \le 0$. C' is a thrice-punctured sphere, so $\chi(C') = -1$. Thus, (7.2.2) becomes $2\delta(C') = w - 1 \ge -1$, a contradiction. □

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