Kashaev-Reshetikhin Invariants for $SL_2(\mathbb{C})$ at Roots of Unity

by

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Abstract

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An important milestone of the theory of knot invariants is the Reshetikhin-Turaev functor introduced in \[\text{[RT]}\]. This construction could generate tangle invariants from quantum groups. Later, Kashaev and Reshetikhin generalize this construction \[\text{[KR1]}\] based on the idea of the holonomy braiding, the braiding defined for $\mathcal{C}$-colored diagrams. The purpose of this work is to have some discussion of this construction. There are three parts in this thesis: first the full description of the construction is provided. Then in the second part, some examples computed via Mathematica are shown. And some properties and theorems are given in the end.
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# 1 Quantum Groups

## 1.1 Hopf Algebras

**Definition (Monoidal Category).** Suppose \( \mathcal{C} \) is a category. We say \( \mathcal{C} \) is a monoidal category if it equips with the following structures

1. The tensor product functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \).

2. The unit object \( 1 \in \mathcal{C} \).

3. \( a : \otimes \circ (\otimes, \text{id}_\mathcal{C}) \Rightarrow \otimes \circ (\text{id}_\mathcal{C}, \otimes) \) is a natural isomorphism.

4. \( \ell : 1 \otimes \text{id}_\mathcal{C} \Rightarrow \text{id}_\mathcal{C} \) is a natural isomorphism.

5. \( r : \text{id}_\mathcal{C} \otimes 1 \Rightarrow \text{id}_\mathcal{C} \) is a natural isomorphism.

such that they all together satisfies

1. The triangle identity: \( \forall X,Y \in \mathcal{C} \),

\[
\begin{array}{c}
(X \otimes 1) \otimes Y \\
\downarrow r_X \otimes \text{id}_Y \\
X \otimes Y
\end{array}
\]

is commutative, where \( f \otimes g := \otimes(f,g) \) for any morphisms \( f, g \) in \( \mathcal{C} \).

2. The pentagon identity: \( \forall X,Y,Z,U \in \mathcal{C} \),

\[
\begin{array}{c}
(X \otimes Y) \otimes (Z \otimes U) \\
\downarrow \text{id}_X \otimes a_{Y,Z,U} \\
(X \otimes (Y \otimes Z)) \otimes U
\end{array}
\]

\[
\begin{array}{c}
\downarrow a_{X,Y,Z} \otimes \text{id}_U \\
X \otimes ((Y \otimes Z) \otimes U)
\end{array}
\]

is commutative.

\( \mathcal{C} \) is called strict if all \( a, \ell, r \) are identities.

**Definition (Rigid Monoidal Category).** Suppose \( (\mathcal{C}, \otimes, 1, a, \ell, r) \) is a monoidal category. It is right rigid if for every \( X \in \mathcal{C} \), there exists its right dual \( X^R \in \mathcal{C} \) and
1. The right-evaluation map \((ev^R)_X \in \mathcal{C}(X \otimes X^R, 1)\).

2. The right-coevaluation map \((coev^R)_X \in \mathcal{C}(1, X^R \otimes X)\).

such that they all together satisfies the triangle identities:

\[
\begin{align*}
&X^R \otimes 1 \xrightarrow{r^R_X} X^R \xleftarrow{\ell^R_X} 1 \otimes X^R \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&X^R \otimes (X \otimes X^R) \xleftarrow{a^{R,R,R}_X} (X^R \otimes X) \otimes X^R \\
&X \otimes 1 \xrightarrow{r_X} X \xleftarrow{\ell_X} 1 \otimes X \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&X \otimes (X^R \otimes X) \xleftarrow{a^{-1}_{X,R,R}} (X \otimes X^R) \otimes X
\end{align*}
\]

Similarly, we say \(\mathcal{C}\) is left rigid if for every \(X \in \mathcal{C}\), there exists its left dual \(X^L \in \mathcal{C}\) and

1. The left-evaluation map \((ev^L)_X \in \mathcal{C}(X^L \otimes X, 1)\).

2. The left-coevaluation map \((coev^L)_X \in \mathcal{C}(1, X \otimes X^L)\).

such that they all together satisfies the triangle identities:

\[
\begin{align*}
&X^L \otimes 1 \xrightarrow{r^L_X} X^L \xleftarrow{\ell^L_X} 1 \otimes X^L \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&X^L \otimes (X \otimes X^L) \xleftarrow{a^{-1}_{X,L,L}} (X^L \otimes X) \otimes X^L \\
&X \otimes 1 \xrightarrow{r_X} X \xleftarrow{\ell_X} 1 \otimes X \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&X \otimes (X^L \otimes X) \xleftarrow{a_{X,L,L}} (X \otimes X^L) \otimes X
\end{align*}
\]

One could verify, as shown in [EGNO], that both left dual and right dual are unique up to unique isomorphisms. And one would say \(\mathcal{C}\) is rigid if it is both left rigid and right rigid.

**Definition** (Braided Monoidal Category). Suppose \((\mathcal{C}, \otimes, 1, a, \ell, r)\) is a monoidal category. It is braided if it equips with a natural isomorphism, called the braiding,

\[B : \otimes \Rightarrow \otimes \circ \tau\]
where $\tau : C \times C \to C \times C$ defined by $(X, Y) \mapsto (Y, X)$ is the usual flipping functor, and satisfies the hexagon identities: $\forall X, Y, Z \in C$,

$$
\begin{align*}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} X \otimes (Y \otimes Z) \xrightarrow{B_{X,Y\otimes Z}} (Y \otimes Z) \otimes X \\
(Y \otimes X) \otimes Z & \xrightarrow{a_{Y,X,Z}} Y \otimes (X \otimes Z) \xrightarrow{id_Y \otimes B_{X,Z}} Y \otimes (Z \otimes X) \\
X \otimes (Y \otimes Z) & \xrightarrow{a_{X,Y,Z}^{-1}} (X \otimes Y) \otimes Z \xrightarrow{B_{X,Y\otimes Z}} Z \otimes (X \otimes Y) \\
X \otimes (Z \otimes Y) & \xrightarrow{a_{X,Z,Y}^{-1}} (X \otimes Z) \otimes Y \xrightarrow{B_{X,Z\otimes id_Y}} (Z \otimes X) \otimes Y
\end{align*}
$$

$C$ is called symmetric if $\forall X, Y \in C$, $B_{Y,X} \circ B_{X,Y} = id_{X \otimes Y}$.

**Proposition 1.1.** Suppose $(C, \otimes, 1, a, \ell, r, B)$ is a braided monoidal category. Then for every $X \in C$, $X^L$ exists if and only if $X^R$ exists and they are isomorphic.

**Proof.** Suppose $X^L$ exists for every $X \in C$. Then we can define $X^R$ to be $X^L$ and have $(ev^R)_X := (ev^L)_X \circ B_{X,L,X}^{-1}$, $(coev^R)_X := B_{X,X^L} \circ (coev^L)_X$. The other direction can be shown in a similar way. \qed

There are some common monoidal categories that we are going to discuss throughout this paper. For instance, $(\text{Vec}_{\text{fd}}(C), \otimes, C, id, id, [x \otimes y \mapsto y \otimes x] = \tau_{X,Y} : X \otimes Y \cong Y \otimes X)$, the category of (finite dimensional) vector spaces over $C$, is a rigid symmetric strict monoidal category, whose left dual and right dual of $V \in \text{Vec}_{\text{fd}}(C)$ are both $V^* := \text{Vec}_C^{(fd)}(V, C)$. For convenience, it is simply denoted by $\text{Vec}_C^{(fd)}$ without confusion. As another example or a generalization, given a ring $R$, the category of left $R$-modules, denoted by $R\text{Mod}$, is a symmetric strict monoidal category. But it is not rigid for general $R$.

Now we would like to define $C$-algebras, coalgebras, bialgebras, etc. Their collection can all be regarded as monoidal categories induced from $\text{Vec}_C^{(fd)}$.

**Definition (Monoid Object).** Suppose $(C, \otimes, 1, a, \ell, r)$ is a monoidal category. Then a monoid object in $C$ is $M \in C$ with $\mu \in C(M \otimes M, M)$ and $\eta \in C(1, M)$ such that

$$
\begin{align*}
(M \otimes M) \otimes M & \xrightarrow{a_{M,M,M}} M \otimes (M \otimes M) \\
M \otimes M & \xrightarrow{\mu} M \\
M & \xrightarrow{\mu} M
\end{align*}
$$
Definition (Algebra). A (finite-dimensional) $\mathbb{C}$-algebra is a monoid object in $\text{Vec}_\mathbb{C}^{(fd)}$. Let's denote $\text{Alg}_\mathbb{C}^{(fd)} := \text{Mon}(\text{Vec}_\mathbb{C}^{(fd)})$ for convenience.

Definition (Coalgebra). A (finite-dimensional) $\mathbb{C}$-coalgebra is a monoid object in $\text{Vec}_\mathbb{C}^{op,(fd)}$. Let's denote $\text{CoAlg}_\mathbb{C}^{(fd)} := \text{Mon}(\text{Vec}_\mathbb{C}^{op,(fd)})$.

We usually denote an object in $\text{Alg}_\mathbb{C}$ by $(A, \mu, \eta)$ and an object in $\text{CoAlg}_\mathbb{C}$ by $(C, \Delta, \epsilon)$. Now we try to explain $\mathbb{C}$-bialgebras are bimonoid objects in $\text{Vec}_\mathbb{C}$.

Definition (Bialgebra). Suppose $H \in \text{Vec}_\mathbb{C}$ that equips with $\mu \in \text{Vec}_\mathbb{C}(H \otimes H, H)$, $\eta \in \text{Vec}_\mathbb{C}(\mathbb{C}, H)$, $\Delta \in \text{Vec}_\mathbb{C}(H, H \otimes H)$, $\epsilon \in \text{Vec}_\mathbb{C}(H, \mathbb{C})$ so that $(H, \mu, \eta, \Delta, \epsilon) \in \text{CoAlg}_\mathbb{C}$. Then $(H, \mu, \eta, \Delta, \epsilon)$ is a $\mathbb{C}$-bialgebra if $\mu \in \text{CoAlg}_\mathbb{C}(H \otimes H, H)$ and $\eta \in \text{CoAlg}_\mathbb{C}(\mathbb{C}, H)$, or equivalently, $\Delta \in \text{Alg}_\mathbb{C}(H, H \otimes H)$ and $\epsilon \in \text{Alg}_\mathbb{C}(H, \mathbb{C})$. The category of (finite-dimensional) $\mathbb{C}$-bialgebras is denoted by $\text{BiAlg}_\mathbb{C}^{(fd)}$, where a morphism is an algebra morphism and a coalgebra morphism at the same time.

This is to say, a bialgebra, which is a bimonoid object in $\text{Vec}_\mathbb{C}$, can be regarded as a monoid object in $\text{CoAlg}_\mathbb{C}$, or equivalently, a monoid object in $\text{Alg}_\mathbb{C}^{op}$.

Definition (Convolution). Given $(A, \mu, \eta) \in \text{Alg}_\mathbb{C}$, $(C, \Delta, \epsilon) \in \text{CoAlg}_\mathbb{C}$ and $f_1, f_2 \in \text{Vec}_\mathbb{C}(C, A)$. The convolution $f_1 \star f_2 \in \text{Vec}_\mathbb{C}(C, A)$ is defined by $\mu \circ (f_1 \otimes f_2) \circ \Delta$.

Definition (Hopf Algebra). Let $(H, \mu, \eta, \Delta, \epsilon) \in \text{BiAlg}_\mathbb{C}$. Consider $S \in \text{Vec}_\mathbb{C}(H, H)$. Then $(H, \mu, \eta, \Delta, \epsilon, S)$ is a $\mathbb{C}$-Hopf algebra if

$$S \star \text{id} = \text{id} \star S = \eta \circ \epsilon$$

Such $S$ is then called an antipode. The category of (finite-dimensional) $\mathbb{C}$-Hopf algebras is denoted by $\text{Hpf}_\mathbb{C}^{(fd)}$, where $f \in \text{Hpf}_\mathbb{C}(H, H')$ if $f \in \text{BiAlg}_\mathbb{C}(H, H')$ such that $f \circ S_H = S_{H'} \circ f$. One can actually check that this condition is true for any bialgebra morphism $f$.

Definition (Quasicocommutative Algebra). Let $(H, \mu, \eta, \Delta, \epsilon) \in \text{BiAlg}_\mathbb{C}$. It is called quasicocommutative if there exists an invertible $R \in H \otimes H$ such that $\forall x \in H$, we have

$$\Delta^{op}(x) = R \Delta(x) R^{-1}$$

where $\Delta^{op} := \tau_{H,H} \circ \Delta$. Then such $R$ is called an universal $R$-matrix.
Definition (Quasitriangular Algebra). Suppose \((H, \mu, \eta, \Delta, \epsilon) \in \text{BiAlg}_C\) is quasicocommutative. It is called quasitriangular (or braided) if it satisfies
\[
(\Delta \otimes \text{id})R = R_{13}R_{23}
\]
\[
(\text{id} \otimes \Delta)R = R_{13}R_{12}
\]
where \(R_{12} = \sum x_i^{(1)} \otimes x_i^{(2)} \otimes 1\), \(R_{13} = \sum x_i^{(1)} \otimes 1 \otimes x_i^{(2)}\), \(R_{23} = \sum 1 \otimes x_i^{(1)} \otimes x_i^{(2)}\) if we write \(R = \sum x_i^{(1)} \otimes x_i^{(2)}\).

Definition (Pseudoquasitriangular Algebra). Suppose \((H, \mu, \eta, \Delta, \epsilon) \in \text{BiAlg}_C\) such that \(H = \mathbb{C}\langle X \rangle[[h]]/\langle R \rangle\) equipped with the induced topology from the \(h\)-adic topology on \(\mathbb{C}\langle X \rangle[[h]]\), where \(R \subset \mathbb{C}\langle X \rangle[[h]]\). Then it is called pseudoquasitriangular if \(\exists R \in H \otimes H\), the topological tensor product, satisfying the conditions to be quasicocommutative and quasitriangular.

### 1.2 The Quantum Groups \(U_q(\mathfrak{sl}_2(\mathbb{C}))\) and \(U_h(\mathfrak{sl}_2(\mathbb{C}))\)

In this paper, the pseudoquasitriangular Hopf algebra that plays a central role is \(U_h = U_h(\mathfrak{sl}_2(\mathbb{C}))\). Therefore in the following we try to show that \(U_h\) is indeed a pseudoquasitriangular Hopf algebra. The argument then can be extended to any complex semisimple Lie algebra \(\mathfrak{g}\).

Recall that \(U = U(\mathfrak{sl}_2(\mathbb{C})) \in \text{Alg}_C\) is generated by \(H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\) satisfying \([X,Y] = H, [H,X] = 2X, [H,Y] = -2Y\), where the 2 shows up because the Cartan matrix of the semisimple Lie algebra \(\mathfrak{sl}_2(\mathbb{C})\) is \([2]\). Now let’s define \(U_q = U_q(\mathfrak{sl}_2(\mathbb{C})) \in \text{Alg}_C\) generated by \(K, K^{-1}, E, F\) satisfying
\[
KK^{-1} = K^{-1}K = 1, \ KE = q^2EK, \KF = q^{-2}FK
\]
\[
[E,F] = (q - q^{-1})^{-1}(K - K^{-1})
\]
where \(q \in \mathbb{C}\setminus\{\pm 1, 0\}\), so that \(q - q^{-1} \in \mathbb{C}\) is invertible. Again, the 2 in the exponent is related to the Cartan matrix of \(\mathfrak{sl}_2(\mathbb{C})\). One could regard \(U_q\) as a deformation of \(U\) in the following sense. Consider \(\hat{U}_q \in \text{Alg}_C\) generated by \(K, K^{-1}, E, F, G\) satisfying
\[
KK^{-1} = K^{-1}K = 1, \ KE = q^2EK, \KF = q^{-2}FK
\]
\[
[E,F] = G, \ (q - q^{-1})G = K - K^{-1}
\]
\[
[E,G] = -q(EK + K^{-1}E), \ [F,G] = q^{-1}(FK + K^{-1}F)
\]
Then it is easy to verify that
\[
U_q \simeq \hat{U}_q, \ \hat{U}_1 \simeq U[K]/\langle K^2 - 1 \rangle, \ U \simeq \hat{U}_1/\langle K - 1 \rangle
\]
where the second isomorphism is constructed by \( E \mapsto X K, F \mapsto Y, K \mapsto K, G \mapsto HK \).

Now we can make \( U_q \in \text{Hpf}_C \) by making the following definitions:

\[
\begin{align*}
\Delta(K) &= K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad \Delta(E) = E \otimes K + 1 \otimes E \\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1, \quad \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0 \\
S(E) &= -E K^{-1}, \quad S(F) = -KF, \quad S(K^{\pm 1}) = K^{\mp 1}
\end{align*}
\]

Then \( U_q \) is neither commutative nor cocommutative. Moreover, the understanding of \( U_q \) as a deformation of \( U \) in \( \text{Alg}_C \) can be restricted into the subcategory \( \text{Hpf}_C \) by equipping the standard Hopf structures on both \( \tilde{U}_q \) and \( U \). Explicitly,

\[
\begin{align*}
\Delta(H) &= H \otimes 1 + 1 \otimes H, \quad \Delta(X) = X \otimes 1 + 1 \otimes X \\
\Delta(Y) &= Y \otimes 1 + 1 \otimes Y, \quad \epsilon(H) = \epsilon(X) = \epsilon(Y) = 0 \\
S(H) &= -H, \quad S(X) = -X, \quad S(Y) = -Y
\end{align*}
\]

So the above Hopf structure on \( U_q \) can be obtained by deforming the Hopf structure on \( U \) in some preferred way.

Nevertheless, \( U_q \) is not really a quasi-triangular Hopf algebra. It is actually true that for any odd \( d \in \mathbb{N} \),

\[
\tilde{U}_q^{(d)} := U_q / \langle E^d, F^d, K^d - 1 \rangle
\]

whose Hopf algebra structures is naturally induced by the natural projection from \( U_q \) to \( \tilde{U}_q^{(d)} \), is quasi-triangular. However, the universal \( R \)-matrix in each \( \tilde{U}_q^{(d)} \otimes \tilde{U}_q^{(d)} \) fails to extend to one in \( U_q \otimes U_q \) because the inverse limit will have a form in formal power series, hence we would like to introduce \( \hat{U}_h := U_h(\mathfrak{sl}_2(\mathbb{C})) \). Drinfeld has shown above in his famous paper \([D]\). Let’s try to give a little bit more details to these.

First we try to construct Drinfeld’s quantum double. Suppose \((H, \mu, \eta, \Delta, \epsilon, S) \in \text{Hpf}_C^{fd}\). Then as shown in \([\text{EGNO}]\), \( S \) must be invertible. From the given Hopf algebra \( H \), there are plenty of ways to obtain new Hopf algebras.

1. If we consider its opposite algebra, while keeping its coalgebra structure the same and equipping it with \( S^{-1} \in \text{End}(H^{op}) \). Then

\[
H^{op} := (H, \mu^{op}, \eta, \Delta, \epsilon, S^{-1}) \in \text{Hpf}_C^{fd}
\]

2. Similarly, if we consider its opposite coalgebra, while keeping its algebra structure the same and equipping it with \( S^{-1} \). Then

\[
H^{cop} := (H, \mu, \eta, \Delta^{op}, \epsilon, S^{-1}) \in \text{Hpf}_C^{fd}
\]

3. If we have both opposite algebra and opposite coalgebra structure at the same time, this gives \( H^{op,cop} := (H, \mu^{op}, \eta, \Delta^{op}, \epsilon, S) \in \text{Hpf}_C^{fd} \).
4. As $\text{dim}_\mathbb{C} H < \infty$, the dual of algebra structure becomes a coalgebra structure on $H^* \otimes H$ and the dual of coalgebra structure becomes an algebra structure. Then one can verify $(H^*, \Delta^*, e^*, \mu^*, \eta^*, S^*) \in \text{H}^{fd}_\mathbb{C}$.

5. Suppose $(H', \mu', \eta', \Delta', \epsilon', S')$ is another finite-dimensional $\mathbb{C}$-Hopf algebra. Then the (non-twisted) tensor product $(H \otimes H', \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon}, \tilde{S}) \in \text{H}^{fd}_\mathbb{C}$, where

$$
\begin{align*}
\tilde{\mu} &= (\mu \otimes \mu') \circ (\text{id}_H \otimes \tau_{H',H} \otimes \text{id}_H') \\
\tilde{\eta} &= (\eta \otimes \eta') \circ \ell_1 = (\eta \otimes \eta') \circ r_1 \\
\tilde{\Delta} &= (\text{id}_H \otimes \tau_{H,H'} \otimes \text{id}_H') \circ (\Delta \otimes \Delta') \\
\tilde{\epsilon} &= \ell_1 \circ (\epsilon \otimes \epsilon') = r_1 \circ (\epsilon \otimes \epsilon') \\
\tilde{S} &= S \otimes S'
\end{align*}
$$

And this definition of tensor product gives $\text{H}^{fd}_\mathbb{C}$ a monoidal category structure and makes it a rigid symmetric strict monoidal category.

6. Drinfeld’s quantum double $D(H)$ is a new Hopf algebra defined as the bi-crossed product (or the inter-twisted tensor product) of $(H^*)^* \otimes H$. It has the same coalgebra structure as the non-twisted tensor product $(H^*)^* \otimes H$, but its algebra structure is inter-twisted and it equips with a modified antipode. (One can verify that the uniqueness of antipodes in a Hopf algebra. Hence in our case, once we determine the bialgebra structure of $D(H)$, we have no choice but to modify the antipode from the simple $\tilde{S} = S \otimes S'$, if it ever exists.) Explicitly, given $f \otimes a, g \otimes b \in (H^*)^* \otimes H$, then $\mu^D : ((H^*)^* \otimes H) \otimes ((H^*)^* \otimes H) \rightarrow (H^*)^* \otimes H$ is defined by

$$(f \otimes a) \otimes (g \otimes b) \mapsto \sum_{(a)(g)} \Delta^*(f \otimes (a' \circ g')) \otimes \mu((a'' \circ g'') \otimes b)$$

where we adapt the Sweedler’s notation by writing $\Delta(a)$ as $\sum_{(a)} a' \otimes a''$ for every $a \in H$ and $(\mu^o)^*(g)$ as $\sum_{(g)} g' \otimes g''$ for every $g \in (H^o)^*$, and $\circ \in \text{CoAlg}^{fd}_\mathbb{C}(H \otimes (H^o)^*, (H^o)^*)$ is the generalized left adjoint representation defined by

$$h \otimes f \mapsto \left[ x \mapsto \sum_{(h)} f(S^{-1}(h'')xh') \right]$$

which makes $(H^o)^* \in \text{HMod}$. One can see this definition is a generalization of the usual left adjoint representation which occurs when $\Delta(x) = x \otimes x$ and $S^{-1}(x) = x^{-1}$. Similarly, $\circ \in \text{CoAlg}^{fd}_\mathbb{C}(H \otimes (H^o)^*, H)$ is the generalized right adjoint representation defined by
\[ h \otimes f \mapsto \sum_{(h)} f(S^{-1}(h'')h')h'' \]

where \(((\Delta \otimes \text{id}_H) \circ \Delta)(h) = ((\text{id}_H \otimes \Delta) \circ \Delta)(h) = \sum_{(h)} h' \otimes h'' \otimes h'''\) is the Sweedler notation by the coassociativity of \(\Delta\). So one can further write \(\sum_{(h)} h' \otimes \cdots \otimes h^{(n)}\) by generalizing this notation a little bit without any confusion. This adjoint action also makes \(H\) a right \((H^\text{op})^*\) module.

Then \(\mu^D\) can be further simplified as

\[
(f \otimes a) \otimes (g \otimes b) \mapsto \sum_{(a)(g)} \left[ x \mapsto \sum_{(x)} f(x')g(S^{-1}(a'(5))a'(3))a'(4)b \right] \otimes a'(4)b
\]

\[
= \sum_{(a)(g)} \left[ x \mapsto \sum_{(x)} f(x')g(S^{-1}(a'(5))a'(3))S^{-1}(a''(3))x''a'') \right] \otimes a'(3)b
\]

\[
= \sum_{(a)(g)} \left[ x \mapsto \epsilon(a'') \sum_{(x)} f(x')g(S^{-1}(a'(4))x''a'') \right] \otimes a''b
\]

Let’s also define \(S^D \in \text{Vec}_{fd}^f((H^\text{op})^* \otimes H, (H^\text{op})^* \otimes H)\) by

\[
f \otimes h \mapsto \sum_{(f)(h)} (S(h'') \circ (S^{-1})^* (f'')) \otimes (S(h') \circ (S^{-1})^* (f'))
\]

Then \(D(H) = ((H^\text{op})^* \otimes H, \mu^D, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon}, S^D) \in \text{Hpf}_{fd}^f\), where \(\tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon}\) are defined exactly the same as in the non-twisted tensor product, with \(\eta' = \epsilon^*, \Delta' = (\mu^\text{op})^*\) and \(\epsilon' = \eta^*\). The notion of bicrossed product of Hopf algebras is motivated by the bicrossed product of groups and extended to the special Hopf algebra, the group algebras. These details can be found in \([K]\).

This construction is particularly useful because \(D(H)\) is quasi-triangular, for every \(H \in \text{Hpf}_{fd}^f\). The universal \(R\)-matrix is

\[
R = \sum_I (1 \otimes e_i) \otimes (f^i \otimes 1)
\]

where \((e_i)_I\) is a basis of \(H\) as a finite-dimensional \(\mathbb{C}\)-vector space and \((f^i)_I\) is its dual basis of \(H^*\). Its inverse \(R^{-1}\), as one can verify, is \(\sum_I (1 \otimes e_i) \otimes (S^*(f^i) \otimes 1)\).
Now we are ready to go back to show $\bar{U}_q^{(d)}$ is quasi-triangular. Let’s only do the case when $d$ is odd, which is the case that is focused in this paper. First one can verify $\bar{U}_q^{(d)}$ has the Poincaré-Birkhoff-Witt basis $\{ E^a K^b F^c \}_{0 \leq a,b,c \leq d-1}$. If we consider the Hopf subalgebra $\bar{B}_q^{(d)}$ to be the vector subspace generated by $\{ E^a K^b \}_{0 \leq a,b,c \leq d-1}$, equipped with the Hopf structure induced from $\bar{U}_q^{(d)}$, then the quantum double $D(\bar{B}_q^{(d)})$ is quasi-triangular and there is a surjective morphism from $D(\bar{B}_q^{(d)})$ to $\bar{U}_q^{(d)}$. To give more details, if we define $\alpha, \beta \in (\bar{B}_q^{(d)})^*$ by $\alpha(E^a K^b) = \delta_{ab} q^{2b}$ and $\beta(E^a K^b) = \delta_{a1}$, then $D(\bar{B}_q^{(d)})$ has a basis $(\alpha^i \beta^j \otimes E^a K^b)_{0 \leq i,j,a,b \leq d-1}$ with the following relations determining its algebra structure

$$(1 \otimes E)(\alpha \otimes 1) = -q^{-2}[1 \otimes 1 - (\alpha \otimes 1)(1 \otimes E) - (\beta \otimes 1)(1 \otimes K)]$$

$$(1 \otimes K)(\alpha \otimes 1) = q^{-2}(\alpha \otimes 1)(1 \otimes K)$$

$$(1 \otimes E)(\beta \otimes 1) = q^{-2}(\beta \otimes 1)(1 \otimes E)$$

$$(1 \otimes K)(\beta \otimes 1) = (\beta \otimes 1)(1 \otimes K)$$

And one can verify $\psi \in \text{Hopf}_c^{fd}(D(\bar{B}_q^{(d)}), \bar{U}_q^{(d)})$ defined by

$$\alpha^i \beta^j \otimes E^a K^b \mapsto \left( \frac{q - q^{-1}}{q^2} \right)^i q^{2(i+j)a-i(i-1)} F^i E^a K^{i+j+b}$$

is surjective. Let’s call $R_D^{(d)}$ the universal $R$-matrix of $D(\bar{B}_q^{(d)})$ defined as in the construction of Drinfeld quantum double. Then $R^{(d)} := (\psi \otimes \psi)(R_D^{(d)})$ is an universal $R$-matrix for $\bar{U}_q^{(d)}$. Explicitly we will have

$$R^{(d)} = \sum_{0 \leq a,b,c \leq d-1} \frac{(q - q^{-1})^c}{d[c]!} q^{c(c-1)/2 + 2a(c-b)} E^c K^a \otimes K^b F^c$$

where $[c]! := [c]_q \cdots [1]_q$ with $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$.

As mentioned before, the universal $R$-matrix defined for each $\bar{U}_q^{(d)}$, unfortunately, won’t extend to one for $U_q$. One has to consider a larger Hopf algebra, which is usually called the Drinfeld-Jimbo algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$.

Let’s write $U_h = U_h(\mathfrak{sl}_2(\mathbb{C}))$, the quantum enveloping algebra for $\mathfrak{sl}_2(\mathbb{C})$, as

$$\mathbb{C} \langle X, Y, H \rangle [[h]]/\langle QS \rangle$$

where $QS$ is the collection of the quantum Chevalley-Serre relations:

$$[X,Y] = \frac{1}{h} \left( e^{\frac{h}{2}X} - e^{-\frac{h}{2}X} \right), \quad [H,X] = 2X, \quad [H,Y] = -2Y$$
(The other two quantum Serre relations are null for the case $g = sl_2(\mathbb{C})$.) Notice that $e^{\frac{h}{2}} - e^{-\frac{h}{2}}$ is not invertible in $\mathbb{C}[[h]]$, but $\frac{1}{h}\left(e^{\frac{h}{2}} - e^{-\frac{h}{2}}\right) = \frac{2}{h} \sinh\left(\frac{h}{2}\right) \in \mathbb{C}[[h]]$ is invertible. Therefore we purposely write the above relation $[X,Y]$ in that way. Its coalgebra structure and antipode is defined by

$$
\Delta_h(H) = H \otimes 1 + 1 \otimes H, \quad \Delta_h(X) = X \otimes e^{\frac{dH}{4h}} + e^{-\frac{dH}{4h}} \otimes X
$$

$\Delta_h(Y) = Y \otimes e^{\frac{dH}{4h}} + e^{-\frac{dH}{4h}} \otimes Y$, $\epsilon_h(H) = \epsilon_h(X) = \epsilon_h(Y) = 0$

with an universal $R$-matrix

$$
R_h = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} \frac{(q - q^{-1})^n}{[n]_q!} q^ {\frac{n(n-1)}{2}} \left(X_q \frac{H}{2}\right)^n \otimes \left(q^{-\frac{H}{2}} Y\right)^n
$$

where $q = e^{\frac{h}{2}}$. As shown in [K], there is a way to construct $U_h$ and its universal $R$-matrix $R_h$ by generalizing the Drinfeld’s quantum double construction from finite dimensional Hopf algebras to topological bialgebras. Moreover, one can regard $U_q$ as a sub-bialgebra of $U_h$ by considering $E = X_q \frac{H}{2}$, $F = q^{-\frac{H}{2}} Y$ and $K = e^{\frac{H}{2}}$. Then one can recover the universal $R$-matrix $R^{(d)}$ of $U^{(d)}_q$ mentioned before from $R_h$.

One can see the above argument and construction should easily be able to generalize to any $U_h(g)$, where $g$ is any complex semisimple Lie algebra.

The reason why a (pseudo)quasitriangular Hopf algebra with invertible antipode is so crucial in the construction of invariant of knot is because its universal $R$-matrix naturally generates a solution to Yang-Baxter equation on its modules, which naturally corresponds to the type III Reidmeister move, which will be introduced in the next section. To be explicit, say $H \in H_{pf}$ is quasitriangular and $R$ is its universal $R$-matrix, then $\forall U, V, W \in H_{Mod}$, we have the so-called Yang-Baxter equation

$$(c_{V,W} \otimes \text{id}_U)(\text{id}_V \otimes c_{U,W})(c_{U,V} \otimes \text{id}_W) = (\text{id}_V \otimes c_{U,V})(c_{U,W} \otimes \text{id}_V)(\text{id}_U \otimes c_{V,W})$$

where $c_{U,V} \in U \otimes V \rightarrow V \otimes U$ is a natural isomorphism defined by $u \otimes v \mapsto \tau_{U,V} (R(u \otimes v))$, and $\tau_{U,V}(u \otimes v) = v \otimes u$.

### 1.3 Poisson-Lie Groups and Lie Bialgebras

In the following, a real manifold always means a smooth real manifold. A complex manifold always means a holomorphic complex manifold. A manifold is either real or complex.

**Definition** (Symplectic Manifold). Suppose $M$ is a manifold. Then $M$ is a symplectic manifold if it is equipped with a symplectic form, which is an $\omega \in \Omega^2(M) = \Gamma_M (\Lambda^2(T^*M))$ satisfying $\forall p \in M, \omega_p$ is non-degenerate and $d\omega = 0$. 

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A symplectic form $\omega$ naturally induces an isomorphism between $\Gamma_M(TM)$ and $\Omega^1(M)$. In this way, for any (Hamiltonian) function $H \in C^\infty(M)$, one can find the associated Hamiltonian vector field $X_H$ so that $i_{X_H} \omega = dH$. In particular, on the symplectic manifold $\left(\mathbb{R}^{2n}, \omega = \sum dq_i \wedge dp_i\right)$, the flow line $\Phi_H$ generated by $X_H$ gives the time evolution of the physics system whose Hamiltonian function is $H$, obeying the Hamilton’s equations.

To describe the same physics system, instead of constructing a symplectic form, one can also just define a Hamiltonian structure, or a Poisson structure on a smooth manifold $M$ as follows.

**Definition (Poisson Manifold).** Suppose $M$ is a manifold. Then $M$ is a Poisson manifold if there is a Poisson bracket $\{ -, - \}$ on $C^\infty(M)$ so that it becomes a Lie algebra, and $\{ f, - \}$ is a derivation for any $f \in C^\infty(M)$. That is, for every $g, h \in C^\infty(M)$,

$$\{ f, gh \} = \{ f, g \} h + g \{ f, h \}$$

It is equivalent to define a Poisson bivector $\pi \in \Gamma_M(\bigwedge^2 TM)$ on $M$ satisfying $[\pi, \pi]_S = 0$, where $[ -, - ]_S$ is the Schouten-Nijenhuis bracket, the unique extension of the Lie algebra $(\Gamma_M(TM), [ -, - ]_S)$ to the (graded) exterior algebra $\Gamma_M(\bigwedge^\bullet TM)$, making it into a Gerstenhaber algebra. Then they are related in the following relationship

$$\pi(df \wedge dg) = \{ f, g \}$$

Given two Poisson manifolds $M_1, M_2$. We say $f : M_1 \to M_2$ is a Poisson map if it is a smooth map and satisfies

$$f^* \{ -, - \}_{M_2} = \{ -, - \}_{M_1} \circ (f^*, f^*)$$

Or equivalently, for every $p \in M_1$,

$$\left(\bigwedge^2(T_pf)\right)(\pi_1)_p = (\pi_2)_{f(p)}$$

Then every symplectic manifold $(M, \omega)$ is a Poisson manifold, whose Poisson structure is defined by $\{ f, g \} := \omega(X_f, X_g)$. Now we try to discuss for a given Lie group $G$, how one could equip a Poisson structure on $G$ so that it is compatible with its group structure.

**Definition (Product Poisson Structure).** Given Poisson manifolds $M_1, M_2$. Then $M_1 \times M_2$ can be endowed with a Poisson structure defined by

$$\{ f, g \}_{M_1 \times M_2} (x, y) := \{ f(-, y), g(-, y) \}_{M_1}(x) + \{ f(x, -), g(x, -) \}_{M_2}(y)$$

which is the unique Poisson structure so that $p_i : M_1 \times M_2 \to M_i$ are Poisson maps and $\forall f_i \in C^\infty(M_i), \{ p_1^* f_1, p_2^* f_2 \} = 0$. 

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Definition (Poisson-Lie Group). A Poisson-Lie group \((G, \pi)\) is a Lie group \(G\) equipped with a Poisson structure \(\pi\) such that the product map \(\mu : G \times G \to G\) is a Poisson map if \(G \times G\) is endowed with the product Poisson structure.

Given a Poisson-Lie group \((G, \pi)\), as \(G\) is a Lie group, we know its tangent space at \(e\) is a Lie algebra, say \(\mathfrak{g} = T_e G\). We would like to know how would the Poisson structure gives more structures on \(\mathfrak{g}\). It turns out that \(\pi\) induces a natural linear Poisson structure on \(\mathfrak{g}\), and this linear Poisson structure is equivalent to a Lie algebra structure on \(\mathfrak{g}^*\), which leads us to Definition of Lie bialgebra.

Definition (Lie Bialgebra). Suppose \(\mathfrak{g} \in \text{Vec}_C^f\), \([\cdot,\cdot]_0 \in \text{Vec}_C^f(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})\) and \([\cdot,\cdot]_{0^*} \in \text{Vec}_C^f(\mathfrak{g}^* \times \mathfrak{g}^*, \mathfrak{g}^*)\). Then \((\mathfrak{g}, [\cdot,\cdot]_0, [\cdot,\cdot]_{0^*})\) is a Lie bialgebra if \((\mathfrak{g}, [\cdot,\cdot]_0), (\mathfrak{g}^*, [\cdot,\cdot]_{0^*})\) and \((\mathfrak{d}, [\cdot,\cdot]_0)\) are all Lie algebras, where \(\mathfrak{d} = \mathfrak{g} \times \mathfrak{g}^*\) and

\[
[(x, \xi), (y, \eta)]_\mathfrak{d} := ([x, y]_0 + (\text{ad}_{g^*})_\mathfrak{g} \xi y - (\text{ad}_{g^*})_\mathfrak{g} \eta x, [\xi, \eta]_0 + (\text{ad}_{\mathfrak{g}^*})_\mathfrak{g} \xi \eta - (\text{ad}_{\mathfrak{g}^*})_\mathfrak{g} \xi \eta).
\]

where \(\text{ad}_{g^*} : \mathfrak{g}^* \to \text{End}(\mathfrak{g})\) is the coadjoint representation of \(\mathfrak{g}^*\), that is, for every \(\zeta, \mu \in \mathfrak{g}^*\) and \(x \in \mathfrak{g}\),

\[
\mu((\text{ad}_{g^*})_\mathfrak{g}(\zeta)(x)) = [\mu, \zeta]_{0^*}(x)
\]

and \(\text{ad}_{\mathfrak{g}} : \mathfrak{g} \to \text{End}(\mathfrak{g}^*)\) is the coadjoint representation of \(\mathfrak{g}\) that satisfies

\[
((\text{ad}_{\mathfrak{g}^*})_\mathfrak{g}(\eta))(y) = \zeta([y, x]_\mathfrak{g})
\]

for all \(\zeta \in \mathfrak{g}^*\) and \(x, y \in \mathfrak{g}\). They are defined in the same way if we replace \(\mathfrak{g}\) by \(\mathfrak{g}^*\) and regard \(\mathfrak{g}^{**} \cong \mathfrak{g}\). \(\mathfrak{d}\) is then called the double of the Lie bialgebra \((\mathfrak{g}, [\cdot,\cdot]_0, [\cdot,\cdot]_{0^*})\).

Notice that a Lie bialgebra is indeed a Lie-theoretical version of bialgebra mentioned in \([1.1]\). Explicitly, given a Lie bialgebra \((\mathfrak{g}, [\cdot,\cdot]_0, [\cdot,\cdot]_{0^*})\), it is equivalent to consider \((\mathfrak{g}, [\cdot,\cdot]_0, \gamma : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g})\) so that \((\mathfrak{g}, \gamma)\) is a Lie coalgebra, that is, \(\gamma\) satisfies the 1-cocycle condition, where \(\mathfrak{g}\) acts on \(\mathfrak{g} \wedge \mathfrak{g}\) by the adjoint representation. Explicitly we have

\[
\gamma([x, y]_\mathfrak{g}) = (\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_y) \gamma(y) - (\text{ad}_y \otimes 1 + 1 \otimes \text{ad}_y) \gamma(x) = [x, \gamma(y)]_{0^*} + [\gamma(x), y]_{0^*},
\]

where \(\gamma\) is the map dual to \([\cdot,\cdot]_{0^*} : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*\), \(\text{ad}_x(y) = [x, y]_\mathfrak{g}\) and \([\cdot,\cdot]_{0^*}\) is the Schouten bracket extending the Lie bracket \([\cdot,\cdot]_0\) on \(\mathfrak{g}\) to \(\wedge^* \mathfrak{g}\). Then the double \(\mathfrak{d}\) should be regarded as a Lie-theoretical version of Drinfeld quantum double and \(\mathfrak{d}\) satisfies the condition to be quasi-triangular, that is, satisfies the classical Yang-Baxter equation.

Proposition 1.2. Suppose \((G, \pi)\) is a Poisson-Lie group. Then \(\pi_e = 0\) and this induces the linearized Poisson structure \(\pi_1\) on \(\mathfrak{g} = T_e G\), i.e.,

\[
\forall F, G \in C^\infty_e(G), \quad d_e \{F, G\} = \pi_1(d_eF \wedge d_eG)
\]
where $C^\infty_e(G)$ is the germ of smooth functions on $G$ at $e$ and $\{\cdot,\cdot\}$ is associated to $\pi$. Then $\pi_1$ naturally corresponds to the unique Lie-algebra structure on $\mathfrak{g}^*$ and it makes $\mathfrak{g}$ a Lie bialgebra.

**Proof.** Since $\mu : G \times G \to G$ is a Poisson map, this implies $\pi_e = \pi_e + \pi_e \to \pi_e = 0$. One then can check $(\ast)$ defines a Poisson structure $\pi_1$ on $\mathfrak{g}$. The Poisson bracket $\{\cdot,\cdot\}_1$ associated to $\pi_1$ is a Lie algebra on $\mathfrak{g}^*$. Finally, we refer to [LPV] to verify that $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\{\cdot,\cdot\}_1)$ is a Lie bialgebra. \qed

Just like in the Lie algebra case, one would then want to ask whether Lie bialgebras can be integrated, that is, one would like to know whether for any finite dimensional Lie bialgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\{\cdot,\cdot\}_{\mathfrak{g}^*})$, there exists a Poisson Lie group such that its tangent space at identity produces it or not. Because we already know there is a (not necessary unique!) connected Lie group $G$ so that $T_eG = \mathfrak{g}$, we reduce to ask whether we can equip $G$ with a compatible Poisson structure. The answer is affirmative and this equipment is unique, as shown below.

**Proposition 1.3.** Suppose $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\{\cdot,\cdot\}_{\mathfrak{g}^*})$ is a finite-dimensional Lie bialgebra and $G$ is the connected and simply connected Lie group such that its Lie algebra is $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$. Then there exists an unique Poisson structure $\pi \in \Gamma_G (\wedge^2TG)$ so that $(G,\pi)$ is a Poisson-Lie group whose Lie bialgebra is $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\{\cdot,\cdot\}_{\mathfrak{g}^*})$.

**Proof.** Let’s consider the double $(\mathfrak{d},[\cdot,\cdot]_{\mathfrak{d}})$ of $\mathfrak{g}$. Then there is a canonical $r$-matrix

$$r = \frac{1}{2} \sum_{i=1}^{d} e_i \wedge f^i \in \mathfrak{d} \wedge \mathfrak{d}$$

where $\{e_i\}$ is a basis of $\mathfrak{g}$ and $\{f^i\}$ is the dual basis of $\mathfrak{g}^*$ with respect to the standard bilinear form $\langle e_i, e_j \rangle = \delta_{ij}$, and $r$ would induce the Lie algebra structure on $\mathfrak{d}^*$ by

$$[\xi, \eta]_{\mathfrak{d}^*}(x) := (\xi \wedge \eta)(\delta r(x)), \quad \forall \xi, \eta \in \mathfrak{d}^*, x \in \mathfrak{d}$$

where $\delta r(x) := \text{ad}_x(r)$ is the 1-coboundary of $r$, making the double $\mathfrak{d}$ a Lie bialgebra and $\mathfrak{g}$ as its Lie sub-bialgebra. Now consider the unique connected and simply connected Lie group $B$ such that its Lie algebra is $\mathfrak{d}$. Then $(B,\pi_r)$ is a Poisson-Lie group, where $\pi_r := r^L - r^R, r^L \in \Gamma_B (\wedge^2TB)$ is the left translation of $r \in \wedge^2 T_eB$ in $B$ and similarly for $r^R$. Then the Poisson-Lie subgroup $(G,\pi_r|_G)$ associated to the Lie sub-bialgebra $\mathfrak{g}$ is the one we are looking for. And the uniqueness is basically an extension of Lie’s third theorem. We refer to [LPV] for more details. \qed

**Definition** (Dual Lie Bialgebra). Given a Lie bialgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\{\cdot,\cdot\}_{\mathfrak{g}^*})$. Then its Lie bialgebra dual is $(\mathfrak{g}^*,[\cdot,\cdot]_{\mathfrak{g}^*},\{\cdot,\cdot\}_{\mathfrak{g}^*})$, where we identify $\mathfrak{g}$ with $\mathfrak{g}^{**}$ by the usual map $x \mapsto [\varphi \mapsto \varphi(x)]$. Since $\mathfrak{g} \in \text{Vec}_C^d$, this identification is an isomorphism.
Notice that because we are in the setting of finite dimensional vector space, the double Lie bialgebra dual of a Lie bialgebra can be identified with itself. Therefore we may say they are dual to each other.

**Definition** (Dual Poisson-Lie Group). Given two Poisson-Lie groups \((G, \pi_G), (H, \pi_H)\). They are dual to each other if their Lie bialgebras, tangent spaces at the identity, are dual to each other. Then we denote \((H, \pi_H)\) as \((G^*, \pi_{G^*})\).

### 1.4 The Dual Pair \(SL_2(\mathbb{C})\) and \((SL_2(\mathbb{C}))^*\)

In the following, we consider the dual pair of Poisson-Lie group \(SL_2(\mathbb{C})\) and \((SL_2(\mathbb{C}))^*\) that is important in this paper.

Since we have discussed the Lie algebra \(\mathfrak{sl}_2(\mathbb{C})\), we would like to discuss the Lie algebra structure on its dual vector space. \((\mathfrak{sl}_2(\mathbb{C}))^*\) has the dual basis \([,]_\mathfrak{sl}(\mathbb{C})^*\), as defined in the Lie bialgebra definition, with the relation

\[
[X^*, Y^*_\mathfrak{sl}(\mathbb{C})^*] = 0, [H^*, X^*_\mathfrak{sl}(\mathbb{C})^*] = \frac{1}{4}X^*, [H^*, Y^*_\mathfrak{sl}(\mathbb{C})^*] = \frac{1}{4}Y^*
\]

Then its double \(\mathfrak{d} = (\mathfrak{sl}_2(\mathbb{C})) \times (\mathfrak{sl}_2(\mathbb{C}))^*\) has the bracket \([,]_\mathfrak{d}\), as defined in the Lie bialgebra definition, with the relation

\[
[H, H^*_\mathfrak{d}] = 0, [H, X^*_\mathfrak{d}] = -2X^*, [H, Y^*_\mathfrak{d}] = 2Y^*
\]

\[
[X, H^*_\mathfrak{d}] = \left(\frac{1}{4}X, -Y^*\right), [X, X^*_\mathfrak{d}] = \left(-\frac{1}{4}H, 2H^*\right), [X, Y^*_\mathfrak{d}] = 0
\]

\[
[Y, H^*_\mathfrak{d}] = \left(\frac{1}{4}Y, X^*\right), [Y, X^*_\mathfrak{d}] = 0, [Y, Y^*_\mathfrak{d}] = \left(-\frac{1}{4}H, -2H^*\right)
\]

where we regard elements of \(\mathfrak{sl}_2(\mathbb{C})\) are in \(\mathfrak{d}\) using the injective Lie algebra morphism \(\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{d}\), defined by \(x \mapsto (x, 0)\). Similarly for elements of \((\mathfrak{sl}_2(\mathbb{C}))^*\). One can check that \([,]_\mathfrak{d}\) is a Lie bracket, hence \((\mathfrak{sl}_2(\mathbb{C}), [,], [[,]_\mathfrak{sl}(\mathbb{C})^*])\) is a Lie bialgebra and its dual is \(((\mathfrak{sl}_2(\mathbb{C}))^*[,], [[,]_\mathfrak{sl}(\mathbb{C})^*], [[,]])\). Then \((SL_2(\mathbb{C}))^*\) is the connected and simply connected Lie group whose Lie algebra is \((\mathfrak{sl}_2(\mathbb{C}))^*\). For our purpose, we would like to understand \((SL_2(\mathbb{C}))^*\) as a Lie subgroup of \(B_+ \times B_-\), where \(B_+\) is the upper Borel subgroup and \(B_-\) is the lower Borel subgroup of \(SL_2(\mathbb{C})\), which makes \(SL_2(\mathbb{C})\) factorizable.

**Definition** (Factorizable). A Lie group \(G\) is said to be (locally) factorizable if there exist a Lie subgroup \(G'\), as an open neighborhood of \(e\), Lie subgroups \(G_+, G_-\), and a Lie subgroup \(G^*\) of \(G_+ \times G_-\), as an open neighborhood of \((e, e)\), such that

\[G^* \subset G_+ \times G_- \to G'\]

defined by \((g_+, g_-) \mapsto g_+g_-^{-1}\) is a covering map in the category of manifolds.
**Definition** (Manin Triple). A Manin triple consists of a Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\), an adjoint invariant non-degenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle\) and two subalgebras \(\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}\) so that they are complementary isotropic, i.e.,

\[
\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}, \quad \langle \mathfrak{a}, \mathfrak{a} \rangle = \langle \mathfrak{b}, \mathfrak{b} \rangle = 0
\]

We would denote such a Manin triple as \(((\mathfrak{g}, [\cdot, \cdot]), \langle \cdot, \cdot \rangle), \mathfrak{a}, \mathfrak{b})\).

Suppose \(G\) is a Lie group whose Lie algebra is \(\mathfrak{g}\). Recall that a bilinear form \(\langle \cdot, \cdot \rangle\) is Ad-invariant if

\[
\langle \operatorname{Ad}_g x, \operatorname{Ad}_g y \rangle = \langle x, y \rangle
\]

for every \(x, y \in \mathfrak{g}\) and \(g \in G\). One then can deduce that \(\langle [x, z], y \rangle = \langle x, [z, y] \rangle\) for every \(z \in \mathfrak{g}\) and any bilinear form satisfying this condition is called adjoint invariant.

One example of a Manin triple is \(((\mathfrak{g}, [\cdot, \cdot]), \langle \cdot, \cdot \rangle), \mathfrak{g}, \mathfrak{g}^*)\), where \(\langle (x, \xi), (y, \eta) \rangle_{\mathfrak{g}} := \xi(y) + \eta(x)\). Conversely, if \(((\mathfrak{g}, [\cdot, \cdot]), \langle \cdot, \cdot \rangle), \mathfrak{a}, \mathfrak{b})\) is a Manin triple, then

\[
\begin{align*}
\langle \mathfrak{a}, [\cdot, \cdot] \rangle_{\mathfrak{a} \times \mathfrak{a}} : [\Psi_{\mathfrak{a}}^{-1}, \Psi_{\mathfrak{a}}^{-1}]_{\mathfrak{a} \times \mathfrak{a}},
\end{align*}
\]

is a Lie bialgebra, where \(\Psi_{\mathfrak{b}} : \mathfrak{b} \simeq \mathfrak{a}^*\) is defined by \(x \mapsto \langle x, \cdot \rangle\). And its dual Lie bialgebra is \((\mathfrak{b}, [\cdot, \cdot]_{\mathfrak{b} \times \mathfrak{b}}, [\Psi_{\mathfrak{a}}^{-1}, \Psi_{\mathfrak{a}}^{-1}]_{\mathfrak{a} \times \mathfrak{a}}\), where \(\Psi_{\mathfrak{b}}\) is defined similarly. This gives a natural one-to-one correspondence between Lie bialgebras and Manin triples.

Now in order to understand \((\mathfrak{sl}_2(\mathbb{C}))^*\) more, we would like to view the construction above in a different way. Notice that in the construction above, we define the Lie algebra structure on \(\mathfrak{g}^*\) pretty artificially. It shall be better understood in the following way.

Consider the upper Borel subalgebra \(\mathfrak{b}_+ = \mathbb{C}X \oplus \mathbb{C}H\) of \(\mathfrak{sl}_2(\mathbb{C})\) with Cartan subalgebra \(\mathfrak{h} = \mathbb{C}H\), and denote the lower Borel subalgebra \(\mathfrak{b}_- = \mathbb{C}Y \oplus \mathbb{C}H\). Then one can check that \(((\mathfrak{d}, [\cdot, \cdot], \langle \cdot, \cdot \rangle), \mathfrak{s}, \mathfrak{t})\) is a Manin triple, where

\[
\begin{align*}
\mathfrak{d} &= \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \\
\langle (x, y), (u, v) \rangle_{\mathfrak{d}} &= \langle [x, u], [y, v] \rangle, \\
\langle (x, y), (u, v) \rangle_{\mathfrak{d}} &= -\frac{1}{2} \langle (x, u)_{\mathfrak{K}} - \langle y, v \rangle_{\mathfrak{K}} \rangle, \\
\mathfrak{s} &= \{(x, x) : x \in \mathfrak{sl}_2\} \cong \mathfrak{sl}_2, \\
\mathfrak{t} &= \{(aX + bH, -bH + cY) \in \mathfrak{b}_+ \oplus \mathfrak{b}_-\}
\end{align*}
\]

and \(\langle x, u \rangle_{\mathfrak{K}} = \operatorname{Tr}(\operatorname{ad}_x \operatorname{ad}_u)\) is the Killing form on \(\mathfrak{sl}_2\). Notice that the bilinear form \(\langle \cdot, \cdot \rangle_{\mathfrak{d}}\) is the easiest way to be defined so that \(\mathfrak{s}\) and \(\mathfrak{t}\) are both isotropic. Then \(\langle \cdot, \cdot \rangle_{\mathfrak{d}}\) induces \(\Psi : \mathfrak{t} \cong \mathfrak{s}^* \cong \mathfrak{sl}_2^*\) and its Lie bialgebra structure. Moreover, this bialgebra structure is isomorphic to the one we have defined before: the diagnoal map identifies \(H, X, Y \in \mathfrak{sl}_2\) with \(e_1 = (H, H), e_2 = (X, X), e_3 = (Y, Y) \in \mathfrak{s}\).

Suppose we define \(f_1 = \frac{-1}{8} (H, -H), f_2 = \frac{1}{2} (0, Y), f_3 = \frac{-1}{2} (X, 0) \in \mathfrak{t}\), then \(\langle f_i, e_j \rangle_{\mathfrak{d}} = \ldots\).
Now one can easily check that under the identification $f_1$ with $H^*$, $f_2$ with $X^*$ and $f_3$ with $Y^*$, their Lie brackets have exactly the same commutation relations.

This understanding makes one able to write

$$(SL_2(\mathbb{C}))^* \cong \left\{ \left[ \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right], \left[ \begin{array}{cc} a^{-1} & 0 \\ c & a \end{array} \right] : a \in \mathbb{C}^*, b, c \in \mathbb{C} \right\} \subset B_+ \times B_-$$

which makes $SL_2(\mathbb{C})$ a factorizable group by combining with the surjective map $B_+ \times B_- \to SL_2(\mathbb{C})$ defined by $(g_+, g_-) \mapsto g_+g_-^{-1}$ because $(SL_2(\mathbb{C}))^* \to SL_2(\mathbb{C})$ is now a generically 2-to-1 covering map, whose image is an open neighborhood of $\left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$ consisting of matrices whose lower-right entry is nonzero.

2 Categories of Tangles and Their Diagrams

2.1 Tangles and Their Diagrams

Definition (Link/Knot). A link $L \subset \mathbb{R}^3$ is a disjoint union of $L_i$, where $L_i = \phi_i(S^1)$ and $\phi_i : S^1 \to \mathbb{R}^3$ are (continuous) embeddings. A knot is a link with only one connected component.

Definition (Tangle). Let $k, \ell \in \mathbb{N}$. A tangle $T \subset \mathbb{R}^2 \times [0, 1]$ of type $(k, \ell)$ is a disjoint union of $L_i$ and $T_j$, where $L_i = \phi_i(S^1)$ and $T_j = \psi_j([0, 1])$ and $\phi_i, \psi_j$ are embeddings so that (1) $L_i \subset \mathbb{R}^2 \times (0, 1)$ (2) $\partial T = \partial_0 T \sqcup \partial_1 T$, where $\partial_0 T = T \cap (\mathbb{R}^2 \times \{0\}) = \{(0, 1, 0), \ldots, (0, k, 0)\}$, $\partial_1 T = T \cap (\mathbb{R}^2 \times \{1\}) = \{(0, 1, 1), \ldots, (0, \ell, 1)\}$.

Then one can easily regard any link $L$ as a tangle of type $(0, 0)$ by considering a link $L' \subset \mathbb{R}^2 \times (0, 1)$ that is isotopic to $L$.

Definition (Orientation/Framing). A tangle $T = (\bigsqcup L_i) \sqcup (\bigsqcup T_j)$ is oriented if it is equipped with a non-singular continuous tangent vector field. A tangle is framed if it is equipped with a non-singular continuous normal vector field in $\mathbb{R}^2 \times [0, 1]$ so that in the endpoints of $T_j$, it is $(1, 0, 0)$. 

Figure 1: Knots and Links
**Definition** (Isotopy Class). An isotopy class of a tangle $T$, $[T]$, is an equivalent class of tangles $T'$ such that $T'$ is isotopic to $T$. If $T$ is oriented or framed, then $T'$ and $T$ are called isotopic if there is a continuous map $H : [0, 1] \times T(\mathbb{R}^2 \times [0, 1]) \to T(\mathbb{R}^2 \times [0, 1])$ so that $H(0, T) = T$, $H(1, T) = T'$ and $H(t, T)$ is an oriented or framed tangle for every $t$. Here we regard an oriented or framed $T$ is in the tangent bundle by abuse of notation.

One could note that if $T$ and $T'$ are isotopic, then they must be of the same type. Throughout this paper, we will only consider oriented and framed tangles. Whenever we mention a tangle $T$, it is always equipped with an orientation and framing even without being explicitly mentioned. Let us denote the set of isotopy classes of oriented and framed tangles of type $(k, \ell)$ by $T_{k,\ell}$. A link $L$ with $n$ component is called trivial if $L \in [O^\otimes n]$. 

Figure 2: Tangle of Type (3, 1)

Figure 3: Oriented Tangle
Definition (Tangle Diagram). A tangle diagram \( \Pi \) of type \((k, \ell)\) is a finite union of immersed \( S^1 \) and \([0, 1]\) in \( \mathbb{R} \times [0, 1] \) so that (1) immersed \( S^1 \) is in \( \mathbb{R} \times (0, 1) \) (2) \( \partial \Pi = \partial_0 \Pi \sqcup \partial_1 \Pi \), where \( \partial_0 \Pi = \Pi \cap (\mathbb{R} \times \{0\}) = \{(1,0), \cdots, (k,0)\} \), \( \partial_1 \Pi = \Pi \cap (\mathbb{R} \times \{1\}) = \{(1,1), \cdots, (\ell,1)\} \) (3) every two immersions intersect transversely and at each intersection, there are only two immersions passing through and it is equipped with the order of immersions, where the first is called the overcrossing one and the second is the undercrossing one.

![Figure 4: Framed Tangle](image)

(a) Positive Intersection  
(b) Negative Intersection

Figure 5: They are both part of a tangle diagram around intersections. And for both of them, the top-right to left-below curve is called overcrossing. And the other is called undercrossing.

The same as in the definition of the tangles, we would like to equip an orientation and a framing on any tangle diagram \( \Pi \). These structures could be understood as
the structures on a tangle $T$ that is very close to $\Pi$. That is, we can get $T$ by, at every intersection of $\Pi$, deforming the overcrossing one a little bit toward the reader and the undercrossing one a little bit away from the reader. Then an orientation and framing can be defined over $T$ and be regarded as extra information of $\Pi$. It should be easy to verify that this is well-defined, irrelevant to what $T$ one chooses. Again, we always assume that every tangle diagram discussed in this paper is oriented and framed.

Now we then are able to discuss isotopies of tangle diagrams: an isotopy of tangle diagrams $\Pi_1, \Pi_2$ is a continuous map $h : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ so that (1) $h(0, \Pi_1) = \Pi_1$, $h(1, \Pi_1) = \Pi_2$ (2) $h(t, \Pi_1)$ is a tangle diagram (3) $h$ continuously deforms orientation and framing and (4) the order of immersions at every intersection is preserved for every $t$, which means none of the intersection can disappear under the continuous deformation by $h$. Then we denote the collection of isotopy classes of tangle diagrams of type $(k, \ell)$ as $T_{k,\ell}^{diag}$.

We should also regard the way how we define orientation and framing as a connection between a tangle $T$ and its associated tangle diagram $\Pi_T$. More explicitly, for any tangle $T$, there is a tangle $T'$ isotopic to $T$ so that the projection of $T'$ onto the $y$-$z$ plane is a tangle diagram, which is denoted by $\Pi_T$. Although $\Pi_T$ is not well-defined, all the possible $\Pi_T$ are isotopic to each other. Therefore we can denote this isotopy class as $\Pi_T \in T_{k,\ell}^{diag}$ and this gives a map

$$Pr : T_{k,\ell} \to T_{k,\ell}^{diag}$$

given by $[T] \mapsto \Pi_T$. Our ultimate goal is trying to understand $T_{k,\ell}$ as much as possible. It is clear that $T_{k,\ell}^{diag}$ should be easier to deal with and the map $Pr$ provides a way to relate them. Actually $Pr$ is a surjective map and we know what the kernel of this map is.

**Theorem 2.1 (Reidemeister).** $T_{k,\ell} \cong T_{k,\ell}^{diag} / \sim$, where $\sim$ are generated by framed Reidemeister moves, as shown in Figure 6, 7 and 8.

**Proof.** We refer its proof to [AB].

\[ \square \]

### 2.2 The Category $T(G)$ of $G$-Tangles

Now we would want to give a procedure to produce a tangle invariant, that is, a surjective map from $T_{k,\ell}$ to some set $S$. (Ideally we would like to have a bijective map and have $S$ easier to understand. But it is certainly hard.) One usual trick, motivated by [RT], is to consider some coloring of tangles and tangle diagrams, and then produce an invariant from such coloring. Let’s give out some definitions first. Suppose $G$ is a simple complex algebraic group.
Figure 6: Framed Reidemeister Move Type I

Figure 7: Framed Reidemeister Move Type II

Figure 8: Framed Reidemeister Move Type III
Definition (G-tangles). A G-tangle of type \( (k, \ell) \) is a tangle \( T \) of type \( (k, \ell) \) equipped with a representation of the fundamental groupoid \( \pi_1(C_T) \) of the complement space \( C_T = \mathbb{R}^2 \times [0, 1] \setminus T \) into \( G \). Explicitly, it is a functor from \( \pi_1(C_T) \) to \( G \), where \( G \) is regarded as a category with single object \( \{*\} \) and \( G(*,*) = G \).

Since the complement space is path-connected, the fundamental groupoid \( \pi_1(C_T) \) is uniquely determined by any fundamental group at any chosen base point. We follow the construction in \([KR1]\) and choose the base point to be \( B = (0, y, 0) \) so that \( T \subset \mathbb{R} \times [y + 1, \infty) \times [0, 1] \), and denote the fundamental group \( \pi_1(C_T, B) \) as \( \pi_1(T) \). Then a representation of the fundamental groupoid of a G-tangle is equivalent to a group homomorphism \( \rho : \pi_1(T) \to G \), which is usually called the monodromy representation.

As another viewpoint, we can consider a trivial \( G \)-bundle over \( \mathbb{R}^2 \times [0, 1] \) and a flat connection \( A_T \in \Omega^1(C_T, g) = \Gamma_{C_T}(T^*C_T \otimes g) \), a \( G \)-equivariant \( g \)-valued one-form on \( C_T \) splitting its tangent space into horizontal and vertical subspaces and whose curvature is zero. Then the associated parallel transport gives a representation of the fundamental groupoid \( \pi_1(C_T) \) into \( G \). In particular, it gives a monodromy representation \( \rho \) of the fundamental group \( \pi_1(T) \) into \( G \). Conversely, as shown in \([Kh]\), it is well-known that one can construct a flat connection on the trivial \( G \)-bundle from a given monodromy representation. Therefore a G-tangle \( (T, \rho) \) can be regarded as a tangle \( T \) with a choice of flat connection on the trivial \( G \)-bundle over the complement space \( C_T \).

In the literature, for any given tangle \( T \) and any closed subgroup \( G \) of \( GL_n(\mathbb{C}) \), the moduli space \( \text{Rep}_T = \text{Hom}(\pi_1(T), G) \) of flat connections on \( C_T \) is called the representation variety. This is an affine variety whose coordinates are realized by the matrix entries of the images of the generators of \( \pi_1(T) \). And the character variety of \( T \) is the GIT quotient of \( \text{Rep}_T \) by \( G \), which is denoted as

\[ \mathcal{M}_T := \text{Rep}_T//G \]

This should be understood as quotienting by the group of gauge transformations. That is, \( G \) acts on \( \text{Rep}_T \) by conjugation, a.k.a., changing of basis on the trivial \( G \)-bundle.

If we consider the restriction of \( \pi_1(C_T) \) on \( \mathbb{R}^2 \setminus \{0\} \setminus \{(0, 1, 0, \cdots, (0, k, 0)\} \), then it is isomorphic to the fundamental group of \( \mathbb{R}^2 \setminus \{(0, 1), \cdots, (0, k)\} \). As before, it is equivalent to consider the fundamental group with any choice of base point. So we may choose \((0, -1)\) as the base point and denote this fundamental group by \( \pi_1(\partial_0 T) \), generated by homotopy paths \( \{\gamma_i\}_{i=1}^k \), where \( \gamma_i \) is any path homotopic to a path starting from \((0, -1)\), encircling \((0, 1), \cdots, (0, i)\) counterclockwisely, and going back to \((0, -1)\). Then the monodromy representation \( \rho : \pi_1(T) \to G \) would induce \( \rho_0 : \pi_1(\partial_0 T) \to G \), which is fully determined by \( h_0 = (\rho_0(\gamma_1), \cdots, \rho_0(\gamma_k)) \in G^k \). Similarly, we have \( \rho_1 : \pi_1(\partial_1 T) \to G \) fully determined by \( h_1 = (\rho_1(\gamma_1), \cdots, \rho_1(\gamma_{\ell})) \in G^\ell \). Hence for every tangle \( T \), the moduli space of \( G \)-coloring of \( T \) is \( \mathcal{M}_T(h_0, h_1) \),
which is the space of flat connections on $C_T$ inducing $h_i$ for $\mathbb{R}^2 \times \{ i \} \setminus \partial i T$, then quotient by the group of gauge transformations. Hence $\mathcal{M}_T(h_0, h_1)$ is a subspace of the character variety $\mathcal{M}_T$.

![Figure 9: Homotopy Path $\gamma_2$](image)

Suppose $T$ is isotopic to another tangle $T'$. Then the isotopy from $T$ to $T'$ induces an isomorphism from the fundamental groupoid $\pi_1(C_T)$ to $\pi_1(C_{T'})$ and also induces an isomorphism from $\mathcal{M}_T(h_0, h_1)$ to $\mathcal{M}_{T'}(h_0, h_1)$. Suppose it sends the flat connection $A_T$ to the flat connection $A_{T'}$. In this situation, we say the two $G$-tangles $(T, A_T)$ and $(T', A_{T'})$ are isotopic and denote this equivalence class by $[T, A_T]$ and the collection of equivalence classes as $\mathcal{M}_{[T]}(h_0, h_1)$. Then as defined in [KRI], the category of $G$-tangles $\mathcal{T}(G)$ consists of objects $\{(\epsilon_i, g_i)\}_{i=1}^n \in 2^{\{\pm 1\} \times \pi}$ and

\[ \mathcal{T}(G) \left( \{(\epsilon_i, g_i)\}_{i=1}^k , \{(\epsilon'_j, g'_j)\}_{j=1}^\ell \right) := \bigsqcup_{[T] \in \mathcal{T}((g_1, \cdots, g_k), (g'_1, \cdots, g'_\ell))} \mathcal{M}_{[T]}((g_1, \cdots, g_k), (g'_1, \cdots, g'_\ell)) \]

where $T_{(\epsilon_i), (\epsilon'_j)}$ is the collection of isotopic classes of tangles of type $(k, \ell)$ so that the boundary component which connects to $(0, i, 0)$ is oriented upward at $(0, i, 0)$ if $\epsilon_i = 1$, downward if $\epsilon_i = -1$, and the boundary component which connects to $(0, j, 1)$ is oriented upward at $(0, j, 1)$ if $\epsilon'_j = 1$, downward if $\epsilon'_j = -1$. The composition of morphisms is defined as follows: suppose

\[ [T_1, A_{T_1}] \in \mathcal{T}(G) \left( \{(\epsilon_i, g_i)\}_{i=1}^k , \{(\epsilon'_j, g'_j)\}_{j=1}^\ell \right) \]

and

\[ [T_2, A_{T_2}] \in \mathcal{T}(G) \left( \{(\epsilon'_j, g'_j)\}_{j=1}^\ell , \{(\epsilon''_i, g''_i)\}_{i=1}^m \right) \]

then their composition $T$ is defined by attaching $T_2$ on the top of $T_1$ with the flat connection $A_T|_{\mathbb{R}^2 \times [0,1] \setminus T_1} = A_{T_1}$ and $A_T|_{\mathbb{R}^2 \times [1,2] \setminus T_2} = A_{T_2}$. Together it is clearly isotopic to a $G$-tangle in $\mathcal{T}(G) \left( \{(\epsilon_i, g_i)\}_{i=1}^k , \{(\epsilon''_i, g''_i)\}_{i=1}^m \right)$. 
2.3 The Category $\mathcal{D}(G)$ of $G$-Colored Diagrams

Now we would like to define $G$-colored tangle diagrams. Unlike $G$-tangles that it is the homotopy paths in the complement space are colored, now we try to color edges of diagrams. In order to relate $G$-colored tangle diagrams and $G$-tangles, we will see that this comes natural if $G$ is factorizable.

Assume that $G$ is factorizable. Then as defined before, $G'$ is covered by $G^*$, whose group structure is given by the product of the group structures on $G_+$ and $G_-$. We denote the group operation and the inverse on $G'$, induced from the covering map, by $*$ and $i(-)$ respectively, to distinguish from the group structure from $G$. We further assume $\tilde{G}$ is the Zariski open neighborhood of $e$ in $G^*$ so that $\tilde{G} \cong G'$ as an algebraic group. Then for every $g \in G'$, there is an unique way to write it as

$$g = g_+ g_-$$

where $(g_+, g_-) \in \tilde{G} \subseteq G_+ \times G_-$. This implies

$$g * h = g_+ h_+ (g_- h_-)^{-1} = g_+ h_+ h_-^{-1} g_-^{-1}$$

$i(g) = g_+^{-1} g_-$

Since what we really need is the unique factorization property in this paper, for convenience, we may simply regard $G'$ as our factorizable group $G$ and say every element in $G$ has the unique factorization.

Given any tangle diagram $\Pi$. A vertex is either a point in the boundary $\partial \Pi$ or an intersection and an edge is a curve connecting two vertices. We denote the collection of edges by $E(\Pi)$.

![Diagram](image)

(a) Positive Intersection  (b) Negative Intersection

Figure 10: $G$-colored Tangle Diagram Relations

**Definition** ($G$-colored diagrams). Suppose $G$ is factorizable. A $G$-colored tangle diagram is a tangle diagram $\Pi$ equipped with a map $c : E(\Pi) \to G$ satisfying

$$x_{a_v} = x_L(x_{c_v}, x_{d_v}), \quad x_{b_v} = x_R(x_{c_v}, x_{d_v})$$

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at every positive intersection \( v \), where \( c(a) \) is denoted by \( x_a \) for any \( a \in E(\Pi) \), and \( x_L, x_R : G \times G \to G \) are defined by

\[
x_L(g, h) = g^-h\,g^{-1}, \quad x_R(g, h) = x_L(g, h)^{-1}g\,x_L(g, h)_+
\]

And at every negative intersection \( w \), we have the relations

\[
x_{cw} = x_L(x_{aw}, x_{bw}), \quad x_{dw} = x_R(x_{aw}, x_{bw})
\]

Notice that \((x_L, x_R) : G \times G \to G \times G\) has an inverse map \((\tilde{x}_L, \tilde{x}_R)\) defined by

\[
\tilde{x}_L(g, h) = g^+h\,g^{-1}, \quad \tilde{x}_R(g, h) = \tilde{x}_L(g, h)^{-1}g\tilde{x}_L(g, h)_-
\]

So it is possible to construct a coloring map \( c \) from bottom-to-top by first assigning the coloring to edges connecting to vertices in \( \partial_0\Pi \) and then the coloring to other edges by using these relations. Similarly it can also be constructed from top-to-bottom.

If \( \Pi \) is isotopic to another tangle diagram \( \Pi' \), then the isotopy from \( \Pi \) to \( \Pi' \) induces an isomorphism from \( \{c : E(\Pi) \to G\} \) to \( \{c' : E(\Pi') \to G\} \). Suppose the isotopy sends \( (\Pi, c) \) to \( (\Pi', c') \). Then we say these two \( G \)-colored tangle diagrams are isotopic. Moreover, one can easily see the \( G \)-coloring is compatible with framed Reidemeister moves type I and type II by our definition. Actually it is also compatible with framed Reidemeister move type III by the following proposition. We refer its proof to [WX].

**Proposition 2.2.** \( \mathcal{R} = (x_L, x_R) \) satisfies the set-theoretical Yang-Baxter equation

\[
\mathcal{R}_{12}
\mathcal{R}_{23}
\mathcal{R}_{12} = \mathcal{R}_{23}
\mathcal{R}_{12}
\mathcal{R}_{23}
\]

where \( \mathcal{R}_{12} = (\mathcal{R}, \text{id}_G) \) and \( \mathcal{R}_{23} = (\text{id}_G, \mathcal{R}) \).

![Figure 11: Morphisms of \( D(G) \)](image-url)
This implies if two tangle diagrams \( \Pi \) and \( \Pi' \) are Reidemeister equivalent, that is, one is isotopic to the other by also applying some finite number of Reidemeister moves, then any coloring map on one of them can be induced to a coloring map on the other. In this situation, we say \((\Pi, c)\) and \((\Pi', c')\) are Reidemeister equivalent and denote this equivalence class by \([\Pi, c]\). Now as defined in \([KR1]\), the category of \(G\)-colored tangle diagrams \( \mathcal{D}(G) \) consists of objects \( \{(\epsilon_i, x_i)\}_{i=1}^n \in 2^{\{\pm 1\} \times G} \) and \( \mathcal{D}(G) \left( \{(\epsilon_i, x_i)\}_{i=1}^k, \{(\epsilon'_j, x'_j)\}_{j=1}^\ell \right) \) is the collection of \([\Pi, c]\), where \( \Pi \) is of type \((k, \ell)\) so that the orientation of the boundary edges are determined by \( \epsilon_i \) and \( \epsilon'_j \) in exactly the same way as \( G \)-tangles, and the boundary edges are colored by \( x_i \) or \( x'_j \) respectively. Finally, the composition of \([\Pi_1, c_1]\) and \([\Pi_2, c_2]\) is Reidemeister equivalent to the gluing of \( \Pi_1 \) and \( \Pi_2 \) whose coloring is induced by \( c_1 \) and \( c_2 \) naturally.

2.4 \( \mathcal{T}(G) \cong \mathcal{D}(G) \)

**Proposition 2.3.** Suppose \( G \) is factorizable. Then \( \mathcal{T}(G) \) and \( \mathcal{D}(G) \) are naturally equivalent.

**Proof.** One can define the functor \( F : \mathcal{T}(G) \to \mathcal{D}(G) \) by

\[
F(\{(\epsilon_i, g_i)\}_{i=1}^k) = \{(\epsilon_i, x_i)\}_{i=1}^k
\]

where \( g_i = (x_1)^{\epsilon_1} \cdots (x_i)^{\epsilon_i} (x_{i+1})^{-\epsilon_i} \cdots (x_n)^{-\epsilon_1} \). Notice that this is well-defined and invertible because of the unique factorization property from that fact that \( G \) is factorizable. And \( F \) sends a morphism \([T, A_T]\) to \([\Pi, c]\), where \( \Pi \in [\Pi_{[T]}] \) is the projection of a representative \( T' \) in the isotopic class \([T]\) onto \( y-z \) plane (\( T' \) has to be chosen so that the projection is a tangle diagram) and \( c \) is induced by the following definition:

For any edge \( a \in E(\Pi) \), consider a point \( p = (x, y, z) \) on \( a \) and consider a homotopy path \( \gamma_p \) such that it starts from \((0, -1, 0)\), then goes to \((x, y - \delta, z)\) by always staying under the tangle, where \( \delta > 0 \), and then goes to \((x, y + \delta, z)\) by encircling \( p \), and goes back to \((0, -1, 0)\) by reflecting the first part of the path through \( y-z \) plane. Then it is easy to adjust \( \gamma_p \) so that the reflection always stays above the tangle. Say the projection of \( \gamma_p \) onto the \( y-z \) plane intersects with edges \( a_1, \ldots, a_i = a \) whose orientations are indicated by \( \epsilon_1, \ldots, \epsilon_i \in \{\pm 1\} \). Then we color \( a \) by using

\[
\rho(\gamma_p) = (x_{a_1})^{\epsilon_1} \cdots (x_{a_i})^{\epsilon_i} (x_{a_{i+1}})^{-\epsilon_i} \cdots (x_{a_n})^{-\epsilon_1}
\]

where \( \rho : \pi_1(T') \to G \) is the monodromy representation associated to \( A_T \), induced by \( A_T \). This is well-defined up to homotopy by using the wall-chamber argument as described in \([KR1]\). It is also clearly independent of the choice of \( p \) on \( a \). And we can figure out the whole coloring map \( c \) by coloring left-to-right.

It is now easy to see the assignment \([T, A_T]\) to \([\Pi, c]\) is invertible due to the unique factorization property again. Hence \( \mathcal{T}(G) \) and \( \mathcal{D}(G) \) are naturally equivalent. \( \square \)
This indicates that to understand isotopy classes of tangles, we can study $\mathcal{D}(G)$ instead of $\mathcal{T}(G)$, where the former is easier to deal with.

3 $G$-Categories

3.1 Braided Groups

We follow by the setup in [KR1] to define $G$-categories for every factorizable group $G$, which equips with a natural braiding $\mathcal{R}: G \times G \to G \times G$.

**Definition** (Braided Group). A group $G$ is called braided if there exists an invertible $\mathcal{R}: G \times G \to G \times G$ satisfying

\[
\begin{align*}
\mu \circ \tau \circ \mathcal{R} &= \mu & (1) \\
\mathcal{R} \circ (\mu, \text{id}_G) &= (\mu, \text{id}_G) \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} & (2) \\
\mathcal{R} \circ (\text{id}_G, \mu) &= (\text{id}_G, \mu) \circ \mathcal{R}_{13} \circ \mathcal{R}_{12} & (3)
\end{align*}
\]

where $\mathcal{R}_{12}$, $\mathcal{R}_{13}$, and $\mathcal{R}_{23}$ are defined as usual.

This is the group-theoretical version of quasitriangular (or braided) algebra. Hence we call $(G, \mu, \mathcal{R})$ a braided group.

**Proposition 3.1.** Suppose $G$ is factorizable. Then $(G, *, \tau \circ \mathcal{R})$ is braided.

**Proof.** Clearly $\tau \circ \mathcal{R}$ is invertible due to our discussion below the definition of $\mathcal{R}$. The first condition becomes $x_L(x, y) * x_R(x, y) = x * y$. This is true because

\[
x_L(x, y) * x_R(x, y) = (x_L(x, y))_+ x_R(x, y)(x_L(x, y))_{-1} = (x_L(x, y))_+ (x_L(x, y))_{-1} x(x_L(x, y))_+ (x_L(x, y))_{-1} = x x_L(x, y) = x x_{-1} x y x_{-1} = x y x_{-1} = x * y
\]

This implies $x_{\pm} y_{\pm} = (x_L(x, y))_+ (x_R(x, y))_{\pm}$. And the second condition becomes $x_L(x, x_L(y, z)) = x_L(x * y, z)$ and $x_R(x, x_L(y, z)) * x_R(y, z) = x_R(x * y, z)$. Let’s verify both of them:

\[
x_L(x * y, z) = (x * y)_- z (x * y)_{-1} = x y z (x * y)_{-1} = x L(x, y z) = x L(x, x_L(y, z))
\]

and

\[
x_R(x * y, z) = (x_L(x * y, z))_+ x_L y x_{-1} (x_L(x * y, z))_+ = (x_L(x, x_L(y, z)))_+ x_L y x_{-1} (x_L(x, x_L(y, z)))_+ = (x_L(x, x_L(y, z)))_+ x_L y x_{-1} x_L(x, x_L(y, z))_+ = (x_L(x, x_L(y, z)))_+ x_L y x_{-1} (x_L(x, x_L(y, z)))_+ = (x_R(x, x_L(y, z)))_+ (x_L(y, z))_+ x_L(y, z) = (x_R(x, x_L(y, z)))_+ (x_L(y, z))_+ x_L(y, z) = (x_R(x, x_L(y, z)))_+ (x_L(y, z))_+ (x_R(x, x_L(y, z)))_{-1}
\]
where the last equality is from $x_+(x_L(y, z))_+ = (x_L(x, x_L(y, z)))_+(x_R(x, x_L(y, z)))_+$, which is a consequence from the first condition. And

$$x_R(x, x_L(y, z)) \ast x_R(y, z) = (x_R(x, x_L(y, z)))(x_R(x, x_L(y, z)))^{-1}$$

Therefore they are the same. The third condition can be proved in a similar way.  

### 3.2 Ribbon $G$-Categories

In the following, for any braided group $G$, we would like to define what structure a ribbon $G$-category would possess. Then in particular when $G$ is factorizable, the category $D(\mathcal{C})$ of $\mathcal{C}$-colored diagrams, which is a generalization of $D(G)$, is a ribbon $(G, \ast)$-category for any ribbon $(G, \ast)$-category $\mathcal{C}$. Moreover, we can define an invariant functor from $D(\mathcal{C})$ to $\mathcal{C}$, which helps us to construct an invariant for any isotopy class of $G$-tangles.

**Definition (Discrete Category).** A category $\mathcal{D}$ is called discrete if $\mathcal{D}(X,Y) = \{\}$ if $X \neq Y$ and $\mathcal{D}(X,X) = \{\text{id}_X\}$.

**Definition (Fibred Category).** A category $\mathcal{C}$ is fibred over a discrete category $\mathcal{D}$ if there exists a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F^{-1}(X) = \{A \in \mathcal{C} : F(A) = X\}$ is nonempty.

Recall that in Section 1.1 we have introduced monoidal categories. A monoidal functor is a functor between monoidal categories preserving monoidal structures. A monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called strong if $1_{\mathcal{D}} \cong F(1_{\mathcal{C}})$ and $F(X) \otimes_{\mathcal{D}} F(Y) \cong F(X \otimes_{\mathcal{C}} Y)$ for every $X,Y \in \mathcal{C}$. And a monoidal natural transformation is a natural transformation between monoidal functors preserving monoidal structures.

**Definition (G-category).** Suppose $G$ is a group. A monoidal category $\mathcal{C}$ is called a $G$-category if there exists a strong monoidal functor $F : \mathcal{C} \rightarrow \text{Arr}(G)$, where we first regard $G$ as a groupoid with a single object $\ast$, then $\text{Arr}(G)$ is defined as a discrete category consisting of objects in $G(\ast, \ast) = G$ and is equipped with a monoidal structure induced by the group operation on $G$.

A $G$-category $\mathcal{C}$ is strict/rigid if $\mathcal{C}$ is a strict/rigid monoidal category. In the following, every monoidal category we consider is strict and rigid, so we may just simply call them monoidal categories without these adjectives. Now we would like to define what a braided $G$-category means.

**Definition (Braided G-category).** Suppose $\mathcal{C}$ is a $G$-category. It is braided if there exists $\mathcal{R} = (x_R, x_L) : G \times G \rightarrow G \times G$ and $B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ such that

1. $(G, \mathcal{R})$ is braided.
2. $B$ is a lift of $\tau \circ \mathcal{R}$ via $F$. Explicitly, this means

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{B} & \mathcal{C} \times \mathcal{C} \\
(F,F) & \downarrow & (F,F) \\
\text{Arr}(G) \times \text{Arr}(G) & \xrightarrow{\tau \circ \mathcal{R}} & \text{Arr}(G) \times \text{Arr}(G)
\end{array}
$$

commutes. We would denote $B(X_a, X_b)$ by $(X_L(X_a, X_b), X_R(X_a, X_b))$ for any $a, b \in G$, where $X_g \in F^{-1}(g)$ for any $g \in G$ and $X_L(X_a, X_b) \in F^{-1}(x_L(a, b))$, $X_R(X_a, X_b) \in F^{-1}(x_R(a, b))$.

3. $B$ is holonomic, that is, it satisfies the generalized hexagon identities

$$
B \circ (\otimes, \text{id}) = (\text{id}, \otimes) \circ (B, \text{id}) \circ (\text{id}, B)
$$

$$
B \circ (\text{id}, \otimes) = (\otimes, \text{id}) \circ (\text{id}, B) \circ (B, \text{id})
$$

4. There exists a natural isomorphism $R_B : \otimes \Rightarrow \otimes \circ B$.

One could see the generalized hexagon identities will be sent to conditions [2] and [3] in the definition of the braided group $G$ via $F$. And $R_B$ would satisfy the hexagon identities in the definition of braided monoidal categories. Moreover, one should be able to see the generalized hexagon identities imply that $R = \tau \circ R_B$ would satisfy the holonomic Yang-Baxter equation

$$
R(x_R(x, x_L(y, z)), x_R(y, z))_{12}R(x, x_L(y, z))_{13}R(y, z)_{23} = R(x_L(x, y), x_L(x_R(x, y), z))_{23}R(x_R(x, y), z)_{13}R(x, y)_{12}
$$

where $R(a, b) : X_a \otimes X_b \rightarrow X_R(X_a, X_b) \otimes X_L(X_a, X_b)$ and $R(a, b)_{12}, R(a, b)_{13}$ and $R(a, b)_{23}$ are defined as usual.

Besides this, one should also notice that $F(X^L_g) = F(X^R_g) = g^{-1}$ because $F$ is a strong monoidal functor. Although $X^L_g$ and $X^R_g$ are usually not the same object, similar to Proposition [1.1], one can identify them as long as $\mathcal{C}$ is braided. However this identification is not canonical, as explained in [S], [FY]. They could be naturally and canonically identified with each other if $\mathcal{C}$ is equipped with a pivotal structure.

**Definition** (Pivotal $G$-category). A pivotal $G$-category is a $G$-category $\mathcal{C}$ equipped with a monoidal natural isomorphism $\mu : \text{id}_\mathcal{C} \Rightarrow (-)^{LL}$. Then by [S], we automatically have $\mu_X = (\mu_X^L)^{-1}$, where $\mu_X^L$ is defined via

$$
X^{LLL} \xrightarrow{\text{id} \otimes (\text{coev}_X^L)} X^{LLL} \otimes X \otimes X^L \xrightarrow{\text{id} \otimes \mu_X \otimes \text{id}} X^{LLL} \otimes X^{LL} \otimes X^L \xrightarrow{(\text{ev}_X)^L \otimes \text{id}} X^L
$$

And $X^R \xrightarrow{\mu_X^R} (X^R)^{LL} \cong X^L$, where the second isomorphism is the unique isomorphism identifies two left duals of $X$. We then use this identification to regard both the left dual and the right dual of $X$ as the same object $X^*$. 

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**Definition** (Ribbon $G$-category). A pivotal braided $G$-category is called ribbon.

Usually in the literature, for instance, [S], a ribbon category is defined by equipping with *twists*. However, one can generate these morphisms by using the pivotal structure and the braiding.

### 3.3 The Category $\mathcal{D}(\mathcal{C})$ of $\mathcal{C}$-Colored Diagrams

Now we try to define $\mathcal{C}$-colored tangle diagrams as a generalization of $G$-colored tangle diagrams, where $(\mathcal{C}, B)$ is a ribbon $(G, R)$-category. We say a tangle diagram $\Pi$ is $\mathcal{C}$-colored if there exists a map $\kappa: E(\Pi) \to \mathcal{C}$ such that

$$\kappa(a_v) = X_L(\kappa(c_v), \kappa(d_v)), \quad \kappa(b_v) = X_R(\kappa(c_v), \kappa(d_v))$$

at every positive intersection $v$ as shown in Figure 10. Similarly at every negative intersection $w$ as shown in Figure 10, we have

$$\kappa(c_w) = X_L(\kappa(a_w), \kappa(b_w)), \quad \kappa(d_w) = X_R(\kappa(a_w), \kappa(b_w))$$

In particular when $G$ is factorizable, $(G, *, \tau \circ R)$ is a braided group and this definition should be thought as a lift of $G$-coloring on the tangle diagram $\Pi$ via $F: \mathcal{C} \to \text{Arr}(G)$.

Now we define the category $\mathcal{D}(\mathcal{C})$ of $\mathcal{C}$-colored diagrams as a category with objects $\{ (\epsilon_i, X_i) \}_{i=1}^n \in 2^{\{\pm 1\} \times \mathcal{C}}$. The morphisms in $\mathcal{D}(\mathcal{C}) \left( \{ (\epsilon_i, X_i) \}_{i=1}^k, \{ (\epsilon'_j, X'_j) \}_{j=1}^\ell \right)$ are all Reidemeister equivalence classes $[\Pi, \kappa]$ of $\mathcal{C}$-colored diagrams satisfying the boundary conditions

1. $a_i$ is oriented upward if $\epsilon_i = 1$, downward if $\epsilon_i = -1$;

   $a'_j$ is oriented upward if $\epsilon'_j = 1$, downward if $\epsilon'_j = -1$.

2. $\kappa(a_i) = X_i$; $\kappa(a'_j) = X'_j$.

where $a_i$ are the edges connecting to $(0, i, 0)$ and $a'_j$ are the edges connecting to $(0, j, 1)$, and the Reidemeister equivalence is defined the same way as in $G$-colored diagrams. Without any surprise, the composition is defined to be induced by gluing of two $\mathcal{C}$-colored diagrams just as before.

**Proposition 3.2.** Suppose $(\mathcal{C}, B, R_B)$ is a ribbon $(G, R)$-category. Then $(\mathcal{D}(\mathcal{C}), B, R_B)$ is a ribbon $(G, R)$-category for some $B$ and $R_B$.

**Proof.** $\mathcal{D}(\mathcal{C})$ has a monoidal structure $\otimes_{\mathcal{D}(\mathcal{C})}$ defined by putting two diagrams next to each other. Explicitly, any representative of $\otimes_{\mathcal{D}(\mathcal{C})}( [\Pi_1, c_1], [\Pi_2, c_2] )$ is Reidemeister equivalent to Figure 12 whose $\mathcal{C}$-coloring is naturally induced by $c_1$ and $c_2$. Also we have $1_{\mathcal{D}(\mathcal{C})} = \phi \in 2^{\{\pm 1\} \times \mathcal{C}}$ and

$$\{ (\epsilon_1, X_1), \cdots, (\epsilon_k, X_k) \}^\ast := \{ (-\epsilon_1, X_1), \cdots, (-\epsilon_k, X_k) \}$$
where the evaluation map and the coevaluation map are the *rainbow tangle diagrams* as shown in Figure 13, whose $C$-coloring is given by $a_i \mapsto X_i$ for the edge $a_i$ connecting the two vertices colored by $(\epsilon_i, X_i)$ and $(-\epsilon_i, X_i)$ respectively.

Figure 12: $\otimes_{\mathcal{D}(\mathbb{C})}([\Pi_1, c_1], [\Pi_2, c_2])$

Figure 13: The Coevaluation Map

Now we try to realize $\mathcal{D}(\mathbb{C})$ as a $G$-category. Say $F : \mathbb{C} \to \text{Arr}(G)$ is the functor making $\mathbb{C}$ a $G$-category. Then consider

$$\tilde{F} : \mathcal{D}(\mathbb{C}) \to \text{Arr}(G)$$

defined by $\{(\epsilon_1, X_1), \cdots, (\epsilon_k, X_k)\} \mapsto F(X_1^{\epsilon_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} X_k^{\epsilon_k})$ and $\phi \mapsto e \in G$, where $X^{-1} := X^*$. One should notice that the condition (1) in the definition of the braided group implies

$$\tilde{F} (\{(\epsilon_1, X_1), \cdots, (\epsilon_k, X_k)\}) = \tilde{F} (\{(\epsilon'_1, X'_1), \cdots, (\epsilon'_k, X'_k)\})$$

if $\mathcal{D}(\mathbb{C}) (\{(\epsilon_1, X_1), \cdots, (\epsilon_k, X_k)\}, \{(\epsilon'_1, X'_1), \cdots, (\epsilon'_k, X'_k)\})$ is non-empty. Hence $\tilde{F}$ makes $\mathcal{D}(\mathbb{C})$ a $G$-category.
Our next step is to define the braiding on $D(C)$. Consider $B : D(C) \times D(C) \to D(C)$ defined by

$$(\{(\epsilon_1, X_1), \ldots, (\epsilon_k, X_k)\}, \{(\eta_1, Y_1), \ldots, (\eta_m, Y_m)\})$$

$$\mapsto \left(\{(\eta_1, \tilde{Y}_1), \ldots, (\eta_m, \tilde{Y}_m)\}, \{((\epsilon_1, \tilde{X}_1), \ldots, (\epsilon_k, \tilde{X}_k)\}\right)$$

where $(\tilde{Y}^1_m, \ldots, \tilde{Y}^m_m, \tilde{X}^1_i, \ldots, \tilde{X}^k_i)$ is given via

$$B_{m-(k+m-1)} \cdots B_{2-(k+1)} B_{1-k}(X^1_{1}, \ldots, X^k_{k}, Y^1_{1}, \ldots, Y^m_{m})$$

Here $B_{i\cdots j} := B_{i(i+1)} \cdots B_{j(j+1)}$ for any $i, j \in \mathbb{N}$ and $R_B$ is defined via the crossing diagram as shown in Figure 14.

![Crossing Diagram](image)

Figure 14: $R_B((\{(\epsilon_i, X_i), (\eta_j, Y_j)\}) = \{(\eta_j, \tilde{Y}_j), (\epsilon_i, \tilde{X}_i)\}$

Finally we leave it to the reader to check all the conditions to be a ribbon $G$-category are satisfied.

3.4 The Construction of Invariants of $G$-Tangles

Let’s discuss about the general theory of constructing invariants of $G$-tangles for any factorizable group $G$. In this section, we would require the assumption that there exists a section of $F$, that is, a collection of objects $\{X_g \in \mathcal{C}\}_{g \in G}$ satisfying $F(X_g) = g$ and

$$B(X_a, X_b) = (X_L(X_a, X_b), X_R(X_a, X_b)) = (X_{x_L(a,b)}, X_{x_R(a,b)})$$

Our first step is an important observation, which states that all the morphisms in $D(C)$ can be obtained by tensoring and composing some elementary diagrams. Hence if we want to define a functor from $D(C)$ to another category, it is sufficient to consider the assignment of these diagrams when we want to know how a functor acts on any morphism. This observation is done in [R].
Proposition 3.3. Suppose \((\mathcal{C}, B, R)\) is a ribbon \((G, *, \tau \circ \mathbb{R})\)-category. Then all morphisms of \((D(\mathcal{C}), B, R_B)\) are Reidemeister equivalent to some compositions and tensors of the following elementary diagrams, where \(X, Y \in \{X_g\}_{g \in G}\).

\[\text{Figure 15: Elementary Diagrams}\]

\[\text{Figure 16: Elementary Diagrams}\]

Proof. The key idea is to observe that the right co-evaluation map \(\phi \to (-1, X) \otimes (+1, X)\) and the right evaluation map \((+1, X) \otimes (-1, X) \to \phi\) can be obtained by composing elementary diagrams in Figure 16. Explicitly, we have

\[\text{Figure 17: } (\text{coev}_R^{+1, X}) : \phi \to (-1, X) \otimes (+1, X)\]
Here $Z \in \{X_g\}$ is a choice so that $\mathcal{R}(x, z) = (x, z)$, where $z = F(Z)$ and $x = F(X)$. An easy computation shows that the only solution to $z$ is $(i(x))^{-1}$, hence $Z = X_{i(x)^{-1}}$.

Recall that a monoidal functor is called braided if it preserves the braiding structure. Similarly, a $G$-braided monoidal functor $H$ between two braided $G$-categories $(\mathcal{D}, F_\mathcal{D} : \mathcal{D} \to \text{Arr}(G), B_\mathcal{D}, R_{B_\mathcal{D}})$ and $(\mathcal{C}, F_\mathcal{C} : \mathcal{C} \to \text{Arr}(G), B_\mathcal{C}, R_{B_\mathcal{C}})$ is a monoidal functor satisfying

\[
F_\mathcal{C} \circ H = F_\mathcal{D} \\
B_\mathcal{C} \circ (H, H) = (H, H) \circ B_\mathcal{D} \\
R_{B_\mathcal{C}} \circ H = H \circ R_{B_\mathcal{D}}
\]

Then the following theorem is a generalization of the theorem about invariants of tangles described in [RT]. This should be a pretty straightforward result from the previous Proposition 3.3.

**Theorem 3.4.** There exists an unique $(G, \ast, \mathcal{R})$-braided monoidal functor

$$\text{Inv} : (\mathcal{D}(\mathcal{C}), B, R_B) \to (\mathcal{C}, B, R_B)$$

such that

1. $\text{Inv} \left( \{(\epsilon_1, X_1), \ldots, (\epsilon_n, X_n)\} \right) = \bigotimes_{i=1}^n X_{\epsilon_i}^{\epsilon_i}$
2. $\text{Inv}(\text{id}_{(\epsilon, X)}) = \text{id}_{X^\epsilon}$
3. $\text{Inv}((\text{ev}^L)^{\ast}_{(\epsilon, X)}) = (\text{ev}^L)_X^{\ast \epsilon}$

Figure 18: $(\text{ev}^R)_{(+1, X)} : (+1, X) \otimes (-1, X) \to \phi$

Here $Y \in \{X_g\}$ is a choice so that $\mathcal{R}(y, x) = (y, x)$, where $y = F(Y)$. An easy computation shows that the only solution to $y$ is $i(x^{-1})$, hence $Y = X_{i(x^{-1})}$. 

![Diagram](image-url)
(4) \( \text{Inv}((\text{coev}^{L,R})(\epsilon,X)) = (\text{coev}^{L,R})_{X^*}. \)

Moreover, one should notice that we have

\begin{align*}
\text{Figure 19: Reformation of Elementary Diagrams}
\end{align*}

Then via the functor \( \text{Inv} \), this implies \( (R_B)_{X_a,X_b^*} = (R_B)_{X_a,X_{i(b)}} \) is

\[
[(\text{ev}^L)_{X_c} \otimes \text{id}_{X_a} \otimes X_b^*) \circ (\text{id}_{X_c^*} \otimes (R_B)_{X_d,X_b} \otimes \text{id}_{X_b^*}) \circ (\text{id}_{X_c^*} \otimes X_d \otimes (\text{coev}^L)_X)]^{-1}
\]

where \((c,a) = \mathcal{R}(d,b)\), and similarly \( (R_B^{-1})_{X_a,X_b^*} = (R_B^{-1})_{X_a,X_{i(b)}} \) is

\[
[(\text{ev}^L)_{X_g} \otimes \text{id}_{X_a} \otimes X_b^*) \circ (\text{id}_{X_g^*} \otimes (R_B^{-1})_{X_h,X_b} \otimes \text{id}_{X_b^*}) \circ (\text{id}_{X_g^*} \otimes X_h \otimes (\text{coev}^L)_X)]^{-1}
\]

where \((h,b) = \mathcal{R}(g,a)\). This indicates both \((\text{ev}^R)_X\) and \((\text{coev}^R)_X\) are fully determined by \((\text{ev}^L)_X\), \((\text{coev}^L)_X\) and \((R_B)_{X,Y}\).

Finally, \( \text{Inv} \) induces a functor \( \text{Inv}' : \mathcal{T}(G) \to \mathcal{C} \) by first using the equivalence of \( \mathcal{T}(G) \) and \( \mathcal{D}(G) \), then obtaining a lift from \( \mathcal{D}(G) \) to \( \mathcal{D}(\mathcal{C}) \) by considering \( \{X_g\}_{g \in G} \), the section of \( F \), and applying \( \text{Inv} \) as the last step.
4 Quantum $\mathfrak{sl}_2(\mathbb{C})$ at Roots of Unity

4.1 $U_q$ and Its Cyclic Representations

Similar to the provided example in [KR1] and [KR2], we consider $U_q$ to be a $\mathbb{C}[q, q^{-1}]$-algebra generated by $E, F, K, K^{-1}$ under the following relations

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = (q - q^{-1})(K - K^{-1})$$

It becomes a Hopf algebra if it is equipped with

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad \Delta(E) = E \otimes K + 1 \otimes E$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K^{\pm 1}) = K^{\mp 1}$$

Note for $q \neq 1$, this algebra is isomorphic to the usual unrestricted quantum group $U_q = U_q(\mathfrak{sl}_2(\mathbb{C}))$. When $q = 1$, later we will see that it is isomorphic to the ring of regular functions on an open neighborhood of $(e, e)$ in $(SL_2(\mathbb{C}))^*$ rather than $U(\mathfrak{sl}_2(\mathbb{C}))$. The difference between $U_q$ and $U_q$ at $q = 1$ is technical.

Throughout this paper, we will only care about the case that $q$ is a primitive $\ell$-th root of unity, where $\ell$ is odd. The behavior of $U_q$ at roots of unity has been studied by De Concini and Kac. One of the key fact is that $U_q$ has a large zenter.

We have the following Proposition and would like to refer its proof to [DC-K].

**Proposition 4.1.** The followings are true:

1. The center $Z$ of $U_q$ is the subalgebra $\langle E^\ell, F^\ell, K^{\pm \ell}, \Omega | P(D, \Omega) = 0 \rangle$, where $\Omega = EF + q^{-1}K + qK^{-1}$ is the Casimir element, $D = E^\ell F^\ell + K^\ell + K^{-\ell}$ and
   $$P(D, \Omega) = \prod_{j=0}^{\ell-1} (\Omega - q^jK - q^{-j}K^{-1}) - E^\ell F^\ell$$

2. The subalgebra $Z_0 = \langle E^\ell, F^\ell, K^{\pm \ell} \rangle$ of $U_q$ is a Hopf subalgebra.

3. $Z_0 \cong U_1$ as a $\mathbb{C}$-Hopf algebra.

4. Consider a neighborhood $\tilde{G}$ of $(e, e)$ in $(SL_2(\mathbb{C}))^*$ identified by

   $$\tilde{G} \cong \left\{ \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ c & a \end{array} \right) : a \in \mathbb{C}^\times, b, c \in \mathbb{C} \right\} \subset \tilde{B}_+ \times \tilde{B}_-$$

   where as algebraic groups $\tilde{B}_\pm \cong B_\pm$, the upper/lower Borel subgroups in $SL_2(\mathbb{C})$ as mentioned in 1.4. Then $\tilde{G}$ is isomorphic to an open neighborhood $G'$ of $e$ in $SL_2(\mathbb{C})$ consisting of matrices whose lower-right entry is nonzero. Moreover, the map $E^\ell \mapsto -\frac{c}{a}$, $K^\ell \mapsto \frac{1}{a}$, $F^\ell \mapsto b$ is an isomorphism of Hopf algebras $Z_0 \rightarrow C(\tilde{G})$, the Hopf algebra of regular functions on $\tilde{G}$. Hence

   $$\text{Spec } Z_0 \cong \tilde{G}$$
Now we would like to study the irreducible representations of this algebra $U_q$. In particular, we care about the cyclic representations as described in [K], [KR1] and [KR2]. We define $V_{\alpha, \beta, \mu} \in \text{Vec}_{\mathbb{C}}^d$ with a basis $\{v_0, \cdots, v_{\ell-1}\}$ as an $U_q$ module by

$$
E v_i = \beta (\mu - \alpha q^{2i+1}) v_{i+1} \\
K v_i = \alpha q^{2i} v_i \\
F v_i = \beta^{-1} (1 - \mu^{-1} \alpha^{-1} q^{-2i+1}) v_{i-1}
$$

where $\alpha, \beta, \mu \in \mathbb{C}^*$ and $v_0 = v_{\ell}$. This expression has the advantage that

$$
\Omega v_i = (EF + q^{-1} K + qK^{-1}) v_i = (\mu + \mu^{-1}) v_i \tag{4}
$$

for every $i \in \mathbb{N}$. This implies

$$
E^\ell v_i = \beta^\ell \left( \prod_{n=1}^{\ell} (\mu - \alpha q^n) \right) v_i \tag{5} \\
K^\ell v_i = \alpha^\ell v_i \tag{6} \\
F^\ell v_i = \beta^{-\ell} \left( \prod_{n=1}^{\ell} (1 - \mu^{-1} \alpha^{-1} q^n) \right) v_i \tag{7}
$$

because $\{q^{2i+1}\}_{i=1}^\ell = \{q^n\}_{n=1}^\ell$. This is why we would like to have the assumption that $\ell$ is odd.

For any central character $x : Z \to \mathbb{C}$ and its restriction on $Z_0$ as $\chi = x|_{Z_0}$, $\chi$ can be regarded as the image of $x$ via the natural map $\text{Spec} Z \to \text{Spec} Z_0$ induced by $Z_0 \subset Z$. And by Proposition 4.1, for any $\chi \in \text{Spec} Z_0$, we can regard it as

$$
((g_\chi)_+, (g_\chi)_-) = \left[ \begin{array}{c} \chi(K^{-\ell}) \\
0 
\end{array} \right], \\
\left[ \begin{array}{c}
\chi(E^\ell) \\
1 
\end{array} \right], \\
\left[ \begin{array}{c}
1 \\
-\chi(E^\ell K^{-\ell}) \chi(K^{-\ell})
\end{array} \right] \in \tilde{G}
$$

as an element in $(SL_2(\mathbb{C}))^*$ or as an element

$$
g_\chi = (g_\chi)_+ (g_\chi)_-^{-1} = \left[ \begin{array}{cc}
\chi(K^{-\ell}) + \chi(E^\ell F^\ell) & \chi(K^\ell F^\ell) \\
\chi(E^\ell) & \chi(K^\ell)
\end{array} \right] \in G' \subset SL_2(\mathbb{C})
$$

For the cyclic representation $V_{\alpha, \beta, \mu}, x_{\alpha, \beta, \mu} : Z \to \mathbb{C}$ and $\chi_{\alpha, \beta, \mu} : Z_0 \to \mathbb{C}$ are defined via (4), (5), (6) and (7). As shown in [DC-K], due to the dimension counting, this representation induces an algebra isomorphism

$$
B_{x_{\alpha, \beta, \mu}} := U_q/I_{x_{\alpha, \beta, \mu}} \cong \text{End}(V_{\alpha, \beta, \mu})
$$

where $I_x$ is the ideal associated to $x$. Explicitly $I_x$ is generated by $E^\ell - x(E^\ell)$, $K^\ell - x(K^\ell), F^\ell - x(F^\ell)$, $\Omega - x(\Omega)$. And one can easily see that $P(D, \Omega) = 0$ as stated in Proposition 4.1 now.
As an another important observation, from Proposition 4.1, Spec \( Z \to \text{Spec} Z_0 \) is a generically \( \ell \)-to-1 map because \( P(\chi(D), \Omega) \in \mathbb{C}[\Omega] \) has degree \( \ell \). One can actually find that \( V_{\alpha,\beta,\mu} \) and \( V_{\alpha,\beta,\mu q^j} \) are representations of \( U_q \) such that

\[
\chi_{\alpha,\beta,\mu} = X_{\alpha,\beta,\mu q^j},
\]
\[
x_{\alpha,\beta,\mu}(\Omega) = \mu + \mu^{-1}
\]
\[
x_{\alpha,\beta,\mu q^j}(\Omega) = \mu q^j + \mu^{-1} q^{-j}
\]

for any \( j \in \mathbb{Z} \). This gives a description of the fiber of any \( \chi_{\alpha,\beta,\mu} \in \text{Spec} Z_0 \) and we have

\[
A_{\chi_{\alpha,\beta,\mu}} := U_q/I_{\chi_{\alpha,\beta,\mu}} \cong \bigoplus_{j=1}^\ell \text{End}(V_{\alpha,\beta,\mu q^j}) \cong \bigoplus_{j=1}^\ell B_{x_{\alpha,\beta,\mu q^j}}
\]

where \( I_{\chi_{\alpha,\beta,\mu}} \), similarly, is the ideal associated to \( \chi_{\alpha,\beta,\mu} \).

Moreover, the eigenvalues \( \lambda_1, \lambda_2 \) of \( g_{\chi_{\alpha,\beta,\mu}} \) would satisfy

\[
\lambda_1 + \lambda_2 = x(D), \quad \lambda_1 \lambda_2 = 1
\]

Notice at the same time that

\[
\mu^\ell + \mu^{-\ell} = P(x(D), x(\Omega)) + x(D) = x(D)
\]

where the first equality simply comes from the observation that both sides are degree \( \ell \) polynomials in \( \mathbb{C}[x(\Omega)] = \mathbb{C} \left[ \mu + \frac{1}{\mu} \right] \), having the same evaluation at the \( \ell \) distinct points \( x(K^\ell)^{1/\ell} q^j \). Hence the eigenvalues of \( g_{\chi_{\alpha,\beta,\mu}} \) are \( \mu^\ell \) and \( \mu^{-\ell} \).

### 4.2 The Construction of the Invariant Functor for \( SL_2(\mathbb{C}) \)

So far we have the factorizable group \( SL_2(\mathbb{C}) \) whose subgroup

\[
G' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C}, d \in \mathbb{C}^\times, ad - bc = 1 \right\}
\]

has the unique factorization property. We also know that \( \text{Spec} Z_0 \cong G' \). Let’s consider the category \( \mathcal{C} \) to be the monoidal category generated by \( \{ A_\chi : \chi \in \text{Spec} Z_0 \} \) with the monoidal structure given by the usual tensor product of representations and \( F : \mathcal{C} \to \text{Arr}(G') \) defined by \( A_\chi \mapsto \chi \) makes \( \mathcal{C} \) a \( G' \)-category. Now we would like to define the braiding on \( \mathcal{C} \) so that \( \mathcal{C} \) becomes a ribbon \( G' \)-category and use the section \( \{ A_\chi : \chi \in \text{Spec} Z_0 \} \) of \( F \) to define the invariant followed by the same procedure in 3.4.

Recall that although \( U_q \) is not quasi-triangular, as explained in 1.2, there is a bigger quasi-triangular Hopf algebra \( U_h = U_h(\mathfrak{sl}_2(\mathbb{C})) \) with the universal \( R \)-matrix

\[
R_h = q^{\frac{\eta_{\mathfrak{sl}_2}}{2}} f \left( q^{-1} \left( Xq^\eta \right) \otimes \left( q^{-\frac{\eta}{2}} Y \right) \right) ; q^2
\]

where \( f(z; q) = \prod_{n=1}^\infty (1 - z q^n) = \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n(n+1)}{2}} z^n}{(1-q) \cdots (1-q^n)} \). Then we can define
Definition. \( \mathcal{R} \in \text{Aut}(U_k^{\otimes 2}[[h]]) \) is given by \( u \otimes v \mapsto R_h(u \otimes v)R_h^{-1} \).

Then one can check \( \mathcal{R} \) is an outer automorphism on \( \hat{U}_q^{\otimes 2} \), where \( \hat{U}_q^{\otimes 2} \) is the division ring of \( U_q^{\otimes 2} \). More Explicitly, we have

**Proposition 4.2.** The following identities are true:

\[
\begin{align*}
\mathcal{R}(E \otimes 1) &= E \otimes K \\
\mathcal{R}(1 \otimes E) &= K \otimes E + E \otimes 1 - (E \otimes K^2)X^{-1} \\
\mathcal{R}(K \otimes 1) &= X(K \otimes 1) \\
\mathcal{R}(1 \otimes K) &= (1 \otimes K)X^{-1} \\
\mathcal{R}(F \otimes 1) &= 1 \otimes F + F \otimes K^{-1} - (K^{-2} \otimes F)X^{-1} \\
\mathcal{R}(1 \otimes F) &= K^{-1} \otimes F
\end{align*}
\]

where \( X = 1 \otimes 1 - qK^{-1}E \otimes FK \in U_q^{\otimes 2} \).

This is a pretty straightforward verification. Moreover, this is the braiding we are looking for.

**Proposition 4.3.** The followings are true:

1. \( \mathcal{R}(Z_0 \otimes Z_0) \subset Z_0 \otimes Z_0 \).
2. \( \mathcal{R}(\Omega \otimes \Omega) = \Omega \otimes \Omega \).
3. \( \mathcal{R} \) induces an isomorphism of algebras

\[
\mathcal{R}_{\chi_0,\chi_1} : A_{\chi_0} \otimes A_{\chi_1} \to A_{\chi_R} \otimes A_{\chi_L}
\]

such that \( B(A_{\chi_0}, A_{\chi_1}) := (A_{\chi_R}, A_{\chi_L}) \) is the pullback of \( \tau \circ \mathcal{R} \), where \( \mathcal{R}(x, y) = (x_L(x, y), x_R(x, y)) \) and

\[
x_L(x, y) = x_\tau yx_\tau^{-1}, \quad x_R(x, y) = (x_L(x, y)_+)^{-1}xx_L(x, y)_+
\]

defined via \( (SL_2(\mathbb{C}))^* \to SL_2(\mathbb{C}) \) given by \( (g_+, g_-) \mapsto g_+g_-^{-1} \).

4. \((G', \ast, \tau \circ B, \tau \circ \mathcal{R})\) is a ribbon \((G', \ast, \tau \circ \mathcal{R})\)-category, where \((G', \ast, i)\) has the group structure induced from \((SL_2(\mathbb{C}))^* \) via the same map.

To understand how \( \mathcal{R} \) acts on \( A_{\chi_0} \otimes A_{\chi_1} \), we would like to give a little bit more details. By [K] or [DC-K], it is well-known that \( A_{\chi} \) has the Poincaré-Birkhoff-Witt basis \( \{ E^aK^bF^c \}_{0 \leq a, b, c \leq \ell-1} \). So \( \mathcal{R} \) is fully determined by Proposition 4.2 and we have

\[
X^{-1} = \frac{\chi_R(K^e)}{\chi_R(K^e) - \chi_R(E^e)\chi_L(K^e)\chi_L(F^e)} \sum_{j=0}^{\ell-1} (qK^{-1}E \otimes KF)^j
\]
4.3 Kashaev-Reshetikhin Invariants of Links

One should notice that the invariant functor we define in the previous section always has quantum dimension zero, which can be seen in the cyclic representation structure, where the quantum dimension is defined by

\[
1 \xrightarrow{(\text{coev})_{A_{\chi_0}}} A_{\chi_0} \otimes A_{\chi_0}^* \xrightarrow{(\text{ev})_{A_{\chi_0}}} 1
\]

This implies the invariant functor would send every link to 0. In order to give a non-trivial invariant of links, one of the topological trick is to apply the invariant functor on the colored tangle obtained by cutting open any colored link at a point. Since the invariant functor outputs the same result for any isotopic colored tangle, it may depend on the choice of the component to do the cut, but is irrelevant to which point we choose on the component.

Moreover, since \( A_{\chi} \) is semisimple with simple \( \mathcal{U}_q \) submodules \( B^i_x \cong \text{End}(V_{\alpha,\beta,\mu q^i}) \) and the branches \( \mu q^i \) of the \( \ell \)th root of \( \mu \) parametrize isomorphic representations \( V_{\alpha,\beta,\mu q^i} \), we can fix a choice of branch around a neighborhood of the identity and get the collection of \( \{ B^i_x \} \) for generic \( \chi \in \text{Spec } Z_0 \). It is a ribbon \( (G', \ast, \tau \circ \Re) \)-category again if we consider the braiding to be the restriction \( \Re \) to \( B^i_x \). This setup has the advantages that (1) \( B^i_x \) has a smaller dimension as a vector space (2) this procedure would send any knot to \( \text{End}_{\mathcal{U}_q \text{Mod}}(B^i_x) \) for some \( x \in \text{Spec } Z_0 \), which is necessarily a number depending on \( x \) by Schur’s lemma. Hence we produce a function depending on \( x \in \text{Spec } Z \). Furthermore, we can change the basis of the trivial \( G \)-bundle on the complement of the tangle to have \( \chi \) upper-triangular without changing the monodromy. In this way, the function would only depend on the parameter \( \mu q^i \).

**Definition** (Kashaev-Reshetikhin invariant). Suppose \([\Pi, c : E(\Pi) \to \{ B_x \}]\) is a Reidemeister equivalent class of colored link diagram. Assume that the colored tangle obtained by cutting open \( \Pi \) at a point on an edge colored by \( B_{x_{\alpha,\beta,\mu}} \), where \( x_{\alpha,\beta,\mu} \in \text{Spec } Z \). Then the Kashaev-Reshetikhin invariant of \([\Pi, c : E(\Pi) \to \{ B_x \}]\) cut open at \( B_{x_{\alpha,\beta,\mu}} \)-colored edge is a number depending on \( \mu \).

We would like to give more details about the general computation. First of all, for every \( x \in \text{Spec } Z \), \( B_x \) has a basis \( \{ E^a K^b \}_{0 \leq a, b \leq \ell - 1} \) obtained from the Poincaré-Birkhoff-Witt basis of \( A_{\chi=x|\mathbb{Z}_0} \) by using the following relation

\[
EF = \Omega - q^{-1}K - qK^{-1}
\]

\[
\Rightarrow E^\ell F = \Omega E^{\ell-1} - q^{-1}E^{\ell-1}K - (qK^{-\ell})E^{\ell-1}K^{\ell-1}
\]

\[
\Rightarrow F = x(E^{\ell})^{-1} [x(\Omega)E^{\ell-1} - q^{-1}E^{\ell-1}K - qx(K^{-\ell})E^{\ell-1}K^{\ell-1}]
\]

if \( x(E^{\ell}) \neq 0 \). Similarly if \( x(F^{\ell}) \neq 0 \), then \( B_x \) has a basis \( \{ K^b F^a \}_{0 \leq a, b \leq \ell - 1} \). Since generically we have \( x(E^{\ell}) \neq 0 \), in all the examples below, we are going to use the
basis \( \{ E^a K^b \}_{0 \leq a, b \leq \ell - 1} \) for \( B_x \) and order it in the colexicographic order. Under this order, we rename the basis as \( \{ e_j \}_{j=0}^{(\ell^2-1)} \). That is to say, \( e_{b+\ell} = E^a K^b \). We will call the dual basis as \( \{ f^j \}_{j=0}^{(\ell^2-1)} \), which is defined via some bilinear form. We will see that the Kashaev-Reshetikhin invariant can be found without further understanding of the dual basis.

To follow the procedure in 3.4, we will only need the definition of \( \text{ev}^L \) and \( \text{coev}^L \).

Let's say

\[
\begin{align*}
(\text{coev}^L)_{B_x} : & 1 \to B_x \otimes B_x^*, & 1 \mapsto e_j \otimes f^j \\
(\text{ev}^L)_{B_x} : & B_x \otimes B_x \to 1, & f^i \otimes e_i \mapsto \delta_i^j
\end{align*}
\]

Then the right-evaluation map is

\[
(\text{ev}^R)_{B_{x_0}} : B_{x_0} \otimes B_{x_0}^* \xrightarrow{\tau \circ \mathcal{R}_{x_0,x_0}(\cdot)} B_{x_1}^* \otimes B_{x_1} \xrightarrow{(\text{ev}^L)_{B_{x_1}}} 1
\]

where \( g_{x_1} = i(g_{x_0}^{-1}) \in G' \) and \( \tau \circ \mathcal{R}_{x_0,x_0}(\cdot) \) can be found via

\[
[(\text{ev}^L)_{B_{x_1}} \otimes \text{id}_{B_{x_0} \otimes B_{x_0}^*}) \circ (\text{id}_{B_{x_1}^*} \otimes (\tau \circ \mathcal{R}_{x_1,x_0}(\cdot) \otimes \text{id}_{B_{x_0}^*})) \circ (\text{id}_{B_{x_1}^*} \otimes \text{coev}^L)_{B_{x_0}}]^{-1}
\]

which can be simplified as \( \tau \circ (\mathcal{R}_{x_1,x_0}^{-1})^{-1} \) and \((-)^{t_2}\) is the partial transpose on the second component. Whereas the right-coevaluation is

\[
(\text{coev}^R)_{B_{x_0}} : 1 \xrightarrow{(\text{coev}^L)_{B_{x_2}}} B_{x_2} \otimes B_{x_2}^* \xrightarrow{\mathcal{R}_{x_0,x_0}^{-1} \circ \tau} B_{x_0}^* \otimes B_{x_0}
\]

where \( g_{x_2} = i(g_{x_0}^{-1}) \in G' \) and \( \mathcal{R}_{x_0,x_0}^{-1} \circ \tau \) can be found via

\[
[(\text{ev}^L)_{B_{x_0}} \otimes \text{id}_{B_{x_2} \otimes B_{x_2}^*}) \circ (\text{id}_{B_{x_2}^*} \otimes (\mathcal{R}_{x_0,x_0}^{-1}(\cdot) \otimes \text{id}_{B_{x_2}^*})) \circ (\text{id}_{B_{x_2}^*} \otimes \text{coev}^L)_{B_{x_2}}]^{-1}
\]

which can be simplified as \( ((\mathcal{R}_{x_0,x_2}^{-1})^{-1} \circ \tau \) and \((-)^{t_1}\) is the partial transpose on the first component.

In particular, we can compute the quantum dimension by considering

\[
1 \xrightarrow{(\text{coev}^L)_{B_{x_0}}} B_{x_0} \otimes B_{x_0}^* \xrightarrow{(\text{ev}^R)_{B_{x_0}}} 1
\]

which gives 0. The calculation of this result when \( \ell = 3 \) can be found in the Mathematica notebook \texttt{RMatrix.nb} in the "qdim" section.

### 4.4 The Twist is the Identity

To compute the Kashaev-Reshetikhin invariant for the twist, we consider

\[
B_{x_0} \xrightarrow{1 \otimes \text{coev}^L} B_{x_0} \otimes B_{x_2} \otimes B_{x_2}^* \xrightarrow{(\tau \circ \mathcal{R}_{x_0,x_2})^{-1} \otimes 1} B_{x_0} \otimes B_{x_2} \otimes B_{x_2}^* \xrightarrow{1 \otimes \text{ev}^R} B_{x_0}
\]
which gives 1 when $\ell = 3$. The calculation of this result appears in the Mathematica notebook \texttt{RMatrix.nb} in the "twist" section. Hence the Kashaev-Reshetikhin invariant is independent of the choice of framing when $\ell = 3$ and we believe this should be true for every $\ell$.

5 Examples

5.1 The Trefoil Knot

Suppose $K$ is the trefoil knot and $T_K$ is the tangle obtained by cutting open $K$ at some point. Then the Wirtinger presentation of $\Pi_1 := \pi_1(\mathbb{R}^2 \times [0, 1] \setminus T_K, (0, -1, 0))$ is $\langle x, w | xwx = wxw \rangle$. For any $SL_2(\mathbb{C})$ representation of the fundamental group, that is, a group homomorphism from $\Pi_1$ to $SL_2(\mathbb{C})$, we denote the image of $x$ and $w$ still as $x$ and $w$ respectively for convenience. Then $w = x_yx^{-1}$ if we color the edges of the trefoil knot as in Figure 20. And from $xwx = wxw$, we can get the following relation between $x$ and $y$

$$x_yx^{-1}x_yx^{-1} = x_yx^{-1}x_yx^{-1} \Rightarrow i(x)^{-1}yi(x)^{-1} = yi(x)^{-1}y$$

Let’s choose $C$ such that $\tilde{z} = C^{-1}i(x)^{-1}C$ is the Jordan form of $i(x)^{-1}$ and call $\tilde{y} = C^{-1}yC$. Then there will be three distinct cases:

(i) $\tilde{z} = \pm I \Rightarrow \tilde{y} = \tilde{z}$.  

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(ii) \( \tilde{z} = \begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix}, M \neq \pm 1 \Rightarrow \tilde{y} = \tilde{z} \) or \( \begin{bmatrix} -(M^3 - M)^{-1} & y_2 \\ -1/(M - \frac{1}{M})^{-2} & \frac{M^3}{M^3 - 1} \end{bmatrix} \). We can simplify it by defining \( Y = \begin{bmatrix} y_2^{1/2} & 0 \\ 0 & y_2^{-1/2} \end{bmatrix} \). Then the change of basis by doing the conjugation by \( Y \) will keep \( \tilde{z} \) unchanged and allow us to assume \( y_2 = 1 \).

(iii) \( \tilde{z} = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \tilde{y} = \tilde{z} \) or \( \begin{bmatrix} y_1 & (y_1 - 1)^2 \\ -1 & 2 - y_1 \end{bmatrix} \). We can simplify it by defining \( Y = \begin{bmatrix} 1 & y_1 - 1 \\ 0 & 1 \end{bmatrix} \). Then the change of basis by doing the conjugation by \( Y \) will keep \( \tilde{z} \) unchanged and allow us to assume \( y_1 = 1 \).

For the situation that \( \tilde{y} = \tilde{z} \), that is, \( y = i(x)^{-1} \), we have \( u_1 = u_2 = x \) and \( v_1 = v_2 = y \). This is the case corresponding to the commutative component of the character variety. We have computed the invariant, as shown in the Mathematica notebook \( \text{RM} \text{Matrix}\.\text{nb} \) in the "trefoilcomm" section, in the following way:

\[
B_x \xrightarrow{1 \otimes \text{coev}^L} B_x \otimes B_y \otimes B_y^* \xrightarrow{(\tau \circ R_{x,y})^3 \otimes 1} B_x \otimes B_y \otimes B_y^* \xrightarrow{1 \otimes \text{coev}^R} B_x
\]

In our computation, when \( \ell = 3 \), the Kashaev-Reshetikhin invariant on the commutative component of the character variety of the trefoil knot is

\[
\left(1 + 2k^2 + k^4 + 2efk + 2efk^3 + e^2f^2k^2\right)c^2 + \left(3k + 3k^3 + 3efk^2\right)c + 3
\]

where \( c = x(\Omega), e = x(E^x), f = x(F^x), k = x(K^x) \). We can simplify it by using the identity \( P(D, \Omega) = 0 \Rightarrow c^3 - 3c = ef + k + \frac{1}{k} \). Then we get

\[
\left(ef + k + \frac{1}{k}\right)^2 c^2 + 3\left(ef + k + \frac{1}{k}\right)c + 3 = c^8 - 6c^6 + 12c^4 - 9c^2 + 3
\]

Replace \( c \) by \( \mu + \mu^{-1} \) as introduced in the cyclic representation of \( \mathcal{U}_q \) in [4.1] we are able to obtain Morrison-Snyder’s formula for the commutative component of the character variety

\[
\mu^8 + 2\mu^6 + 4\mu^4 + 5\mu^2 + 7 + 5\mu^{-2} + 4\mu^{-4} + 2\mu^{-6} + \mu^{-8}
\]

One could notice that this function is unchanged under the transformation \( \mu \mapsto \mu^{-1} \).

For the geometric component of the character variety, which is the situation that \( \tilde{y} \neq \tilde{z} \), we consider:

\[
B_x \xrightarrow{1 \otimes \text{coev}^L} B_x \otimes B_y \otimes B_y^* \xrightarrow{(\tau \circ R_{x,y})^3 \otimes 1} B_{u_1} \otimes B_{v_1} \otimes B_y^* \xrightarrow{(\tau \circ R_{u_1,v_1})^3 \otimes 1} B_{u_2} \otimes B_{v_2} \otimes B_y^* \xrightarrow{1 \otimes \text{coev}^R} B_x
\]
where in case (ii), \( M \) and \( \frac{1}{M} \) are the eigenvalues of \( i(x)^{-1} \), which are also the eigenvalues of \( x \), hence \( M = \mu^\ell \) as introduced in 4.1 and we have

\[
y = C \begin{bmatrix}
-\frac{(M^3 - M)^{-1}}{1 - (M - \frac{1}{M})^2} & \frac{1}{M^3 - 1} \\
-1 & \frac{1}{M^2 - 1}
\end{bmatrix} C^{-1}
\]

Here \( C = \begin{bmatrix} 1 - kM^{-1} & 1 - kM \\ e & e \end{bmatrix} \) if \( e \neq 0 \). If \( e = 0 \), we will have different \( C = \begin{bmatrix} 1 & f \\ 0 & k - k^{-1} \end{bmatrix} \) and \( M = \frac{1}{k} \). One should notice that in this case \( y \) and \( i(x)^{-1} \) still have exactly the same eigenvalues and we would have

\[
y(\Omega) = i(x)^{-1}(\Omega) = x(\Omega) = c
\]

when we choose the same branch.

In case (iii), for \( \tilde{z} = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), we have

\[
y = C \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} C^{-1}
\]

with \( C = \begin{bmatrix} \mp k & 1 \\ e & e \end{bmatrix} \) for \( e \neq 0 \). If \( e = 0 \), then \( C = \begin{bmatrix} \pm f & \pm f(1 \mp k)^{-1} \\ 1 \mp k^{-1} & 1 \mp k^{-1} \end{bmatrix} \).

The computation of the Kashaev-Reshetikhin invariant on the geometric component for the case (ii) when \( \ell = 3 \) can be found in the "trefoilgeom" section of the Mathematica notebook RMatrix.nb. The result matches Morrison-Snyder’s formula

\[
\frac{3(1 + \mu^2 + \mu^4)^2}{\mu^4}
\]

One could notice this invariant is still unchanged under the transformation \( \mu \mapsto \mu^{-1} \).

### 5.2 The Figure Eight Knot

Suppose \( K \) is the figure eight knot and \( T_K \) is the tangle obtained by cutting open \( K \) at some point. Then the Wirtinger presentation of its fundamental group \( \Pi_1 := \pi_1(\mathbb{R}^2 \times [0,1] \setminus T_K,(0,-1,0)) \) is

\[
\langle x, w | w^{-1} x w^{-1} x w^{-1} x w^{-1} x w = x \rangle
\]

For any \( SL_2(\mathbb{C}) \) representation of \( \Pi_1 \), again we denote the image of \( x \) and \( w \) still as \( x \) and \( w \) respectively for convenience. Then \( w = x y x^{-1} \) if we color the edges of the figure eight knot as in Figure 21. And from the relation in the Wirtinger presentation, we can get the following relation between \( x \) and \( y \)

\[
y^{-1} i(x)^{-1} y i(x) y i(x)^{-1} y^{-1} i(x) y = i(x)^{-1}
\]
Figure 21: The Tangle $T_K$ Associated to the Figure Eight Knot

As in the trefoil knot case, we would like to choose $C$ such that $\tilde{z} = C^{-1} i(x)^{-1} C$ is the Jordan form of $i(x)^{-1}$ and call $\tilde{y} = C^{-1} y C$. Then there will be four distinct ways to color the figure eight knot:

(i) $\tilde{z} = \pm I \Rightarrow \tilde{y} = \tilde{z}$.

(ii) $\tilde{z} = \begin{bmatrix} M & 0 \\ 0 & \frac{1}{M} \end{bmatrix}$, $M \neq \pm 1, \pm \frac{1 \pm \sqrt{5}}{2} \Rightarrow \tilde{y} = \tilde{z}$ or

$$\begin{bmatrix} g_{\pm}(M) & y_2 \\ y_3 & M + M^{-1} - g_{\pm}(M) \end{bmatrix}$$

where $y_2$ is arbitrary, $y_3$ is chosen such that the determinant is 1 and

$$g_{\pm}(M) = \frac{M^4 + M^2 - 1 \pm \sqrt{M^8 - 2M^6 - M^4 - 2M^2 + 1}}{2M(M^2 - 1)}$$

Again we may assume $y_2 = 1$ under some change of basis.

(iii) $\tilde{z} = \begin{bmatrix} M & 0 \\ 0 & \frac{1}{M} \end{bmatrix}$, $M = \pm \frac{1 \pm \sqrt{5}}{2} \Rightarrow \tilde{y} = \tilde{z}$ or

$$\begin{bmatrix} g_{\pm}(M) & y_2 \\ y_3 & M + M^{-1} - g_{\pm}(M) \end{bmatrix}$$

or $\begin{bmatrix} M & y_2 \\ 0 & \frac{1}{M} \end{bmatrix}$ or $\begin{bmatrix} M & 0 \\ y_3 & \frac{1}{M} \end{bmatrix}$. Similarly we may assume $y_2 = 1$ or $y_3 = 1$.  

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(iv) \[ \tilde{z} = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \tilde{y} = \tilde{z} \text{ or} \]
\[ \begin{bmatrix} y_1 & y_2 \\ \frac{1}{2} \pm \frac{\sqrt{3}i}{2} & 2 - y_1 \end{bmatrix} \]

where \( y_2 \) is chosen such that the determinant is 1. We may assume \( y_1 = 1 \).

For the situation that \( \tilde{y} = \tilde{z} \), that is, \( y = i(x)^{-1} \), we have \( u_1 = v_2 = w_2 = w_3 = x \), \( v_1 = y = i(x)^{-1} \) and \( u_2 = v_3 = i(x^{-1}) \). This is the case corresponding to the commutative component of the character variety. We have computed the invariant, which appeared in the Mathematica notebook RMatrix.nb in the "figureeightcomm" section, as follows

\[
\begin{align*}
B_x \xrightarrow{1 \otimes \text{coev}^t} & B_x \otimes B_{i(x)^{-1}} \otimes B_{i(x)^{-1}}^* \xrightarrow{f_1} B_x \otimes B_{i(x)^{-1}} \otimes B_{i(x)^{-1}}^* \xrightarrow{f_2} B_x \otimes B_x^* \otimes B_x \\
& \xrightarrow{f_3} B_{i(x)^{-1}}^* \otimes B_{i(x^{-1})} \otimes B_x \xrightarrow{f_4} B_{i(x^{-1})}^* \otimes B_{i(x^{-1})} \otimes B_x \xrightarrow{\text{ev}^b \otimes 1} B_x
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= (\tau \circ R_{x,i(x)^{-1}})^{-1} \otimes 1 \\
f_2 &= 1 \otimes (\tau \circ (R_{x,i(x)^{-1}}^\ell)^{-1}) \\
f_3 &= (\tau \circ (R_{i(x^{-1}),x}^{-1})^t)^{-1} \otimes 1 \\
f_4 &= 1 \otimes (\tau \circ R_{i(x^{-1}),x})
\end{align*}
\]

In our computation, when \( \ell = 3 \), the Kashaev-Reshetikhin invariant on the commutative component of the character variety of the figure eight knot is

\[
\left[ \left( \frac{1 + k^2 + e f k}{k} \right)^2 + 16 \right] c^2 + 18 \left( \frac{k + k^3 + e f k^2}{k^2} \right) c + \left[ 4 \left( \frac{1 + k^2 + e f k}{k} \right)^2 + 1 \right]
\]

where \( c = x(\Omega) \), \( e = x(F^\ell) \), \( f = x(F^\ell) \), \( k = x(K^\ell) \). If we simplify as in the trefoil knot computation by the identity \( c^3 - 3c = ef + k + \frac{1}{k} \), we would get \( (c^4 - c^2 + 1)^2 \), or in terms of \( \mu \) as introduced in 4.1, we have

\[
\mu^8 + 6 \mu^6 + 19 \mu^4 + 36 \mu^2 + 45 + 36 \mu^{-2} + 19 \mu^{-4} + 6 \mu^{-6} + \mu^{-8}
\]

Morrison and Snyder’s computation is for the geometric component of the character variety when case (ii) happens, we try to duplicate their result as follows. Consider

\[
y = C \left[ \begin{array}{cc} g_\pm(M) & 1 \\
               h_\pm(M) & M + M^{-1} - g_\pm(M) \end{array} \right] C^{-1}
\]

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where \( M = \mu^\ell \) and \( \frac{1}{M} \) are eigenvalues of \( x \), which are eigenvalues of \( y \) as well, and

\[
h_{\pm}(M) = \frac{-1 + 3M^2 - M^4 \pm \sqrt{M^8 - 2M^6 - M^4 - 2M^2 + 1}}{2(M^2 - 1)^2}
\]

and \( C = \begin{bmatrix} 1 - kM^{-1} & 1 - kM \\ e & e \end{bmatrix} \) if \( e \neq 0 \). If \( e = 0 \), we will have different \( C = \begin{bmatrix} 1 \\ 0 \\ \frac{f}{k - k^{-1}} \end{bmatrix} \) and \( M = \frac{1}{k} \). Then one can find the computation of the Kashaev-Reshetikhin invariant on the geometric component for this situation when \( \ell = 3 \) in the "figureeightgeom" section of the Mathematica notebook \texttt{RMatrix.nb}. It has the formula as follows

\[
3\mu^8 + 9\mu^6 + 21\mu^4 + 30\mu^2 + 36 + 30\mu^{-2} + 21\mu^{-4} + 9\mu^{-6} + 3\mu^{-8}
\]

### 5.3 The Hopf Link

Suppose \( L \) is the Hopf link and \( T_L \) is the tangle obtained by cutting open \( L \) at some point. Then the Wirtinger presentation of its fundamental group is

\[
\Pi_1 := \pi_1(\mathbb{R}^2 \times [0, 1] \setminus T_L, (0, -1, 0)) = \langle x, w | xw = wx \rangle
\]

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For any $SL_2(C)$ representation of $\Pi_1$, again we denote the image of $x$ and $w$ still as $x$ and $w$ respectively for convenience. Then $w = x^{-1}yx^{-1}$ if we color the edges of the Hopf link as in Figure [22]. And from the relation in the Wirtinger presentation, we get

$$x_+yx^{-1} = x_+yx_+x^{-1} \Rightarrow i(x)^{-1}y = yi(x)^{-1}$$

As before, we choose $C$ such that $\tilde{\zeta} = C^{-1}i(x)^{-1}C$ is the Jordan form of $i(x)^{-1}$ and call $\tilde{y} = C^{-1}yC$. Then there will be three possibilities:

(i) $\tilde{\zeta} = \pm I \Rightarrow \tilde{y}$ can be anything.

(ii) $\tilde{\zeta} = \begin{bmatrix} M & 0 \\ 0 & \frac{1}{M} \end{bmatrix}$, $M \neq \pm 1 \Rightarrow \tilde{y} = \begin{bmatrix} N & 0 \\ 0 & \frac{1}{N} \end{bmatrix}$. 

(iii) $\tilde{\zeta} = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \tilde{y} = \begin{bmatrix} \pm 1 & y_2 \\ 0 & \pm 1 \end{bmatrix}$.

where $M$ and $\frac{1}{M}$ are eigenvalues of $i(x)^{-1}$, and $N$ and $\frac{1}{N}$ are eigenvalues of $y$. Hence $M$ and $\frac{1}{M}$ are eigenvalues of $x$ and we have $M = \mu^\ell$ as introduced in 4.1.

Let’s first consider the case $\tilde{\zeta} = \tilde{\zeta}$, i.e., $y = i(x)^{-1}$. Then as shown in the Mathematica notebook $\text{RMatrix.nb}$ in the "HopfSpecial" section, when $\ell = 3$, the Kashaev-Reshetikhin invariant for the Hopf link can be found via

$$B_x \xrightarrow{1 \otimes \text{coev}^L} B_x \otimes B_y \otimes B_y^* \xrightarrow{(\tau \circ R_{x,y})^2 \otimes 1} B_x \otimes B_y \otimes B_y^* \xrightarrow{1 \otimes \text{ev}^R} B_x$$

and it has value

$$1 + c \left( ef + k + \frac{1}{k} \right) + c^2 = c^4 - 2c^2 + 1 = \mu^4 + 2\mu^2 + 3 + 2\mu^{-2} + \mu^{-4}$$

where $c = x(\Omega) = \mu + \mu^{-1}$, $e = x(E^\ell)$, $f = x(F^\ell)$, $k = x(K^\ell)$ is the same definition as before.

Now let’s consider case (ii). We have

$$y = C \begin{bmatrix} N & 0 \\ 0 & N^{-1} \end{bmatrix} C^{-1}$$

with $C = \begin{bmatrix} 1 - kM^{-1} & 1 - kM \\ e & e \end{bmatrix}$ under the assumption that $e \neq 0$. The Kashaev-Reshetikhin invariant in this case, when $\ell = 3$ is

$$\frac{1}{\mu^4}(1 + \mu^2 + \mu^4)^2$$

One can check this computation in the "Hopf" section of $\text{RMatrix.nb}$. 47
For the last situation case (iii). Suppose $e \neq 0$, then we have $f = \frac{2-k-\frac{1}{k}}{e} = \frac{-(k-1)^2}{ek}$ and $y = C \left[ \begin{array}{cc} \pm 1 & y_2 \\ 0 & \pm 1 \end{array} \right] C^{-1}$, where $C = \left[ \begin{array}{cc} 1 & \mp k \\ e & \mp 1 \end{array} \right]$. We find the invariant in this case is 9 if we choose the branch that $\mu = 1$, the invariant is 0 if we choose the other branches, and this computation appears in the "HopfHyper" section of RMatrix.nb.

Therefore one can see that no matter in which case, the computation when $\ell = 3$ always gives the same result. Notice this number is independent of the choice of $y$, hence this fits in the framework of [GPT] and possibly suggests $B_x$ are all ambidextrous objects. It would certainly require more understanding of $B_x$ as simple $U_q$-bimodules to verify this guess.

6 Properties and Theorems

Proposition 6.1. The Kashaev-Reshetikhin invariant $I(\mu)$ satisfies $I(\mu) = I(\mu^{-1})$.

Proof. It is easy to check that $x_{\alpha,\beta,\mu} = x_{\alpha,\beta',\frac{1}{\mu}}$ for every $x \in \text{Spec } Z$, where

$$\beta' = \left[ \prod_{n=1}^{\ell} \frac{\mu - \alpha q^n}{\frac{1}{\mu} - \alpha q^n} \right]^{\frac{1}{\ell}} \beta$$

Hence $\mathcal{R}_{x_{\alpha_1,\beta_1,\mu_1}, x_{\alpha_2,\beta_2,\mu_2}} = \mathcal{R}_{x_{\alpha_1,\beta'_{1}, \frac{1}{\mu_1}}, x_{\alpha_2,\beta'_{2}, \frac{1}{\mu_2}}}$. Since our construction as indicated in 4.3 is fully determined by $\mathcal{R}$, this implies $I(\mu)$ stays unchanged under the transformation $\mu \mapsto \frac{1}{\mu}$. \hfill \Box

Not only one can see from the examples that the invariant has the property above, but also one could notice that the invariant we have found is always a meromorphic function. We try to show this result for one class of knots: 2-bridge knots. The examples that we have computed are all 2-bridge links. We will refer to [Ka], [G] for the following introduction to 2-bridge knots.

Definition (Bridge Presentation). Suppose $\Pi$ is a link diagram associated to a link $L$. We say $\Pi$ is a bridge presentation of $L$ if there exists $B_i \subset \Pi$ such that

$$\Pi \setminus \bigcup_{i=1}^{m} B_i = \bigcup_{j=1}^{m} C_j$$

satisfying $B_i$ is always an overcrossing for every crossing on it. $B_i$ are usually called the overbridges and $C_j$ are called the underbridges of $L$. The bridge number of $L$ is the minimal $m$ of all the possible knot diagrams associated to $L$.

Definition (2-Bridge Link). $L$ is a 2-bridge link if it has bridge number 2.
Notice that if the bridge number of a knot $K$ is 1, then it is easy to see that $K$ is isotopic to an unknot. So 2-bridge knots are in some sense the easiest non-trivial knots to deal with. Indeed, we are going to see that it has an easy Wirtinger presentation of its knot group.

First let us try to classify all the 2-bridge links. It is well-known that all the 2-bridge links are isotopic to $L_{p/q}$ for some relative prime $p, q \in \mathbb{Z}$ with $q > p > 0$ and $2 \nmid p$, where $L_{p/q}$ is a link whose projection onto $y$-$z$ plane is isotopic to Schubert’s normal form built up in the following procedure:

1. Draw two overbridges $B_1 = \{0\} \times [0, 1]$ and $B_2 = \{1\} \times [0, 1]$.

2. For every $0 \leq i \leq q$, draw underbridges with slope $\frac{p}{q}$ starting from $(0, \frac{i}{q})$.
   (There could be duplicate arcs with the same endpoints in the following construction. We then only keep one of them.)

   (1) If $0 \leq i \leq q - 2p$, then arrive $(1, \frac{p+i}{q}) \in B_2$ first and keep drawing an arc around to $(0, \frac{2p+i}{q})$.

   (2) If $q - 2p \leq i \leq q - p$, then arrive $(1, \frac{p+i}{q}) \in B_2$ and keep drawing an arc around to $(0, \frac{2q-2p-i}{q})$.

   (3) If $q - p < i \leq q$, then it won’t reach $B_2$ and we would keep drawing an arc around to $(1, \frac{2q-p-i}{q})$. 

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3. For every $1 \leq i < q - p$, draw underbridges starting from $\left(1, \frac{i}{q}\right)$ to $\left(0, \frac{p+i}{q}\right)$.

4. For every $0 \leq i < p$, draw underbridges starting from $\left(1, \frac{i}{q}\right)$ to $\left(0, \frac{p-i}{q}\right)$.

Hence one can check that Figure 23 indicates that the trefoil knot is $L_{1/3}$. Then as proved in [G], we have

**Lemma 6.2.** $\pi_1(K_{p/q}) = \langle x, w : xz = zw \rangle$, where

$$z = w^{e_1}x^{e_2} \cdots w^{e_{q-2}}x^{e_{q-1}}$$

and $e_i = (-1)^{\left\lfloor \frac{ip}{q} \right\rfloor}$.

The other ingredient that is required is the famous Newton-Puiseux Theorem. We refer its proof and our particular usage to [N].

**Definition (Puiseux Series).** A Puiseux series is a series $x^m f(x^{1/n})$ for some $f \in \mathbb{C}[[x]]$ and $n \in \mathbb{N}, m \in \mathbb{Z}$. It is called analytic if $f$ is analytic. We denote the ring of Puiseux series as $\mathbb{C}\langle\langle x \rangle\rangle$ and the subring of analytic Puiseux series as $\mathbb{C}\langle\{x\}\rangle$.

**Lemma 6.3** (Newton-Puiseux). $\mathbb{C}\langle\langle x \rangle\rangle$ and $\mathbb{C}\langle\{x\}\rangle$ are algebraically closed fields.

Then the main theorem is now stated as follows.

**Theorem 6.4.** For any 2-bridge knot, the Kashaev-Reshetikhin invariant is a meromorphic function on each component of the character variety.

**Proof.** For any 2-bridge knot, we know the knot group has a presentation like

$$\langle x, w : xz = zw \rangle = \langle x, w : x^{\gamma_i^x}w^{\gamma_i^w}x^{\gamma_i^x}w^{\gamma_i^w} \cdots w^{\gamma_i^w} = 1 \rangle$$

where $\gamma_i^x, \gamma_i^w \in \{\pm 1\}$. As before, we call the image of $x, w$ still as $x, w$ in $SL_2(\mathbb{C})$ under any $SL_2(\mathbb{C})$ representation, then the character variety is characterized by

$$i(x)^{\gamma_i^x}y^{\gamma_i^y}i(x)^{\gamma_i^x}y^{\gamma_i^y} \cdots y^{\gamma_i^w} = 1 \quad (8)$$

where $y = xw^{-1}$ and $\gamma_i^x, \gamma_i^w \in \{\pm 1\}$. By some change of basis, we may assume

$$i(x)^{-1} = \begin{pmatrix} \mu^l & 0 \\ 0 & \mu^{-l} \end{pmatrix}, \quad y = \begin{pmatrix} y_1 & -1 + y_1y_4 \\ 1 & y_4 \end{pmatrix}$$

Moreover, one could see that

$$y_1 + y_4 = \text{tr}(y) = \text{tr}(xw^{-1}) = \text{tr}(w) = \text{tr}(x^{-1}xz) = \text{tr}(x) = \text{tr}(i(x)) = \mu^l + \mu^{-l}$$

Hence the condition $(8)$ would produce equations

$$p_i(\mu, y_1) = 0$$

where $p_i \in \mathbb{C}(\mu)[y_1]$. By Newton-Puiseux Theorem, any solution $y_1(\mu) \in \mathbb{C}\langle\{x\}\rangle$, which means both $y_1(\mu)$ and $y_4(\mu)$ are meromorphic functions on each component. Since the invariant is fully determined by $\mathcal{R}$ and now we know all the entries are meromorphic functions of $\mu$, the invariant is also a meromorphic function.
There is no reason why 2-bridge knots are the only special knots satisfying this property. So we would guess that for any knot, the Kashaev-Reshetikhin invariant is always a meromorphic function on each component of the character variety.

7 Mathematica Code in RMatrix.nb

7.1 Code Comments

In this code, we will remember every element

\[ \sum_{i=1}^{m} C_i E^{a_i} K^{b_i} \otimes E^{c_i} K^{d_i} \in B_x \otimes B_y \]

as a \( m \)-by-5 matrix such that each row has entries \((C_i, a_i, b_i, c_i, d_i)\), where \(C_i \neq 0\) for every \(i\). And we will define a function \texttt{Multi} to perform the multiplication for every pair of 5-tuples to describe the multiplication operation on \(B_x \otimes B_y\). And for any element \(\chi \in \text{Spec } \mathbb{Z}_0\), we will restore it as a 3-tuple \((\chi(E^\ell), \chi(K^\ell), \chi(F^\ell))\). As shown in 4.3, the computation is mainly based on the description of \(\mathcal{R} : B_x \otimes B_y \rightarrow B_{xR} \otimes B_{xL}\).

Thus in the setup section, we will first try to restore \(\mathcal{R}\) as a symbolic matrix in \(GL_{\ell^4}(\mathbb{C}[e_{xR}, k_{xR}, f_{xR}, e_{xL}, k_{xL}, f_{xL}])\) by choosing the same basis \(\{E^a K^b \otimes E^c K^d\}\) on both domain and codomain, where \(e_x = x(E^\ell), k_x = x(K^\ell), f_x = x(F^\ell)\) for any \(x \in \text{Spec } \mathbb{Z}\). We fix the branch by choosing \(\mu\) as one of the \(\ell\)th root of one of the eigenvalue of \(\chi = x|_{\mathbb{Z}_0}\) continuously. And because of the Proposition 4.2, \(\mathcal{R}\) will only depend on \(x_L(\Omega)\), which is denoted as \(\text{Cm}\) in the code. Moreover, by Proposition 4.3, we know that \(x_R(\Omega) = x(\Omega)\) and \(x_L(\Omega) = y(\Omega)\). So it is easy to fix the branch while during computation.

One could notice that the computation of the invariant would involve product of \(\ell^4\)-by-\(\ell^4\) symbolic matrices. This would in general take hours to do the computation. Fortunately, \(\mathcal{R}\) is a sparse matrix. Hence in the code we will actually restore \(\mathcal{R}\) as a \(k\)-by-3 matrix such that each row has entries \((\mathcal{R}_{ij}, i, j)\) only when \(\mathcal{R}_{ij} \neq 0\). And we could implement the symbolic matrix product easily, and refer [DM1], [DM2] to implement the symbolic matrix inverse algorithm.

Also one could simplify the computation by assuming \(x \in SL_2(\mathbb{C})\) is an upper-triangular or a lower-triangular matrix, just as what we did in the examples above. In this code we don’t use this simplification, but we indeed see that the invariant is only relevant to \(\mu\).
7.2 Setup Section

(*Initial assumptions*)
\(\ell = 3;\)
\[LthRoot = \{\eta^\wedge{n} \rightarrow \eta^\wedge{\text{Mod}[n, \ell]}\};\]

(*Functions for Sparse Inverse Algorithm*)
\[\text{MTA}[T, O, M, RC, RCL, SIZE] :=\]
\[
\text{Module}\left\{\begin{array}{l}
A = \text{ConstantArray}[0, \{\text{SIZE}, \text{SIZE}\}], \\
B = \text{ConstantArray}[0, \{\text{SIZE}, \text{SIZE}\}], \\
i, j, m, \text{PosA} = 0, \text{PosB} = 0, k = 1, \text{Judge} = 0, \text{Modi}, \\
\text{For}[i = 1, i \leq \text{SIZE}, i++, \\
\text{If}[O[[i]] == 0, \text{PosB}++; B[[k, \text{PosB}]] = i;];]
\text{While}[\text{Judge} == 0, \\
\text{For}[i = 1, i \leq \text{SIZE}, i++, \\
\text{For}[j = 1, j \leq \text{PosB}, j++, \\
\text{If}[\text{MemberQ}[T[[i]], B[[k, j]]], \text{PosA}++; \\
A[[k, i]] = 1; \text{Break}[];];]
\text{PosB} = 0;
\text{For}[i = 1, i \leq \text{SIZE}, i++, \\
\text{If}[A[[k, i]] == 1, \text{PosB}++; B[[k + 1, \text{PosB}]] = M[[i]];];]
\text{k}++; \text{PosA} = 0;
\text{For}[i = 1, i \leq \text{RCL}, i++, \\
\text{For}[j = 1, j \leq \text{PosB}, j++, \\
\text{If}[\text{MemberQ}[T[[\text{RC}[[i]]]], B[[k, j]]], \text{Judge} = 1; \text{Break}[];];]
\text{If}[\text{Judge} == 1, \text{Break}[];];]
\text{Modi} = \{\{\text{RC}[[i]], B[[k, j]]\}\};
\text{While}[\text{k} > 1, \\
\text{k}--;]
\text{For}[i = 1, i \leq \text{SIZE}, i++, \\
\text{If}[A[[k, i]] == 1, \text{If}[B[[k + 1, j]] == M[[i]], \text{Break}[];];]
\text{For}[j = 1, j \leq \ell^4, j++, \\
\text{If}[\text{MemberQ}[T[[i]], B[[k, j]]], \text{Break}[];];]
\text{Modi} = \text{Flatten}[\text{List}[\text{Modi}, \{\{i, B[[k, j]]\}\}], 1]; \text{Modi};]
\right.\]

\[\text{BTA}[T, \text{SIZE}] :=\]
\[
\text{Module}\left\{\begin{array}{l}
Q = \text{Range}[\text{SIZE}], P, PP, RT, B = \text{ConstantArray}[1, \text{SIZE}], \\
\text{BSIZE}, BL = \text{SIZE}, A = \text{Range}[\text{SIZE}], L, \\
\text{Transversal} = \text{ConstantArray}[1, \text{SIZE}], \text{Pos} = 1, \text{Index}, \text{BPos} = 0, \\
\text{Induction} = 0, i, j, k, m, \text{Looped} = 0, \text{TEMP}, \\
\text{Index} = \text{Transversal}[\text{Pos}]; RT = T; \\
\text{While}[\text{SIZE} > \text{Induction}, \\
\right.\]
\]

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L = Length[RT[[Index]]];
If[L == 0,
    For[j = 1, j \leq BL - BPos, j++,
        L = Length[RT[[j]]];
        For[k = 1, k \leq L, k++,
            If[RT[[j, k]] == Index,
                RT[[j]] = Flatten[List[RT[[j, 1 ;; k - 1]], RT[[j, k + 1 ;; L]]], 1];
                Break[];]
            If[RT[[j, k]] > Index, RT[[j, k]] = RT[[j, k]] - 1;]];
    For[k = 1, k \leq L - 1, k++,
        If[RT[[j, k]] > Index, RT[[j, k]] = RT[[j, k]] - 1;]];
    RT = Flatten[List[RT[[1 ;; Index - 1]], RT[[Index + 1 ;; BL - BPos]]], 1];
    P = Range[SIZE]; Index = BPos + Index; BSIZE = B[[Index]];
    For[k = 1, k \leq BSIZE, k++,
        P[[A[[Index]] - BSIZE + k]] = k + Induction;]
    For[k = Induction + 1, k \leq A[[Index]] - BSIZE, k++,
        P[[k]] = k + BSIZE;]
    For[k = 1, k \leq SIZE, k++,
        If[k == 1, A[[k]] = B[[k]], A[[k]] = A[[k - 1]] + B[[k]];]
        If[Pos \neq 1,
            Pos--;]
        For[j = 1, j \leq Pos, j++,
            If[Transversal[[j]] > Index - BPos,
                Transversal[[j]] = Transversal[[j]] - 1;]];
        Index = Transversal[[Pos]];]
    If[Pos == 1, Index = 1];
    If[L \neq 0,
        Index = RT[[Index, 1]];]
    For[i = 1, i < Pos, i++,
        If[Index == Transversal[[i]], Looped = 1; Break[];]];
    If[Looped == 1,
        For[j = i, j \leq Pos, j++,
            P = Range[SIZE];
            Index = BPos + Transversal[[j]];
            BSIZE = B[[Index]];]
For \( k = 1, k \leq \text{BSIZE}, k++, \)
\[
P[[A[[\text{Index}]] - \text{BSIZE} + k]] = k + \text{Induction};
\]
For \( k = \text{Induction} + 1, k \leq A[[\text{Index}]] - \text{BSIZE}, k++, P[[k]] = k + \text{BSIZE};\)
For \( k = 1, k \leq \text{SIZE}, k++, Q[[k]] = P[[Q[[k]]]];\)
For \( k = \text{Index}, k > \text{BPos} + 1, k--,
\]
\[
B[[k]] = B[[k - 1]];\]
\[
B[[\text{BPos} + 1]] = \text{BSIZE};\]
For \( k = \text{BPos} + 1, k \leq \text{Index}, k++,\)
\[
\text{If}[k == 1, A[[k]] = B[[k]];]
\]
\[
\text{If}[k \neq 1, A[[k]] = A[[k - 1]] + B[[k]];]
\]
\[
\text{PP} = \text{Range}[	ext{BL} - \text{BPos}];\]
\[
\text{PP}[[\text{Transversal}[[j]]]] = 1;\]
For \( k = 1, k < \text{Transversal}[[j]], k++,\)
\[
\text{PP}[[k]] = k + 1;\]
For \( k = 1, k \leq \text{BL} - \text{BPos}, k++\)
\[
\text{L} = \text{Length}[	ext{RT}[[k]]];\]
\[
\text{For}[m = 1, m \leq \text{L}, m++,
\]
\[
\text{RT}[[k, m]] = \text{PP}[[\text{RT}[[k, m]]]];\]
\[
\text{TEMP} = \text{RT}[[\text{Transversal}[[j]]]];\]
\[
\text{For}[k = \text{Transversal}[[j]], k > 1, k--,
\]
\[
\text{RT}[[k]] = \text{RT}[[k - 1]];\]
\[
\text{RT}[[1]] = \text{TEMP};\]
For \( k = 1, k \leq \text{Pos}, k++,\)
\[
\text{Transversal}[[k]] = \text{PP}[[\text{Transversal}[[k]]]];\]
For \( j = 1, j \leq \text{BL} - \text{BPos}, j++,\)
\[
\text{L} = \text{Length}[	ext{RT}[[j]]];\]
\[
\text{For}[k = 1, k \leq \text{L}, k++,
\]
\[
\text{If}[	ext{MemberQ}[	ext{Transversal}[[i ;; \text{Pos}]],
\]
\[
\text{RT}[[i, k]],
\]
\[
\text{RT}[[j, k]] = 1,
\]
\[
\text{RT}[[j, k]] = \text{RT}[[j, k]] - \text{Pos} + i;\]
\]
\[
\text{RT}[[j]] = \text{DeleteDuplicates}[	ext{RT}[[j]]];\]
\[
\text{RT}[[1]] = \text{DeleteDuplicates}[
\]
\[
\text{Flatten}[	ext{Table}[	ext{RT}[[1 + j]], \{j, 0, \text{Pos} - i\}, 1]];\]
\[
\text{L} = \text{Length}[	ext{RT}[[1]]];\]
\[
\text{For}[j = 1, j \leq \text{L}, j++,
\]
\[
\text{If}[	ext{RT}[[1, j]] == 1,
\]
\[
\text{RT}[[1]] = \text{Flatten}[	ext{List}[	ext{RT}[[1, 1 ;; j - 1]],
\]
\[
\text{RT}[[1, j + 1 ;; \text{L}]], 1]];\]
\]
\[
\text{Break}[];\]
BL = BL + i - Pos;
For[j = 2, j ≤ BL - BPos, j++,
    RT[[j]] = RT[[j + Pos - i]]];
RT = RT[[1 ;; BL - BPos]];  
B[[BPos + 1]] = Sum[B[[j]],
    {j, BPos + 1, BPos + 1 + Pos - i}];
If[BPos == 0, A[[1]] = B[[1]],
    A[[BPos + 1]] =
    A[[BPos]] + B[[BPos + 1]]];
For[j = BPos + 2, j ≤ BL, j++,
    B[[j]] = B[[j + Pos - i]];  
    A[[j]] = A[[j - 1]] + B[[j]]];
For[j = 1, j < i, j++,
    Transversal[[j]] = Transversal[[j]] - Pos + i];
Pos = i; Index = 1;
Transversal[[Pos]] = 1; Looped = 0;
If[Looped ≠ 1,
    Pos++;
    Transversal[[Pos]] = Index];]
B = B[[1 ;; BL]]; {Q, B}];

(*Tools*)
SparseToMtx[S_, R_, C_] :=
    Module[{M = ConstantArray[0, {R, C}], i, L = Length[S]},
        For[i = 1, i ≤ L, i++, M[[S[[i, 2]], S[[i, 3]]]] = S[[i, 1]]; M];
StdSparse[M_] :=
    Module[{A, L = ConstantArray[1, ℓ^4], TotalL = 0, NowL, i, j, P = 1},
        For[i = 1, i ≤ ℓ^4, i++, L[[i]] = Length[M[[i]]];  
            TotalL = TotalL + L[[i]]];
        A = ConstantArray[0, TotalL, 3];
        For[i = 1, i ≤ ℓ^4, i++,
            NowL = L[[i]];
            For[j = 1, j ≤ NowL, j++,
                A[[P, 1]] = M[[i, j, 1]];  
                A[[P, 2]] = 1 + M[[i, j, 2]] + M[[i, j, 3]]*ℓ + M[[i, j, 4]]*ℓ^2 +
                    M[[i, j, 5]]*ℓ^3;
                A[[P, 3]] = i; P++; ]; A];
SparseRemoveZero[M_] := Module[{L, Pos, NONZERO, NewM, i},
        L = Length[M]; NONZERO = ConstantArray[0, L]; Pos = 0;
        For[i = 1, i ≤ L, i++,
            ...]
If[TrueQ[M[[i, 1]] == 0], Pos++; NONZERO[[Pos]] = i;];
NewM = M[[!NONZERO[[1 ;; Pos]]]]; NewM;

SparseMulti[M1, M2] :=
Module[{L1, L2, L = 0, M, i, j, k, R, C, FLAG, NONZERO, P, Temp},
  L1 = Length[M1]; L2 = Length[M2];
  M = ConstantArray[0, {L1*L2, 3}];
  For[i = 1, i ≤ L1, i++,
    For[j = 1, j ≤ L2, j++,
      If[M1[[i, 3]] == M2[[j, 1]],
        FLAG = 0; Temp = M1[[i, 1]]*M2[[j, 1]];
        R = M1[[i, 2]]; C = M2[[j, 2]];
        For[k = 1, k ≤ L, k++,
          If[M[[k, 2]] == R && M[[k, 3]] == C,
            FLAG = 1; Break[];];
        ];
        If[FLAG == 0,
          L++; M[[L, 1]] = Temp; M[[L, 2]] = R;
          M[[L, 3]] = C, M[[k, 1]] = M[[k, 1]] + Temp];
      ];
    ];
  ];
  M = M[[1 ;; L]]; M

SparseInverse[M, SIZE] :=
Module[{P = ConstantArray[0, SIZE], PP = ConstantArray[0, SIZE], Q,
  InvP = ConstantArray[0, SIZE], InvQ = ConstantArray[0, SIZE],
  Mark = ConstantArray[1, SIZE + 1], L, TotalL = Length[M],
  Tree, i, j, k, Pos = 1, MaxTrans = ConstantArray[0, SIZE],
  Occupy = ConstantArray[0, SIZE], Index, Modification, TempL,
  RowCompl = ConstantArray[0, SIZE], ArrangedM, Inv},
  ArrangedM = SortBy[M, #[[3]] &];
  For[i = 1, i ≤ SIZE && Pos ≤ TotalL, ,
    If[ArrangedM[[Pos, 3]] == i, Pos++, i++; Mark[[i]] = Pos];];
  Mark[[SIZE + 1]] = Pos;
  L = Table[Mark[[i + 1]] - Mark[[i]], {i, SIZE}];
  Tree = Table[ArrangedM[[Mark[[i]] ;; Mark[[i + 1]] - 1, 2]], {i, SIZE}];
  Pos = 0;
  For[i = 1, i ≤ SIZE, i++,
    For[j = 1, j ≤ L[[i]], j++;]
Index = Tree[[i, j]]; If[Occupy[[Index]] == 0, MaxTrans[[i]] = Index; Occupy[[Index]] = 1; Break[]]; If[j == L[[i]] + 1, Pos++; RowCompl[[Pos]] = i];
While[Pos \[NotEqual] 0, Modification = MTA[Tree, Occupy, MaxTrans, RowCompl, Pos, SIZE]; TempL = Length[Modification]; For[i = 1, i \[LessEqual] TempL, i++, MaxTrans[[Modification[[i, 1]]]] = Modification[[i, 2]];]; Occupy[[Modification[[TempL, 2]]]] = 1; Index = Modification[[1, 1]]; For[i = 1, i \[LessEqual] Pos, i++, If[RowCompl[[i]] == Index, Break[]];]; For[j = i, j < Pos, j++, RowCompl[[j]] = RowCompl[[j + 1]]]; Pos--;]; For[i = 1, i \[LessEqual] SIZE, i++, P[[MaxTrans[[i]]]] = i]; For[i = 1, i \[LessEqual] SIZE, i++, For[j = i, j \[LessEqual] L[[i]], j++, Tree[[i, j]] = P[[Tree[[i, j]]]]];]; For[i = 1, i \[LessEqual] SIZE, i++, RowCompl = ConstantArray[0, L[[i]] - 1]; For[j = 1, j \[LessEqual] L[[i]], j++, If[Tree[[i, j]] == i, Break[], RowCompl[[j]] = Tree[[i, j]]];]; For[j++, j \[LessEqual] L[[i]], j++, RowCompl[[j - 1]] = Tree[[i, j]]]; Tree[[i]] = RowCompl;]; {Q, Occupy} = BTA[Tree, SIZE]; For[i = 1, i \[LessEqual] SIZE, i++, PP[[i]] = Q[[P[[i]]]]]; For[i = 1, i \[LessEqual] TotalL, i++, For[j = 1, j \[LessEqual] L[[i]], j++, Tree[[i, j]] = PP[[Tree[[i, j]]]]];]; For[i = 1, i \[LessEqual] TotalL, i++, For[j = i, j \[LessEqual] L[[i]], j++, If[Tree[[i, j]] == i, Break[], Tree[[i, j]] = Tree[[i, j]]];]; For[j++, j \[LessEqual] L[[i]], j++, Tree[[i, j]] = Tree[[i, j]]]; Tree[[i]] = Tree[[i]]]; TempL = Length[Occupy]; Pos = 1; Modification = Flatten[List[{0}, Occupy], 1]; For[i = 2, i \[LessEqual] TempL + 1, i++, Modification[[i]] = Modification[[i]] + Modification[[i - 1]]]; P = ConstantArray[0, {SIZE, SIZE}]; For[i = 1, i \[LessEqual] TotalL, i++, P[[ArrangedM[[i, 2]], ArrangedM[[i, 3]]]] = ArrangedM[[i, 1]]]; For[i = 1, i \[LessEqual] TempL, i++, P[[Modification[[i]] + 1 ;; Modification[[i + 1]], Modification[[i]] + 1 ;; Modification[[i + 1]]]] = Simplify[PolynomialMod[Simplify[Inverse[P[[Modification[[i]] + 1 ;; Modification[[i + 1]], Modification[[i]] + 1 ;; Modification[[i + 1]]]]]]];
Modification[[i]] + 1 ;; Modification[[i + 1]]]} /. LthRoot] /. LthRoot, Sum[η^n, {n, 0, ℓ - 1}]];]

For[i = 1, i < TempL, i++,
    For[j = i + 1, j < TempL, j++,
        P[[Modification[[i]] + 1 ;; Modification[[i + 1]],
            Modification[[j]] + 1 ;; Modification[[j + 1]]]] =
            Simplify[PolynomialMod[Simplify[
                -P[[Modification[[i]] + 1 ;; Modification[[i + 1]],
                    Modification[[i]] + 1 ;; Modification[[i + 1]]]],
                    P[[Modification[[j]] + 1 ;; Modification[[j + 1]]]],
                    Modification[[j]] + 1 ;; Modification[[j + 1]]]]/. LthRoot] /. LthRoot, Sum[η^n, {n, 0, ℓ - 1}]];];

For[i = TempL - 1, i > 1, i--,
    For[j = TempL, j > i, j--,
        P[[Modification[[i]] + 1 ;; Modification[[i + 1]],
            Modification[[j]] + 1 ;; Modification[[j + 1]]]] =
            Simplify[PolynomialMod[Simplify[
                P[[Modification[[i]] + 1 ;; Modification[[i + 1]],
                    Modification[[j]] + 1 ;; Modification[[j + 1]]]],
                    P[[Modification[[j]] + 1 ;; Modification[[j + 1]]]],
                    Modification[[j]] + 1 ;; Modification[[j + 1]]]]/. LthRoot] /. LthRoot, Sum[η^n, {n, 0, ℓ - 1}]];];

For[k = 1, k < j - i, k++,
        P[[Modification[[i]] + 1 ;; Modification[[i + 1]],
            Modification[[j]] + 1 ;; Modification[[j + 1]]]] =
            Simplify[PolynomialMod[Simplify[
                P[[Modification[[i]] + 1 ;; Modification[[i + 1]],
                    Modification[[j]] + 1 ;; Modification[[j + 1]]]],
                    P[[Modification[[j-k]] + 1 ;; Modification[[j-k+1]]]],
                    Modification[[j]] + 1 ;; Modification[[j+1]]]]/. LthRoot] /. LthRoot, Sum[η^n, {n, 0, ℓ - 1}]];];

TempL = 0; Inv = ConstantArray[0, {SIZE^2, 3}];

For[i = 1, i < SIZE, i++,
    For[j = 1, j < SIZE, j++,
        If[TrueQ[P[[i, j]] == 0], ,
        TempL++;
        Inv[[TempL, 1]] = P[[i, j]];
        Inv[[TempL, 2]] = i;
        Inv[[TempL, 3]] = j;];];

Inv = Inv[[1 ;; TempL]];
For[i = 1, i ≤ SIZE, i++, InvP[[PP[[i]]]] = i];
For[i = 1, i ≤ SIZE, i++, InvQ[[Q[[i]]]] = i];
For[i = 1, i ≤ TempL, i++,
  Inv[i, 2] = InvQ[[Inv[i, 2]]];
  Inv[i, 3] = InvP[[Inv[i, 3]]];]; Inv;

R12[M_] := Module[{L = Length[M], R12, i, j, JL, Jl},
  R12 = ConstantArray[0, L*\ell^2, 3];
  For[j = 0, j < \ell^2, j++,
    JL = j*L; Jl = j*\ell^4;
    For[i = 1, i ≤ L, i++,
      R12[[JL + i, 1]] = M[[i, 1]];  
      R12[[JL + i, 2]] = M[[i, 2]] + Jl;
      R12[[JL + i, 3]] = M[[i, 3]] + Jl];]; R12];

R23[M_] := Module[{L = Length[M], R23, i, j, Dig1, Dig2, R, C, Index},
  R23 = ConstantArray[0, L*\ell^2, 3];
  For[i = 1, i ≤ L, i++,
    Dig1 = IntegerDigits[M[[i, 2]] - 1, \ell^2, 2];
    Dig2 = IntegerDigits[M[[i, 3]] - 1, \ell^2, 2];
    R = Dig1[[2]]*\ell^2 + Dig1[[1]]*\ell^4;
    C = Dig2[[2]]*\ell^2 + Dig2[[1]]*\ell^4; Index = (i - 1)*\ell^2;
    For[j = 1, j ≤ \ell^2, j++,
      R23[[Index + j, 1]] = M[[i, 1]];  
      R23[[Index + j, 2]] = R + j;
      R23[[Index + j, 3]] = C + j*\ell^4];]; R23];

R13[M_] := Module[{L = Length[M], R13, i, j, Dig1, Dig2, R, C, Index},
  R13 = ConstantArray[0, L*\ell^2, 3];
  For[i = 1, i ≤ L, i++,
    Dig1 = IntegerDigits[M[[i, 2]] - 1, \ell^2, 2];
    Dig2 = IntegerDigits[M[[i, 3]] - 1, \ell^2, 2];
    R = 1 + Dig1[[2]] + Dig1[[1]]*\ell^4;
    C = 1 + Dig2[[2]] + Dig2[[1]]*\ell^4; Index = (i - 1)*\ell^2;
    For[j = 0, j < \ell^2, j++,
      R13[[Index + j + 1, 1]] = M[[i, 1]];  
      R13[[Index + j + 1, 2]] = R + j*\ell^2;
      R13[[Index + j + 1, 3]] = C + j*\ell^4];]; R13];

P[M_] := Module[{L = Length[M], P, i, Dig},
  P = ConstantArray[0, {L, 3}];
  For[i = 1, i ≤ L, i++,
    ...]
P[i, 1] = M[i, 1]; P[i, 3] = M[i, 3];
Dig = IntegerDigits[M[i, 2] - 1, \(\ell^2\), 2];
P[i, 2] = 1 + Dig[1] + Dig[2]*\(\ell^2\);]

RP[M_] := Module[{L = Length[M], P, i, Dig},
P = ConstantArray[0, {L, 3}];
For[i = 1, i \leq L, i++,
P[i, 1] = M[i, 1]; P[i, 2] = M[i, 2];
Dig = IntegerDigits[M[i, 3] - 1, \(\ell^2\), 2];
P[i, 3] = 1 + Dig[1] + Dig[2]*\(\ell^2\);]
]

T1[M_] := Module[{L = Length[M], T1, i, Dig1, Dig2},
T1 = ConstantArray[0, {L, 3}];
For[i = 1, i \leq L, i++,
T1[i, 1] = M[i, 1];
Dig1 = IntegerDigits[M[i, 2] - 1, \(\ell^2\), 2];
Dig2 = IntegerDigits[M[i, 3] - 1, \(\ell^2\), 2];
T1[i, 2] = 1 + Dig2[2] + Dig1[1]*\(\ell^2\);
T1[i, 3] = 1 + Dig1[2] + Dig2[1]*\(\ell^2\);]
]

T2[M_] := Module[{T2, L = Length[M], i, Dig1, Dig2},
T2 = ConstantArray[0, {L, 3}];
For[i = 1, i \leq L, i++,
T2[i, 1] = M[i, 1];
Dig1 = IntegerDigits[M[i, 2] - 1, \(\ell^2\), 2];
Dig2 = IntegerDigits[M[i, 3] - 1, \(\ell^2\), 2];
T1[i, 2] = 1 + Dig1[2] + Dig2[1]*\(\ell^2\);
T1[i, 3] = 1 + Dig2[2] + Dig1[1]*\(\ell^2\);]
]

INITIALIZE3[L_] := Module[{INPUT = ConstantArray[1, {L^4, 3}]},
For[i = 0, i < L^2, i++,
For[j = 0, j < L^2, j++,
INPUT[[i*L^2 + j + 1]] = j*L^4 + j*L^2 + i + 1;
INPUT[[i*L^2 + j + 1, 3]] = i + 1;];]
]

EVAL3[OUTV_, L_] := Module[{INVAR = ConstantArray[0, {L^2, L^2}]},
For[i = 1, i \leq L^2, i++,
For[j = 1, j \leq L^2, j++,
For[m = 0, m < L^2, m++,
INVAR[[j, i]] = INVAR[[j, i]] + OUTV[[m*L^4 + m*L^2 + j, i]]];];]
]

INVAR];
(*For \(B_x \otimes B_y\))

\[
\text{Multi}[A_1, A_2] := \text{Module}[\{A, B, C, D, E\}, \\
A = A_1[1] \ast A_2[1] \ast \eta \ast (2 \ast A_1[3] \ast A_2[2] + 2 \ast A_1[5] \ast A_2[4]); \\
B = A_1[2] + A_2[2]; \\
C = A_1[3] + A_2[3]; \\
D = A_1[4] + A_2[4]; \\
E = A_1[5] + A_2[5]; \\
\{A, B, C, D, E\}];
\]

\[
\text{Simple}[C, y, x] := \text{Module}[\{Q, a, b, c, d, e\}, \\
Q = \text{Quotient}[C[2; 5], \ell]; \\
\{b, c, d, e\} = C[2; 5] - \ell \ast Q; \\
\{a, b, c, d, e\}];
\]

\[
\text{Compare}[V_1, V_2] := \text{Module}[\{C = 0, i\}, \\
\text{For}[i = 4, i > 0, i-- , \text{If}[V_1[i] > V_2[i], C = 1; \text{Break[]}]; \\
\text{If}[V_1[i] < V_2[i], C = 2; \text{Break[]}]; C];
\]

\[
\text{BinarySearch}[\text{Tree}, \text{Vect}] := \text{Module}[\{\text{BS, BP, LR, P}\}, \\
\text{BS} = 0; \text{P} = 1; \text{LR} = 0; \\
\text{While}[P != 0, LR = \text{Compare}[	ext{Tree}[P, 2 ;; 5], \text{Vect}[2 ;; 5]]; \\
\text{BP} = P; \text{If}[\text{LR} == 0, \text{BS} = 1; \text{Break[]}], \\
\text{P} = \text{Tree}[P, \text{LR} + 5]; \text{BS, BP, LR};]
\]

\[
\text{RealMulti}[M_1, M_2, y, x] := \\
\text{Module}[\{M, \text{BinaryTree, L, BS, BP, LR, Temp, L1, L2, i, j, NONZERO, P}\}, \\
L1 = \text{Length}[M_1]; L2 = \text{Length}[M_2]; \\
\text{BinaryTree} = \text{ConstantArray}[0, \{L1*L2, 7\}]; L = 0; \\
\text{For}[i = 1, i \leq L1, i++, \\
\text{For}[j = 1, j \leq L2, j++, \\
\text{Temp} = \text{Simple}[\text{Multi}[M_1[i], M_2[j]], y, x]; \\
\text{If}[L == 0, L++; \text{BinaryTree}[1, 1 ;; 5] = \text{Temp}, \\
\text{BS, BP, LR}] = \text{BinarySearch}[	ext{BinaryTree}, \text{Temp}]; \\
\text{If}[\text{BS} != 0, \text{BinaryTree}[\text{BP}, 1] = \\
\text{BinaryTree}[\text{BP}, 1] + \text{Temp}[1], \\
L++; \\
\text{BinaryTree}[L, 1 ;; 5] = \text{Temp}; \\
\text{BinaryTree}[\text{BP}, \text{LR} + 5] = \text{L};];]; \\
\text{BinaryTree}[1 ;; L, 1] = \text{Simplify}[\text{PolynomialMod}[ \\
\text{BinaryTree}[1 ;; L, 1]/. \text{LthRoot}, \text{Sum}[\eta^n, \{n, 0, \ell - 1\}]]]; \\
\text{NONZERO} = \text{ConstantArray}[0, \text{L}]; \text{P} = 0; \\
\text{For}[i = 1, i < L + 1, i++ , \\
\text{If}[\text{TrueQ}[\text{BinaryTree}[i, 1] == 0], P++; \text{NONZERO}[\text{P}] = 1]; \\
M = \text{BinaryTree}[[\text{NONZERO}[1 ;; \text{P}], 1 ;; 5]]; M];
\]
(*For $\chi \in SL_2(\mathbb{C})$*)

\[
\text{Cyclic}[A, B, C] :=
\text{PolynomialMod}\{\text{B}*\text{Product}[C-A*\eta^n, \{n,0,\ell-1\}], A^\ell, B^{\ell-1}\text{Product}[1-C^{\ell-1}A^{\ell-1}*\eta^n, \{n,0,\ell-1\}]\},
\text{Sum}[\eta^n, \{n,0,\ell-1\}];
\]

\[
\text{Braid}[x, y] := \text{Module}\{\{xL, xR, gxP, gxM, gyP, gyM, gL, gLP, gR\},
gxP = \{\{1/x[2], x[3]\}, \{0, 1\}\};
gxM = \{\{1, 0\}, \{-x[1]/x[2], 1/x[2]\}\};
gyP = \{\{1/y[2], y[3]\}, \{0, 1\}\};
gyM = \{\{1, 0\}, \{-y[1]/y[2], 1/y[2]\}\};
gL = \text{gxM}.\text{gyP}.\text{Inverse}[gyM].\text{Inverse}[gxM];
xL = \text{Simplify}[\{gL[2, 1], gL[2, 2], gL[1, 2]/gL[2, 2]\}];
gLP = \{\{1/xL[2], xL[3]\}, \{0, 1\}\};
gR = \text{Inverse}[gL].\text{gxP}.\text{Inverse}[gxM].\text{gL};
xR = \text{Simplify}[\{gR[2, 1], gR[2, 2], gR[1, 2]/gR[2, 2]\}]; \{xL, xR\};
\]

\[
\text{BraidInv}[x, y] := \text{Module}\{\{xL, xR, gxP, gxM, gyP, gyM, gL, gLM, gR\},
gxP = \{\{1/x[2], x[3]\}, \{0, 1\}\};
gxM = \{\{1, 0\}, \{-x[1]/x[2], 1/x[2]\}\};
gyP = \{\{1/y[2], y[3]\}, \{0, 1\}\};
gyM = \{\{1, 0\}, \{-y[1]/y[2], 1/y[2]\}\};
gL = \text{gxM}.\text{gyP}.\text{Inverse}[gyP].\text{gyM}.\text{Inverse}[gxM];
xL = \text{Simplify}[\{gL[2, 1], gL[2, 2], gL[1, 2]/gL[2, 2]\}];
gLM = \{\{1, 0\}, \{-xL[1]/xL[2], 1/xL[2]\}\};
gR = \text{Inverse}[gL].\text{gxP}.\text{Inverse}[gxM].\text{gL};
xR = \text{Simplify}[\{gR[2, 1], gR[2, 2], gR[1, 2]/gR[2, 2]\}]; \{xL, xR\};
\]

\[
\text{BraidT}[x, y] := \text{Module}\{\{xL, xR, gxP, gxM, gyP, gyM, gL, gLM, gR\},
gxP = \{\{1/x[2], x[3]\}, \{0, 1\}\};
gxM = \{\{1, 0\}, \{-x[1]/x[2], 1/x[2]\}\};
gyP = \{\{1/y[2], y[3]\}, \{0, 1\}\};
gyM = \{\{1, 0\}, \{-y[1]/y[2], 1/y[2]\}\};
gL = \text{gxP}.\text{Inverse}[gyP].\text{gyM}.\text{Inverse}[gxM];
xL = \text{Simplify}[\{gL[2, 1], gL[2, 2], gL[1, 2]/gL[2, 2]\}];
gLM = \{\{1, 0\}, \{-xL[1]/xL[2], 1/xL[2]\}\};
gR = \text{gL}.\text{gxP}.\text{Inverse}[gxM].\text{Inverse}[gL];
xR = \text{Simplify}[\{gR[2, 1], gR[2, 2], gR[1, 2]/gR[2, 2]\}]; \{xL, xR\};
\]

\[
\text{BraidTInv}[x, y] := \text{Module}\{\{xL, xR, gxP, gxM, gyP, gyM, gL, gLM, gR\},
gxP = \{\{1/x[2], x[3]\}, \{0, 1\}\};
gxM = \{\{1, 0\}, \{-x[1]/x[2], 1/x[2]\}\};
gyP = \{\{1/y[2], y[3]\}, \{0, 1\}\};
gyM = \{\{1, 0\}, \{-y[1]/y[2], 1/y[2]\}\};
gL = \text{gxP}.\text{Inverse}[gyP].\text{gyM}.\text{Inverse}[gxP];
xL = \text{Simplify}[\{gL[2, 1], gL[2, 2], gL[1, 2]/gL[2, 2]\}];
gLP = \{\{1/xL[2], xL[3]\}, \{0, 1\}\};
gR = \text{gL}.\text{gxP}.\text{Inverse}[gxM].\text{Inverse}[gL];
xR = \text{Simplify}[\{gR[2, 1], gR[2, 2], gR[1, 2]/gR[2, 2]\}]; \{xL, xR\};
\]
LDual[x_] := Module[{gxP, gxM, g, Invx, InvgP, InvgM, LDg, LDx},
gxP = {{1/x[[2]], x[[3]]}, {0, 1}}; gxM = {{1, 0}, {-x[[1]]/x[[2]], 1/x[[2]]}};
g = gxP.Inverse[gxM]; g = Inverse[g];
Invx = Simplify[{g[[2, 1]], g[[2, 2]], g[[1, 2]]/g[[2, 2]]};
InvgP = {{1/Invx[[2]], Invx[[3]]}, {0, 1}};
InvgM = {{1, 0}, {-Invx[[1]]/Invx[[2]], 1/Invx[[2]]}};
LDg = Inverse[InvgP].InvgM;
LDx = Simplify[{LDg[[2, 1]], LDg[[2, 2]], LDg[[1, 2]]/LDg[[2, 2]]};
]

RDual[x_] := Module[{gxP, gxM, RDg, RDx},
gxP = {{1/x[[2]], x[[3]]}, {0, 1}}; gxM = {{1, 0}, {-x[[1]]/x[[2]], 1/x[[2]]}};
RDg = Inverse[gxM].gxP;
RDx = Simplify[{RDg[[2, 1]], RDg[[2, 2]], RDg[[1, 2]]/RDg[[2, 2]]};
]

(*For R*)

OneFm[x_] := 

Key[x_, y_] := RealMulti[OneFm[x], {{η^(-1)*y[[2]]}^(-1), 1, ℓ - 1, 0}];

X[x_, y_] := Flatten[List[{{1, 0, 0, 0, 0}}, Key[x, y].{-1, 0, 0, 0, 0}, {0, 1, 0, 0, 0},
{0, 0, 1, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}}], 1];

InvX[x_, y_] := Module[{InvX, Rec, I1, L},
InvX = Flatten[List[{{1, 0, 0, 0, 0}}, Key[x, y], 1];
Rec = Key[x, y];
For[I1 = 2, I1 < ℓ, I1++, Rec = RealMulti[Rec, Key[x, y], y, x];
InvX = Flatten[List[InvX, Rec, 1]]];

L = Length[InvX];
InvX[[1 ;; L, 1]] = Simplify[InvX[[1 ;; L, 1]]/(1 - x[[2]]*x[[3]]*y[[1]]/y[[2]])];
InvX];

M1000 = {{1, 1, 0, 0, 1}};
M0100[x_, y_] := RealMulti[X[x, y], {{1, 0, 1, 0, 0}}];
M0010[x_, y_] := Flatten[List[{{1, 0, 1, 1, 0}}, {{1, 1, 0, 0, 0}},
RealMulti[{{1, 1, 0, 0, 2}}, InvX[x, y], y, x],
{0, 0, 1, 0, 0}, {0, 0, 0, 1, 0}, {0, 0, 1, 0, 1}, {0, 0, 0, 0, 1}}], 1];
M001[x_, y_] := RealMulti[{{1, 0, 0, 0, 1}}];
RSparseMatrix[C_] := 
Module[{x, y, Ms, PriorR, R, Temp, J1, J2, J3, J4, P, L, f},
  x = C[[1]]; y = C[[2]];
  Ms = List[M1000, M0100[x, y], M0010[x, y], M0001[x, y]];
  PriorR = Flatten[List[ConstantArray[0, {ℓ, 1, 5}],
                         ConstantArray[0, {ℓ^4 - 1, ℓ^4, 5}], 1]];
  For[J1 = 0, J1 < ℓ, J1++, PriorR[[J1 + 1]] = {{1, J1, 0, 0, J1}}];
  L = ConstantArray[1, ℓ^4]; P = ℓ + 1;
  For[J2 = 1, J2 < ℓ, J2++,
    Temp = RealMulti[PriorR[[P - ℓ, 1 ;; L[[P - ℓ]]]], Ms[[2]], y, x];
    L[[P]] = Length[Temp]; PriorR[[P, 1 ;; L[[P]]]] = Temp;
    P = P + ℓ;]
  P = ℓ + 2;
  For[J2 = 1, J2 < ℓ, J2++,
    For[J1 = 1, J1 < ℓ, J1++,
      Temp = RealMulti[Ms[[1]], PriorR[[P - ℓ, 1 ;; L[[P - ℓ]]]], y, x];
      L[[P]] = Length[Temp]; PriorR[[P, 1 ;; L[[P]]]] = Temp;
      P++;]
    P--;]
  For[J3 = 1, J3 < ℓ, J3++,
    Temp = RealMulti[PriorR[[P - ℓ^2, 1 ;; L[[P - ℓ^2]]]], Ms[[3]], y, x];
    L[[P]] = Length[Temp]; PriorR[[P, 1 ;; L[[P]]]] = Temp;
    P = P + ℓ^2;]
  P = ℓ^2 + ℓ + 1;
  For[J3 = 1, J3 < ℓ, J3++,
    For[J2 = 1, J2 < ℓ, J2++,
      Temp = RealMulti[Ms[[2]], PriorR[[P - ℓ, 1 ;; L[[P - ℓ]]]], y, x];
      L[[P]] = Length[Temp]; PriorR[[P, 1 ;; L[[P]]]] = Temp;
      P = P + ℓ;]
    P = ℓ^2 + 2;]
  For[J3 = 1, J3 < ℓ, J3++,
    For[J2 = 0, J2 < ℓ, J2++,
      For[J1 = 1, J1 < ℓ, J1++,
        Temp = RealMulti[Ms[[1]],
                         PriorR[[P - 1, 1 ;; L[[P - 1]]]], y, x];
        L[[P]] = Length[Temp]; PriorR[[P, 1 ;; L[[P]]]] = Temp;
        P++;]
      P++;];
  ];
\( P = \ell^3 + 2; \)
\( P = \ell^3 - \ell^2, J4++; \)
\( \text{Temp} = \text{RealMulti}[[P - \ell^3, 1 ;; L[[P - \ell^3]]], Ms[[4]], y, x]; \)
\( L[[P]] = \text{Length}[	ext{Temp}]; \text{PriorR}[[P, 1 ;; L[[P]]]] = \text{Temp}; \)
\( P = P + \ell; \)
\( P = \ell^3 + 2; \)
\( \text{For}[J4 = 1, J4 < \ell, J4++; \)
\( \text{For}[J3 = 0, J3 < \ell, J3++; \)
\( \text{For}[J2 = 0, J2 < \ell, J2++; \)
\( \text{For}[J1 = 1, J1 < \ell, J1++; \)
\( \text{Temp} = \text{RealMulti}[Ms[[1]], \text{PriorR}[[P - 1, 1 ;; L[[P - 1]]]], y, x]; \)
\( L[[P]] = \text{Length}[	ext{Temp}]; \)
\( \text{PriorR}[[P, 1 ;; L[[P]]]] = \text{Temp}; \)
\( P++; \); \)
\( P++; \); \)
\( P++; \); \)
\( R = \text{Table}[	ext{PriorR}[[i, 1 ;; L[[i]]]], \{i, \ell^4\}; \text{StdSparse}[R]]; \)

(*Useful \( \mathcal{R} \) matrices*)
\( x = \text{Cyclic}[a, b, \mu]; \)
\( \text{LDx} = \text{LDual}[x]; \text{RDx} = \text{RDual}[x]; \)
\( \text{RL} = \text{SparseRemoveZero}[[\text{Simplify}[[\text{RSparseMatrix}[[\text{LDx}, x]]]/.\{\text{Cm} \to \mu + 1/\mu\}]]; \)
\( \text{RR} = \text{SparseRemoveZero}[[\text{Simplify}[[\text{RSparseMatrix}[[x, \text{RDx}]]]/.\{\text{Cm} \to \mu + 1/\mu\}]]; \)
\( \text{InvRL} = \text{SparseInverse}[\text{RL}, \ell^4]; \)
\( \text{InvRR} = \text{SparseInverse}[\text{RR}, \ell^4]; \)
\( \text{LEVL} = \text{RP}[	ext{SparseInverse}[\text{T1}[\text{InvRL}], \ell^4]]; \)
\( \text{REVL} = \text{RP}[	ext{SparseInverse}[\text{T1}[\text{InvRR}], \ell^4]]; \)
\( \text{LCOEVL} = \text{P}[	ext{SparseInverse}[\text{T2}[\text{RL}], \ell^4]]; \)
\( \text{RCOEVL} = \text{P}[	ext{SparseInverse}[\text{T2}[\text{RR}], \ell^4]]; \)

(*Verify Yang-Baxter Equations*)
\( \text{YBL}[[x, y, z]] := \text{Module}[[\{\text{xLxy, xRxy, xLxRxyz, A, B, C}\}, \{\text{xLxy, xRxy}\} = \text{Braid}[x, y]; \)
\( \{\text{xLxRxyz, C}\} = \text{Braid}[xRxy, z]; \)
\( \{A, B\} = \text{Braid}[xLxy, xLxRxyz]; \{A, B, C\}]; \)
\( \text{YBR}[[x, y, z]] := \text{Module}[[\{\text{yLz, xRy, xRxxLyz, A, B, C}\}, \{\text{yLz, xRy}\} = \text{Braid}[y, z]; \)
\( \{A, xRxxLyz\} = \text{Braid}[x, xLyz]; \)
\( \{B, C\} = \text{Braid}[xRxxLyz, xRy]; \{A, B, C\}]; \)
RYBL := Module[{x, y, z, xLxy, xRxy, xLxRxyz, A, B, C, R0, R1, R2, R},
   x = Cyclic[a1, b1, µ1];
   y = Cyclic[a2, b2, µ2];
   z = Cyclic[a3, b3, µ3];
   {xLxy, xRxy} = Braid[x, y];
   {xLxRxyz, C} = Braid[xRxy, z];
   {A, B} = Braid[xLxy, xLxRxyz];
   R0 = Simplify[R12[P[RSparseMatrix[{x, y}]]].{Cm → µ2 + µ2^(-1)}];
   R1 = Simplify[R23[P[RSparseMatrix[{xRxy, z}]]].{Cm → µ3 + µ3^(-1)}];
   R2 = Simplify[R12[P[RSparseMatrix[{xLxy, xLxRxyz}]]].
     {Cm → µ3 + µ3^(-1)}];
   R = SparseMulti[R1, R0];
   R = SparseMulti[R2, R]; R];

RYBR := Module[{x, y, z, xLyz, xRyz, xRxxLyz, A, B, C, R0, R1, R2, R},
   x = Cyclic[a1, b1, µ1];
   y = Cyclic[a2, b2, µ2];
   z = Cyclic[a3, b3, µ3];
   {xLyz, xRyz} = Braid[y, z];
   {xRxxLyz, C} = Braid[x, xLyz];
   {A, B} = Braid[xRxxLyz, xRyz];
   R0 = Simplify[R23[P[RSparseMatrix[{y, z}]]].{Cm → µ3 + µ3^(-1)}];
   R1 = Simplify[R12[P[RSparseMatrix[{x, xLyz}]]].{Cm → µ3 + µ3^(-1)}];
   R2 = Simplify[R23[P[RSparseMatrix[{xRxxLyz, xRyz}]]].
     {Cm → µ2 + µ2^(-1)}];
   R = SparseMulti[R1, R0];
   R = SparseMulti[R2, R]; R];

7.3 Examples Section

(*qdim*)
INPUT = ConstantArray[1, {ℓ^2, 3}];
For[j = 1, j < ℓ^2, j ++, INPUT[[j + 1, 2]] = 1 + j + j * ℓ^2];
OUTPUT = SparseMulti[LCOEVL, INPUT];
OUTV = SparsetoMtx[OUTPUT, ℓ^4, 1];
QDIM = 0;
For[j = 0, j < ℓ^2, j ++, QDIM = Simplify[QDIM + OUTV[[1 + j + j * ℓ^2, 1]]]]; QDIM
(*twist*)
R1 = R12[RIP[InvRR]];
R2 = R23[RCOEVL];
INPUT = INITIALIZE3[ℓ];
OUTPUT = SparseMulti[R1, INPUT];
OUTPUT = SparseMulti[R2, OUTPUT];
OUTV = SparsetoMtx[OUTPUT, ℓ^6, ℓ^2];
TWIST = Simplify[EVAL3[OUTV, ℓ]]

(*trefoilcomm*)
R1 = R12[P[RR]];
R2 = R23[RCOEVL];
INPUT = INITIALIZE3[ℓ];
OUTPUT = Simplify[SparseMulti[R1, INPUT]];
OUTPUT = Simplify[SparseMulti[R1, OUTPUT]];
OUTPUT = Simplify[SparseMulti[R1, OUTPUT]];
OUTPUT = Simplify[SparseMulti[R2, OUTPUT]];
OUTV = SparsetoMtx[OUTPUT, ℓ^6, ℓ^2];
TREFOIL = Simplify[EVAL3[OUTV, ℓ]]

(*trefoilgeom*)

\[
gy = \text{Simplify}\left[\left\{\left\{1 - x[[2]]/\mu^\ell, 1 - x[[2]]*\mu^\ell\right\}, \left\{x[[1]], x[[1]]\right\}\right\}
\right.
\]

\[
\text{Inverse}\left[\left\{\left\{1 - x[[2]]/\mu^\ell, 1 - x[[2]]*\mu^\ell\right\}, \left\{x[[1]], x[[1]]\right\}\right\}\right]
\]

y = Simplify[gy[[2, 1]], gy[[2, 2]], gy[[1, 2]]/gy[[2, 2]]];
ay = y[[2]]^((1/ℓ));
by = Simplify[PolynomialMod[
    y[[1]]/Product[\mu - ay*n, \{n, 0, ℓ - 1\}], Sum[n, \{n, 0, ℓ - 1\}]];
{u, v} = Simplify[Braid[x, y]];
{u2, v2} = Simplify[Braid[u, v]];
{u3, v3} = Simplify[Braid[u2, v2]];
R1 = R12[P[SparseRemoveZero[Simplify[RSparseMatrix[{u, v}] /. \{Cm \to \mu + 1/\mu\}]]]];
R2 = R12[P[SparseRemoveZero[Simplify[RSparseMatrix[{u2, v2}] /. \{Cm \to \mu + 1/\mu\}]]]];
R3 = R12[P[SparseRemoveZero[Simplify[RSparseMatrix[{x, y}] /. \{Cm \to \mu + 1/\mu\}]]]];
R4 = R23[SparseRemoveZero[Simplify[LCOEVL /. \{a \to ay, b \to by\}]]];
INPUT = INITIALIZE3[ℓ];
OUTPUT = SparseMulti[R1, INPUT];
OUTPUT = SparseMulti[R2, OUTPUT];
OUTPUT = SparseMulti[R3, OUTPUT];
OUTPUT = SparseMulti[R4, OUTPUT];
OUTV = SparsetoMtx[OUTPUT, ℓ^6, ℓ^2];
TREFOIL = Simplify[EVAL3[OUTV, ℓ]]
\(*\text{figureeightcomm}\*)
R1 = R12[R][P[InvRR]];  
R2 = R23[R][COEVL];  
R3 = R12[LEVL];  
R4 = R23[P[RL]];  
INPUT = INITIALIZE3[\ell];  
OUTPUT = SparseMulti[R1, INPUT];  
OUTPUT = SparseMulti[R2, OUTPUT];  
OUTPUT = SparseMulti[R3, OUTPUT];  
OUTPUT = SparseMulti[R4, OUTPUT];  
OUTV = SparsetoMtx[OUTPUT, \ell_6, \ell_2];  
FIGUREEIGHT = Simplify[EVAL3[OUTV, \ell]]

\(*\text{figureeightgeom}\*)
GP[x_] := (x^4 + x^2 + 1 + \sqrt{x^8 - 2 x^6 - x^4 - 2 x^2 + 1})/(2 x^*(x^2 - 1));  
HP[x_] := (-1 + 3 x^2 - x^4 - \sqrt{x^8 - 2 x^6 - x^4 - 2 x^2 + 1})/(2 (x^2 - 1)^2);  
gy = Simplify[{{1 - x[2][\mu], 1 - x[2][\mu]}, \{x[1][\mu], x[1][\mu]\}}];  
{GP[\mu][\tau], \{HP[\mu][\tau], \mu[\tau] + \mu[\tau] - GP[\mu][\tau]\}}];  
y = Simplify[gy[2, 1], gy[2, 2], gy[1, 2]/gy[2, 2]];  
{u1, v1} = Simplify[BraidInv[x, y]];  
{v2, u2} = Simplify[BraidT[v1, y]];  
{u2, v3} = Simplify[BraidTInv[u1, v2]];  
R1 = R12[SparseInverse[P[SparseRemoveZero][Simplify[RSparseMatrix[{x, y}]/.\{\text{Cm} \rightarrow \mu + 1/\mu\}]], \ell^4]];  
R2 = R23[P[SparseInverse[T2[SparseRemoveZero][Simplify[RSparseMatrix[{v2, v1}]/.\{\text{Cm} \rightarrow \mu + 1/\mu\}], \ell^4]]];  
R3 = R12[SparseInverse[P[T1[SparseInverse[SparseRemoveZero][Simplify[RSparseMatrix[{v3, v2}]/.\{\text{Cm} \rightarrow \mu + 1/\mu\}], \ell^4]], \ell^4]];  
R4 = R23[P[SparseRemoveZero][Simplify[RSparseMatrix[{u2, x}]/.\{\text{Cm} \rightarrow \mu + 1/\mu\}]], \ell^4]];  
INPUT = INITIALIZE3[\ell];  
OUTPUT = SparseMulti[R1, INPUT];  
OUTPUT = SparseMulti[R2, OUTPUT];  
OUTPUT = SparseMulti[R3, OUTPUT];  
OUTPUT = SparseMulti[R4, OUTPUT];  
OUTV = SparsetoMtx[OUTPUT, \ell^6, \ell^2];  
FIGUREEIGHT = Simplify[EVAL3[OUTV, \ell]]

\(*\text{HopfSpecial}\*)
R1 = R12[P[RR]];  
R2 = R23[R][COEVL];
INPUT = INITIALIZE3[\ell];
OUTPUT = SparseMulti[R1, INPUT];
OUTPUT = SparseMulti[R1, OUTPUT];
OUTPUT = SparseMulti[R2, OUTPUT];
OUTV = SparsetoMtx[OUTPUT, \ell^6, \ell^2];
HOPF = Simplify[EVAL3[OUTV, \ell]]

(*Hopf*)
gy = Simplify[\{1 - x[[2]]/\mu, 1 - x[[2]]*\mu\}, \{x[[1]], x[[1]]\}];
   Inverse[\{1 - x[[2]]/\mu, 1 - x[[2]]*\mu\}, \{x[[1]], x[[1]]\}];
y = Simplify[\{gy[[2, 1]], gy[[2, 2]], gy[[1, 2]]/gy[[2, 2]]\}];
ay = y[[2]]^\ell/(1/\ell);
by = Simplify[PolynomialMod[
   y[[1]]/Product[\mu - ay*\eta^n, \{n, 0, \ell - 1\}], Sum[\eta^n, \{n, 0, \ell - 1\}];
   \{u, v\}] = Simplify[Braid[Newx, y]];]
R1 = R12[P[Simplify[RSparseMatrix[\{u, v\} /. \{Cm \rightarrow \mu + 1/\mu\}]]];
R2 = R12[P[Simplify[RSparseMatrix[\{x, y\} /. \{Cm \rightarrow \mu + 1/\mu\}]]];
R3 = R23[Simplify[LCOEVL /. \{a \rightarrow ay, b \rightarrow by, \mu \rightarrow \nu\}];
INPUT = INITIALIZE3[\ell];
OUTPUT = SparseMulti[R1, INPUT];
OUTPUT = SparseMulti[R2, OUTPUT];
OUTPUT = SparseMulti[R3, OUTPUT];
OUTV = SparsetoMtx[OUTPUT, \ell^6, \ell^2];
HOPF = Simplify[EVAL3[OUTV, \ell]]

(*HopfHyper*)
Branch = \{\mu \rightarrow 1\}; (*Could replace 1 by \epsilon^{2\pi mi/\ell} for other branches*)
x = x /. Branch;
gy = Simplify[\{1 - x[[2]], 1\}, \{x[[1]], x[[1]]\}];
   Inverse[\{1 - x[[2]], 1\}, \{x[[1]], x[[1]]\}];
y = Simplify[\{gy[[2, 1]], gy[[2, 2]], gy[[1, 2]]/gy[[2, 2]]\}];
ay = y[[2]]^\ell/(1/\ell);
by = Simplify[PolynomialMod[
   y[[1]]/Product[\mu - ay*\eta^n, \{n, 0, \ell - 1\}], Sum[\eta^n, \{n, 0, \ell - 1\}];
   \{u, v\}] = Braid[x,y, \{u, v\}] /. Branch];
R1 = R12[P[RSparseMatrix[\{u, v\} /. Branch]];];
R2 = R12[P[RSparseMatrix[\{x, y\} /. Branch]];];
R3 = R23[Simplify[LCOEVL /. \{a \rightarrow ay, b \rightarrow by\} /. Branch]];}
INPUT = INITIALIZE3[\ell];
OUTPUT = SparseMulti[R3, SparseMulti[R2, SparseMulti[R1, INPUT]]];
OUTV = SparsetoMtx[OUTPUT, \ell^6, \ell^2];
HOPF = Simplify[EVAL3[OUTV, \ell]]

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References


